Minimal homogeneous submanifolds in euclidean spaces

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Abstract

We prove that minimal (extrinsically) homogeneous submanifolds of the euclidean space are totally geodesic. As an application, we obtain that a complex homogeneous submanifold of $\mathbb{C}^N$ must be totally geodesic.

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1 Introduction

The theory of minimal immersions into spheres is very well developed [L], [C2], [S], [DW]. There is a beautiful method, using eigenfunctions of the Laplacian, for constructing minimal equivariant immersions of compact homogeneous spaces into spheres [T], [W]. In particular, Hsiang [H] has constructed orbits of subgroups of isometries of the sphere which are minimal

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(see also [H-L]). In this paper we consider the analogous problem for the euclidean space.

A (extrinsically) homogeneous submanifold of the euclidean space is a submanifold which is an orbit of a Lie subgroup of isometries of the euclidean space. The following theorem shows that in the euclidean spaces there are only trivial minimal homogeneous submanifolds.

Theorem 1.1 *A (extrinsically) homogeneous minimal submanifold of the euclidean space must be totally geodesic.*

We remark that the homogeneity hypothesis cannot be weakened, since there exist minimal submanifolds of the euclidean space with cohomogeneity 1 and they are not totally geodesic. For instance, we can take a minimal surface of revolution, or the complex submanifold of $\mathbb{C}^2$ defined by the equation $z^2 + w^2 = 1$. We also remark that in the case that the submanifold is (extrinsically) symmetric (i.e. has parallel second fundamental form) the result is due to D. Ferus [F, Lemma 4].

It is a well known result that a complex immersed submanifold of $\mathbb{C}^N$ is minimal [S, Th. 3.1.2], [KN, pp. 380]. On the other hand, Calabi [C1] has shown that complex isometric immersions are rigid. A simple consequence of these two facts is the following corollary, which was in fact the starting question of this paper [D].

Corollary 1.2 *A complex isometric immersion from a complex homogeneous space into $\mathbb{C}^N$ must be totally geodesic.*

In other words, such an isometric immersion can not exist unless the immersed manifold is an affine space. A special case of this corollary, for symmetric bounded domains, is contained in [B, Th. 13].

As another application of our theorem we obtain the following improvement of the corollary in [O2, pp. 2928] (see also [O1])

Corollary 1.3 *Let M^n $(n \geq 2)$ be a homogeneous irreducible submanifold of the euclidean space with parallel mean curvature vector H. Then $H \neq 0$ and M is contained in a sphere, where it is either minimal or it is an orbit of the isotropy representation of a simple symmetric space.*

It is interesting to note that our result plays an important role in the proof of the same result in the hyperbolic space (i.e. a minimal homogeneous submanifold of the hyperbolic space must be totally geodesic, see [DO]). On the other hand, there exist nontrivial homogeneous minimal hypersurfaces in complex hyperbolic spaces or in more general symmetric spaces of negative curvature see [Be].

2 Homogeneous submanifolds of the euclidean space

We say that an orbit $G.v$ of $\mathbb{R}^N$ is *reducible* if $G.v = M_1 \times M_2$ (Riemannian product) where M_1, M_2 are nontrivial factors and $i = i_1 \times i_2$ where i is the natural inclusion of $G.v$ in $\mathbb{R}^N$ and $i_1 : M_1 \to \mathbb{R}^{N_1}$, $i_2 : M_2 \to \mathbb{R}^{N_2}$ are isometric immersions and $N = N_1 + N_2$. If $G.v$ is a reducible submanifold, then each factor is also a homogeneous submanifold of the corresponding euclidean space.

We need the following stronger version of the theorem in [O2, appendix] (see also [V]). Roughly speaking, it says that (non compact) homogeneous submanifolds of the euclidean space are generalized helicoids.

Theorem 2.1 *Let $M = G.v$ be a homogeneous irreducible submanifold of $\mathbb{R}^N$, where G is a Lie subgroup of the isometry group $I(\mathbb{R}^N)$ of $\mathbb{R}^N$. Then, the universal cover $\tilde{G}$ of G splits as $K \times \mathbb{R}^k$, where K is a compact simply connected Lie group. Moreover, the representation ρ of $K \times \mathbb{R}^k$ into $I(\mathbb{R}^N)$ is equivalent to $\rho_1 \oplus \rho_2$, where ρ_1 is a representation of $K \times \mathbb{R}^k$ into $SO(\mathbb{R}^d)$ and ρ_2 is a linear map of $\mathbb{R}^k$ into $\mathbb{R}^e$, (N=d+e), regarding $\mathbb{R}^e$ as its group of translations.*

