On submanifolds whose shape operator is unipotent

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Abstract

The object of this article is to characterize submanifolds $M \subset \mathbb{R}^n$ of the Euclidean space whose shape operator A^ξ satisfies the equation $(A^\xi)^2 = k\|\xi\|^2 Id$, where $k > 0$ is constant.

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1 Introduction.

Characterization of submanifolds of the Euclidean space by imposing conditions on its extrinsic or intrinsic invariants is a classical subject of the differential geometry ([7], [2]) e.g. minimal submanifolds, isometric immersions of real space forms, isoparametric submanifolds, etc.

The shape operator A^ξ of a submanifold $M \subset \tilde{M}$ is called *unipotent* if it satisfies the equation

$$(A^\xi)^2 = k\|\xi\|^2 Id,$$

where ξ belongs to the normal bundle $T^\perp M$ of the submanifold M. The following theorem was proved in [1, Theorem 14].

Theorem *Let M^m be a Kähler submanifold of the complex space form $\mathbb{S}^n_c = \mathbb{C}P^n, \mathbb{C}^n, \mathbb{C}H^n$ $(n > m > 1)$ with $c = 4, 0, -4$ and A^ξ its shape operator. Then, the following conditions are equivalent:*

(i) $(A^{\xi})^2 = f\|\xi\|^2 Id$, *where* f *is a positive function on* M,

(ii) $c = 4$ *and* M *is an open subset of the complex quadric* $Q^m := \{[z_0 : \cdots : z_{m+1}] \in \mathbb{C}P^{m+1} \mid z_0^2 + \cdots + z_{m+1}^2 = 0\}$ *. In particular* $f = (\frac{4}{c})^2$ *is a constant.*

The object of this article is to give a similar characterization for submanifolds of the Euclidean space. We get the following result.

Theorem 1.1 *Let* $M^m \subset \mathbb{R}^n$ *be a submanifold of the Euclidean space and let* A^{ξ} *its shape operator. Then, the following facts are equivalent:*

(i) $(A^{\xi})^2 = k\|\xi\|^2 Id$, *where* $k > 0$ *is constant;*

(ii) M^m *is an open subset of a hypersphere or an open subset of the Veronese embedding of the real, complex, quaternionic or Cayley projective planes in the sphere* S^{3m+1} *where* $m = 1, 2, 4, 8$, *i.e. a focal manifold of an isoparametric hypersurface of a sphere with three distinct principal curvatures.*

It is interesting to remark that the proof uses techniques related with the normal holonomy group of the given submanifold. Normal holonomy groups were classified by C. Olmos [6] and they are very useful for the study of the geometry of homogeneous submanifolds [2].

2 Submanifolds with constant principal curvatures.

A submanifold $M \subset \mathbb{R}^n$ has *constant principal curvatures* if the shape operator $A^{\xi(t)}$ has constant eigenvalues for any $\nabla^{\perp}$-parallel normal vector field $\xi(t)$ along any piecewise differentiable curve. If in addition the normal bundle is flat i.e. $R^{\perp} \equiv 0$ then M is called *isoparametric* [7]. We will need the following result.

Theorem 2.1 [5] *A submanifold* M *of Euclidean space has constant principal curvatures if and only if is either isoparametric or a focal manifold to an isoparametric submanifold.*

For a discussion of focal manifolds of an isoparametric submanifold, its related Coxeter group, etc, see [2, Chapter 5].

3 Proof of Theorem 1.1.

It is well-known that (ii) implies (i) (see [3] or [4] for details).

Before dwelling into the proof that (i) implies (ii), we briefly explain the main ideas. In first place we will show that the submanifold M has principal constant curvatures. Then, from Theorem 2.1 there exists an isoparametric submanifold N such that $M = N_\xi$ i.e. M is a parallel submanifold of N. Then we will show that N is a hypersurface of a hypersphere. Since N is a hypersurface it follows that M is obtained from N by focalizing an eigendistribution i.e. $M = N_{\frac{\xi}{\lambda}}$ for a constant principal curvature λ of N, see [2, pag.119, Example 4.2].

Proof of Theorem 1.1. Since $A^{\xi(t)}$ satisfies a polynomial equation then the shape operator $A^{\xi(t)}$ with respect to any $\nabla^\perp$-parallel normal section $\xi(t)$ has constant eingenvalues. So the submanifold has constant principal curvatures.
Then, as a consequence of Theorem 2.1, either M is isoparametric or is a focal manifold of an isoparametric submanifold. If M is isoparametric then it is not difficult to see that M is a piece of an hypersphere.
Assume now that M is a focal submanifold of an isoparametric submanifold N. Then from the proof of Theorem 2.1 it follows that N is a holonomy tube $(M)_{\xi_p}$ through a principal vector $\xi_p \in T_p^\perp M$. We claim that N is a isoparametric hypersurface of a hypersphere. In fact, from [2, pag.126, Remark 4.4.13] it is enough to show that the codimension of a principal orbit $Hol_p(\nabla^\perp).\xi_p$ is one. Indeed, if it is greater than one, we will obtain a contradiction. Note first that the Ricci equation implies $[A^\xi, A^\eta] = 0$ for any two normal vectors ξ, η to a principal orbit of the action of the normal holonomy group. So, we can perform a simultaneous diagonalization of the family $\{A^\xi : \xi \in$ the normal space to a principal orbit $\}$. Let $\lambda(\xi) = \langle r, \xi \rangle$ be a common eigenvalue. Thus, if $\eta \perp r$ is in the normal space of the principal orbit then $(A^\eta)^2 \neq kId$ and we get a contradiction.
Thus, N is a isoparametric hypersurface of a hypersphere and the submanifold M is a focal submanifold of N i.e. $M = N_{\frac{\xi}{\lambda}}$ where ξ is the unit normal of N and λ a constant principal curvature of N. Since the shape operator A^ξ of M has two eigenspaces for any normal vector ξ then the "Tube formula" (see [2, pag.121, Lemma 4.4.7]) implies that the shape operator of N must have three principal curvatures.
So from a theorem of Elie Cartan [3] it follows that M must be one of the cited embeddings (for a beautiful proof of this Cartan's Theorem see [4]). □

References

[1] Alekseevsky, D.V. and Di Scala, A.J.: *The normal holonomy group of Kähler*

submanifolds, Proc. London Math. Soc. (3) 89 (2004) 193-216.

[2] Berndt, J. ; Console, S. and Olmos, C.: *Submanifolds and holonomy,* Research Notes in Mathematics 434, Chapman & Hall/CRC, 2003.

[3] Cartan, E.: *Sur des familles remarquables d'hypersurface isoparamétriques dans les espaces sphériques,* Math. Z. 45 (1939) 335-367.

[4] Console, S. and Olmos, C.: *Clifford systems, algebraically constant second fundamental form and isoparametric hypersurfaces* Manuscripta Math. 97 (1998) 335-342.

[5] Heintze, E. ; Olmos, C. and Thorbergsson G.: *Submanifolds with constant principal curvatures and normal holonomy groups,* Internat. J. Math. 2 (1991) 167-175.

[6] Olmos, C.: *The normal holonomy group,* Proc. Amer. Math. Soc. 110 (1990) 813-818.

[7] Palais, R. and Terng, C.-L.: *Critical point theory and submanifold geometry,* Lectures Notes in Math., vol. 1353, Springer-Verlag, Berlin, 1988.

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