

Generalized Wiener-Hopf Equations for Wedge problems involving arbitrary linear media

V. Daniele*

G. Lombardi*

Abstract – This paper provides new functional equations in angular regions that turn useful to study wedge problems in presence of arbitrary linear media. The enforcement of the boundary conditions on these equations reduces the wedge problems to Generalized Wiener-Hopf (GWHE) equations that can be approached with standard solution techniques. This procedure is briefly illustrated in this paper.

1 FUNCTIONAL EQUATIONS IN ANGULAR REGION

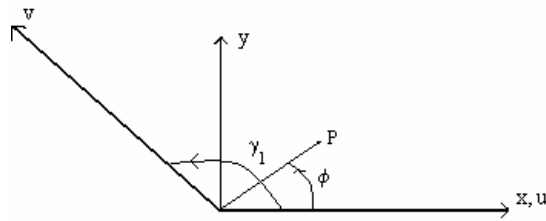


Figure 1: Angular region ($0 \leq \varphi \leq \gamma_1$)

We consider time harmonic electromagnetic fields with a time dependence specified by the factor $e^{j\omega t}$ which is omitted. The electromagnetic field is studied in the angular region indicated in figure 1 that is defined by the aperture angle γ_1 , ($0 \leq \varphi \leq \gamma_1$). This region is filled by an arbitrary homogeneous medium where the electromagnetic field is characterized by the following constitutive relations

$$\begin{aligned} \mathbf{D} &= \boldsymbol{\varepsilon} \cdot \mathbf{E} + \boldsymbol{\xi} \cdot \mathbf{H} \\ \mathbf{B} &= \boldsymbol{\zeta} \cdot \mathbf{E} + \boldsymbol{\mu} \cdot \mathbf{H} \end{aligned} \quad (1)$$

In equation (1) \mathbf{D} , \mathbf{B} are the induction fields; \mathbf{E} , \mathbf{H} are the electromagnetic fields and the dyadic electromagnetic parameters $\boldsymbol{\varepsilon}$, $\boldsymbol{\xi}$, $\boldsymbol{\zeta}$, $\boldsymbol{\mu}$ are known.

Without loss of generality we assume the z dependence of the electromagnetic field \mathbf{E} and \mathbf{H} specified by the factor $e^{-j\alpha_0 z}$ which is omitted. Cartesian coordinates $\{x, y, z\}$, polar coordinates $\{\rho, \varphi, z\}$ and oblique coordinates $\{u, v, z\}$ are used to describe the problem in term of GWHE.

With reference to figure 1, the oblique Cartesian coordinates u and v are defined by (with $\gamma_1 = \gamma$):

$$u = x - y \cot \gamma, \quad v = \frac{y}{\sin \gamma} \quad (2)$$

$$x = u + v \cos \gamma, \quad y = v \sin \gamma$$

Through a very cumbersome procedure that is not reported here, we obtain the following representation (5) of the Laplace transform of the field (3-4).

$$\psi_t(u, v) = \begin{pmatrix} E_z(u, v) \\ E_x(u, v) \\ H_z(u, v) \\ H_x(u, v) \end{pmatrix} \quad (3)$$

$$\tilde{\psi}_t(\eta, v) = \int_0^\infty e^{j\eta u} \psi_t(u, v) du \quad (4)$$

$$\begin{aligned} \tilde{\psi}_t(\eta, v) &= u_1 v_1 \cdot \tilde{\psi}_t(\eta, 0) e^{-\lambda_1 v} + \\ &+ u_2 v_2 \cdot \tilde{\psi}_t(\eta, 0) e^{-\lambda_2 v} + \\ &+ u_1 v_1 \cdot T \cdot \int_0^v e^{-\lambda_1(v-v_1)} \psi_\tau(v_1) dv_1 + \\ &u_2 v_2 \cdot T \cdot \int_0^v e^{-\lambda_2(v-v_1)} \psi_\tau(v_1) dv_1 + \\ &- u_3 v_3 \cdot T \cdot \int_v^\infty e^{-\lambda_3(v-v_1)} \psi_\tau(v_1) dv_1 + \\ &- u_4 v_4 \cdot T \cdot \int_v^\infty e^{-\lambda_4(v-v_1)} \psi_\tau(v_1) dv_1 \end{aligned} \quad (5)$$

The vector $\psi_\tau(v)$ represents the components of the electromagnetic field that are tangential to the boundaries defined by the half line $\varphi = \gamma$ or $u = 0$.

