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Passive elasto-magnetic suspensions: nonlinear models and experimental outcomes

E. Bonisoli, A. Vigliani

Dipartimento di Meccanica - Politecnico di Torino
C.so Duca degli Abruzzi, 24 - 10129 Torino - ITALY
E-mail: *alessandro.vigliani@polito.it*

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Abstract *The paper presents a passive elasto-magnetic suspension based on rare-earth permanent magnets: the dynamical system is described with theoretical and numerical nonlinear models, whose results are validated through experimental comparison. The goal is to minimize the dependence on mass of the natural frequency of a single degree of freedom system. For a system with variable mass, static configuration and dynamical behaviour are compared for classic linear elastic systems, for purely magnetic suspensions and for a combination of the two. In particular the dynamics of the magneto-mechanic interaction is predicted by use of nonlinear and linearised models and experimentally observed through a suitable single degree of freedom test rig.*

1 Introduction

It is well known that in traditional single degree of freedom (sdof) elastic systems the natural frequency is inversely proportional to the square root of mass. For many applications, such as for suspensions, it may be interesting to develop passive systems whose natural frequency is independent of mass. In the automotive field some examples can be found where it is interesting to have a frequency that does not depend on the load (Sternberg, 1976; Bunne and Jable, 1996); the authors suggest the use of air suspensions to achieve this goal. Fujita et al. (1997) propose to use passive permanent magnets in order to improve the dynamic performances of passenger seats. In fact, whereas for a single degree of freedom system with linear elastic springs a relevant decrease of resonance with respect to a mass increment is evinced, a magnetic spring allows to obtain the opposite effect, due to the nonlinear nature of the magnetic repulsive force.

The original contribution of this research lays in the idea of coupling traditional springs with magnets: the aim is to minimize the dependence on mass of the natural frequency of a sdof system with variable mass, thus obtaining an adaptive passive suspension, practically resonance-invariant with respect to mass changes.

The need for passive magnetic forces suitable for the purpose implies the usage of rare-earth permanent magnets, sintered from Samarium-Cobalt or Neodymium-Iron-Boron; nowadays these materials permit to reach the highest residual magnetic induction and hysteresis energy (Coey, 2002) and are applied in passive mechanical field such as viscous-type dissipative elements in magnetic dampers and eddy current brakes (Nagaya et al., 1984), or as nonlinear magneto-elastic springs in magnetic bearings, suspensions and levitation devices (Yonnet, 1978). Permanent magnets allow enhancing the characteristics of reliability, thermal stability and proportionality of viscous damping elements, practically in absence of mechanical friction (Nagaraj, 1988); moreover stiffening and damping properties may be easily modified by varying the air-gap between the magnets (Bonisoli and Vigliani, 2003).

The paper presents a theoretical model that describes the dynamic behaviour of a sdof system equipped with a traditional linear elastic spring coupled to a magnetic spring. Magnetic interactions are estimated by means of non linear empirical formulas derived through a magnetic model based on the analogy of the equivalent currents method (Nagaraj, 1988; Bonisoli and Vigliani, 2003).

The dynamics of magneto-mechanics interaction is analysed by use of nonlinear and linearised models. Thus, static configuration and dynamic behaviour are evaluated for classic linear elastic systems, for purely magnetic suspensions and for a combination of the two. State space and frequency response are investigated respectively for non-zero initial conditions and by applying sweep excitations in order to underline the influence of nonlinearities on the system response. By using the linearised system, the frequency response is computed for different values of geometrical and inertial properties of the system. Then the minimization of the resonance variability of the model is discussed.

Theoretical and numerical results are then verified and compared with the experimental data obtained from a suitable sdof test rig in order to verify the attainment of the desired goal of minimising the natural frequency variation with respect to mass changes.

2 Magneto-elastic adaptive suspension

The magneto-elastic suspension shown in Fig.1 is a sdof system with the elastic element consisting in a traditional linear spring mounted in parallel with a nonlinear magnetic spring. Its dynamic behaviour is described by the following nonlinear equation:

$$m\ddot{x} + k\,(x - l_0) + F_m + m\,g = 0 \tag{1}$$

where x is the position of the suspended mass, k is the linear spring stiffness, l_0 is the spring rest length, $g = 9.81$ m/s^2 is the gravity constant, m is the mass of the system and the magnetic repulsive force F_m is modelled through an empirical formula, having form

$$F_m = -\frac{A}{(x + B)^n}, \tag{2}$$

where parameters A, B and n may be evaluated through magnetic models based on equivalent currents method (Nagaraj, 1988; Bonisoli and Vigliani, 2003) or, alternatively, A, B may be experimentally determined by a least square error approach and n is an integer, set equal to 3 (Bonisoli and Vigliani, 2003).

