Modeling Edge Singularities in the Method of Moments

Roberto D. Graglia[1], Guido Lombardi[1], Donald R. Wilton[2,3], and William A. Johnson[3]

[1] Politecnico di Torino, Torino, Italy
[2] University of Houston, Houston, TX, USA
[3] SANDIA Nat. Labs., Albuquerque, NM, USA

Introduction

The method of moments (MoM) is widely used to discretize and numerically solve integral equations modeling complex electromagnetic structures made of impenetrable materials (perfectly conducting or metallic structures). However, it is rather difficult to numerically model structures containing edges because of the large dynamic range of the solution in the neighborhood of edges. Indeed it is well known that near the edge of a wedge, surface current and charge densities, as well as electromagnetic fields, are generally singular [1],[2]. To clarify the problem, let us consider for a moment the fields at angular frequency ω in the neighborhood of a straight metal wedge of internal wedge angle α immersed in free space with electric permittivity and magnetic permeability equal to ε_0 and μ_0, respectively (see Fig. 1).

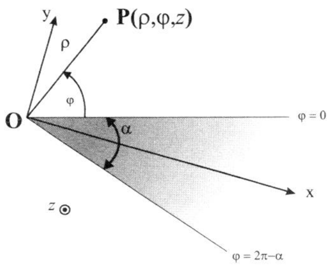

Figure 1: Straight perfectly conducting wedge with aperture angle α and local longitudinal axis z.

In a polar reference frame (ρ, ϕ, z) with origin at the edge of the wedge and with the z-axis parallel to the edge itself, the surface charge ρ_s and current densities $\boldsymbol{J}_s$ on a face of the wedge in the vicinity of the edge take, when expanded in a power series, the following forms (only the leading terms are reported):

$$\boldsymbol{J}_s = (A + j\omega Bz)\rho^{\nu-1}\,\hat{\boldsymbol{z}} + (C + j\omega D\rho^{\nu})\,\hat{\boldsymbol{\rho}} \tag{1}$$

$$\rho_s = -(B + \nu D)\rho^{\nu-1} \tag{2}$$

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where A, B, C, D are coefficients that depend on the excitation. The surface charge is found from the current density by the continuity equation. The constant component in the radial part of the surface current, when present, does not pose any numerical modeling problem. Indeed, when combined with the surface current from the opposite face for a zero-thickness metal edge, the constant terms cancel since no charge can accumulate at the edge. For a perfectly conducting wedge of internal wedge angle α, one has $\nu = \pi/(2\pi - \alpha)$. The smallest value, $\nu = 1/2$, occurs for a half plane ($\alpha = 0$), while $\nu = 1$ represents a flat plane. The current component parallel to the metal edge and the charge density are, in general, infinite at the edge. The surface current component normal to the edge is finite, but is still singular in the sense that its derivative does not exist there.

Many electromagnetic structures of practical engineering interest contain conducting (or even penetrable) edges and, in the vicinity of these edges, the surface charge and the field behavior can be singular. Unfortunately, though the singular behavior of the electromagnetic field or current is isolated near the vicinity of the edge, the usual non-singular (regular) bases require the expensive use of a dense mesh in the neighborhood of edges in order to accurately model the fields, even when high-order (regular) vector bases are used. Indeed, the difficulty is that no polynomial adequately approximates the algebraic behavior of fractional orders ν present in the factor ρ^{ν} in (1); even worse, no polynomial can approximate the infinite behavior of the factors $\rho^{\nu-1}$ in (1, 2).

State of the Art and Open Problems

In order to model singular algebraic behaviors, it is much more efficient to introduce and use singular vector basis functions able to precisely model the singular edge behavior of fields and currents, rather than using regular vector functions on very dense meshes. In this respect, the state of the art is well summarized in [3, 4], and references therein. The wedge faces in the neighborhood of the edge profile are meshed using edge singularity quadrilaterals and/or two types of singularity triangles: the edge (e) and the vertex (v) singularity triangle, with local edge numbering schemes as shown in Fig. 2. There is no need for considering vertex singularity quadrilaterals since the vertex singularity triangle may serve as the only element-filler needed for

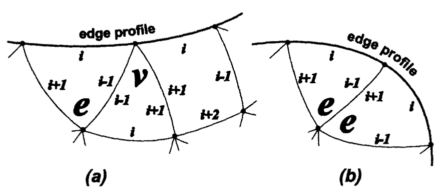

Figure 2: a) Local edge-numbering scheme used for edge singularity quadrilaterals and edge (e) and vertex (v) singularity triangles. Notice that the element edges $i \pm 1$ always depart from the edge profile. b) Although two edge singularity triangles can have an edge in common, the basis functions cannot model a corner singularity.

