Perturbations of Symmetric Constraints in Eigenvalue Problems for Variational Inequalities

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1 Introduction

Let us consider a nonlinear eigenvalue problem of the form

$$\begin{cases} (\lambda, u) \in \mathbb{R} \times \mathbb{K} \\ \int_\Omega \left[DuD(v-u) + p(x,u)(v-u) \right] dx \geq \lambda \int_\Omega u(v-u)\, dx \quad \forall v \in \mathbb{K} \ , \\ \int_\Omega u^2 \, dx = \rho^2 \end{cases} \tag{1.1}$$

where Ω is a bounded open subset of $\mathbb{R}^n$, $\mathbb{K}$ is a convex closed subset of $H_0^1(\Omega)$ of the form

$$\mathbb{K} = \left\{ u \in H_0^1(\Omega) : \psi_1 \leq u \leq \psi_2 \right\} ,$$

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p is a given nonlinearity and $\rho > 0$.

If $p(x, \cdot)$ is odd, $\psi_1 = -\psi_2$ and suitable qualitative conditions are satisfied, it has been shown that (1.1) admits a sequence (λ_j, u_j) of solutions with $\lambda_j \to +\infty$ (see [6, 8, 18]).

Therefore one can ask what happens, if (1.1) is subjected to a non-symmetric perturbation. More precisely, one can expect that the number of solutions of the perturbed problem becomes greater and greater, as the perturbed problem approaches the original symmetric problem.

This type of result has been proved in [13], for a perturbation of (1.1) of the form

$$\begin{cases} (\lambda, u) \in \mathbb{R} \times \mathbb{K} \\ \int_\Omega \, [DuD(v-u) \, + \, (p(x,u) + q_h(x,u))\,(v-u)] \; dx \, + \\ + \, < \mu_h, v-u > \geq \, \lambda \int_\Omega \, u(v-u) \, dx \quad \forall v \in \mathbb{K} \\ \int_\Omega \, u^2 \, dx \; = \; \rho^2 \end{cases}$$

where q_h and μ_h become smaller and smaller in a suitable sense.

The purpose of this paper is to get the same result for a perturbation of (1.1) of the form

$$\begin{cases} (\lambda, u) \in \mathbb{R} \times \mathbb{K}_h \\ \int_\Omega \, [DuD(v-u) \, + \, p(x,u)(v-u)] \; dx \, \geq \, \lambda \int_\Omega \, u(v-u) \, dx \quad \forall v \in \mathbb{K}_h \; . \\ \int_\Omega \, u^2 \, dx \; = \; \rho^2 \end{cases} \tag{1.2}$$

Here $(\mathbb{K}_h)$ is a sequence of convex closed subsets of $H^1_0(\Omega)$ convergent to $\mathbb{K}$ in the sense of Mosco [21]. Our main contribution (Theorem (3.12)) asserts that for every $m \in \mathbb{N}$ there exists $\overline{h} \in \mathbb{N}$ such that for every $h \geq \overline{h}$ problem (1.2) admits at least m solutions.

In the case of equations, results of this kind have been obtained in [1, 4, 17]. Moreover, in some situations, the perturbed problem has still infinitely many solutions (see [3, 4, 5, 19, 22, 26]).

For variational inequalities, situations of this type have been considered in bifurcation problems (see, for instance, [11, 12, 14, 20, 23, 24]). However, in that case the limit problem (1.1) has a very particular structure ($\mathbb{K}$ is a convex cone and $p(x,\cdot)$ is linear). Moreover, it is $\mathbb{K}_h = t_h\widetilde{\mathbb{K}}$ with $t_h \to +\infty$ and $\widetilde{\mathbb{K}}$ a fixed closed convex set.

As an example, suppose that $\psi \in H^1(\Omega)$ and consider two sequences (ψ_h) and (φ_h) in $H^1(\Omega)$ such that

$$\varphi_h \le 0 \le \psi_h \qquad \text{a. e.}$$

$$\lim_h \psi_h = \psi\,, \qquad \lim_h \varphi_h = -\psi$$

in the strong topology of $H^1(\Omega)$. Then the case in which

$$\mathbb{K} \;=\; \left\{u \in H^1_0(\Omega) :\; -\psi \;\le\; u \;\le\; \psi \;\text{ a. e.}\right\}\,,$$

$$\mathbb{K}_h \;=\; \left\{u \in H^1_0(\Omega) :\; \varphi_h \;\le\; u \;\le\; \psi_h \;\text{ a. e.}\right\}$$

can be treated by our approach, even if it has not the structure of a bifurcation problem.

In the next section, we modify the notion of essential value, introduced in [13], to get a tool suitable for our purposes. The most important section is the third one, where we prove the main results.

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2 Essential values of continuous functionals

In the following X will denote a metric space endowed with the metric d and $f : X \to \mathbb{R}$ a continuous function. If $b \in \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$, let us set

$$f^b = \{u \in X :\; f(u) \le b\}\,.$$

For the topological notions involved in this section, the reader is referred to [25].

(2.1) Definition. Let $a, b \in \overline{\mathbb{R}}$ with $a \le b$. The pair $\left(f^b, f^a\right)$ is said to be *trivial*, if for every neighbourhood $[\alpha', \alpha'']$ of a and $[\beta', \beta'']$ of b $\left(\alpha', \alpha'', \beta', \beta'' \in \overline{\mathbb{R}}\right)$ there exists a continuous map $\mathcal{H} : f^{\beta'} \times [0,1] \to f^{\beta''}$ such that

$$\mathcal{H}(x,0) = x \quad \forall x \in f^{\beta'},$$

$$\mathcal{H}\left(f^{\beta'} \times \{1\}\right) \subseteq f^{\alpha''},$$

$$\mathcal{H}\left(f^{\alpha'} \times [0,1]\right) \subseteq f^{\alpha''}.$$

(2.2) *Remark.* If $\alpha < \alpha'$ in the above definition, we can suppose, without loss of generality, that $\mathcal{H}(x,t) = x$ on $f^{\alpha} \times [0,1]$. Actually, it is sufficient to substitute $\mathcal{H}(x,t)$ with $\mathcal{H}\left(x, t\vartheta(x)\right)$, where $\vartheta : f^{\beta'} \to [0,1]$ is a continuous function with $\vartheta(x) = 0$ for $f(x) \le \alpha$ and $\vartheta(x) = 1$ for $f(x) \ge \alpha'$.

