Autoparallel distributions and splitting theorems

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Abstract

We study some links between autoparallel distributions and the factorization of a riemannian manifold. Finally, we prove a splitting theorem for Lie groups with biinvariant metrics.

Key Words: autoparallel distributions, De Rham splitting theorem, riemannian products.

1 Introduction

Splitting theorems are very important in the study of intrinsic and extrinsic geometry of riemannian manifolds. De Rham theorem [KN] in the intrinsic case, and Moore's Lemma [M] in the extrinsic one, are well-known examples. Splitting theorems play also an important role in the theory of isoparametric submanifolds, see for example [PT], [HL].

In many situations it is possible to construct (locally) two autoparallel distributions, spanning the tangent bundle, which are perpendicular modulo the intersection. If the intersection is trivial then both distributions must be parallel and so the riemannian manifold splits. In this case the hypothesis of orthogonality cannot be omitted (see first example in section 2).
In [D], the author shows that (see Proposition 2.1 below) a riemannian manifold splits if it has two autoparallel nontrivial distributions satisfying a curvature condition. Existence of two autoparallel distributions does not imply in general, that the manifold splits.

In this article we prove the following theorem which shows that, in homogeneous case, this condition is, however, sufficient to imply the splitting of the involved manifold.

Theorem 1.1 *Let G be a Lie group and $\langle , \rangle$ a biinvariant metric on G. If there exist two $G-$invariant (nontrivial) autoparallel distributions $\mathcal{D}_1, \mathcal{D}_2$ such that:*

i) $TG = \mathcal{D}_1 + \mathcal{D}_2$

ii) $\mathcal{D}_1, \mathcal{D}_2$ are orthogonal modulo intersection.

then G is not simple. In particular, the riemannian manifold $(G, \langle , \rangle)$ is a product of Lie groups with biinvariant metrics (i.e. $G = G_1 \times G_2$ and $\langle , \rangle = \langle , \rangle_1 \times \langle , \rangle_2$).

2 Autoparallel distributions and factorization

Let $(M, \langle , \rangle)$ be a riemannian manifold and ∇ the Levi-Civita connection. A distribution $\mathcal{D}$ is autoparallel if $\nabla_X Y \in \mathcal{D}$ for all vector fields $X, Y \in \chi(\mathcal{D})$. The distribution $\mathcal{D}$ is called *parallel* if $\nabla_X Y \in \mathcal{D}$ for all vector fields $Y \in \chi(\mathcal{D})$ and X an arbitrary vector field. An autoparallel distribution is integrable (zero torsion property) with totally geodesic leaves. Conversely, the tangent spaces to the leaves of a totally geodesic foliation define an autoparallel distribution.

As we pointed out in the introduction one has:

Proposition 2.1 *If (locally) $TM = \mathcal{D}_1 \oplus \mathcal{D}_2$, orthogonal sum and both distributions are autoparallel, then M splits locally.*

We include the proof, since it is difficult to find it in the mathematical literature. In fact, if $X \in \chi(\mathcal{D}_1)$ and $Y, Z \in \chi(\mathcal{D}_2)$ then

$$0 = Y\langle X, Z\rangle = \langle \nabla_Y X, Z\rangle + \langle X, \nabla_Y Z\rangle = \langle \nabla_Y X, Z\rangle$$

which implies that $\mathcal{D}_1$ is parallel. Then we can use the well-known De Rham theorem [KN] to decompose M.

First example . If we omit the hypothesis of perpendicularity in the above Proposition 2.1 then the proposition is false. In fact, take any surface S of non zero curvature. Then it is clear that the manifold is not locally a product. Around any point $p \in S$ it is possible to define an autoparallel distribution $\mathbf{F}_p$ given by the radial direction (i.e. $\mathbf{F}_p$ is generated by the radial vector field in normal coordinates around p). Thus, if p, q are sufficient close these two distributions are complementary and the manifold does not split. $\square$

Remark 2.2 *The hyperboloid of one sheet given by $x^2 + y^2 - z^2 = 0$ in the euclidean space $\mathbb{R}^3$ is a double ruled surface. So, there are two complementary (non perpendicular!) totally geodesic distributions and the hyperboloid does not split.*

In [D] the author shows that the hypothesis on the intersection can be deleted introducing conditions on the behavior of the curvature tensor. More precisely:

Proposition 2.3 *[D] Let M be a Riemannian manifold and let $\mathcal{T}_1$ and $\mathcal{T}_2$ be autoparallel distributions spanning TM which are orthogonal modulo the intersection $\mathcal{T}_1 \bigcap \mathcal{T}_2$ ($\mathcal{T}_1 \neq TM \neq \mathcal{T}_2$). Assume that the curvature tensor $R_{XY} = 0$ if X lies in $\mathcal{T}_1$ and Y lies in $\mathcal{T}_2$. Then, for each $p \in M$ there exists a nontrivial subspace of $\mathcal{T}_1(p)$ which contains $\mathcal{T}_2^{\perp}$ and is invariant under the local holonomy group Φ_p^{loc}. In particular M is locally reducible at each point.*

