Reducibility of complex submanifolds of the complex euclidean space

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Abstract

Let M be a simply connected complex submanifold of $\mathbb{C}^N$. We prove that M is irreducible, up a totally geodesic factor, if and only if the normal holonomy group acts irreducibly. This is an extrinsic analogue of the well-known De Rham decomposition theorem for a complex manifold. Our result is not valid in the real context, as it is shown by many counter-examples.

1 Introduction

In the last few years the holonomy group of the normal connection turned out to be a very important tool for studying submanifold geometry in Euclidean space, or more generally in Hilbert space [O1], [O2], [O3], [HOT], [HL1]. Although normal holonomy groups are even simpler than the Riemannian ones, the restriction they impose on the geometry of arbitrary submanifolds is, in general, weaker. Thus, it is natural to combine the knowledge of such groups with simple geometric invariants as, for instance, in the case

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of isoparametric submanifolds [T], [PT], or submanifolds with constant principal curvatures [HOT].

The main purpose of this article is to prove the following decomposition theorem.

Theorem 1.1 *A complex isometric full immersion of a simply connected complete Kähler manifold $f : M \to \mathbb{C}^N$ is irreducible, up a totally geodesic factor, if and only if the normal holonomy group acts irreducibly.*

This can be interpreted as an extrinsic analogue of the well-known De Rham decomposition theorem for Kähler manifolds (in both cases local versions also hold). A general decomposition theorem for arbitrary real submanifolds does not hold, since the normal holonomy group is invariant under conformal diffeomorphisms of the ambient space. There are also examples of irreducible orbits of s-representations with reducible normal holonomy groups [HO], and it is an open problem whether there exist other homogeneous examples or not (see [O2, Section 7]). Observe that non totally geodesic complex submanifolds of $\mathbb{C}^N$ can never be (extrinsically) homogeneous, due to the fact that they are minimal [D].

2 Preliminaries and basic facts

Let $f : M \to \mathbb{C}^N$ be a complex (immersed) submanifold. M is said to be *reducible* if it is the product of two isometric immersions, and M is said to be *full* when it is not contained in a proper affine hyperplane, regarding M as a submanifold of the Euclidean space $\mathbb{R}^{2N} = \mathbb{C}^N$.

Let R,$R^{\perp}$,A and α denote the Riemannian curvature tensor, the normal curvature tensor, the shape operator, and the second fundamental form, respectively. They are related by the well-known identities of Gauss, Codazzi and Ricci. Moreover, since f is a complex immersion, the following relations hold [KN, pp.175]: $\alpha(X, JY) = \alpha(JX, Y) = J\alpha(X, Y)$. Equivalently

$$A_\xi J = -JA_\xi = -A_{J\xi}. \tag{1}$$

Let $\nu(M)$ be the normal bundle of M, endowed with the normal connection $\nabla^{\perp}$. Observe that the normal bundle of M, as well as the tangent bundle, have a complex structure J which is invariant under parallel transport. Thus, the respective holonomy groups act by complex orthogonal endomorphisms. Let us decompose $\nu(M) = \nu_0(M) \oplus \nu_s(M)$, where $\nu_0(M)$ is the maximal $\nabla^\perp$-parallel subbundle of $\nu(M)$ which is flat and $\nu_s(M)$ is the orthogonal complement. Namely, $\nu_0(M)_p$ consists of those vectors of the normal space at p which are fixed by the restricted normal holonomy group ${}^\perp\Phi_p^*$(see [O2]). The following lemma generalizes the well-known fact that there exist no nontrivial complex hypersurfaces of $\mathbb{C}^N$ with flat normal bundle(see [SN, Theorem 7]).

Lemma 2.1 *Let M be a complex submanifold of $\mathbb{C}^N$. Then M is full if and only if $\nu_0(M)$ is trivial.*

Proof. It is clear that if $\nu_0(M)$ is trivial then M is full. Conversely, let $\xi \neq 0$ be a $\nabla^\perp$-parallel local section of $\nu_0(M)$. Then, $\langle R^\perp(X,Y)\xi, J\xi\rangle = 0$ and so, by the Ricci identity and equation (1) we have that $[A_\xi, A_{J\xi}] = -J2A_\xi^2 = 0$, which implies $A_\xi = 0$. Then, by [E] M is not full.