(2.3) Theorem. *Let $a, c, d, b \in \overline{\mathbb{R}}$ with $a < c < d < b$. Let us assume that the pairs $\left(f^b, f^c\right)$ and $\left(f^d, f^a\right)$ are trivial.*

Then the pair $\left(f^b, f^a\right)$ is trivial.

Proof. Let $[\alpha', \alpha'']$ be a neighbourhood of a and $[\beta', \beta'']$ a neighbourhood of b. Without loss of generality, we can assume $\alpha'' < c$ and $\beta' > d$. Moreover, let $c < \gamma < d$. There exists a continuous map $\mathcal{H}_1 : f^{\beta'} \times [0,1] \to f^{\beta''}$ such that $\mathcal{H}_1(x,0) = x \quad \forall x \in f^{\beta'}$, $\mathcal{H}_1\left(f^{\beta'} \times \{1\}\right) \subseteq f^{\gamma}$, $\mathcal{H}_1\left(f^{\alpha''} \times [0,1]\right) \subseteq f^{\gamma}$ and such that $\mathcal{H}_1(x,t) = x$ on $f^{\alpha'} \times [0,1]$. Moreover there exists a continuous map $\mathcal{H}_2 : f^{\gamma} \times [0,1] \to f^{\beta'}$ such that $\mathcal{H}_2(x,0) = x \quad \forall x \in f^{\gamma}$, $\mathcal{H}_2\left(f^{\gamma} \times \{1\}\right) \subseteq f^{\alpha''}$, $\mathcal{H}_2\left(f^{\alpha'} \times [0,1]\right) \subseteq f^{\alpha''}$. If we define $\mathcal{H} : f^{\beta'} \times [0,1] \to f^{\beta''}$ by

$$\mathcal{H}(u,t) = \begin{cases} \mathcal{H}_1(u, 2t) & 0 \le t \le \frac{1}{2} \\ \mathcal{H}_2\left(\mathcal{H}_1(u,1), 2t-1\right) & \frac{1}{2} \le t \le 1 \end{cases},$$

it turns out that $\mathcal{H}$ is a continuous map with the required properties, therefore the thesis follows. ■

(2.4) Definition. A real number c is said to be *an essential value* of f, if for every $\varepsilon > 0$ there exist $a, b \in]c-\varepsilon, c+\varepsilon[$ with $a < b$ such that the pair $\left(f^b, f^a\right)$ is not trivial.

(2.5) *Remark.* The set of essential values of f is closed in $\mathbb{R}$.

(2.6) Theorem. *Let $a, b \in \overline{\mathbb{R}}$ with $a < b$. Let us assume that f has no essential value in $]a, b[$.*

Then the pair $\left(f^b, f^a\right)$ is trivial.

Proof. For a slightly different notion of essential value, the assertion is proved in [13, Theorem (2.5)]. Taking into account Theorem (2.3) , the same argument works in the present context. ■

Now let us recall a notion from [9, 15].

(2.7) Definition. For every $u \in X$ let us denote by $|df|(u)$ the supremum of the σ's in $[0, +\infty[$ such that there exist $\delta > 0$ and a continuous map $\mathcal{H} : B_\delta(u) \times [0, \delta] \to X$ with

$$d(\mathcal{H}(v, t), v) \leq t\,,$$

$$f(\mathcal{H}(v, t)) \leq f(v) - \sigma t\,.$$

The extended real number $|df|(u)$ is called *the weak slope* of f at u.

If X is a Finsler manifold of class C^1 and f a function of class C^1, it turns out that $|df|(u) = \|df(u)\|$ for every $u \in X$.

Let us point out that the above notion has been independently introduced also in [16].

(2.8) Definition. An element $u \in X$ is said to be *a critical point* of f, if $|df|(u) = 0$. A real number c is said to be *a critical value* of f, if there exists a critical point $u \in X$ of f such that $f(u) = c$. Otherwise c is said to be *a regular value* of f.

(2.9) Definition. Let c be a real number. The function f is said to satisfy *the Palais - Smale condition at level* c ($(PS)_c$ in short), if every sequence (u_h) in X with $|df|(u_h) \to 0$ and $f(u_h) \to c$ admits a subsequence (u_{h_k}) converging in X.

(2.10) Theorem. *Let c be an essential value of f. Let us assume that X is complete and f satisfies $(PS)_c$.*

Then c is a critical value of f.

Proof. Again the result is proved in [13, Theorem (2.10)] for a slightly different notion of essential value, but the same argument works in our case. ■

(2.11) Theorem. *Let E be a normed space, D a symmetric subset of E with respect to the origin with $0 \notin D$ and $f : D \to \mathbb{R}$ an even continuous function. Let us assume that D is non-empty and $k-$connected for every $k \geq 0$. For every $h \geq 1$ let us set*

$$c_h = \inf_{C \in \Gamma_h} \sup_{u \in C} f(u)\,,$$

where Γ_h is the family of compact subsets of D of the form $\varphi(S^{h-1})$ with $\varphi : S^{h-1} \to D$ continuous and odd.