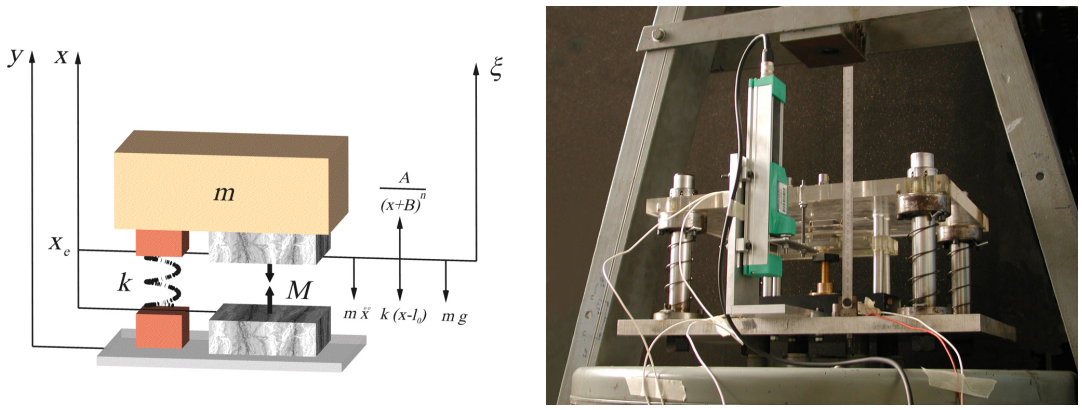

Figure 1: Model (left) and experimental set-up (right) of the elasto-magnetic suspension

The static equilibrium configuration x_e can be easily determined:

$$x_e = -B + \sqrt[n]{\frac{A}{m\,g}(1+\alpha)} = l_0 - \frac{m\,g}{k}\frac{\alpha}{1+\alpha}, \tag{3}$$

where α is the non-dimensional elasto-magnetic ratio between the elastic force F_k and the magnetic force F_m computed at the equilibrium position, i.e.:

$$\alpha = \frac{F_k}{F_m} = -\frac{k\,(x_e - l_0)\,(x_e + B)^n}{A}. \tag{4}$$

Hence, obviously, for $\alpha = +\infty$ the system has only a traditional elastic spring ($A = 0$), while for $\alpha = 0$ only a magnetic spring is present ($k = 0$). Positive values of α correspond to configurations where the elastic spring is compressed (i.e. $x_e < l_0$), while negative values match the conditions in which large magnetic repulsive forces lead the elastic spring to work in extension (i.e. $x_e > l_0$).

3 Nonlinear analysis

Eq.(1) is a nonlinear differential equation that can be studied by means of well-known nonlinear analysis techniques (Guckenheimer and Holmes, 1990; Nayfeh and Mook, 1979). A qualitative analysis highlights that the conservative mechanical system described in eq.(1) has potential energy $F(x)$:

$$F(x) = \int f(x)\,dx = \frac{k}{2}\,(x - l_0)^2 + \frac{1}{n-1}\frac{A}{(x_e + B)^{n-1}} + m\,g\,x + D \tag{5}$$

where D is an integration constant. It is evident that, for all the values of the positive integer n, in the range $x \geq 0$ the potential energy $F(x)$ has a minimum: therefore each level curve $\dot{x}$ vs. x of the phase diagram consists of a single closed trajectory surrounding the centre (corresponding to the minimum of $F(x)$, where the curve degenerates into a singular point). The stability analysis also reveals that the system has only one stable (in the sense of Lyapunov) equilibrium point in the field where physical solutions are acceptable (i.e. $x \geq 0$); in fact, it holds $\left.\frac{df(x)}{dx}\right|_{x=x_e} > 0$.

