Equilibrium Analysis of Cellular Neural Networks

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***Abstract*— Cellular neural networks are dynamical systems, described by a large set of coupled nonlinear differential equations. The equilibrium point analysis is an important step for understanding the global dynamics and for providing design rules. We yield a set of sufficient conditions (and a simple algorithm for checking them) ensuring the existence of at least one stable equilibrium point. Such conditions give rise to simple constraints, that extend the class of CNN, for which the existence of a stable equilibrium point is rigorously proved. In addition, they are suitable for design and easy to check, because they are directly expressed in term of the template elements.**

***Index Terms*— Nonlinear circuits, cellular neural networks, equilibrium analysis, stable equilibrium points.**

I. Introduction

CELLULAR Neural Networks (CNNs) are analog dynamic processors, that have found several applications for the solution of complex computational problems [1]-[6]. A CNN can be described as an array of identical nonlinear dynamical systems (called cells), that are locally interconnected. In most applications the connections are specified through space-invariant templates.

CNNs are modeled by large systems of coupled nonlinear differential equations, that have been mainly studied through extensive computer simulations. As far as the dynamic behavior is concerned, CNNs can be divided in two main classes: stable CNNs, with the property that each trajectory (with the exception of a set of measure zero) converges towards an equilibrium point; unstable CNNs, that exhibit at least one attractor, that is not a stable equilibrium point. Due to the complex CNN mathematical model, so far a complete characterization of the two classes above is not available [7].

A preliminary step for investigating CNN dynamics is the equilibrium point analysis: in fact the existence of at least one stable equilibrium point is a necessary condition for the CNN stability, whereas the absence of stable equilibria is a sufficient condition for instability. CNN equilibrium points have been studied in several papers, that have provided important conditions, concerning the existence of stable equilibrium points [8]-[14]. Most of these contributions, however, also apply to general networks and do not really exploit the two main characteristics of a CNN, i.e. the local connectivity and the space-invariance structure.

In this paper we provide a set of sufficient conditions, ensuring the existence of at least one stable equilibrium point. They are expressed in term of the template elements and, hence, they are very easy to check and to exploit for CNN design.

A simple algorithm is given for checking the proposed conditions and it is used for performing a detailed comparison with previous results reported in the literature.

Through this comparison we show that they include most of the sufficient conditions previously reported in the literature and considerably extend the class of CNNs for which the existence of a stable equilibrium point is rigorously proved [8]-[13].

II. Space-invariant CNNs

We consider CNNs composed by $N \times M$ cells arranged on a regular grid. We denote the position of a cell with two indexes (k, l) with $1 \leq k \leq N$ and $1 \leq l \leq M$ and we assume that cell $(1, 1)$ is located in the upper left corner and cell (N, M) is located in the lower right corner.

The network dynamics is governed by the following normalized state equations

$$\begin{aligned} \dot{x}_{kl} &= -x_{kl} + \sum_{|n|\leq r, |m|\leq r} A_{nm}\, y_{k+n,l+m} + \\ &\quad + \sum_{|n|\leq r, |m|\leq r} B_{nm}\, u_{k+n,l+m} + I \end{aligned} \tag{1}$$

where x_{kl} and u_{kl} represent the state-voltage and the input voltage of cell (k, l); y_{kl} is the output voltage, defined through the following piecewise linear expression:

$$y_{kl} = f(x_{kl}) = \frac{1}{2}\left(|x_{kl} + 1| - |x_{kl} - 1|\right) \tag{2}$$

Finally r denotes the neighborhood of interaction of each cell; A_{nm} and B_{nm} are the elements of the linear templates $\boldsymbol{A}$ and $\boldsymbol{B}$, that are assumed to be space-invariant

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and I is the bias term. The description of the structure is completed by the specification of the boundary conditions, that we assume to be null.

An alternative and useful expression for the state equations of a CNN is obtained by ordering the cells in some way (e.g. by rows or by columns) and by repacking the state, the input, the output variables and the bias terms into the vectors $\boldsymbol{x}$, $\boldsymbol{u}$, $\boldsymbol{y}$ and $\hat{\boldsymbol{I}}$. The following compact form is obtained:

$$\dot{\boldsymbol{x}} = -\boldsymbol{x} + \hat{\boldsymbol{A}}\boldsymbol{y} + \hat{\boldsymbol{B}}\boldsymbol{u} + \hat{\boldsymbol{I}} \tag{3}$$

where matrices $\hat{\boldsymbol{A}}$ and $\hat{\boldsymbol{B}}$ are obtained through the templates $\boldsymbol{A}$ and $\boldsymbol{B}$, as explained in [15].