Then $\Gamma_h \neq \emptyset$ for every $h \geq 1$ and we have

$$\sup_h c_h \leq \sup \{c \in \mathbb{R} : c \text{ is an essential value of } f\}$$

with the convention $\sup \emptyset = -\infty$.

Proof. In [13, Theorem (2.12)] it is shown that $\Gamma_h \neq \emptyset$ for every $h \geq 1$.

Let us set

$$\gamma = \sup \{c \in \mathbb{R} : c \text{ is an essential value of } f\}\,.$$

It is readily seen that $c_1 = \inf\limits_D f$ is an essential value of f or $c_1 = -\infty$. Therefore $c_1 \leq \gamma$. By contradiction let us assume that $\sup\limits_h c_h > \gamma$. Hence there exists $h \geq 1$ such that $c_h \leq \gamma < c_{h+1}$. Let $a, \alpha, \alpha', \alpha'' \in \mathbb{R}$ be such that $\gamma < \alpha < \alpha' < a < \alpha'' < c_{h+1}$. There exists $\varphi : S^{h-1} \to D$ continuous and odd with $\varphi(S^{h-1}) \subseteq f^\alpha$ and there exists a homotopy $\mathcal{H} : S^{h-1} \times [0,1] \to D$ between φ and a constant map. Since $a > \gamma$, f has no essential value in $]a, +\infty[$. By Theorem (2.6) the pair (D, f^a) is trivial. Let

$$\beta = \max \left\{ f(\mathcal{H}(x,t)) : x \in S^{h-1},\, t \in [0,1] \right\}.$$

Then there exists a continuous map $\eta : f^\beta \times [0,1] \to D$ such that $\eta(x,0) = x \ \ \forall x \in f^\beta$, $\eta\left(f^\beta \times \{1\}\right) \subseteq f^{\alpha''}$, $\eta\left(f^{\alpha'} \times [0,1]\right) \subseteq f^{\alpha''}$ and $\eta(x,t) = x$ on $f^\alpha \times [0,1]$. Let us define $\mathcal{K} : S^{h-1} \times [0,1] \to f^{\alpha''}$ by $\mathcal{K}(x,t) = \eta(\mathcal{H}(x,t),1)$. Then $\mathcal{K}$ is a homotopy between $\varphi : S^{h-1} \to f^{\alpha''}$ and a constant map. By [17, Lemma VI.4.5] there exists $\psi : S^h \to f^{\alpha''}$ continuous and odd. This is absurd, as $\alpha'' < c_{h+1}$. ■

3 Multiplicity of solutions for non-symmetric variational inequalities

Let Ω be a bounded open subset of $\mathbb{R}^n$ with $n \geq 3$, let $p : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that

$$p(x,-s) = -p(x,s),$$

$$s\,p(x,s) \geq 0,$$

$$|p(x,s)| \leq a(x) + b|s|^r$$

with $a \in L^{\frac{2n}{n+2}}(\Omega)$, $b \in \mathbb{R}$, $0 < r < \frac{n+2}{n-2}$ and let $\rho > 0$. For every $h \in \overline{\mathbb{N}} := \mathbb{N} \cup \{+\infty\}$ let $\mathbb{K}_h$ be a convex closed subset of $H^1_0(\Omega)$ with $0 \in \mathbb{K}_h$ such that the sequence $(\mathbb{K}_h)$ is convergent to $\mathbb{K}_\infty$ in the sense of Mosco [21]. This means that the following two properties are satisfied:

a) if $h_j \to +\infty$, $u_j \in \mathbb{K}_{h_j}$ and u_j is weakly convergent to u in $H_0^1(\Omega)$, then $u \in \mathbb{K}_\infty$;

b) for every $u \in \mathbb{K}_\infty$ there exists a sequence (u_h) strongly convergent to u in $H_0^1(\Omega)$ with $u_h \in \mathbb{K}_h$.

In the following $\|\cdot\|_p$ will denote the norm in $L^p(\Omega)$ and $\|\cdot\|_{1,p}$ the norm in $W^{1,p}(\Omega)$.

We are concerned with the family of nonlinear eigenvalue problems

$$\begin{cases} (\lambda, u) \in \mathbb{R} \times \mathbb{K}_h \\ \int_\Omega [DuD(v-u) \,+\, p(x,u)(v-u)]\, dx \,\geq\, \lambda \int_\Omega u(v-u)\, dx \quad \forall v \in \mathbb{K}_h \\ \int_\Omega u^2\, dx \,=\, \rho^2 \end{cases} . \tag{3.1}$$

Problems (3.1) have a variational structure. Let us set

$$S_\rho \,=\, \left\{ u \in L^2(\Omega) : \int_\Omega u^2\, dx \,=\, \rho^2 \right\}$$

and let us define for every $h \in \overline{\mathbb{N}}$ the functional $f_h : \mathbb{K}_h \cap S_\rho \to \mathbb{R}$ by

$$f_h(u) \,=\, \frac{1}{2} \int_\Omega |Du|^2\, dx \,+\, \int_\Omega P(x,u)\, dx\,,$$

where $P(x,s) = \int_0^s p(x,t)\, dt$. In the following, the set $\mathbb{K}_h \cap S_\rho$ will be endowed with the H_0^1-metric.

Let us recall a definition from [7].

(3.2) Definition. Let C be a convex subset of a Banach space X, let M be a hypersurface in X of class C^1, let $u \in C \cap M$ and let $\nu(u) \in X'$ be a unit normal vector to M at u. The sets C and M are said to be *tangent at* u, if we have either

$$< \nu(u), v-u > \leq 0 \qquad \forall v \in C$$

or

$$< \nu(u), v-u > \geq 0 \qquad \forall v \in C\,,$$

where $< \cdot, \cdot >$ is the pairing between X' and X.