It is defined by:

$$\psi_\tau(v) = \begin{pmatrix} E_z(0, v) \\ E_v(0, v) \\ H_z(0, v) \\ H_v(0, v) \end{pmatrix} \quad (6)$$

In equation (5) the four scalars $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ and the four vectors u_1, u_2, u_3, u_4 are respectively the eigenvalues and the eigenvectors of a matrix M_e of

* Dipartimento di Elettronica, Politecnico di Torino, C.so Duca degli Abruzzi 24, 10129 Torino, Italy. E-mail: vito.daniele@polito.it; guido.lombardi@eln.polito.it

order four that depends on η , ω , α_o , γ and on the components of the dyadic electromagnetic parameters $\boldsymbol{\varepsilon}$, $\boldsymbol{\xi}$, $\boldsymbol{\zeta}$, $\boldsymbol{\mu}$ of the medium. The expression of M_e was obtained by MATHEMATICA® and it is not reported here.

The eigenvalues represent wave numbers and are related to plane waves that propagate in suitable directions in the arbitrary medium that fills the angular region.

In a general linear passive medium two eigenvalues (λ_1 , λ_2) have positive real parts and two eigenvalues (λ_3 , λ_4) have negative real parts.

In addition to the eigenvectors u_1 , u_2 , u_3 , u_4 , equation (5) introduces also the reciprocal vectors v_1 , v_2 , v_3 , v_4 . These vectors satisfy the equations:

$$v_j \cdot u_i = \delta_{ji}, \quad i, j = 1, 2, 3, 4 \quad (7a)$$

or

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = u_1 v_1 + u_2 v_2 + u_3 v_3 + u_4 v_4 \quad (7b)$$

where δ_{ji} is the Kronecker symbol.

The matrix T (see equation 5) is a constant matrix that, like M_e , depends on η , ω , α_o , γ and on the material (dyadic electromagnetic parameters $\boldsymbol{\varepsilon}$, $\boldsymbol{\xi}$, $\boldsymbol{\zeta}$, $\boldsymbol{\mu}$). Also T has been obtained by using MATHEMATICA® and its components are not reported here.

For $v=0$ equation (5) yields:

$$\begin{aligned} \tilde{\psi}_t(\eta, 0) &= u_1 v_1 \cdot \tilde{\psi}_t(\eta, 0) + u_2 v_2 \cdot \tilde{\psi}_t(\eta, 0) + \\ &- u_3 v_3 \cdot T \cdot \int_0^\infty e^{\lambda_3 v_1} \psi_\tau(v_1) dv_1 + \\ &- u_4 v_4 \cdot T \cdot \int_0^\infty e^{\lambda_4 v_1} \psi_\tau(v_1) dv_1 \end{aligned} \quad (8)$$

Multiplying equation (8) by $v_j \cdot$ ($j=1, 2, \dots, 4$) and taking into account (7), we obtain the following four equations:

$$v_1 \cdot \tilde{\psi}_t(\eta, 0) = v_1 \cdot \tilde{\psi}_t(\eta, 0) \quad (9a)$$

$$v_2 \cdot \tilde{\psi}_t(\eta, 0) = v_2 \cdot \tilde{\psi}_t(\eta, 0) \quad (9b)$$

$$v_3 \cdot \tilde{\psi}_t(\eta, 0) = -v_3 \cdot T \cdot \tilde{\psi}_\tau(-j\lambda_3) \quad (9c)$$

$$v_4 \cdot \tilde{\psi}_t(\eta, 0) = -v_4 \cdot T \cdot \tilde{\psi}_\tau(-j\lambda_4) \quad (9d)$$

where $\tilde{\psi}_\tau(\alpha)$ is the Laplace transform:

$$\tilde{\psi}_\tau(\alpha) = \int_0^\infty e^{j\alpha v} \psi_\tau(v) dv \quad (10)$$