III. EQUILIBRIUM POINTS

For the sake of simplicity, we assume that the input and the bias terms are null; however we remark that the results presented in this paper also apply, with slight modifications, to the case of constant inputs and non-zero boundary conditions. In the following, with the term *saturation region* we indicate a linear region of the state space where all the output voltages y_{kl} are saturated (i.e. $\forall\ k,\ l:\ |x_{kl}| > 1$). A saturation region will be described by a matrix, containing as entries the output voltage values (i.e. $+1$ or -1). We also assume $r = 1$, that is the template $\boldsymbol{A}$ is represented by a 3×3 matrix, as required in most applications. Furthermore, the central element of $\boldsymbol{A}$ is greater than one (i.e. $A_{0,0} > 1$), to ensure that stable equilibrium points are located in saturation regions [1]. We have:

$$\boldsymbol{A} = \begin{bmatrix} A_{-1,-1} & A_{-1,0} & A_{-1,1} \\ A_{0,-1} & A_{0,0} & A_{0,1} \\ A_{1,-1} & A_{1,0} & A_{1,1} \end{bmatrix} \tag{4}$$

Under the above assumptions, the $N \times M$ CNN turns out to be described by the following state equations:

$$\dot{x}_{kl} \ = \ -\ x_{kl} + \sum_{|n| \le r, |m| \le r} A_{nm}\ y_{k+n,l+m} \tag{5}$$

In this Section we will prove a set of sufficient conditions for the existence of at least one stable equilibrium point in a CNN described by equation (5).

The proof is carried out according to the following strategy. As a first step, we reduce the problem of the existence of a generic equilibrium point, to the study of the conditions under which a suitable space-variant CNN (described by a space-variant template) exhibits the equilibrium point, where all cell outputs are saturated to $+1$ (Propositions 1, 2 and 3). As a second step we introduce the definition of periodic saturation region and periodic equilibrium point (Definitions 1 and 2) and of periodic space-variant templates (Propositions 4 and 5). As a third step we exploit the concept of space-variant template for investigating the conditions of existence of periodic equilibrium points (Propositions 6 and 7). Based on such Propositions, we finally derive a set of sufficient conditions for the existence of at least one stable equilibrium point in a generic space-invariant CNN (Proposition 8). Such conditions are then simplified into a simple algorithm, that can be easily checked by examining the template elements.

Proposition 1: Let $S = \{y_{kl}:\ (1 \le k \le N,\ 1 \le l \le M)\}$ be a saturation region. Let g_{kl} be a constant function defined as follows:

$$g_{kl} = \begin{cases} \ \ 1 & \text{iff} \quad y_{kl} \ = \ \ \ 1 \\ -1 & \text{iff} \quad y_{kl} \ = \ -1 \end{cases} \tag{6}$$

Let us consider the space-variant CNN (SVCNN), associated to (5) with respect to the saturation region S, described by the following equations:

$$\dot{w}_{kl} \ = \ -\ w_{kl} + \sum_{|n| \le r, |m| \le r} P_{kl,nm}\ z_{k+n,l+m} \tag{7}$$

where $z_{kl} = f(w_{kl})$, and $P_{kl,nm}$ are the entries of the following space-variant feedback template $\boldsymbol{P}_{kl}$:

$$\boldsymbol{P}_{kl}:\ P_{kl,nm} = g_{kl}\ A_{nm}\ g_{k+n,l+m} \tag{8}$$

A sufficient and necessary condition in order that the CNN (5) presents an equilibrium point in the saturation region S is that the SVCNN (7) exhibits an equilibrium point in the saturation region $S' = \{z_{kl}:\ (\forall\ k, l,\ z_{kl} = 1)\}$.

Proof: We prove that the system of equations (7) is equivalent to (5) under the state transformation $w_{kl} = g_{kl}\, x_{kl}$ that implies, according to (2), $z_{kl} = g_{kl}\, y_{kl}$. In fact, since $\dot{w}_{kl} = g_{kl}\ \dot{x}_{kl}$, using (5) and substituting $g_{kl}x_{kl}$ and y_{kl} with w_{kl} and $g_{kl}^{-1} z_{kl}$ respectively, we obtain the system (7):

$$\begin{aligned} \dot{w}_{kl} &= g_{kl}\ \dot{x}_{kl} \\ &= g_{kl}\ \left(-x_{kl} + \textstyle\sum_{|n| \le r, |m| \le r} A_{nm}\ y_{k+n,l+m}\right) \\ &= -\ w_{kl} + \\ &\quad + \sum_{|n| \le r, |m| \le r} \left(g_{kl}\ A_{nm}\ g_{k+n,l+m}^{-1}\right)\ z_{k+n,l+m} \\ &= \ -\ w_{kl} + \textstyle\sum_{|n| \le r, |m| \le r} P_{kl,nm}\ z_{k+n,l+m} \end{aligned} \tag{9}$$

where

$$P_{kl,nm} = g_{kl}\ A_{nm}\ g_{k+n,l+m}^{-1} = g_{kl}\ A_{nm}\ g_{k+n,l+m} \tag{10}$$