The sets C and M are said to be *tangent*, if they are tangent at some point of $C \cap M$.

Let us set

$$D = \left\{ \, (h,u) \in \overline{\mathbb{N}} \times S_\rho : \; u \in \mathbb{K}_h \text{ and } \mathbb{K}_h \text{ and } S_\rho \text{ are not tangent at } u \right\} .$$

In the following, D will be endowed with the topology induced by $\overline{\mathbb{N}} \times L^2(\Omega)$.

(3.3) Theorem. *For every $\tilde{\varepsilon} > 0$ there exists a continuous map*

$$\eta : D \to H^1_0(\Omega)$$

such that for every $(h,u) \in D$ we have

$$\eta(h,u) \in \mathbb{K}_h \, ,$$

$$\int_\Omega u(\eta(h,u) - u)\, dx \; > \; 0 \, ,$$

$$\|\eta(h,u) - u\|_2 \; \leq \; \tilde{\varepsilon} \, ,$$

$$\frac{1}{2}\|D\eta(h,u)\|_2^2 \; \leq \; \frac{1}{2}\|Du\|_2^2 \; + \; \tilde{\varepsilon} \, .$$

Proof. For every $(h,u) \in D$ let us denote by $\Sigma(h,u)$ the set of σ's in $]0,+\infty[$ such that there exists $u^+ \in \mathbb{K}_h$ with

$$\int_\Omega u(u^+ - u) dx \; > \; \sigma \, , \; \|u^+ - u\|_2 \; < \; \tilde{\varepsilon} \, , \; \frac{1}{2}\|Du^+\|_2^2 \; < \; \frac{1}{2}\|Du\|_2^2 \; + \; \tilde{\varepsilon} \, .$$

Because of the definition of D, for every $(h,u) \in D$ we can find $u^+ \in \mathbb{K}_h$ with $\int_\Omega u(u^+ - u)dx > 0$. By substituting u^+ with $(1-t)u + tu^+$ for some $t \in]0,1[$, we can also suppose that $\|u^+ - u\|_2 < \tilde{\varepsilon}$, and $\frac{1}{2}\|Du^+\|_2^2 < \frac{1}{2}\|Du\|_2^2 + \tilde{\varepsilon}$. Therefore $\Sigma(h,u)$ is a non-empty interval in $\mathbb{R}$.

Moreover, let us consider $\sigma \in \Sigma(\infty,u)$ and let us choose $u^+ \in \mathbb{K}_\infty$ according to the definition of $\Sigma(\infty,u)$. Let (u_h^+) be a sequence converging to u^+ in $H_0^1(\Omega)$ with $(u_h^+) \in \mathbb{K}_h$. Then it is readily seen that $\sigma \in \Sigma(h,v)$ for every (h,v) sufficiently close to (∞,u) in D.

Now it is easy to see that, for every $(h,u) \in D$ and for every $\sigma \in \Sigma(h,u)$, we have $\sigma \in \Sigma(k,v)$ whenever (k,v) is sufficiently close to $(h,u) \in D$. Therefore there exists a continuous function $\sigma : D \to]0,+\infty[$ such that $\sigma(h,u) \in \Sigma(h,u)$.

For every $(h,u) \in D$ let us denote by $\mathcal{F}(h,u)$ the set of u^+'s in $\mathbb{K}_h$ such that

$$\int_\Omega u(u^+ - u)dx \geq \sigma(h,u)\,,\ \|u^+ - u\|_2 \leq \tilde{\varepsilon}\,,\ \frac{1}{2}\|Du^+\|_2^2 \leq \frac{1}{2}\|Du\|_2^2 + \tilde{\varepsilon}\,.$$

Then $\mathcal{F}(h,u)$ is a non-empty closed convex subset of $H_0^1(\Omega)$.

Let $(\infty,u) \in D$, $u^+ \in \mathcal{F}(\infty,u)$ and $\varepsilon > 0$. Let $\hat{u}^+ \in K_\infty$ be related to $\sigma(\infty,u)$, as in the definition of $\Sigma(\infty,u)$. By substituting $\hat{u}^+$ with $(1-t)u^+ + t\hat{u}^+$ for some $t \in]0,1[$, we can suppose that $\|\hat{u}^+ - u^+\|_{1,2} < \frac{\varepsilon}{2}$. Let $(\hat{u}_h^+)$ be a sequence converging to $\hat{u}^+$ in $H_0^1(\Omega)$ with $\hat{u}_h^+ \in \mathbb{K}_h$. Then it is readily seen that $\|\hat{u}_h^+ - u^+\|_{1,2} < \varepsilon$ and $\hat{u}_h^+ \in \mathcal{F}(h,v)$ for every (h,v) sufficiently close to (∞,u) in D.

Now it is easy to see that the multifunction $\{(h,u) \longmapsto \mathcal{F}(h,u)\}$ is lower semicontinuous on D. By Michael selection theorem [2, Theorem (1.11.1)] there exists a continuous map $\eta : D \to H_0^1(\Omega)$ such that $\eta(h,u) \in \mathcal{F}(h,u)$ and the thesis follows. ■

(3.4) Lemma. *Let us assume that $\mathbb{K}_\infty$ and S_ρ are not tangent. Then for every $b \in \mathbb{R}$ and $\hat{\varepsilon} > 0$ there exists a function $\eta : D \to H_0^1(\Omega)$ as in Theorem (3.3) such that*

$$\frac{1}{2}\int_\Omega |Dv|^2\,dx + \int_\Omega P(x,v)\,dx \leq f_\infty(u) + \hat{\varepsilon}$$

whenever

$$v = \rho \frac{(1-t)u + t\eta(\infty, u)}{\|(1-t)u + t\eta(\infty, u)\|_2}$$

with $u \in f^b_\infty$, $t \in [0,1]$.