Second example . If we omit the hypothesis on the curvature tensor in the above Proposition 2.3 then the proposition is false. To give an example we recall the *half plane* model of the hyperbolic space H^n. In this model, the hyperbolic space H^n $:= \{p \in \mathbb{R}^n : p = (x_1, \ldots, x_n), x_n > 0\}$ is endowed with the metric $ds^2 := \frac{\langle , \rangle}{x_n^2}$. The totally geodesic submanifolds are the semispaces parallels to the x_n-axis and the upper part of spheres which meets orthogonally the hyperplane $x_n = 0$. The hyperbolic space H^n has constant negative sectional curvature and then it is a locally irreducible riemannian space. Define the following two foliations of H^n: $\mathbf{D}_1 := \{p \in \mathbb{R}^n : p = (x_1, \ldots, x_n), x_1 = cte, x_n > 0\}$ and $\mathbf{D}_2 := \{p \in \mathbb{R}^n : p = (x_1, \ldots, x_n), x_2 = cte, x_n > 0\}$.
Then, the two distributions: $\mathcal{D}_1 := T\mathbf{D}_1$ and $\mathcal{D}_2 := T\mathbf{D}_2$ are (nontrivial) totally geodesic, orthogonal modulo intersection and $TH^n = \mathcal{D}_1 + \mathcal{D}_2$. This shows that without curvature assumptions proposition 2.3 is false. Observe that if one also avoids the hypothesis of nontriviality of the distributions then Proposition 2.3 is trivially false as shown by any ruled non developable surface.
It is interesting to note that there also exists examples in positive curvature. In fact, let $i : S^n \times S^n = M \times N \to S^{2n+1}(\sqrt{2})$ be the standard immersion of the product of spheres into a big dimensional sphere. As $i(S^n \times S^n)$ is an hypersurface of $S^{2n+1}(\sqrt{2})$ we can define the parallel submanifolds $i_t : S^n \times S^n \to S^{2n+1}(\sqrt{2})$, namely $i_t(x) := i(x)cos(t) + \sqrt{2}\xi(x)sin(t)$ where ξ is a (unitary) normal vector field along $i(S^n \times S^n)$. Note that i_t is also a product immersion of a product of spheres of different radius. Thus, we have $i_t(S^n \times S^n) = M_t \times N_t \subset S^{2n+1}(\sqrt{2})$. Then it is not difficult to verify that (locally): $TS^{2n+1}(\sqrt{2}) = TM_t \oplus TN_t \oplus \nu(M_t \times N_t)$ and that the two distributions given by $\mathcal{D}_1 := TM_t \oplus \nu(M_t \times N_t)$, $\mathcal{D}_2 := TN_t \oplus \nu(M_t \times N_t)$ are autoparallel and orthogonal modulo intersection, where $\nu(M_t \times N_t)$ is the normal bundle of the submanifold $M_t \times N_t$. $\square$

Remark 2.4 *Another way to construct examples of irreducible riemannian manifolds with the above properties is the following. Look for functions f, g such that the following metric $\langle , \rangle$ defined in an open subset of $\mathbb{R}^3$ verifies: (1) $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ and $\frac{\partial}{\partial z}$ are perpendicular. (2) $\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial x} \rangle = g(x,y)$, $\langle \frac{\partial}{\partial z}, \frac{\partial}{\partial z} \rangle = f(y,z)$ and $\langle \frac{\partial}{\partial y}, \frac{\partial}{\partial y} \rangle = 1$. (3) The distributions spanned by $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ and by $\frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ are autoparallel. (4) The curvature tensor of $\langle , \rangle$ does not have nullity.*

Finally, we give the proof of Theorem 1.1.

Proof of Theorem 1.1 . The following proof is due to J.J. Bigeón and J. Vargas. Let $\mathfrak{g}$ be the Lie algebra of G. It is well-known that $G-$ invariant distributions $\mathcal{D}$ are in $1-1$ correspondence with subspaces $V := \mathcal{D}(e)$ of $\mathfrak{g}$, where e is the identity of G. Moreover, $\mathcal{D}$ is autoparallel (resp. parallel) if and only if V is a subalgebra (resp. an ideal) of $\mathfrak{g}$ (i.e. $[V,V] \subset V$ (resp. $[V,\mathfrak{g}] \subset V$)). Thus, $i)$ and $ii)$ imply that $\mathfrak{g} = A \oplus B \oplus C$ (nontrivial orthogonal sum) and $A \oplus B$, $B \oplus C$ are Lie subalgebras of $\mathfrak{g}$. Assume that $\mathfrak{g}$ is simple. We claim that $I := C + [C,C] + [C,[C,C]] + \dots$ is a nontrivial ideal of $\mathfrak{g}$ (Note that $C \subset I \subset B \oplus C$). In fact, $[C,I] \subset I$ by construction. In order to finish the proof it is sufficient to prove that $[A,I] \subset I$ and $[B,I] \subset I$. Let $K(x,y)$ be the Killing form of $\mathfrak{g}$. As G is simple it is well-known that K is a multiple of $\langle , \rangle$. Then for $a \in A, b \in B$ and $c \in C$:

$$K(b,[a,c]) = K([b,a],c) = 0$$

because $A \oplus B$ is a Lie subalgebra and $A \oplus B \perp C$. So, we obtain that $[A,C] \subset A \oplus C$. Then, for $a_1, a_2 \in A$ and $c \in C$ we have that:

$$K(a_1,[a_2,c]) = K([a_1,a_2],c) = 0$$

which implies that $[A,C] \subset C$. Now, from Jacobi identity, it is standard to conclude $[A,I] \subset I$. Finally, for $b_1, b_2 \in B$ and $c \in C$ we have that:

$$K(b_1,[b_2,c]) = K([b_1,b_2],c) = 0$$

which implies that $[B,C] \subset C$. Again, from Jacobi identity, it is standard to conclude $[B,I] \subset I$. □

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