Fredholm Factorization for Wedge Problems

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Introduction

Recently the diffraction by arbitrary impenetrable wedges has been reduced to the factorization of matrices of order four [1]. This paper provides an efficient and general factorization technique that is based on the solution of a Fredholm integral equation of second kind.

Wiener- Hopf solution of the problem

Figure 1 illustrates the problem of the diffraction by a plane wave at skew incidence on an impenetrable wedge immersed in a medium with permittivity ε and permeability μ.

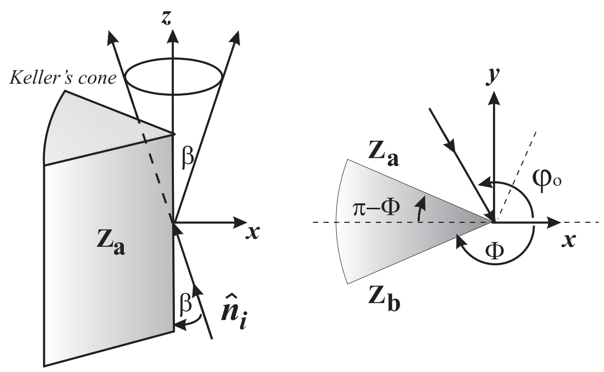

Figure 1: Geometry of the problem

The incident field is constituted by a plane wave having the following longitudinal components:

$$E_z^i = E_o e^{j\tau_o \rho\cos(\varphi-\varphi_o)} e^{-j\alpha_o z} \qquad H_z^i = H_o e^{j\tau_o \rho\cos(\varphi-\varphi_o)} e^{-j\alpha_o z} \tag{1}$$

where β and φ_0 are the zenithal and azimuthal angle of the direction of the plane wave $\hat{n}_i$ and $k=\omega\sqrt{\mu\varepsilon}$, $\alpha_0 = k\cos\beta$, $\tau_0 = k\sin\beta$.

The tangential fields are related on the boundaries of the wedge $\varphi = +\Phi$ (a-face) and $\varphi = -\Phi$ (b-face) through the Leontovich conditions:

$$\begin{bmatrix} E_z(\rho,\Phi) \\ E_\rho(\rho,\Phi) \end{bmatrix} = Z_a \begin{bmatrix} H_\rho(\rho,\Phi) \\ -H_z(\rho,\Phi) \end{bmatrix}, \quad \begin{bmatrix} E_z(\rho,-\Phi) \\ E_\rho(\rho,-\Phi) \end{bmatrix} = -Z_b \begin{bmatrix} H_\rho(\rho,-\Phi) \\ -H_z(\rho,-\Phi) \end{bmatrix} \tag{2}$$

where the matrices $Z_{a,b} = Z_o \begin{bmatrix} z_{11}^{a,b} & z_{12}^{a,b} \\ z_{21}^{a,b} & z_{22}^{a,b} \end{bmatrix}$ depends on the wedge material and

$Z_o = \sqrt{\mu / \varepsilon}$ is the free space impedance.

The Wiener-Hopf formulation [1,4] of this problem yields the solution:

$$\bar{X}_+(\bar{\eta}) = \bar{G}_+^{-1}(\bar{\eta}))\bar{G}_+(\bar{\eta}_o)\frac{\bar{T}_o}{\bar{\eta} - \bar{\eta}_o} \quad (3)$$

where:

$$V_{z+}(\eta,\varphi) = \int_0^\infty E_z(\rho,\varphi)e^{j\eta\rho}d\rho, \qquad I_{z+}(\eta,\varphi) = \int_0^\infty H_z(\rho,\varphi)e^{j\eta\rho}d\rho \quad (4)$$

$$V_{\rho+}(\eta,\varphi) = \int_0^\infty E_\rho(\rho,\varphi)e^{j\eta\rho}d\rho, \qquad I_{\rho+}(\eta,\varphi) = \int_0^\infty H_\rho(\rho,\varphi)e^{j\eta\rho}d\rho \quad (5)$$

$$\bar{X}_+(\bar{\eta}) = \left| V_{z+}(\eta,0) \quad V_{\rho+}(\eta,0) \quad Z_o I_{z+}(\eta,0) \quad Z_o I_{\rho+}(\eta,0) \right|^t \quad (6)$$

and $\eta = \eta(\bar{\eta}) = -\tau_o \cos[\frac{\Phi}{\pi}[\arccos[-\frac{\bar{\eta}}{\tau_o}]]$.