Proof. By contradiction, let us assume that there exist $b \in \mathbb{R}$, $\hat{\varepsilon} > 0$, $u_j \in f^b_\infty$, $t_j \in [0,1]$ and a sequence of functions $\eta_j : D \to H^1_0(\Omega)$ such that $\|\eta_j(\infty, u_j) - u_j\|_2 \leq \frac{1}{j}$, $\frac{1}{2}\|D\eta_j(\infty, u_j)\|_2^2 \leq \frac{1}{2}\|Du_j\|_2^2 + \frac{1}{j}$ and

$$\frac{1}{2}\int_\Omega |Dv_j|^2\,dx + \int_\Omega P(x, v_j)\,dx > \frac{1}{2}\int_\Omega |Du_j|^2\,dx + \int_\Omega P(x, u_j)\,dx + \hat{\varepsilon}$$

with

$$v_j = \rho \frac{(1-t_j)u_j + t_j\eta_j(\infty, u_j)}{\|(1-t_j)u_j + t_j\eta_j(\infty, u_j)\|_2} .$$

Up to a subsequence, (u_j) is weakly convergent in $H^1_0(\Omega)$ to some $u \in \mathbb{K}_\infty \cap S_\rho$. Hence we have that $\eta_j(\infty, u_j) \rightharpoonup u$ in $H^1_0(\Omega)$. It follows that $[(1-t_j)u_j + t_j\eta_j(\infty, u_j)] \rightharpoonup u$ in $H^1_0(\Omega)$, hence $v_j \rightharpoonup u$ in $H^1_0(\Omega)$. On the other hand, from $\|(1-t_j)u_j + t_j\eta_j(\infty, u_j)\|_2 \geq \rho$ we deduce that

$$\frac{1}{2}\int_\Omega |Du_j|^2\,dx + \int_\Omega P(x, u_j)\,dx + \hat{\varepsilon} < \frac{1}{2}\int_\Omega |Dv_j|^2\,dx + \int_\Omega P(x, v_j)\,dx \leq$$

$$\leq \frac{1}{2}\int_\Omega |Du_j|^2\,dx + \frac{1}{j} + \int_\Omega P(x, v_j)\,dx .$$

For j sufficiently large we get a contradiction and the thesis follows. ■

For every $h \in \overline{\mathbb{N}}$ let us denote by $\pi_h : H^1_0(\Omega) \to \mathbb{K}_h$ the orthogonal projection in $H^1_0(\Omega)$ on the closed convex set $\mathbb{K}_h$.

(3.5) Lemma. *Let us assume that* $\mathbb{K}_\infty$ *and* S_ρ *are not tangent. Let* $b \in \mathbb{R}$, $\hat{\varepsilon} > 0$ *and* $\eta : D \to H^1_0(\Omega)$ *be as in the previous lemma. Moreover, if* $u \in \mathbb{K}_\infty \cap S_\rho$ *and* $\pi_h(\eta(\infty, u)) \neq 0$, *let*

$$P_h(u) = \rho \frac{\pi_h(\eta(\infty, u))}{\|\pi_h(\eta(\infty, u))\|_2} .$$

Then there exists $\overline{h} \in \mathbb{N}$ *such that the following facts hold:*

a) for every $h \geq \overline{h}$ *the sets* $\mathbb{K}_h$ *and* S_ρ *are not tangent at* u*, whenever* $u \in f_h^{b+2\hat{\varepsilon}}$*;*

b) for every $h, k \in \overline{\mathbb{N}}$ *with* $h, k \geq \overline{h}$ *and* $u \in f_k^b$ *we have*

$$\|\pi_h(\eta(k,u))\|_2 > \rho \,,$$

$$f_h\left(\rho \frac{\pi_h(\eta(k,u))}{\|\pi_h(\eta(k,u))\|_2}\right) \leq f_k(u) + 2\hat{\varepsilon}\,;$$

c) for every $h \geq \overline{h}$*,* $u \in f_\infty^b$ *and* $t \in [0,1]$ *we have*

$$\|(1-t)\eta(\infty, P_\infty(u)) + t\pi_\infty(\eta(h, P_h(u)))\|_2 > \rho \,,$$

$$f_\infty\left(\rho \frac{(1-t)\eta(\infty, P_\infty(u)) + t\pi_\infty(\eta(h, P_h(u)))}{\|(1-t)\eta(\infty, P_\infty(u)) + t\pi_\infty(\eta(h, P_h(u)))\|_2}\right) \leq f_\infty(u) + 4\hat{\varepsilon}\,.$$

Proof. Let us prove property $a)$. By contradiction, let us assume that there exist $h_k \to +\infty$ and $u_k \in f_{h_k}^{b+2\hat{\varepsilon}}$ such that $\mathbb{K}_{h_k}$ and S_ρ are tangent at u_k. Since $0 \in \mathbb{K}_{h_k}$, we have

$$\int_\Omega u_k(v - u_k)\, dx \leq 0 \qquad \forall v \in \mathbb{K}_{h_k}$$

and, up to a subsequence, (u_k) is weakly convergent in $H_0^1(\Omega)$ to some $u \in \mathbb{K}_\infty \cap S_\rho$. Let $v \in \mathbb{K}_\infty$. There exists $v_h \in \mathbb{K}_h$ such that $v_h \to v$ in $H_0^1(\Omega)$. Therefore for every $k \in \mathbb{N}$ we have

$$\int_\Omega u_k\,(v_{h_k} - u_k)\; dx \leq 0\,,$$

which implies

$$\int_\Omega u(v-u)\, dx \leq 0\;:$$

a contradiction, because $\mathbb{K}_\infty$ and S_ρ are not tangent.