Remark 2.2 *Observe that the complex structure J of the normal bundle lies in the normal holonomy group. In fact, normal holonomy groups act as s-representations [O3] and for such representations this is a well-known fact (see [H, pp.375 Theorem 4.5]). Namely, in a Hermitian symmetric space the complex structure J at a given point, preserves the curvature tensor and so it is given as the differential of some isometry which fixes that point (see also [O2, Lemma 5.2]).*

Assume that $f : M \to \mathbb{C}^N$ is full, hence, by Lemma 2.1 $\nu(M) = \nu_s(M)$. If the normal holonomy group of M does not act irreducibly on the normal space, then the normal bundle $\nu(M)$ decomposes orthogonally as $\nu(M) = \nu_1(M) \oplus \nu_2(M)$, where $\nu_1(M)$ and $\nu_2(M)$ are $\nabla^\perp$-parallel nontrivial subbundles.

The following lemma will be crucial for our purposes.

Lemma 2.3 *Let ξ_1,ξ_2 be local sections of $\nu_1(M)$ and $\nu_2(M)$ respectively. Then $A_{\xi_1}.A_{\xi_2} = A_{\xi_2}.A_{\xi_1} = 0$.*

Proof. We obtain, from the Ricci identity and the fact that both $\nu_1(M)$ and $\nu_2(M)$ are parallel, that $0 = J[A_{\xi_1}, A_{J\xi_2}] + [A_{\xi_1}, A_{\xi_2}] = 2A_{\xi_1}.A_{\xi_2}$. The last equality follows from equation (1).

We will also need the following complex versions of Moore lemma [M].

Lemma 2.4 (Local complex version of Moore lemma) *A complex submanifold of $\mathbb{C}^N$ is locally reducible at p if and only if there exists a complex parallel distribution $\mathcal{H}$ in a neighborhood of p such that $\alpha(\mathcal{H}, \mathcal{H}^\perp) = 0$.*

Proof. The essential part in Moore's proof is that the affine spaces generated by the leaves of $\mathcal{H}$ (as well as those of $\mathcal{H}^\perp$) are all parallel and orthogonal to the leaves of $\mathcal{H}^\perp$. In our case, these affine spaces are complex, since the leaves are complex submanifolds of $\mathbb{C}^N$. Taking into account this observation, the proof follows similarly to that of Moore for the real case.

Lemma 2.5 (Global complex version of Moore lemma) *Suppose that M_1,M_2 are connected complex riemannian manifolds and that*

$$f : M_1 \times M_2 \to \mathbb{C}^N$$

is an isometric complex immersion of the riemannian product. If the second fundamental form verifies

$$\alpha(TM_1, TM_2) = 0$$

then

$$f = f_1 \times f_2 : M_1 \times M_2 \to \mathbb{C}^{N_1} \times \mathbb{C}^{N_2} = \mathbb{C}^N$$

Proof. This follows from [M] as in the proof of Lemma 2.4.

3 Invariant autoparallel distributions

Let us assume, as at the end of the previous section, that $f : M \to \mathbb{C}^N$ is a full isometric immersion, and let us decompose $\nu(M) = \nu_1(M) \oplus \nu_2(M)$ into $\nabla^\perp$-parallel subbundles. Let, for $p \in M$,

$$\mathcal{D}_i(p) = \bigcap_{\xi \in \nu_i(M)_p} ker(A_\xi) \qquad i = 1, 2.$$

It is standard to show that both $\mathcal{D}_1$ and $\mathcal{D}_2$ define C^∞-distributions in an open and dense subset U of M (i.e. $\mathcal{D}_1$ and $\mathcal{D}_2$ are linear subbundles of $TU \to U$).

Lemma 3.1 *The distributions $\mathcal{D}_1$, $\mathcal{D}_2$ have the following properties*

(i) $\mathcal{D}_1 + \mathcal{D}_2 = TM$,
(ii) $\mathcal{D}_1$ *and* $\mathcal{D}_2$ *are autoparallel,*
(iii) $\alpha(\mathcal{D}_1, \mathcal{D}_2) = 0$ *and* $\alpha(\mathcal{D}_i, TM) \subset \nu_j(M)$ *for* $i \neq j$,
(iv) $R(\mathcal{D}_1, \mathcal{D}_2) = 0$.