For the problem at hand the constants $\bar{T}_o$, $\bar{\eta}_o$ assume the following expressions:

$$\bar{T}_o = \frac{\pi}{\Phi}\frac{\sin\frac{\pi}{\Phi}\varphi_o}{\sin\varphi_o}\begin{vmatrix} jE_o \\ j\dfrac{\alpha_o\cos\varphi_o E_o + kZ_o\sin\varphi_o H_o}{\tau_o} \\ jZ_o H_o \\ j\dfrac{\alpha_o Z_o\cos\varphi_o H_o - k\sin\varphi_o E_o}{\tau_o}\end{vmatrix} \text{ and } \bar{\eta}_o = -\tau_o\cos\frac{\pi}{\Phi}\varphi_o$$

and the matrix $\bar{G}_+(\bar{\eta})$ is the plus factorized matrix of the matrix kernel

$$\bar{G}(\bar{\eta}) = \bar{G}_-(\bar{\eta})\bar{G}_+(\bar{\eta}), \qquad \bar{G}(\bar{\eta}) = \begin{vmatrix} \frac{g_{11}}{d^a} & \frac{g_{12}}{d^a} & \frac{g_{13}}{d^a} & \frac{g_{14}}{d^a} \\ \frac{g_{21}}{d^a} & \frac{g_{22}}{d^a} & \frac{g_{23}}{d^a} & \frac{g_{24}}{d^a} \\ \frac{g_{31}}{d^b} & \frac{g_{32}}{d^b} & \frac{g_{33}}{d^b} & \frac{g_{34}}{d^b} \\ \frac{g_{41}}{d^b} & \frac{g_{42}}{d^b} & \frac{g_{43}}{d^b} & \frac{g_{44}}{d^b} \end{vmatrix} \quad (7)$$

where:

$g_{11} = -k\,n\,z_{11}^a\alpha_o\eta - m\eta\alpha_o^2 - k^2\eta\xi + kmz_{12}^a\alpha_o\xi - kz_{22}^a\xi\,\tau_o^2$, $g_{12} = -k\,n\,z_{12}^a\tau_o^2 - m\,\alpha_o\tau_o^2$,

$g_{13} = k\,n\,\alpha_o\eta - m\eta z_{12}^a\alpha_o^2 - k^2nz_{12}^a\xi - km\alpha_o\xi + z_{22}^a\eta\alpha_o\,\tau_o^2$,

$g_{14} = k\,n\,\tau_o^2 - z_{12}^a m\,\alpha_o\tau_o^2 + z_{22}^a\tau_o^4$,

$g_{21} = k\,(n\eta - m\xi)\alpha_o z_{11}^a + (\eta\,\alpha_o + k\,z_{21}^a\xi)\tau_o^2$, $g_{22} = k\,nz_{11}^a\tau_o^2 + \tau_o^4$,

$g_{23} = m\,z_{11}^a\alpha_o^2\eta + k^2n\,z_{11}^a\xi - z_{21}^a\alpha_o\eta\tau_o^2 + k\xi\tau_o^2$, $g_{24} = mz_{11}^a\alpha_o\tau_o^2 - z_{21}^a\tau_o^4$,

$d^a = k^2n^2\,z_{11}^a + m^2\,z_{11}^a\alpha_o^2 + kn(1+\Delta^a)\tau_o^2 - m(z_{12}^a + z_{21}^a)\alpha_o\tau_o^2 + z_{22}^a\tau_o^4$, $\Delta_z^a = z_{11}^a z_{22}^a - z_{12}^a z_{21}^a$

d^b, Δ_z^b, $g_{31}, g_{32}, g_{33}, g_{34}, g_{41}, g_{42,} g_{43}, g_{44}$ assume respectively the same expression of d^a, $\Delta_z^a, g_{11}, g_{12}, -g_{13}, -g_{14}, g_{21}, g_{22,} -g_{23}, -g_{24}$ except for the substitution of the superscript ***a*** with the superscript ***b***.

The functions ξ, m and n depends on $\overline{\eta}$ and are defined by:

$$\begin{cases} \xi = \xi(\overline{\eta}) = -\tau_o \sin[\frac{\Phi}{\pi}[\arccos[-\frac{\overline{\eta}}{\tau_o}]] \\ m = m(\overline{\eta}) = \tau_o \cos[\frac{\Phi}{\pi}[\arccos[-\frac{\overline{\eta}}{\tau_o}] + \Phi] \\ n = n(\overline{\eta}) = \tau_o \sin[\frac{\Phi}{\pi}[\arccos[-\frac{\overline{\eta}}{\tau_o}] + \Phi] \end{cases} \qquad (8)$$

In several important cases the matrix $\overline{G}(\overline{\eta})$ can be factorized in closed form [2]. For instance, this property is verified for the whole class of problems that have been solved with the Malyuzhinets-Sommerfeld technique.