Let us prove property $b)$. First of all, by contradiction, let us assume that there exist $h_j \to +\infty$, $k_j \to +\infty$ and $u_j \in f_{k_j}^b$ such that

$$\left\|\pi_{h_j}\,(\eta(k_j, u_j))\right\|_2 \leq \rho\,.$$

Up to a subsequence, (u_j) is weakly convergent in $H_0^1(\Omega)$ to some $u \in \mathbb{K}_\infty \cap S_\rho$. Consequently, $(\eta(k_j, u_j))$ is strongly convergent in $H_0^1(\Omega)$ to $\eta(\infty, u)$. Let v_h be a sequence converging to $\eta(\infty, u)$ in $H_0^1(\Omega)$ with $v_h \in \mathbb{K}_h$. We have that

$$\left\|\pi_{h_j}\left(\eta(k_j, u_j)\right) - \eta(k_j, u_j)\right\|_{1,2} \leq \|v_{h_j} - \eta(k_j, u_j)\|_{1,2}\,.$$

Therefore $\pi_{h_j}(\eta(k_j, u_j)) \to \eta(\infty, u)$ in $H_0^1(\Omega)$, which implies $\|\eta(\infty, u)\|_2 \leq \rho$. This is absurd, as $\int_\Omega u(\eta(\infty, u) - u)dx > 0$.

Now, by contradiction, let us assume that there exist $h_j \to +\infty$, $k_j \to +\infty$ and $u_j \in f_{k_j}^b$ such that

$$f_{h_j}\left(\rho \frac{\pi_{h_j}(\eta(k_j, u_j))}{\|\pi_{h_j}(\eta(k_j, u_j))\|_2}\right) > f_{k_j}(u_j) + 2\hat{\varepsilon}\,.$$

Up to a subsequence, (u_j) is weakly convergent in $H_0^1(\Omega)$ to some $u \in \mathbb{K}_\infty \cap S_\rho$. As in the previous argument, it follows $\pi_{h_j}(\eta(k_j, u_j)) \to \eta(\infty, u)$ in $H_0^1(\Omega)$. Since

$$f_\infty(u) \leq \liminf_j f_{kj}(u_j)\,,$$

by Lemma (3.4) we get a contradiction.

Let us prove property c). First of all, by contradiction, let us assume that there exist $h_k \to +\infty$, $u_k \in f_\infty^b$ and $t_k \in [0, 1]$ such that

$$\|(1 - t_k)\eta(\infty, P_\infty(u_k)) + t_k \pi_\infty(\eta(h_k, P_{h_k}(u_k)))\|_2 \leq \rho\,.$$

Up to a subsequence, (u_k) is weakly convergent in $H_0^1(\Omega)$ to some $u \in \mathbb{K}_\infty \cap S_\rho$. As in the proof of property b), we have that $\pi_{h_k}(\eta(\infty, u_k)) \to \eta(\infty, u)$ in $H_0^1(\Omega)$. It follows $P_{h_k}(u_k) \to P_\infty(u)$ and $\eta(h_k, P_{h_k}(u_k)) \to \eta(\infty, P_\infty(u))$ in $H_0^1(\Omega)$. As in the proof of b), we get a contradiction.

Finally, by contradiction, let us assume that there exist $h_k \to +\infty$, $u_k \in f_\infty^b$ and $t_k \in [0, 1]$ such that

$$f_\infty\left(\rho \frac{(1 - t_k)\eta(\infty, P_\infty(u_k)) + t_k \pi_\infty(\eta(h_k, P_{h_k}(u_k)))}{\|(1 - t_k)\eta(\infty, P_\infty(u_k)) + t_k \pi_\infty(\eta(h_k, P_{h_k}(u_k)))\|_2}\right) > f_\infty(u_k) + 4\hat{\varepsilon}\,.$$

Up to a subsequence, (u_k) is weakly convergent in $H_0^1(\Omega)$ to some $u \in \mathbb{K}_\infty \cap S_\rho$. As in the previous argument, we have $(1-t_k)\eta(\infty, P_\infty(u_k)) + t_k\pi_\infty(\eta(h_k, P_{h_k}(u_k))) \to \eta(\infty, P_\infty(u))$ in $H_0^1(\Omega)$. Therefore by Lemma (3.4) we get a contradiction. ■

(3.6) Lemma. *Let us assume that $\mathbb{K}_\infty$ and S_ρ are not tangent and let $b \in \mathbb{R}$ and $\hat{\varepsilon} > 0$.*

Then there exists $\overline{h} \in \mathbb{N}$ and, for every $h \geq \overline{h}$, two continuous maps

$$P_h : f_\infty^b \to \mathbb{K}_h \cap S_\rho\,, \qquad Q_h : f_h^{b+\hat{\varepsilon}} \to \mathbb{K}_\infty \cap S_\rho$$

such that $f_h(P_h(u)) \leq f_\infty(u) + \hat{\varepsilon}$, $f_\infty(Q_h(v)) \leq f_h(v) + \hat{\varepsilon}$ for every $u \in f_\infty^b$, $v \in f_h^{b+\hat{\varepsilon}}$ and such that $Q_h \circ P_h : f_\infty^b \to f_\infty^{b+2\hat{\varepsilon}}$ is homotopic to the inclusion map $f_\infty^b \to f_\infty^{b+2\hat{\varepsilon}}$ by a homotopy $\mathcal{H} : f_\infty^b \times [0,1] \to f_\infty^{b+2\hat{\varepsilon}}$ such that

$$\forall (u,t) \in f_\infty^b \times [0,1] : \qquad f_\infty(\mathcal{H}(u,t)) \leq f_\infty(u) + 2\hat{\varepsilon}\,.$$