To test the system dynamic behaviour, let a harmonic oscillation be applied to the basis of the model, i.e. the basis moves with law $y = y_0 \cos(\Omega t)$. Hence, the system dynamic equilibrium, in presence of a dissipative term λ, is described by

$$\ddot{\xi} + \lambda\dot{\xi} + \frac{k}{m}\xi - \frac{A}{m\,(x_e + B)^n}\left[\left(\frac{\xi}{x_e + B} + 1\right)^{-n} - 1\right] = -\ddot{y}, \tag{6}$$

that can be studied with nonlinear analysis techniques. Feeny et al. (2001) analysed a similar magneto-elastic system by means of the harmonic-balance method; instead the authors adopt here a multiple scales nonlinear method (Nayfeh and Mook, 1979). To this aim, eq.(6) can be approximated with:

$$\ddot{\xi} + \lambda\dot{\xi} + a\,\xi + b\,\xi^2 + c\,\xi^3 = \Omega^2 y_0 \cos(\Omega t)\,, \tag{7}$$

where

$$a = \frac{k}{m} + \frac{nA}{m\left(x_e + B\right)^{n+1}}, \quad b = -\frac{n(n+1)A}{2m\left(x_e + B\right)^{n+2}}, \quad c = \frac{n(n+1)(n+2)A}{6m\left(x_e + B\right)^{n+3}} \tag{8}$$

Higher order terms d_i, which obviously depend only on the magnetic effect, can be evaluated through expression:

$$d_i = (-1)^{i-1} \frac{A \prod_{r=0}^{i-1} (n+r)}{i!\, m\left(x_e + B\right)^{n+i}}, \qquad i > 1. \tag{9}$$

The natural frequency of the nonlinear system described by eq.(7) is

$$\omega = \sqrt{a}\left(1 + \frac{9ac - 10b^2}{24a^2}\varepsilon^2\nu^2\right) + o(\varepsilon^3), \tag{10}$$

thus proving its dependence on motion amplitude (parameters ε and ν depend on initial conditions and on motion amplitude). Nonlinear analysis also allows to determine the presence of superharmonic and subharmonic resonances and to investigate jump phenomena. In particular, the conditions for jump to occur are given by the following relation between parameters a, b and c:

$$p = \frac{9ac - 10b^2}{24a^2} = \frac{n(n+1)(n+2)A}{16\left(x_e + B\right)^{n+3}} \frac{k - \dfrac{n(2n-1)A}{3(n+2)\left(x_e + B\right)^{n+1}}}{k + \dfrac{nA}{\left(x_e + B\right)^{n+1}}}. \tag{11}$$

If parameter p is positive, then nonlinearities have a hardening effect; for negative values of p a softening behaviour takes place, while a null value of p corresponds to the condition for which the effects of quadratic and cubic nonlinearities cancel each other. In particular, if only the traditional elastic spring is present ($\alpha \to +\infty$, A = 0), parameter p is thoroughly null, while a purely magnetic system presents a softening behaviour for $n > 1/2$.

For the elasto-magnetic system parameter p does not possess a constant sign, as shown in Fig.2 (plotted in the case of $k = 2098$ N/m for the different values of n found in the literature), hence proving a complex dynamical behaviour. For little values of the air gap, p is negative because $\alpha \cong 0$ and the nonlinear system shows softening characteristics; conversely, for large air gaps, $p \cong 0$, because for $\alpha \to +\infty$ the system magnetic nonlinearities play a very poor role and the system behaviour is almost linear. In the intermediate range, three regions can be identified: a first one where parameter p shows a positive maximum, corresponding to a hardening behaviour; a second region where p is negative (softening system); finally, a third region where the quadratic and cubic terms of nonlinearities approximately annihilate each other ($p \cong 0$), thus leading to a linear behaviour. It is worth noting that while the qualitative nonlinear behaviour of the forces does not depend on the value of n, yet both the value x_e at which the hardening - softening passage takes place and the maximum of parameter p are influenced by the value of n.

