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Implementation of Higher-Order Two-Dimensional Singular Elements in FEM codes and Numerical Results

G. Lombardi

Abstract – **This paper presents strategies and schemes to implement higher-order two dimensional singular elements in Finite Elements codes. Accurate numerical results are presented and the efficiency of the schemes is discussed.**

1 INTRODUCTION

This paper shows how higher-order two-dimensional singular curl-conforming vector bases [1-5] can be implemented in existing Object-Oriented Finite Elements codes.

This procedure can be extended to 3-D FEM codes and to Object-Oriented Method of Moments applications using higher-order singular div-conforming functions [3-6].

We refer in particular to the bases used in [3-5].

In these papers the basis functions incorporate the edge conditions [7-9] and are obtained as the product of lowest-order functions times Silvester-Lagrange interpolatory polynomials [10].

In general singular elements must be fully compatible with the standard, high-order regular vector functions used in adjacent elements [11].

The use of singular high-order bases provides more accurate and efficient numerical solutions of differential problems [4-5].

Sample numerical results confirm the faster convergence of these bases on wedge problems.

The use of singular elements is required to achieve high-precision results in particular problems concerning different application fields, ranging from electromagnetic compatibility to radio astronomy equipments, low power signal measures and scattering problems.

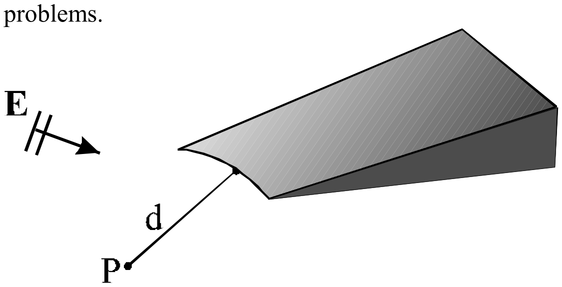

Figure 1. A curved metallic wedge.

2 HIGHER-ORDER TWO-DIMENSIONAL SINGULAR BASIS FUNCTIONS IN FEM

Singular elements have to model the field's behavior in the neighborhood of the edge of the wedge for any order of singularity ν. The coefficient ν is known a priori as described in [7-9] and it is frequency independent [12]. It depends on the material and geometrical properties of the wedge through the aperture angle α, Figure 2.

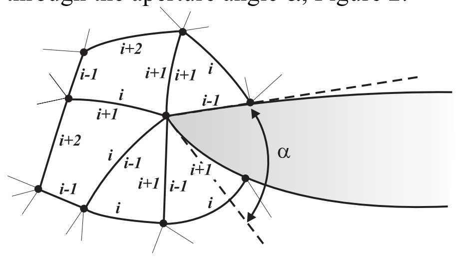

Figure 2. Cross-sectional view of the region around a sharp 2-D curved edge: triangular and quadrilateral curved singular elements.

Singular elements must model the dynamic and the static behavior of the fields near the edge as described by Van Bladel and Meixner [7-9].

The principal properties of the basis functions described in [3-5] are:

- to model the singular behavior of the field and the behavior of its curl (static and dynamic)
- minimum number of degrees of freedom
- higher order singular functions according to Meixner series
- conformity to regular elements of the same regular order
- curl conforming

The basis functions of [3-5] have been used for the numerical tests discussed in the following sections.

3 IMPLEMENTATION

3.1 FEM formulation

In order to test the capabilities of singular basis functions we used the 2D FEM formulation presented in [13] where the unknown is the electric field.

Dipartimento di Elettronica, Politecnico di Torino, C.so Duca degli Abruzzi 24, 10129 Torino, Italy,
e-mail: guido.lombardi@eln.polito.it.

$$\iint_S \left[(\nabla_t \times \boldsymbol{T}_t)_z \mu_{zz}^{\mathrm{inv}} (\nabla_t \times \boldsymbol{E}_t)_z + (\nabla_t T_z + jk_z \boldsymbol{T}_t) \cdot \underline{\underline{\boldsymbol{\nu}}}_t \cdot (\nabla_t E_z + jk_z \boldsymbol{E}_t) + (\nabla_t T_z + jk_z \boldsymbol{T}_t) \cdot \boldsymbol{\upsilon}_t (\nabla_t \times \boldsymbol{E}_t)_z + (\nabla_t \times \boldsymbol{T}_t)_z \tilde{\boldsymbol{\upsilon}}_t \cdot (\nabla_t E_z + jk_z \boldsymbol{E}_t) - k_0^2 (\boldsymbol{T}_t \cdot \underline{\underline{\boldsymbol{\epsilon}}}_t \cdot \boldsymbol{E}_t + T_z \epsilon_{zz} E_z) \right] dS + \int_\gamma \left[\boldsymbol{T} \cdot \hat{\boldsymbol{n}} \times \left(\underline{\underline{\boldsymbol{\mu}}}_r^{-1} \cdot \nabla \times \boldsymbol{E} \right) \right] d\gamma = 0 \qquad (1)$$

Equation (1) is the Galerkin form of the finite-element method applied to the transverse vector Helmholtz equation for shielded waveguides.

