

Polar actions on Berger spheres.

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Dedicated to Dmitri V. Alekseevsky on the occasion of his sixty-fifth birthday.

Abstract

The object of this article is to study a torus action on a so called Berger sphere. We also make some comments on polar actions on naturally reductive homogeneous spaces. Finally, we prove a rigidity-type theorem for Riemannian manifolds carrying a polar action with a fix point.

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1 Introduction.

In a generic Riemannian manifold (M, g) there are neither Killing vector fields (i.e. infinitesimal isometries) nor non trivial totally geodesic submanifolds. The concept of *polar actions* is a good example where both objects come together nicely. A Lie subgroup of isometries $G \subset I(M, g)$ acts polarly on (M, g) if there exists a connected closed submanifold Σ meeting all G -orbits orthogonally. Notice that the sections Σ are totally geodesic submanifolds of (M, g) . So, both Killing vector fields and totally geodesic submanifolds fit together in the setting of polar actions. Polar actions have been considered by several authors, see for example [6],[10], [1].

Let G act polarly on (M, g) and let \tilde{g} be another metric on M . Of course, G does not act isometrically on (M, \tilde{g}) in general. Even in case that the G -action is still by isometries on (M, \tilde{g}) the change of the metric, usually, changes also the second fundamental form of submanifolds of M . Thus, a totally geodesic submanifold of

(M, g) is not in general a totally geodesic submanifold of (M, \tilde{g}) . Indeed, if the G -action on (M, \tilde{g}) is still polar then there are no reasons to think that the *old* section Σ should be still a section after changing the metric.

The first part of this paper is dedicated to give a proof (and some generalizations to naturally reductive spaces) of a theorem stated in [5]. Namely, the S^1 -action on a Berger sphere $(S^3, \text{Berger}(\varepsilon))$ given by $\theta.(z, w) := (z, e^{i\theta}w)$ is polar if and only if $\varepsilon = 1$ i.e. the Berger sphere is just a standard sphere. Our proof is a straightforward corollary of the fact that there are no totally geodesic surfaces in a three dimensional Berger sphere different from the standard one. Also we will show that there is, up to isometry, just one locally polar action on a Berger sphere $(S^3, \text{Berger}(\varepsilon \neq 1))$, namely the diagonal action of the torus T^2 .

The proof in [5] is not correct since the authors make the confusion about *old* and *new* section explained above.

It is interesting to note that the above non-polar S^1 action on a fixed Berger sphere ($\varepsilon > 1$) has fixed points $p \in S^3$. Thus, S^1 acts also in any geodesic sphere around a fixed point p . Since any geodesic sphere has dimension 2, this action is locally polar (actually polar) on any geodesic sphere around p . This contrasts with the well-known fact that in Euclidean spaces (and real space forms) a G -action with a fixed point $p \in \mathbb{R}^n$ is polar if and only if is polar in just one (and then in any) sphere centered at $p \in \mathbb{R}^n$.

Polar actions with fixed points in general homogeneous spaces were studied in [8]. In the last part of this article we give a generalization of Theorem 3 in [8]. Namely, we get

Theorem 1.1 *Let M be a homogeneous Riemannian manifold. Let G be a Lie group of isometries acting polarly on M . If the G -action on M has a fixed point and a section for the G -action is a compact locally symmetric space then, M is locally symmetric.*

Historical Remark. According to É. Cartan it was G. Ricci-Curbastro who first observed the interplay between totally geodesic submanifolds and isometries. Let us quote from Cartan's book this *théorème remarquable* [4, pp. 122, 107]:

S'il existe dans l'espace de Riemann une famille à un paramètre de plans, leurs trajectoires orthogonales établissent entre les différents plans de la famille une correspondance ponctuelle isométrique.

2 Preliminary results.

Let M be a Riemannian manifold and let $G \subset I(M)$ be a connected Lie subgroup of isometries. The action of G on M is called *polar* if there exists a closed embedded submanifold $\Sigma \subset M$, called a section, such that every G -orbit hits Σ perpendicularly. The G -action is called *locally polar* if the distribution given by the normal spaces to the principal orbits is integrable. A polar action is locally polar but the converse it is not true. For a detailed discussion of both definitions see [7, p. 6 and Appendix A.].

It is known that if G acts polarly then the section Σ is a totally geodesic submanifold of M (see [11] or [1] for a detailed explanation). Indeed, note that if the action is locally polar then the integral leaves of the normal distribution are totally geodesic as a consequence of the Killing equation.

In [12] it is proved that the existence of a totally geodesic hypersurface in an irreducible and simply connected naturally reductive homogeneous space M implies that M has constant sectional curvature. As a consequence we get the following proposition.

Proposition 2.1 *Let M be a simply connected and irreducible naturally reductive homogeneous space. Let X be a Killing vector field and let ϕ^X be the corresponding monoparametric Lie group of isometries. If ϕ^X acts locally polar on M then M has constant sectional curvature.*

Let $S^{2n-1} = \{(z_1, \dots, z_n) \in \mathbb{C}^n : |z_1|^2 + \dots + |z_n|^2 = 1\}$ be the standard unit sphere in $\mathbb{C}^n = \mathbb{R}^{2n}$.

Let us now change the standard metric (i.e. the Riemannian structure) on the sphere S^{2n-1} just by changing (by a positive constant ε) the length of the Hopf vector field $H(z_1, z_2, \dots, z_n) = (iz_1, iz_2, \dots, iz_n)$. This new family of metrics $(S^{2n-1}, \text{Berger}(\varepsilon))$ on the sphere, are called Berger spheres.