Equations (9a) and (9b) do not give any new information. Conversely equations (9c) and (9d) provide two functional equations that relate suitable Laplace transforms of the electromagnetic field components that are tangential to the two boundaries $\varphi = 0$ (or $v=0$) and $\varphi = \gamma$ (or $u=0$).

For arbitrary media the equations (9c) and (9d) are very complicated. Simpler equations are derived with homogeneous media where $\boldsymbol{\varepsilon}$, $\boldsymbol{\mu}$ are scalars and $\boldsymbol{\xi}$, $\boldsymbol{\zeta}$ vanish. In this case the eigenvalues λ_1 , λ_2 , λ_3 , λ_4 degenerate: $\lambda_1 = \lambda_2 = -\lambda_3 = -\lambda_4 = -jm$. Equations (9c) and (9d) become:

$$\xi V_{z+}(\eta, 0) - \frac{\tau^2}{\omega \varepsilon} I_{\rho+}(\eta, 0) - \frac{\alpha_o \eta}{\omega \varepsilon} I_{z+}(\eta, 0) = \quad (11a)$$

$$= -n V_{z+}(-m, \gamma) - \frac{\tau^2}{\omega \varepsilon} I_{\rho+}(-m, \gamma) + \frac{\alpha_o m}{\omega \varepsilon} I_{z+}(-m, \gamma)$$

$$\xi I_{z+}(\eta, 0) + \frac{\tau^2}{\omega \mu} V_{\rho+}(\eta, 0) + \frac{\alpha_o \eta}{\omega \mu} V_{z+}(\eta, 0) = \quad (11b)$$

$$-n I_{z+}(-m, \gamma) + \frac{\tau^2}{\omega \mu} V_{\rho+}(-m, \gamma) - \frac{\alpha_o m}{\omega \mu} V_{z+}(-m, \gamma)$$

where:

$$V_{z+}(\alpha, \varphi) = \int_0^\infty E_z(\rho, \varphi) e^{j\alpha \rho} d\rho \quad (12a)$$

$$I_{z+}(\alpha, \varphi) = \int_0^\infty H_z(\rho, \varphi) e^{j\alpha \rho} d\rho \quad (12b)$$

$$V_{\rho+}(\alpha, \varphi) = \int_0^\infty E_\rho(\rho, \varphi) e^{j\alpha \rho} d\rho \quad (12c)$$

$$I_{\rho+}(\alpha, \varphi) = \int_0^\infty H_\rho(\rho, \varphi) e^{j\alpha \rho} d\rho \quad (12d)$$