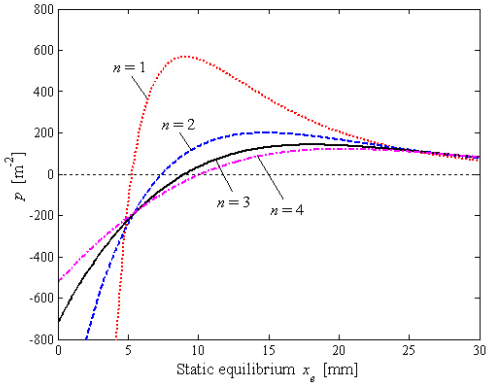

Figure 2: Nonlinear parameter p vs. x_e

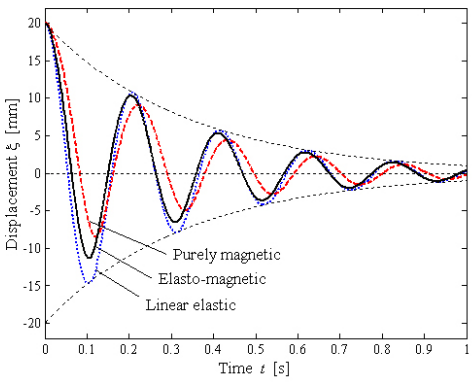

Figure 3: System free response from non-zero initial conditions

Due to the presence of quadratic and cubic nonlinearities, the dynamic system shows superharmonic resonances taking place at twice and three times the first natural frequency. Moreover the quadratic term b is responsible for the drift ξ_d of the centre of the motion, showing that oscillations are not centred in the stable equilibrium point.

Fig.3 shows the system free response from non-zero initial conditions in the three cases under analysis (purely elastic, purely magnetic and a combination of the two), whose parameters are set to possess the same linearised natural frequency $f_n = \omega_n/2\pi = 4.84$ Hz, in presence of a viscous damping, that is set equal to the value experimentally determined. For an initial displacement $\xi = 20$ mm, it is clearly visible the non-symmetric behaviour in the elasto-magnetic and purely magnetic system: this effect decreases with the oscillations amplitude, thus proving that nonlinearities depend on motion amplitude. In the given configuration, the elasto-magnetic system presents a hardening behaviour with natural frequency larger than that shown by the traditional elastic system, while the purely magnetic system has a softening behaviour. According to eq.(10), the analytical model predicts respectively a 6% increase of frequency for the hardening system and a 11% decrease for the softening case.

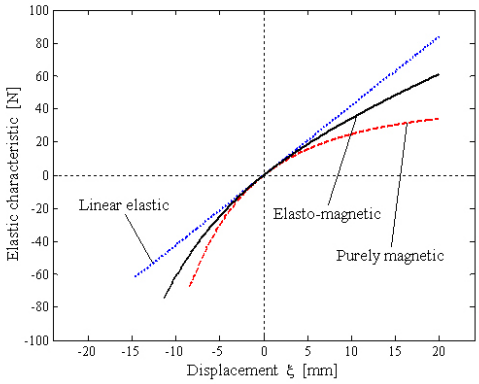

Figure 4: Comparison of elastic characteristics

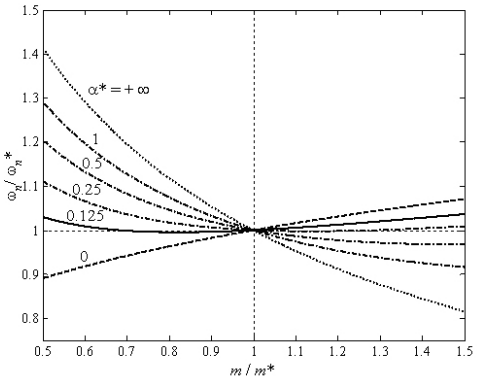

Figure 5: Variability of the natural frequency with mass

Fig.4 shows the elastic characteristics respectively for linear elastic, magneto-elastic and purely magnetic system: it is evident the hypothesis under which the models are compared, i.e. that the system presents the same tangent in the static equilibrium configuration (the natural frequency is the same for all the linearised models). When the magneto-elastic characteristic is approximated through a polynomial expansion, some difficulties arise in describing the softening effects associated with small air gaps. In fact, odd Taylor's expansions (third or fifth order) can describe only locally the softening behaviour, because the curves become hardening for large displacement; on the contrary, even expansions (fourth or, even worse, second order) are stable only locally. For hardening nonlinear systems, however, third and fifth order expansions give satisfactory results in matching the non symmetric behaviour.