Theorem 2.2 [3] *The Berger spheres $(S^{2n-1}, \text{Berger}(\varepsilon))$ are naturally reductive Riemannian homogeneous spaces. Indeed, they are geodesic spheres (of a convenient radius) in a complex space form.*

We recall also the following proposition.

Proposition 2.3 *A Berger sphere $(S^{2n-1}, \text{Berger}(\varepsilon))$ has constant sectional curvature if and only if $\varepsilon = 1$ i.e. the Berger sphere is just the standard one.*

We summarize these results as follows.

Theorem 2.4 *If the flow of a Killing vector field $X \neq 0$ of a Berger sphere $(S^{2n-1}, \text{Berger}(\varepsilon))$ acts locally polar then $\varepsilon = 1$.*

3 Polar actions on Berger spheres.

In this section we give an application of the results of the previous section.

Let $T^{n-1} = \underbrace{S^1 \times S^1 \times \cdots \times S^1}_{n-1 \text{ times}}$ act on $S^{2n-1} \subset \mathbb{C}^n$, endowed with the standard metric, in the following way

$$(e^{i\theta_2}, \dots, e^{i\theta_n}).(z_1, z_2, \dots, z_n) := (z_1, e^{i\theta_2} z_2, \dots, e^{i\theta_n} z_n).$$

In other words, the action is diagonal, it fixes the first coordinate, and is just a rotation of the other coordinates.

Then is easy to check that this action (by isometries) is polar and a section is given by:

$$\Sigma := S^{2n-1} \cap \{(z_1, z_2, \dots, z_n) \in \mathbb{C}^n : \operatorname{Im}(z_2) = \operatorname{Im}(z_3) = \cdots = \operatorname{Im}(z_n) = 0\}.$$

Theorem 3.1 *Let $\varepsilon > 0$ be a real number. If the above T^{n-1} -action is locally polar on $(S^{2n-1}, \operatorname{Berger}(\varepsilon))$ then $\varepsilon = 1$, i.e. the action is (locally) polar just on the standard sphere.*

Proof. Let $\sigma \in U(n)$ be the diagonal matrix $\sigma := \operatorname{diag}(1, 1, \underbrace{-1, \dots, -1}_{n-2 \text{ times}})$. Then,

σ is an isometry of $(S^{2n-1}, \operatorname{Berger}(\varepsilon))$. Thus, the set F_σ of fixed points of σ is a totally geodesic submanifold of $(S^{2n-1}, \operatorname{Berger}(\varepsilon))$. Indeed, it is not difficult to check that $F_\sigma = S^{2n-1} \cap \{(z_1, z_2, 0, 0, \dots, 0) \in \mathbb{C}^n\}$. Also note that the induced Riemannian metric on F_σ is just a Berger sphere i.e. $F_\sigma = (S^3, \operatorname{Berger}(\varepsilon))$. Finally, note that the torus T^{n-1} leaves F_σ invariant and the induced action is exactly the S^1 -action on S^3 given by $(z_1, e^{i\theta_2} z_2)$.

If the torus T^{n-1} acts locally polar on $(S^{2n-1}, \operatorname{Berger}(\varepsilon))$ then the S^1 -action is also locally polar on $F_\sigma = (S^3, \operatorname{Berger}(\varepsilon))$. Thus, by Theorem 2.4 it follows that $\varepsilon = 1$ as we claim. \square

Up to conjugation, the torus $T^2 = \operatorname{diag}(e^{i\theta_1}, e^{i\theta_2})$ is the unique Lie subgroup of isometries of $(S^3, \operatorname{Berger}(\varepsilon \neq 1))$ which is not transitive on S^3 and acts with codimension 1. So, we get the following theorem.

Theorem 3.2 *On a Berger sphere $(S^3, \operatorname{Berger}(\varepsilon \neq 1))$ there is just one locally polar action, up to isometry. Namely, the action of the torus T^2 given by diagonal multiplication.*

4 Polar actions with fixed points.

In [8] the authors studied G -actions with fixed points and they proved the following theorem.

Theorem 4.1 [8, Theorem 3] *Let M be a compact Riemannian homogeneous space. Let G be a compact Lie group acting on M isometrically and polarly; if the G -action on M has a fixed point and the section for the G -action is flat or a rank one symmetric space then M is a locally symmetric.*

Remark 4.2 *Notice that our Theorem 2.4 for $n = 2$ follows also from the above theorem. Indeed, a section for the S^1 -actions is a totally geodesic homogeneous submanifold of dimension 2 (i.e. a symmetric space of rank one or a flat surface). This forces the Berger sphere to be locally symmetric and this occurs just when $\varepsilon = 1$.*

Proof of Theorem 1.1 Let $p \in M$ be a fixed point of the G -action. Then, it is standard to see that any geodesic $\gamma(t)$ passing through p is contained in a section. Since sections are compact symmetric spaces any geodesic is contained in a compact flat. Thus, M is a compact Riemannian homogeneous space all of whose geodesics are contained in a compact flat. So we can apply [9] to get that the universal covering \widetilde{M} splits as $\widetilde{M} = \mathbb{R}^n \times C_1 \times \cdots \times C_k \times S$, where S is a symmetric space and the manifolds C_l are manifolds all of whose geodesics are closed. Since \widetilde{M} is homogeneous any factor C_l is homogeneous too. Then, from [2, pp. 194, 7.47] we get that the manifolds C_l are rank one homogeneous symmetric spaces. \square

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