Proof. It is not difficult to show that $T_pM = \sum_{\xi \in \nu_1(M)_p} Im(A_\xi) \oplus \mathcal{D}_1(p)$ and hence $T_pM = \mathcal{D}_1(p) + \mathcal{D}_2(p)$, since $\sum_{\xi \in \nu_1(M)} Im(A_\xi) \subset \mathcal{D}_2$ due to Lemma 2.3. This shows part (i). We now prove part (ii). Let X, Y be tangent vectors fields in U, such that X lies in $\mathcal{D}_i$ and let ξ_i be a section of $\nu_i(M)$. By the Codazzi identity $\nabla_X(A_{\xi_i}Y) = \nabla_Y(A_{\xi_i}X) - A_{\nabla^\perp_Y \xi_i}X - A_{\xi_i}(\nabla_Y X) + A_{\nabla^\perp_X \xi_i}Y - A_{\xi_i}(\nabla_X Y)$ we obtain $\nabla_X(A_{\xi_i}Y) = -A_{\xi_i}(\nabla_Y X) + A_{\nabla^\perp_X \xi_i}Y - A_{\xi_i}(\nabla_X Y)$ which is a linear combination of images of shape operators of sections in $\nu_i(M)$. This shows that $\nabla_X Z \in \mathcal{D}_i^\perp$ if $Z \in \mathcal{D}_i^\perp$ (because $\mathcal{D}_i^\perp = \sum_{\xi \in \nu_i(M)_p} Im(A_\xi)$). This implies that $\mathcal{D}_i$ is autoparallel, since $\langle \mathcal{D}_i, \mathcal{D}_i^\perp \rangle = 0$. We consider next part (iii). If $X_i \in \mathcal{D}_i$ and $\xi_j \in \nu_j(M)$ $(i,j = 1,2)$, then $\langle \alpha(X_1, X_2), \xi_j \rangle = \langle A_{\xi_j} X_1, X_2 \rangle = 0$, which implies the first part of (iii). The second part is similar. Part (iv) is a consequence of part (iii) and the Gauss identity.

Remark 3.2 *Observe that* $\mathcal{D}_1$*and* $\mathcal{D}_2$ *are integrable with totally geodesic leaves which are Kähler manifolds.*

It is easy to see that two complementary (orthogonal) autoparallel distributions must be parallel (see e.g. [O2, pp.624]). This implies that the local holonomy group Φ_p^{loc} has an invariant subspace. Observe that this is not anymore true in general if the distributions do intersect. But, in our situation, we have the additional property that the curvature tensor R_{XY} vanishes if X lies in $\mathcal{D}_1$ and Y lies in $\mathcal{D}_2$, as it follows from Lemma 3.1.

We will need the following general result.

Proposition 3.3 *Let* M *be a Riemannian manifold and let* $\mathcal{T}_1$ *and* $\mathcal{T}_2$ *be autoparallel distributions spanning* TM *which are orthogonal modulo the intersection* $\mathcal{T}_1 \bigcap \mathcal{T}_2$ *(*$\mathcal{T}_1 \neq TM \neq \mathcal{T}_2$*). Assume that the curvature tensor* $R_{XY} = 0$ *if* X *lies in* $\mathcal{T}_1$ *and* Y *lies in* $\mathcal{T}_2$*. Then, for each* $p \in M$ *there exists a nontrivial subspace of* $\mathcal{T}_1(p)$ *which contains* $\mathcal{T}_2^\perp$ *and is invariant under the local holonomy group* Φ_p^{loc}*. In particular* M *is locally reducible at each point.*