meshing in the neighborhood of the edge profile. The local edge-numbering scheme sketched in Fig. 2 has been chosen so as to associate the $i \pm 1$ labels with the element edges departing from the edge profile. Lowest order bases are first constructed on the edge and vertex singularity elements. Higher order bases can then be generated from the lowest order one by techniques similar to that of [3]. The basis functions defined on the singular elements (i.e., on the elements attached to the edge of the wedge) must incorporate the edge conditions and be able to approximate the unknown fields in the neighborhood of the edge of a wedge for any order of the singularity coefficient ν (see eq. (1)), supposed given and known *a priori*. Several ways to derive *singular* and *complete* lowest-order vector bases have been investigated by the authors. We suggest that the lowest order divergence-conforming bases must satisfy the following fundamental requirements:

1. Both the bases and their divergence are complete to lowest order in the sense that they model *all* terms of (1) and (2) (*completeness and singularity modeling properties*).
2. The bases must exactly model the null space of the divergence operator, i.e., there must exist a linear combination of the bases on an element with an identically vanishing divergence (*null-space modeling property*).
3. The bases must be fully conformable (continuous) along all edges with the lowest order bases defined in adjacent elements (*conformity property*).

The above requirements are not completely independent of one another, and they may be generalized to higher order elements. Higher order bases are typically derived from lower order ones by multiplying them by polynomials complete to successively higher orders. In this process, subsets of dependent bases are generated that must be recognized and eliminated. The identification of dependency relations among the bases aids in recognizing these subsets. A significant complication for singular bases, however, is that higher order terms of the form of (1) and (2) but with $\nu_m = m\nu$ replacing ν *also* appear as terms in the wedge series expansion. Each of these additional singular terms of different order appears to require a separate multiplicative polynomial for modeling current and charge up to a prescribed order, and this leads to a proliferation of "additive" bases in the edge current representation. It is instructive to note that as the wedge degenerates to a flat plate ($\alpha = \pi, \nu = 1$), all such terms degenerate into the usual simple polynomial forms of higher order regular elements. Not only does it appear that these higher order ν_m expansions are necessary in a fundamentally-based formulation, but they can be the dominant terms if the excitation is such (see Fig. 3) that leading terms in the representation vanish[1],[5].

In the paper, the authors explore various possible approaches for generating lowest order and higher order bases for modeling surface currents and their divergence for moment method application to integral equations. The bases developed are defined on *curved triangular* and *quadrilateral* elements. All the bases are conveniently defined in parent element coordinates, and each expansion function spans one or two patches. For example, the bases given in [3] are either associated with an edge

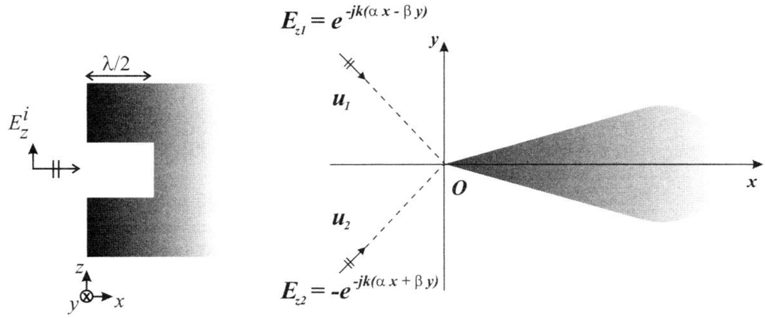

Figure 3: There are special source configurations that do not excite lower order wedge field-singularities [1]. Left: a linearly polarized plane wave normally incident on a $\lambda/2$ metal groove yields finite fields. Right: two linearly polarized plane waves incident on a cylindrical "aircraft wing" along the direction $\boldsymbol{u}_1 = \alpha\hat{\boldsymbol{x}} - \beta\hat{\boldsymbol{y}}$, $\boldsymbol{u}_2 = \alpha\hat{\boldsymbol{x}} + \beta\hat{\boldsymbol{y}}$ yield finite fields also on the wing edge.

of the mesh, or they are edgeless. Edgeless functions are represented in [3] by the symbol **V**, obtained by overturning the symbol **Λ**.

In the presentation we review the properties of existing bases. In particular we present the solution of several problems that at the moment are still considered open. For example, we present a technique used to perform self-term integration on curved singular elements based on a modification of the algorithm given in [6]. We also discuss methods for obtaining singular basis functions from lowest order bases, and dependency relations that arise in their construction. The problem of scattering by a circular disk illuminated by a plane wave serves to illustrate the use of singular vector bases.

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