$$\xi = \sqrt{\tau_o^2 - \eta^2} \quad (12e)$$

$$m = -\eta \cos \gamma + \sqrt{\tau_o^2 - \eta^2} \sin \gamma \quad (12f)$$

$$\tau_o = \sqrt{\omega^2 \mu \varepsilon - \alpha_o^2} \quad (12g)$$

$$n = -\xi \cos \gamma - \eta \sin \gamma = \sqrt{\tau_o^2 - m^2} \quad (12h)$$

2 A SIMPLE APPLICATION

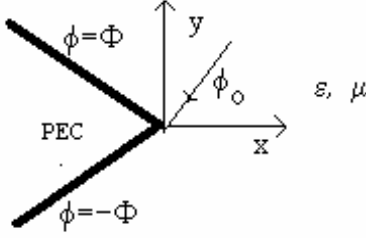


Figure 2: PEC wedge with arbitrary aperture angle Φ

The functional equations (9c) and (9d) yield a system of equations that provides a closed mathematical formulation of the wedge problems. As a simple application, we consider the case of a perfectly electric conducting (PEC) wedge excited by a E- polarized plane wave ($\partial/\partial z = 0$, $\alpha_o = 0$) (see figure 2) and immersed in a isotropic medium. By assuming the angular region defined by $0 \leq \varphi \leq \Phi$ ($\gamma = \Phi$) and taking into account the boundary condition $V_{z+}(-m, \Phi) = 0$, the equation (11a) becomes:

$$\xi V_{z+}(\eta, 0) - \omega \mu I_{\rho+}(\eta, 0) = -\omega \mu I_{\rho+}(-m, \Phi) \quad (13)$$

and (11b) may be ignored since the unknowns of this equation are vanishing.

Similar considerations for the angular region $-\Phi \leq \varphi \leq 0$ yield the functional equation:

$$\xi V_{z+}(\eta, 0) + \omega \mu I_{\rho+}(\eta, 0) = \omega \mu I_{\rho+}(-m, -\Phi) \quad (14)$$

Equation (13) and (14) provide a complete mathematical description of the problem. Notice that these equations involve the two functions $V_{z+}(\eta, 0)$ and $I_{\rho+}(\eta, 0)$, that are regular in η -upper half plane, and the two functions $I_{\rho+}(-m, \Phi)$ and $I_{\rho+}(-m, -\Phi)$, that are regular in m -lower half plane. Using (12f), $\Phi = \pi$ yields $m = \eta$; (13) and (14) become a system of classical Wiener-Hopf equations that are well studied in the literature. Conversely, when $\Phi \neq \pi$ equations (13) and (14) constitute a system of two generalized Wiener-Hopf equations (GWHE).

This problem has been solved in [1] by extending the factorization technique developed for classical Wiener-Hopf equations.

3 SOME TECHNIQUES FOR SOLVING GENERALIZED W-H EQUATIONS

Wedge problems present many homogeneous angular regions wherein the functional equations (9c) and (9d) hold. By enforcing the boundary conditions relative to the different angular regions, we obtain GWHE with the following form:

$$\sum_{j=1}^n G_{ij}(\eta) X_{j+}(\eta) = -Y_{i+}[-m_i(\eta)] \quad (15)$$

($i=1, 2, \dots, n$)

where the unknowns $X_{j+}(\alpha)$ and $Y_{i+}(\alpha)$ are plus functions i.e functions regular in the half-plane $\text{Im}[\alpha] \geq 0$.

The presence of media where more types of waves propagate involves different $m_i(\eta)$ functions since they are related to different wave numbers. In simple but fundamental cases the $m_i(\eta)$ functions become only one: for example the simple problem presented in section 2.

It is remarkable that in presence of one wave number a suitable mapping reduces the GWHE to Classical W-H equations that can be solved with the classical factorization methods [1], [2]. In particular all the wedge problems, that have been solved in closed form by the Malyuzhinets-Sommerfeld method, yield matrix kernels $G_{ij}(\eta)$ that can be factorized in closed forms since they are of the Daniele-Khrapkov form [2].

No analytical solution techniques exist for problems involving different wave numbers: for instance, for dielectric wedges or impenetrable wedges immersed in anisotropic media.

In the following subsections we shortly describe some techniques to obtain approximate solutions.