Fredholm factorization of the matrix kernel $\overline{G}(\overline{\eta})$

By using the technique introduced in [3] the factorized matrix $\overline{G}_+(\overline{\eta})$ can be expressed by:

$$\overline{G}_+(\overline{\eta}) = \frac{1}{\overline{\eta} - \overline{\eta}_p} \left| X_{1+}(\overline{\eta}), X_{2+}(\overline{\eta}), X_{3+}(\overline{\eta}), X_{4+}(\overline{\eta}) \right|^{-1} \qquad (10)$$

where $\overline{\eta}_p$ is an arbitrary point with negative imaginary part and the functions $X_{i+}(\overline{\eta}), \{i = 1,2,3,4\}$ satisfy the following Fredholm integral equation:

$$\overline{G}(\overline{\eta}) X_{i+}(\overline{\eta}) + \frac{1}{2\pi j} \cdot \int_{-\infty}^{\infty} \frac{\left[\overline{G}(x) - \overline{G}(\overline{\eta})\right] X_{i+}(x)}{x - \overline{\eta}} dx = \frac{R_i}{\overline{\eta} - \overline{\eta}_p}, \quad \mathrm{Im}[\overline{\eta}_p] < 0 \qquad (11)$$

with the vector constant R_i given by the canonical basis for the 4D space.

Since $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| \frac{\overline{G}(x) - \overline{G}(\overline{\eta})}{x - \overline{\eta}} \right|^2 dx d\overline{\eta}$ is bounded [3], (11) is a Fredholm equation of second kind where it is applicable the well-known in literature powerful solution technique.

We experienced [4] that the convergence of approximate solutions considerably increases when we solve the integral equation in the ***t*** plane defined by the mapping $\overline{\eta} = \overline{\eta}(t) = -\tau_o \cos(j\, t - \frac{\pi}{2})$.

Numerical validation

To ascertain the correctness of our new methodology we have chosen a well known in literature test case to compare our solution with alternative method [5]: the bistatic far field amplitude evaluation for skew incidence on an impedance half plane. Figure 2 reports the GTD Diffraction Coefficient for Ez component

($D_E(\varphi) = s_E(\varphi - \pi) - s_E(\varphi + \pi)$, where $s_E(w)$ is the Sommerfeld function) for the test case with the following problem parameters: k=1, the incident field φ_0=5π/6, $\beta = \pi/3$, E_{zo}=1, H_{zo}=0, the aperture angle Φ=π, the integration parameters A=5, h=1 for the discretization of equation (11) after the transformation in the ***t*** plane. Peaks of the GTD Diffraction Coefficients are for $\varphi = \varphi_0 - \pi$ (incident field) and for $\varphi = 2\Phi - \varphi_0 - \pi$ (reflected field).

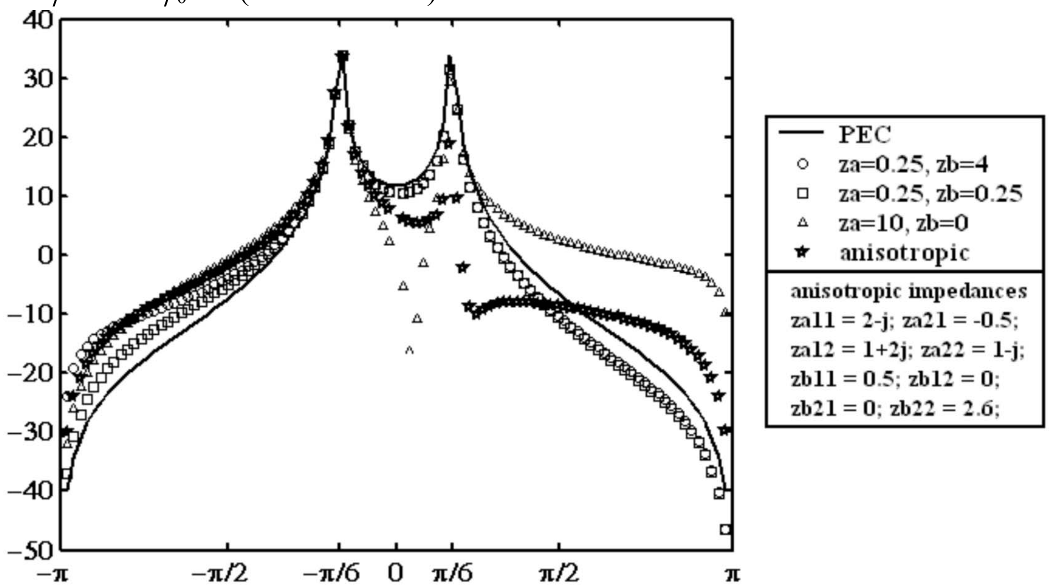

Figure 2: Amplitude of the GTD Diffraction Coefficient (dB)

Several other applications of this technique to wedge problems have been reported in [6]. New examples and convergence tests concerning with new canonical wedge problems will be illustrated in the oral presentation of the paper.

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