Proof. Let us consider $(b + \hat{\varepsilon})$ and $\hat{\varepsilon}/2$. Let $\eta : D \to H_0^1(\Omega)$ be as in Lemma (3.4) and let $\overline{h} \in \mathbb{N}$ be as in Lemma (3.5). According to Lemma (3.5), for every $h \in \overline{\mathbb{N}}$ with $h \geq \overline{h}$ let us set

$$\forall u \in f_\infty^b : \qquad P_h(u) = \rho \frac{\pi_h(\eta(\infty, u))}{\|\pi_h(\eta(\infty, u))\|_2}\,,$$

$$\forall v \in f_h^{b+\hat{\varepsilon}} : \qquad Q_h(v) = \rho \frac{\pi_\infty(\eta(h, v))}{\|\pi_\infty(\eta(h, v))\|_2}\,.$$

By Lemma (3.5) it is readily seen that P_h and Q_h are well defined, continuous and satisfy $f_h(P_h(u)) \leq f_\infty(u) + \hat{\varepsilon}$, $f_\infty(Q_h(v)) \leq f_h(v) + \hat{\varepsilon}$ for every $u \in f_\infty^b$, $v \in f_h^{b+\hat{\varepsilon}}$. Now let us define $\mathcal{H}_0 : f_\infty^b \times [0,1] \to f_\infty^{b+\hat{\varepsilon}}$ by

$$\mathcal{H}_0(u,t) = \rho \frac{(1-t)u + t\eta(\infty, u)}{\|(1-t)u + t\eta(\infty, u)\|_2}\,.$$

Then $\mathcal{H}_0(u,0) = u$ and, by Lemma (3.4) , we have $f_\infty(\mathcal{H}_0(u,t)) \leq f_\infty(u) + \hat{\varepsilon}$. Essentially in the same way, we can define $\mathcal{H}_1 : f_\infty^b \times [0,1] \to f_\infty^{b+2\hat{\varepsilon}}$ by

$$\mathcal{H}_1(u,t) = \rho \frac{(1-t)P_\infty(u) + t\eta(\infty, P_\infty(u))}{\|(1-t)P_\infty(u) + t\eta(\infty, P_\infty(u))\|_2} .$$

Thus, $\mathcal{H}_1(u,0) = \mathcal{H}_0(u,1)$ and $f_\infty(\mathcal{H}_1(u,t)) \leq f_\infty(u) + 2\hat{\varepsilon}$.

Finally, let us define $\mathcal{H}_2 : f_\infty^b \times [0,1] \to f_\infty^{b+2\hat{\varepsilon}}$ by

$$\mathcal{H}_2(u,t) = \rho \frac{(1-t)\eta(\infty, P_\infty(u)) + t\pi_\infty(\eta(h, P_h(u)))}{\|(1-t)\eta(\infty, P_\infty(u)) + t\pi_\infty(\eta(h, P_h(u)))\|_2} .$$

By Lemma (3.5) , $\mathcal{H}_2$ is well defined, continuous, with $f_\infty(\mathcal{H}_2(u,t)) \leq f_\infty(u) + 2\hat{\varepsilon}$. Moreover, $\mathcal{H}_2(u,0) = \mathcal{H}_1(u,1)$ and $\mathcal{H}_2(u,1) = Q_h(P_h(u))$. The proof is complete. ■

Now we can prove a perturbation theorem concerning the essential values of f_∞ . Remark that, so far, $\mathbb{K}_\infty$ is any convex closed subset of $H_0^1(\Omega)$ with $0 \in \mathbb{K}_\infty$.

(3.7) Theorem. *Let us assume that $\mathbb{K}_\infty$ and S_ρ are not tangent. Let $c \in \mathbb{R}$ be an essential value of f_∞ .*

Then for every $\varepsilon > 0$ there exists $\overline{h} \in \mathbb{N}$ such that for every $h \geq \overline{h}$ the functional f_h has an essential value in $]c-\varepsilon, c+\varepsilon[$.

Proof. By contradiction, let us assume there exist $\varepsilon > 0$ and $h_k \to +\infty$ such that f_{h_k} has no essential value in $]c-\varepsilon, c+\varepsilon[$.

Let $a, b \in]c-\varepsilon, c+\varepsilon[$ with $a < b$. Let us prove that the pair $\left(f_\infty^b, f_\infty^a\right)$ is trivial. Let $[\alpha', \alpha'']$ be a neighbourhood of a and $[\beta', \beta'']$ be a neighbourhood of b. Since f_{h_k} has no essential value in $]a, b[$, the pair $\left(f_{h_k}^b, f_{h_k}^a\right)$ is trivial by Theorem (2.6) . Let $a', a'', b', b'' \in \mathbb{R}$ be such that $\alpha' < a' < a < a'' < \alpha''$ and $\beta' < b' < b < b'' < \beta''$. For every $k \in \mathbb{N}$ there exists a continuous function $\mathcal{K}_k : f_{h_k}^{b'} \times [0,1] \to f_{h_k}^{b''}$ such that $\mathcal{K}_k(u,0) = u$, $\mathcal{K}_k\left(f_{h_k}^{b'} \times \{1\}\right) \subseteq f_{h_k}^{a''}$, $\mathcal{K}_k\left(f_{h_k}^{a'} \times [0,1]\right) \subseteq f_{h_k}^{a''}$. Let $\hat{\varepsilon} > 0$ be such that $\alpha' + \hat{\varepsilon} \leq a'$, $a'' + \hat{\varepsilon} \leq \alpha''$, $\beta' + \hat{\varepsilon} \leq b'$, $b'' + \hat{\varepsilon} \leq \beta''$.