The solution is reduced to a generalized eigenvalue problem that can be solved through iterative methods.

3.2 FEM Integration tool

The major problem to implement singular elements is the integration tool development.

Equation (1) shows that several stiffness elements are integrals of functions with a singular behavior.

By using the same notation of [11], the problem is to perform high precision integration of singular functions of the following type:

$$\frac{(1-\xi_i)^\delta F(\boldsymbol{\xi}, \nabla\boldsymbol{\xi})}{[J(\boldsymbol{\xi})]^k} \qquad (2)$$

where i is the singular vertex (Figure 2), $\boldsymbol{\xi} = (\xi_i, \xi_{i+1}, \xi_{i-1})$, δ is the singular coefficient of the stiffness integrand (related to ν as $n\nu + m$ with n, m $\in \boldsymbol{Z}$), $J(\boldsymbol{\xi})$ is the Jacobian, $k \in \boldsymbol{N}$, $F(\xi, \nabla\xi)$ is a combination of parent coordinates and their gradients. For curved patches $J(\boldsymbol{\xi})$ is a polynomial of the ξ variables.

Several methods are suitable to integrate singular functions.

3.2.1 Full numerical integration

Gauss integration schemes are used for FEM codes involving non-singular integrands, as polynomials.

In these schemes the singularity is treated using adaptive numerical schemes, as Gauss-Kronrod [14], or standard numerical integration schemes after variable transformation as Double Exponential.

3.2.2 Analytical-Numerical integration

Higher precision results can be obtained by mixing analytical and numerical integration.

We elaborate the singular term of the integrand by adding and subtracting a constant Jacobian version of the integrand ($J = J(\boldsymbol{\xi})$ for ξ at the singular vertex i):

$$(1-\xi_i)^\delta F(\boldsymbol{\xi}, \nabla\boldsymbol{\xi}) \left\{ \frac{1}{[J(\boldsymbol{\xi})]^k} - \frac{1}{J^k} \right\} + \frac{(1-\xi_i)^\delta F(\boldsymbol{\xi}, \nabla\boldsymbol{\xi})}{J^k} \qquad (3)$$

The first term is non singular and its integration is performed through standard numerical methods. The second integrand is singular and its integration is performed by use of the following known analytical result:

$$I_T(\delta, a, b, c) = \iint_T (1-\xi_i)^\delta \, \xi_i^a \, \xi_{i+1}^b \, \xi_{i-1}^c \, \mathrm{d}T = \frac{a!\; b!\; c!}{(b+c+1)! \prod_{q=0}^{a} (q+b+c+2+\delta)} \qquad (4)$$

with $a,b,c \in \boldsymbol{N}$ and $\delta > -(b+c+2)$.

4 C++ OBJECT ORIENTED PROGRAMMING IN FEM

Highly specialized finite element codes are nowadays available for research, education and practical engineering applications.

Special applications, as the one presented in this paper, require special elements, material models, special solvers according to the particular Partial Differential Equation (PDE) model. In such situations the user is interested implementing new elements to obtain stable algorithms and to provide new facilities with the least possible effort.

This requirements are well satisfied by Object-Oriented Programming (OOP) in which related data and methods are collected in one entity: the object.

Object-oriented programming is therefore a way to improve the program structure of finite element codes, making modication and extension simpler [15].

Object-oriented programming supports modularity and data structuring by organizing the problem in objects with very few internal bindings.

Objects can inherit functionality from each other, thus using the same code for several types of PDE problems.

For the numerical parts procedural programming could still be used, therefore for FEM applications one of the best choice among programming languages is C++.

C++ includes C standard language which is an efficient procedural language and it includes OOP features [16], too.

4.1 Object-oriented computational electromagnetics

The object-oriented design for Computational Electromagnetics is useful to abstract the key concepts of FEM [15] and MoM application [17].

OOP computer codes are easily modified and extended to treat new classes of problems with incremental features, such as new elements, new computer algebra, etc.

Our C++ object-oriented code computes the modal longitudinal wavenumbers k_z at a given frequency *f* as well as the modal fields.

A symbolic representation of the singular FEM integrals is implemented to integrate the singular functions by adding up analytic integral results., as shown in section 3.2.2. This technique is highly effective and does not require complex programming to provide integral results to machine precision.

Numerical integration of the singular functions applied on integrand as (2) yields slightly less accurate results.

5 NUMERICAL RESULTS

Our first numerical test consider the circular vaned waveguide already studied in [2], that is a circular homogeneous waveguide of radius a with a zero-thickness radial vane extending to its center.

The problem is formulated as presentend in section 3.1 to study with higher-order singular curl-conforming elements the eigenmodes of the waveguide.

This structure is of interest because the analytical exact closed-form solution of this problem is known in the literature through Bessel functions ($\mathbf{J_{m/2}}$, $\mathbf{J'_{m/2}}$) in terms of TM and TE eigenmodes.

The first subscript labelling these modes is *m*; the second subscript *n* indicates the order of the zero, as usual.