Proof. By the theorem in [O2, Appendix], we just need to show that any representation $\rho : \mathbb{R}^k \to I(\mathbb{R}^N)$ is equivalent to a direct sum $\rho_1 \oplus \rho_2$, where ρ_1 is a representation of $\mathbb{R}^k$ into $SO(\mathbb{R}^d)$ and ρ_2 is a linear map of $\mathbb{R}^k$ into $\mathbb{R}^e$ ($N = d + e$). The Lie algebra $\mathcal{L}(I(\mathbb{R}^N))$ is the semidirect product $\mathcal{L}(SO(N)) \ltimes \mathbb{R}^N$, where the bracket is defined by $[(A, v), (B, u)] = ([A, B], A(u) - B(v))$, and the exponential is given by $exp(t.(A, v))(p) = e^{t.A}.(p - c) + c + t.d$ for $d \in \ker(A)$ and $v = d - A(c)$.

We are going to show that there exists a common c for the "rotational" part of the Lie algebra $\mathcal{L}(\rho(\mathbb{R}^N))$. Let $\mathcal{R}$ be the projection of $\mathcal{L}(\rho(\mathbb{R}^N))$

in $\mathcal{L}(SO(\mathbb{R}^N))$. The abelian family $\mathcal{R}$ of skew symmetric endomorphisms can be diagonalized simultaneously in $\mathbb{C}$. Now let λ_i $(i = 1, 2, \ldots, n)$ be the different non zero linear functionals associated to each eigenspace. The set $\mathcal{O} = \{R \in \mathcal{R} : \lambda_i(R) \neq 0 \text{ for all } i\}$ is open and dense. It is not hard to show that there exists a basis $w_1 = (R_1, d_1 - R_1(c_1)), \ldots, w_r = (R_r, d_r - R_r(c_r))$ of $\mathcal{L}(\rho(\mathbb{R}^N))$ such that R_i belongs to $\mathcal{O}$ for all $i = 1, \ldots, r$. This implies that $d_i \in V = ker(R_j) = \bigcap_{j=1,\ldots,r} ker(R_j)$ $(i, j = 1, \ldots, r)$. By the bracket formula we obtain that $R_i(R_j(c_i - c_j)) = 0$ $(i, j = 1, \ldots, r)$ and this implies in turn $c_i = c_j$ for $i, j = 1, \ldots, r$. By fixing the origin at c_1 we deduce that ρ is equivalent to $\rho_1 \oplus \rho_2$, where ρ_1 is a representation of $\mathbb{R}^k$ into $SO(V^\perp)$ and ρ_2 is a linear map of $\mathbb{R}^k$ into V.

Now we can prove our principal result.

Proof of Theorem 1.1. Without loss of generality we may assume that the homogeneous submanifold $G.p$ is irreducible (see [O2, section 1]). By Theorem 2.1 we can choose a basis of $\mathcal{L}(G)$ of the form $(A_1, d_1), \ldots, (A_n, d_n)$ where $d_i \in ker(A_i) = V$ $(i = 1, \ldots, n)$. Moreover, we can choose this basis in such a way that $(A_1, d_1), \ldots, (A_r, d_r)$ belong to the isotropy subalgebra of G at p (and so $d_1, \ldots, d_r = 0$) and $(A_{r+1}, d_{r+1}).p = A_{r+1}(p) + d_{r+1}, \ldots, (A_n, d_n).p = A_n(p) + d_n$ form an orthonormal basis of $T_p(G.p)$. Let us decompose $p = p_1 + p_2$, with $p_1 \in V^\perp$ and $p_2 \in V$. Set $\gamma_i(t) = e^{tA_i}.p + t.d_i$ $i = 1, \ldots, n$. We observe that p_1 is a normal vector to $G.p$ at p. We then claim that p_1 must be zero. In fact, if α is the second fundamental form, then $0 = \sum_{i=1}^r \langle \alpha(\dot\gamma_i, \dot\gamma_i), p_1 \rangle = \sum_{i=1}^r \langle A_i^2.(p_1), p_1 \rangle = \sum_{i=1}^r -\langle A_i(p_1), A_i(p_1) \rangle$. This implies $p_1 = 0$, as $A_i(p_1) = 0$ for all i and $p \in V$, and we conclude that the orbit is totally geodesic.

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