Now let $\overline{h}$, P_h and Q_h be related to β'' and $\hat{\varepsilon}$ as in Lemma (3.6) and let $k \in \mathbb{N}$ be such that $h_k \geq \overline{h}$. Let us define $\mathcal{H} : f_\infty^{\beta'} \times [0,1] \to f_\infty^{\beta''}$ by

$$\mathcal{H}(u,t) \;=\; Q_{h_k}\left(\mathcal{K}_k(P_{h_k}(u),t)\right) .$$

Of course $\mathcal{H}\left(f_\infty^{\beta'} \times \{1\}\right) \subseteq f_\infty^{\alpha''}$ and $\mathcal{H}\left(f_\infty^{\alpha'} \times [0,1]\right) \subseteq f_\infty^{\alpha''}$. By Lemma (3.6) $\mathcal{H}(\cdot,0) : \left(f_\infty^{\beta'}, f_\infty^{\alpha'}\right) \to \left(f_\infty^{\beta''}, f_\infty^{\alpha''}\right)$ is homotopic to the inclusion map. Therefore the pair $\left(f_\infty^b, f_\infty^a\right)$ is trivial.

We conclude that c is not an essential value of $f_\infty :$ a contradiction. ■

(3.8) Theorem. *Let us assume that $\mathbb{K}_\infty$ and S_ρ are not tangent.*

Then for every $b \in \mathbb{R}$ there exists $\overline{h} \in \mathbb{N}$ such that for every $h \geq \overline{h}$ the following facts hold:

a) for every $u \in f_h^b$ there exist $\lambda \in \mathbb{R}$ and $\eta \in H^{-1}(\Omega)$ such that $\|\eta\| = |df_h|(u)$ and

$$\int_\Omega [DuD(v-u)+p(x,u)(v-u)]\,dx \;\geq\; \lambda\int_\Omega u(v-u)\,dx + <\eta, v-u> \qquad \forall v \in \mathbb{K}_h\,;$$

b) the function f_h verifies $(PS)_c$ for every $c \leq b$.

Proof. Let $b \in \mathbb{R}$. By Lemma (3.5) there exists $\overline{h} \in \mathbb{N}$ such that for every $h \geq \overline{h}$ the sets $\mathbb{K}_h$ and S_ρ are not tangent at u, for every $u \in f_h^b$. Then the argument is the same of [13, Theorem (3.10)]. ■

Now let us consider the case in which

$$\mathbb{K}_\infty \;=\; \left\{u \in H_0^1(\Omega) :\, -\psi(x) \leq \tilde{u}(x) \leq \psi(x) \text{ cap. q.e. in } \Omega\right\}\,, \tag{3.9}$$

where $\psi : \Omega \to [0,+\infty]$ is a quasi-lower semicontinuous function such that $\int_\Omega \psi^2\,dx > \rho^2$ and $\tilde{u}$ is a quasi-continuous representative of u. For notions and results related to capacities, the reader is referred to [10].

Let us recall a characterization from [13].

(3.10) Theorem. *The following facts hold:*

a) given $u \in \mathbb{K}_\infty \cap S_\rho$, the sets $\mathbb{K}_\infty$ and S_ρ are tangent at u, if and only if

$$\tilde{u}(x) \neq 0 \Longrightarrow |\tilde{u}(x)| = \psi(x) \qquad \text{cap. q.e. in } \Omega\,;$$

b) the sets $\mathbb{K}_\infty$ and S_ρ are tangent, if and only if there exists a measurable subset E of Ω such that the function $\psi\chi_E$ is quasi-continuous and belongs to $H_0^1(\Omega) \cap S_\rho$.

(3.11) Theorem. *Let us assume that $\mathbb{K}_\infty$ and S_ρ are not tangent. Then the functional $f_\infty : \mathbb{K}_\infty \cap S_\rho \to \mathbb{R}$ admits a sequence (d_h) of essential values with $d_h \to +\infty$.*

Proof. Also this result is proved in [13, Theorem (3.9)] with a slightly different notion of esssential value. Taking into account Theorem (2.11) the same argument can be repeated in our situation. ■

Now we can prove our main result.

(3.12) Theorem. *Let us assume that $\mathbb{K}_\infty$ has the form (3.9) and that $\mathbb{K}_\infty$ and S_ρ are not tangent. Then for every $m \in \mathbb{N}$ there exists $\overline{h} \in \mathbb{N}$ such that for every $h \geq \overline{h}$ the problem (3.1) has at least m solutions $(\lambda_1, u_1), \cdots, (\lambda_m, u_m)$ with $u_1, \cdots, u_m$ all distinct.*

Proof. By Theorem (3.11) we can find m distinct essential values $d_1 < \cdots < d_m$ of f_∞. Let $b = d_m + 1$ and let $\varepsilon \in]0, 1[$ be such that $2\varepsilon < d_i - d_{i-1}$ for every i. By Theorem (3.7) there exists $\overline{h}_1 \in \mathbb{N}$ such that for every $h \geq \overline{h}_1$ the functional f_h has an essential value in every $]d_i - \varepsilon, d_i + \varepsilon[$, hence it has at least m distinct essential values in $] - \infty, d_m + \varepsilon[$. Let us choose $\overline{h}_2 \in \mathbb{N}$ according to Theorem (3.8) and let us set $\overline{h} = \max\{\overline{h}_1, \overline{h}_2\}$. If $h \geq \overline{h}$, f_h has m distinct critical values in $] - \infty, d_m + \varepsilon[$ by Theorems (2.10) and (3.8) , hence m distinct critical points $u_1, \cdots, u_m$. By Theorem

(3.8) there exist $\lambda_1, \cdots, \lambda_m \in \mathbb{R}$ such that (λ_i, u_i) is a solution of (3.1) , and the assertion follows. $\blacksquare$

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