Even values of *m* correspond to modes supported also by a circular waveguide, although the vane suppresses all the TM_{0n} circular waveguide modes. The modal fields exhibiting a ν=1/2 singularity at the edge of the vane are those of the TE_{1n} and the TM_{1n} modes. The calculated transverse electric field topographies of the first two singular modes (TE_{11} and TM_{11}) are reported in Figure 3.

Figure 4 reports the percentage error in the computed square values k^2_z of the longitudinal wavenumbers versus the number of unknowns. The comparison is among regular elements (p=3) [11] and singular elements.

Singular elements provides a noticeable improvement also in the regular mode results, since any lack of precision in the stiffness matrix coefficients yields to errors on all modes, see Figure 5.

Table 1 shows the matrix fill-in time versus the number of extra DOF's required to study the problems of Figure 4 with singular elements and for p = 3 (order of regular elements) on Pentium IV Xeon@2.4 GHz machine.

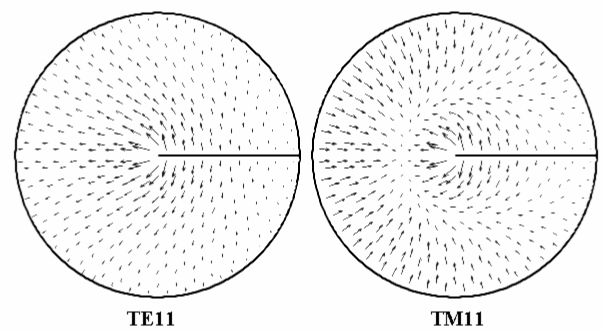

Figure 3. Transverse electric field component of the first two singular modes supported by the circular vaned waveguide.

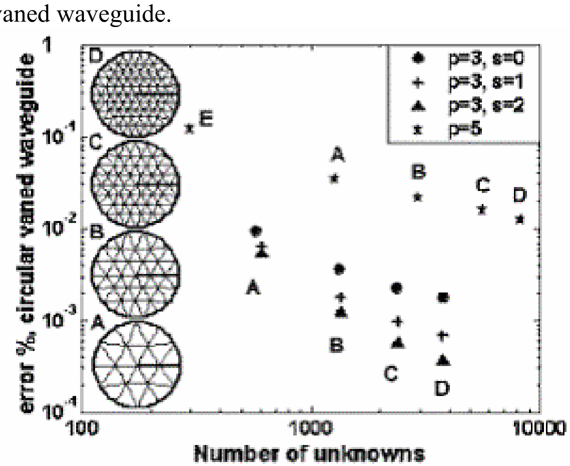

Figure 4. Percentage errors for the first modes of the circular vaned waveguide at $k_o a$ = 11 (a is the WG radius). Average error on the first twenty modes. Mesh A, B, C, D, E consists respectively of 24, 54, 96, 150, and 6 triangular curved elements, with only six singular elements near the edge.

6 CONCLUSIONS

This paper presents the benefits of using of vector bases that incorporate the edge conditions: optimal treatment of singular behavior, highly improved numerical precision, no need of mesh refinement, low increase of the Matrix fill-in time. Implementation features are described in the context of Object-Oriented Finite Element Analysis.

Acknowledgments

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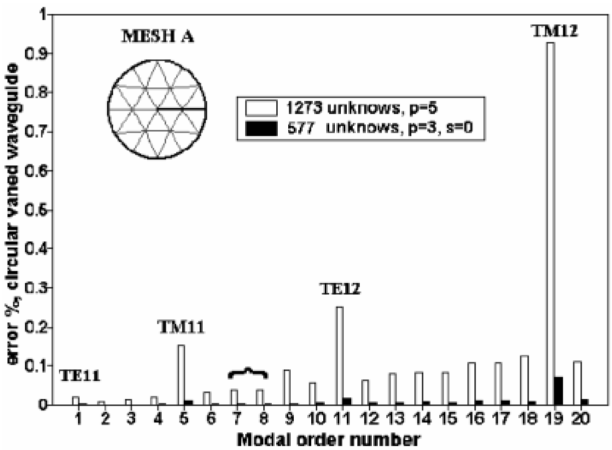

Figure 5. Percentage error in the computed square value of the longitudinal wavenumber (k^2_z) for each of the first twenty modes of the circular vaned waveguide at $k_o a$ = 11.

mesh		time	time	add.	time	add.	time	Add.
type	n ele	reg	s=0	%	S=1	%	s=2	%
A	24	10.84	11.75	1.08	13.16	1.21	21.50	1.98
B	54	22.64	24.03	1.06	25.80	1.14	36.91	1.63
C	96	43.56	44.78	1.03	46.86	1.08	56.48	1.30
D	150	67.50	67.69	1.00	71.53	1.06	79.70	1.18
extra unknown			16		50		102	

Table 1: Absolute matrix fill-in times (seconds) and percentage of additional time versus the number of extra DOF's required to study the problems of Figure 4 with singular elements for p=3 compared to regular elements for p = 3 on Pentium IV Xeon@2.4 GHz.

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