3.1 Reduction of GWHE to Fredholm integral equation of second kind

Using the theory developed for generalized Hilbert problem in chapter 4 of [3], we reduce equations (15) to the following not singular integral equation which is a Fredholm integral equation of the second kind

$$X_+(\eta) + \frac{1}{2\pi j} \int_{-\infty}^{\infty} \left[G^{-1}(\eta) \cdot \tilde{G}^{-1}(\eta, u) - \frac{1}{u - \eta} \right] X_+(u) du = 0 \quad (16)$$

where the unknown is

$$X_+(\eta) = \begin{bmatrix} X_{1+}(\eta) \\ \dots \\ X_{n+}(\eta) \end{bmatrix}$$

The $G(\eta)$ is the matrix with entries $G_{ij}(\eta)$ and $\tilde{G}(\eta, u)$ is the matrix with entries $\tilde{G}_{ij}(\eta, u)$:

$$\tilde{G}_{ij}(\eta, u) = G_{ij}(u) \frac{1}{m_i(u) - m_i(\eta)} \frac{dm_i(u)}{du} \quad (17)$$

This procedure ensures that the problem is well posed and allows for the use of well known numerical schemes to get the solutions.

3.2 The moment method for solving the GWHE

An approximate but very general method for solving the GWHE equations is the moment method. The unknowns $X_{j+}(\eta)$ are written in term of “expansion” functions $\Psi_{r+}^j(\eta)$ ($r=1,2,..$):

$$X_{j+}(\eta) = \sum_r C_r^j \Psi_{r+}^j(\eta) \quad (18)$$

Substituting (18) in equation (15) yields:

$$\sum_{j=1}^n G_{ij}(\eta) \sum_r C_r^j \Psi_{r+}^j(\eta) = Y_{i+}[-m_i(\eta)] \quad (19)$$

($i=1,2,..n$) or

$$\sum_{j=1}^n G_{ij}[\eta(m_i)] \sum_r C_r^j \Psi_{r+}^j[\eta(m_i)] = Y_{i+}[-m_i] \quad (20)$$

($i=1,2,..n$)

We can obtain the eliminations of the minus variables $Y_{i+}[-m_i]$ by using the Parseval theorem after the introduction of the “test” functions $\Phi_{s+}^j(m_i)$ ($s=1,2,..$):

$$\int_{-\infty}^{\infty} \Phi_{s+}^j(-m_i) \cdot Y_{i+}(-m_i) dm_i = 0 \quad (21)$$

Consequently the projection of (20) on the set of test functions, eliminates $Y_{i+}[-m_i]$ and yields the following equation for the unknowns C_r^j :

$$\sum_{j=1}^n \sum_r M_{sr}^{ij} C_r^j = 0 \quad (22)$$

where:

$$M_{sr}^{ij} = \int_{-\infty}^{\infty} \Phi_{s+}^j(-m_i) \cdot G_{ij}[\eta(m_i)] \Psi_{r+}^j[\eta(m_i)] dm_i \quad (23)$$

The homogeneous equations (22) hide the source terms. These terms can be explicitly written by considering the known geometrical optic contributions for the particular given problem.

The successful application of the moment method depends on a suitable choice of the sets of expansion and testing functions. This choice must be accomplished on the basis of physical considerations.

3.3 Use of rational approximant of the kernel $G_{ij}(\eta)$

The presence of a rational kernel $G_{ij}(\eta)$ in (15) allows for a closed form solution of these equations. This property suggests the introduction of rational approximate kernels.

Efficient solutions are obtained by rational approximate kernels when their definitions do not lose the physical properties and the mathematical consistence of the problem.

Acknowledgement

This work was supported by the Italian Ministry of Education, University and Research (MIUR) under the FIRB grant RBAU01M9PF: “Vector expansion functions for singular fields”.

References

- [1] V.G. Daniele: “New analytical Methods for wedge problems”, 2001 International Conference on Electromagnetics in Advanced Applications (ICEAA01), September 10-14 2001, Torino (Italy), pp.385- 393
- [2] V.G. Daniele, "The Wiener-Hopf technique for impenetrable wedges having arbitrary aperture angle", SIAM Journal of Applied Mathematics, vol.63, n.4, pp.1442-1460, 2003
- [3] N.P. Vekua, *System of singular integral equations*, Noordhoff, the Netherlands, 1967, pp.173-203.