Proof. Let $q \in M$ and let $v \in \mathcal{T}_2(q)^\perp$, $v \neq 0$. Then, by hypothesis it follows that $v \in \mathcal{T}_1(q)$. We will show that $\Phi_q^{loc}.v \subset \mathcal{T}_1(q)$, which implies that the linear span of $\Phi_q^{loc}.v$ is an invariant nontrivial subspace of $\mathcal{T}_1(q)$ (Then, by the theorem of De Rham, we get the local reducibility at q). This is equivalent to show that $\Omega.v \subset \mathcal{T}_1(q)$ for some open neighborhood Ω of the identity in Φ_q^{loc}. In fact, for $u \in \mathcal{T}_1^\perp$, the function $g \to \langle u, g.v\rangle$ is analytic on Φ_q^{loc} and vanishes on Ω. Hence it must vanish identically on Φ_q^{loc}, which in term implies the invariance of the linear span of $\Phi_q^{loc}.v$. Let γ be a short (piecewise differentiable) loop through q and let τ_γ be the parallel transport along γ. Let us now consider the integrable distribution $\mathcal{T}_1$ and its orthogonal complement $\mathcal{T}_1^\perp \subset \mathcal{T}_2$. Since $R_{\mathcal{T}_1^\perp, \mathcal{T}_1} = 0$, there exist curves γ_1 tangent to $\mathcal{T}_1$ and γ_2 tangent to $\mathcal{T}_1^\perp$ such that $\tau_\gamma = \tau_{\gamma_1} \circ \tau_{\gamma_2}$. In fact, by Frobenius theorem, we can construct a submersion π from an appropriate neighborhood V of q such that the fibers are (locally) the leaves of $\mathcal{T}_1$. Our assertion follows now from the Lemma in [O1, Appendix], after defining the horizontal distribution to be $\mathcal{T}_1^\perp$. Observe that γ_1 (resp. γ_2) lies in the leaf $L_1(q)$ (resp. $L_2(q)$) of $\mathcal{T}_1$ (resp. $\mathcal{T}_2$) through q. Since $L_2(q)$ is totally geodesic and $v \perp \mathcal{T}_2(q)$ the parallel transport τ_{γ_2} with respect to the Levi-Civita connection in the ambient space coincides with the parallel transport $\tau_{\gamma_2}^\perp$ of the normal connection of $L_2(q)$. Thus, $\tau_{\gamma_2}.v$ belongs to $(\mathcal{T}_2(\gamma_2(q)))^\perp \subset \mathcal{T}_1(\gamma_2(q))$, hence $\tau_{\gamma_1}(\tau_{\gamma_2}.v)$ belongs to $\mathcal{T}_1(q)$ as $L_1(q)$ is totally geodesic. Thus, we have shown that the parallel transport $\tau_\gamma.v$ along short loops belongs to $\mathcal{T}_1(q)$. Since the parallel transport along short loops contains an open neighborhood of the identity of Φ_q^{loc} ([EO, Appendix]) we conclude that $\Phi_q^{loc}.v \subset \mathcal{T}_1(q)$.

Remark 3.4 *A completely different (and extrinsic) approach to the above splitting proposition can be done following some of the ideas in [HL2].*

4 Reducibility of complex submanifolds

We keep the assumptions and notation of the first part of §3. We have, by [E], that $\mathcal{D}_1 \neq TM \neq \mathcal{D}_2$, since we assume that f is full. Let U be the open and dense subset of M where $\mathcal{D}_1$ and $\mathcal{D}_2$ are C^∞-distributions. By Lemma 3.1 and Proposition 3.3, for each $p \in U$ there exists a nontrivial subspace of $\mathcal{D}_1(p)$ invariant with respect to the local holonomy group. Let, for $p \in U \subset M$, $\mathcal{H}(p)$ be the maximal Φ_p^{loc}-invariant subspace of $\mathcal{D}_1(p)$. It is standard to show that $\mathcal{H}$ is a C^∞-distribution near any point in U.

Lemma 4.1 *Under the assumptions in this section, we have*

(i) $\mathcal{H}$ *is a complex distribution,*
(ii) $\alpha(\mathcal{H}, \mathcal{H}^{\perp}) = 0$.

Proof. If $X_p \in \mathcal{H}(p)$ then $\Phi_p^{loc}.J(X_p) = J(\Phi_p^{loc}.X_p)$ because J is parallel. This implies, by the maximality of $\mathcal{H}(p)$, that $\Phi_p^{loc}.J(X_p) \subset \mathcal{H}(p)$. This proves part (i). Part (ii) follows from Lemma 3.1 and the fact that $\mathcal{H}^{\perp} \subset \mathcal{D}_2$ (see Proposition 3.3).

Now we can prove our principal result.