4 Linearised analysis

When studying the dynamics of the system in the neighbourhood of the static equilibrium position x_e, equation (1) can be linearised; thus its natural frequency ω_n can be expressed as a function of the characteristics of both magnetic and elastic springs:

$$\omega_n = \sqrt{\frac{k}{m} + \frac{nA}{m\,(x+B)^{n+1}}} = \sqrt{\frac{k}{m} + \frac{nA}{m}\left[\frac{A}{mg}\,(1+\alpha)\right]^{-\frac{n+1}{n}}}\,. \tag{12}$$

It is worth noting that the presence of an elastic spring alone obviously implies $\omega_n \propto 1/\sqrt{m}$, while a system equipped with only a magnetic spring is characterised by an opposite behaviour, its natural frequency being directly proportional to a function of the system mass; in particular, it holds: $\omega_n \propto \sqrt[2n]{m}$, since a change in the system mass is compensated by the strongly nonlinear magnetic force, whose slope increases with smaller air gap between permanent magnets.

Hence, through a suitable setting of the elastic and magnetic springs, it is possible to design an adaptive suspension for which the linearised system natural frequency variability with mass is minimized (solid line in Fig.5). Moreover, the plot in Fig.5 shows that mass changes equal to

$\pm 50\%$ lead to natural frequency variations of $+5\%$, while for a system equipped only with elastic springs the frequency changes would be in the range $-20\% \div +40\%$.

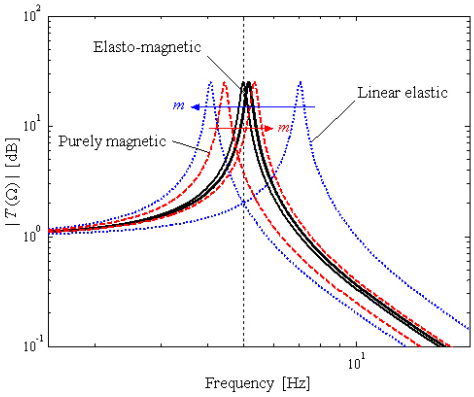

Figure 6: Transmissibility of diffetrent models

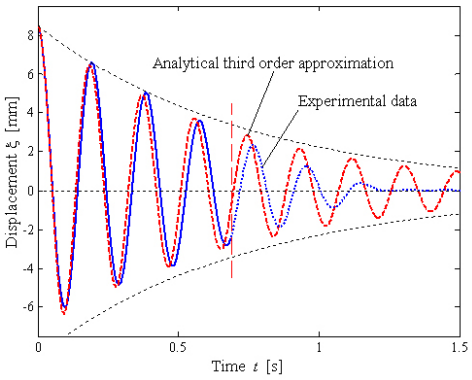

Figure 7: System free response for the elasto-magnetic suspension

Let now the system undergo vibrations caused by a harmonic oscillation $y = y_0 \cos(\Omega t)$; then, the non-dimensional transmissibility of the linearised model is

$$T(\Omega) = \frac{\xi_0}{y_0} = \frac{1 + \jmath\, 2\zeta \dfrac{\Omega}{\omega_n}}{1 - \dfrac{\Omega^2}{\omega_n^2} + \jmath\, 2\zeta \dfrac{\Omega}{\omega_n}}, \tag{13}$$

that is plotted in Fig.6: it is evident that the system with elasto-magnetic springs behaves like a passive adaptive suspension, practically independent on mass changes. The plots refer to the system used for experimental comparison, with design frequency $f_n^\star = 5$ Hz ($m^\star = 8.15$ kg, $A = 1.43 \cdot 10^{-3}$ Nm^3, $B = 19.7$ mm, $n = 3$, $k = 2098$ N/m, $l_0 = 19$ mm, $x_e^\star \approx 10$ mm). Three plots for each kind of suspension (elastic, magnetic and elasto-magnetic) are represented, for different values of the mass, i.e.: 0.5 $m^\star$, $m^\star$, 1.5 $m^\star$ (in the elasto-magnetic case, the plots are evaluated with $\alpha^\star = 0.348$).

5 Experimental comparison

Numerical results are validated by means of an experimental rig (Fig.1) devoted to study the system frequency response and transmissibility. The test bench is a sdof system constituted of three parallel plates: an upper plexiglass plate free to move in a vertical plane along four cylindrical bars and in which permanent magnets can be placed; a lower aluminium plate that can be fixed to a reference plane or to a shaker; an intermediate smaller plexiglass plate used as housing for the opposite permanent magnets and whose vertical position can be adjusted by means of a nut-screw in order to obtain the desired air gap between the magnets. The upper plate can be sustained by four traditional springs, by two (or more) magnets in repulsion, or by a combination of the two systems. The bench also permits to vary the air gap between the magnets, thus allowing to vary the magnetic contribution to the total elastic force. It is worth noting that the materials of the bench are chosen to prevent magnetic interferences to influence the system dynamics: in fact, aluminium plates, that are sensitive to magnetic fields, are positioned at a distance large enough to avoid magnetic interactions. The bench is used to test the dynamic behaviour in the range 2-16 Hz: the signals from two accelerometers (fixed to the lower and upper plates) and from a resistive displacement transducer, measuring the relative displacement between the two free parallel plates, are used to build the transmissibility response.