Proof of Theorem 1.1. It is clear that if $f : M \to \mathbb{C}^N$ is a product of non totally geodesic immersions, then $\Phi^{\perp}$ acts reducibly. Let us assume, conversely, that this action is reducible, that is $\nu(M) = \nu_1(M) \oplus \nu_2(M)$ where $\nu_1(M)$ and $\nu_2(M)$ are $\nabla^{\perp}$-parallel subbundles. Define C^{∞}-distributions $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{H}$ in U as in the previous section, where U is the open and dense set where $\mathcal{D}_1$ and $\mathcal{D}_2$ are defined. If $p \in U$, then it follows from Lemma 4.1 and Lemma 2.4 that M is locally reducible at p. Thus, by using the complex De Rham decomposition theorem [KN, pp.171], Lemma 2.5 and a standard argument involving analiticity we obtain the reducibility of the immersion.

Corollary 4.2 *Let $f : M \to \mathbb{C}^N$ be a complex 1-1 isometric immersion of a complete Kähler manifold M. Then f is irreducible, up a totally geodesic factor, if and only if the normal holonomy group acts irreducibly.*

Proof. It is clear that if $f : M \to \mathbb{C}^N$ is a product of non totally geodesic immersions, then $\Phi^{\perp}$ acts reducibly. Assume that the normal holonomy group acts reducibly. Let $\tilde{f} : \tilde{M} \to \mathbb{C}^N$ be the natural isometric immersion from the universal cover of M. Then, by Theorem 1.1 the immersion $\tilde{f}$ is a product and so $\tilde{f}(\tilde{M}) = f(M)$ is a product. Now it is not difficult to show that f is reducible since it is one to one.

Remark 4.3 *It is well-known that if $M_0 \times M_1 \times \ldots \times M_k$ is the De Rham decomposition of a simply connected, complete Kähler manifold M then each M_i is a Kähler manifold in a natural manner and the isometry between M and $M_0 \times M_1 \times \ldots \times M_k$ is holomorphic (see [KN, pp.171]). The same holds for complex isometric immersions. Namely, if a complex full isometric immersion f is a product of real immersions then each irreducible non totally geodesic factor is a complex submanifold, as it follows from Remark 2.2.*

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References

[D] Di Scala, A.J.: *Minimal homogeneous submanifolds*, preprint (1998).

[E] Erbacher, J.: *Reduction of the codimension of an isometric immersion*, J. Differential Geometry **5** (1971) pp. 333-340.

[EO] Eschenburg, J.-H. and Olmos, C.: *Rank and symmetry of Riemannian manifolds*, Comment. Math. Helvetici, **69**, (1994), pp. 483-499.

[HL1] Heintze, E. and Liu, X.: *Homogeneity of infinite dimensional isoparametric submanifolds,* to appear in Annals of Mathematics.

[HL2] Heintze, E. and Liu, X.: *A splitting theorem for isoparametric submanifolds in Hilbert space,* J. Differential Geometry, **45** (1997) pp. 319-335.

[HO] Heintze, E. ; Olmos, C.: *Normal holonomy groups and s-representations,* Indiana University. Math. J. **41** (1992) pp. 869-874.

[HOT] Heintze, E. ; Olmos, C. and Thorbergsson G.: *Submanifolds with constant principal curvatures and normal holonomy groups,* Internat. J. Math. (2) **2** (1991) pp. 167-175.

[H] Helgason S.: *Differential geometry, Lie groups and symmetric spaces*, Academic Press, New York, (1978).

[KN] Kobayashi, S. and Nomizu, K.: *Foundations of differential geometry*, Vol II, Interscience Publishers, (1969).

[M] Moore, J.D.: *Isometric immersions of Riemannian products,* J. Differential Geometry, **5** (1971) pp. 159-168.

[O1] Olmos, C.: *Isoparametric submanifolds and their homogeneous structures,* J. Differential Geometry **38** (1993) pp. 225-234.

[O2] Olmos, C.: *Homogeneous submanifolds of higher rank and parallel mean curvature,* J. Differential Geometry **39** (1994) pp. 605-627.

[O3] Olmos, C.: *The normal holonomy group,* Proc. Amer. Math. Soc., **110** (1990), pp. 813-818.

[PT] Palais, R. and Terng, C.-L.: *Critical point theory and submanifold geometry,* Lectures Notes in Math., vol. 1353, Springer-Verlag, Berlin, 1988.

[SN] Smyth, S. and Nomizu, K.: *Differential geometry of complex hipersurfaces II,* J. Math. Soc. Japan , Vol 20, No. 3, (1968), pp. 498-521.

[T] Terng, C.-L.: *Isoparametric submanifolds and their Coxeter groups,* J. Differential Geometry **21** (1985) pp. 79-107.

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