Different configurations are tested (with one or two magnetic pairs) to underline the effects of nonlinearities on the system dynamic behaviour; moreover also the suspended mass may be varied, adding calibrated weights. Relative velocity $\dot{\xi}$ is determined as the numerical derivative of the relative displacement with suitable filtering of the reconstructed signal. Different tests are driven on the test bench to investigate the dynamical behaviour of the suspension: the system natural characteristics are evaluated by means of the analysis of non-zero initial conditions free response, while the system response to external excitations is studied with frequency sweep oscillations of

the reference plate. The analysis of the experimental data underlines the relevant effect of nonlinearities, in good agreement with the analytical results. Fig.7 plots the time series of relative displacement ξ from non-zero initial conditions when the system is equipped with a traditional linear elastic springs and with two magnetic pairs. Under the hypothesis that dissipative phenomena are viscous, damping can be estimated through the well-known logarithmic decrement method applied to one or more maxima. With decreasing amplitude the non symmetric behaviour of the oscillations in the negative semi-plane vanishes, thus confirming the displacement dependent nature of the nonlinearity. For small amplitude also Coulomb friction phenomena arise; the validity range of the viscous type model for the damping is bounded with dashed vertical lines.

In Fig.8 an example of the jump phenomena is described: the continuous line corresponds to the sweep with increasing frequency, while the dashed line is relative to the decreasing frequency sweep. This plot proves the hardening nature of the nonlinearities of the elasto-magnetic system ($m = 8.15$ kg, $k = 2098$ N/m, $l_0 = 19$ mm, $A = 1.43 \cdot 10^{-3}$ Nm^3, $B = 19.7$ mm, $n = 3$, $y_0 = 0.254$ mm, computed value of $p = 6.2$ m^{-2}) and the existence of a dynamical instability area in the frequency range 5.3-6.1 Hz. It is worth noting the correspondent amplitude jump of nearly one decade, clearly observable thanks to the limited damping of the system and to the quasi-stationary frequency variation. The softening behaviour of the purely magnetic suspension ($m = 9.25$ kg, $A = 2.26 \cdot 10^{-3}$ Nm^3, $B = 18.8$ mm, $n = 3$, $y_0 = 0.127$ mm, computed value of $p = -488.6$ m^{-2}) is plotted in Fig.9 (left). Finally, for the elasto-magnetic case with $y_0 = 0.127$ mm, Fig.9 (right) shows superharmonic resonances, approximately at twice and three times the natural frequency.

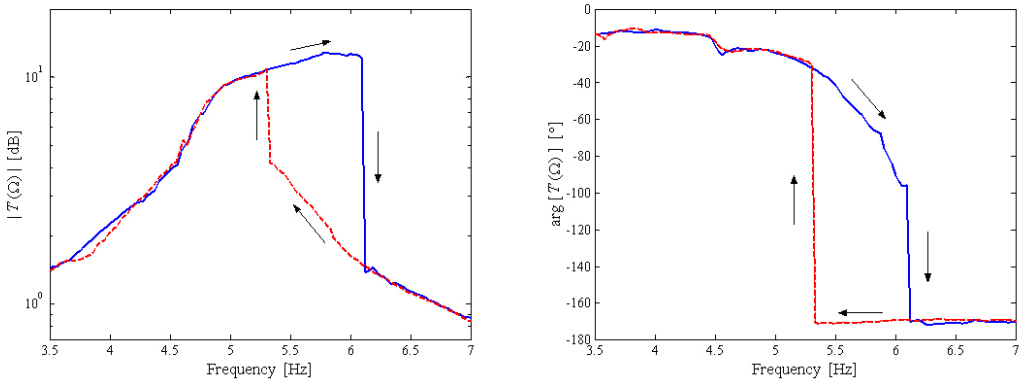

Figure 8: Experimental hardening jump modulus (left) and phase (right)

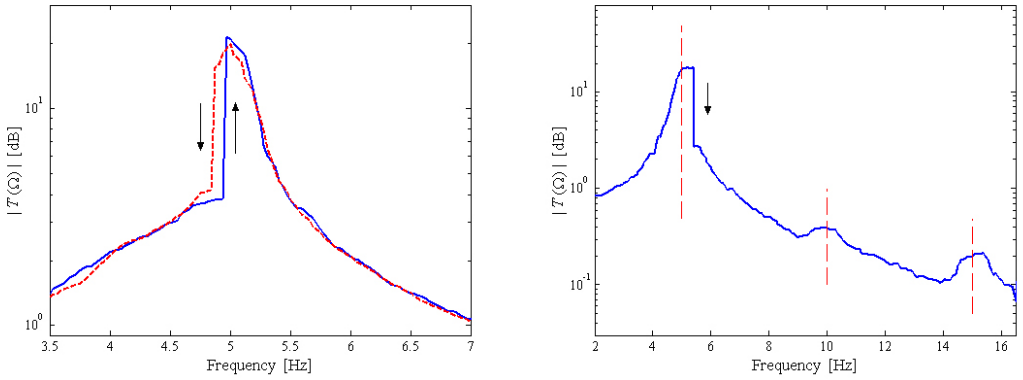

Figure 9: Experimental softening jump modulus (left) and superharmonic resonances (right)

5.1 Linearised model: numeric-experimental comparison

The analytic-experimental comparison of the resonance-invariant properties of the elasto-magnetic suspension is obtained by identifying the natural frequency ω_n and the viscous damping parameter ζ of the linearised system. Fig.10 and 11 show the results for the experimental design configuration $m^\star = 8.15$ kg, $k = 2098$ N/m, $l_0 = 19$ mm, $A = 1.43 \cdot 10^{-3}$ Nm^3, $B = 19.7$ mm, $n = 3$, $\alpha^\star = 0.348$, $f_n^\star = 5$ Hz. The dependence of natural frequency on the system mass is drastically reduced for the

elasto-magnetic suspension, in excellent accordance with the numerical model (solid line in fig.10). Furthermore, transmissibility $T(\Omega)$ proves that a suspension equipped with elasto-magnetic springs effectively behaves like a passive adaptive suspension, practically independent on mass changes. The three plots are relative to the system possessing respectively the design mass $m^\star$ and mass values about $\pm 40\%$ with respect to $m^\star$.

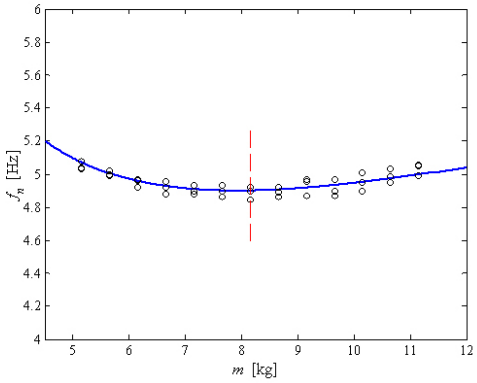

Figure 10: Natural frequency vs. mass for the elasto-magnetic suspension (circles refer to experimental data)

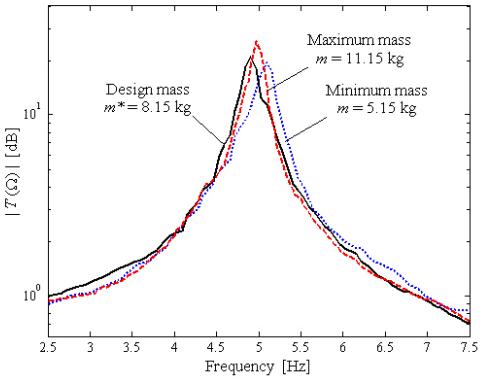

Figure 11: Experimental transmissibility of the elasto-magnetic suspension

6 Conclusions

The proposed model of magneto-elastic suspension proves to possess a natural frequency nearly insensitive to considerable mass changes ($\pm 50\%$). Numerical and experimental results are in excellent accordance; moreover also the results derived from nonlinear dynamic analysis are experimentally verified. Hence it is reasonable to believe that these results can be applied to complex passive systems, in which it is important to keep a constant value of the natural frequency independently of the system mass. Such innovative suspensions represent an interesting basis for the development of passive adaptive application in automotive or aerospace field, especially for systems subject to strong mass changes.

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