Since $z_{kl} = g_{kl}\, y_{kl}$, it is derived that, according to (6), an equilibrium point of the CNN (5) in S corresponds to a point of the SVCNN (7), located in S', i.e. the thesis of Proposition 1. ■

In Table I we report a set of 9 operators (denoted by C), where $\boldsymbol{A}$ is the generic template defined in (4).

Proposition 2: A SVCNN described by the state equation (7) presents an equilibrium point in the saturation region $S' = \{z_{kl} : \ (\forall\, k, l,\ z_{kl} = 1)\}$ if and only if the space-variant template (8) fulfills the set $\mathcal{E}$ of constraints defined in Table II.

Proof: The necessary and sufficient condition in order that an equilibrium point exists in S' is:

$$\forall\, k, l : \quad \dot{w}_{kl}\,|_{w_{kl}=1} > 0 \tag{11}$$

By using (7), the set of conditions C, and by considering the boundary conditions (that are assumed to be null) it is easily verified that (11) exactly corresponds to the set of constraints $\mathcal{E}$ on the template $\boldsymbol{P}_{kl}$ reported in Table II. ■

Proposition 3: A space-invariant CNN, described by equation (5) and template (4), exhibits at least one stable equilibrium point, if and only if there exists a saturation region $\mathcal{S}$ and a function g_{kl} (defined as in (6)), such that the corresponding SVCNN (described by (7) and (8)) satisfies the set of constraints $\mathcal{E}$ defined in Proposition 2.

Proof: The thesis is a direct consequence of Proposition 1 and Proposition 2 and relies on the fact that, under the assumption $A_{00} > 1$, all stable equilibrium points are located in saturation regions. ■

Definition 1: A saturation region $\mathcal{S}$ of a $N \times M$ CNN is said to be periodic of period (T_k, T_l) if and only if, $\forall\, k, l$ such that $1 \le k \le N - T_k$, $1 \le l \le M - T_l$, then:

$$y_{k+T_k, l+T_l} = y_{kl} \tag{12}$$

Definition 2: An equilibrium point of a $N \times M$ CNN is said to be periodic of period (T_k, T_l), if it belongs to a saturation region $\mathcal{S}$, that is periodic of period (T_k, T_l).

Proposition 4: If a saturation region $\mathcal{S}$ is periodic of period (T_k, T_l), then also the function g_{kl} defined in (6) and the space-variant template $P_{kl,nm}$ defined in (7) are periodic of period (T_k, T_l).

Proof: Note that (6) implies $g_{kl} = y_{kl}$. Then g_{kl} is periodic of period (T_k, T_l) and, owing to (7), the same property holds for $P_{kl,nm}$. ■

Proposition 5: The set of space-variant templates (hereafter denoted by $\mathcal{T}_S$) associated to a saturation region $\mathcal{S}$ of period (T_k, T_l) is finite and its cardinality is at most equal to $T_k T_l$.

Proof: It is a direct consequence of the fact that, according to Proposition 4, the space-variant template $P_{kl,nm}$ is periodic of period (T_k, T_l). ■

Proposition 6: A sufficient and necessary condition in order that a $N \times M$ CNN, described by (5), presents a stable equilibrium point of period (T_k, T_l) is that there exists a saturation region $\mathcal{S}$ of period (T_k, T_l) and a corresponding space-variant template $P_{kl,nm}$ of period (T_k, T_l) that satisfy the constraints $\mathcal{E}$ defined in Proposition 2.

Proof: It is a direct consequence of Proposition 2 and Proposition 4. ■

Since any $N \times M$ saturation region can be considered periodic (by assuming in the worst case $N = T_k$ and $M = T_l$), Propositions 3 and 6 imply:

Proposition 7: A sufficient and necessary condition in order that a $N \times M$ CNN, described by (5), admits of at least one stable equilibrium point is that it exhibits a periodic equilibrium point of period (T_k, T_l), for some T_k and T_l.

We will show that the sufficient part of Proposition 7 allows one to considerably extend the set of sufficient conditions for the existence of stable equilibrium points in space-invariant CNNs [8]-[13]. In order to do that, we define the following subclass of periodic equilibrium points.

Definition 3: A (T_k, T_l) periodic equilibrium point is said to be *simple* if and only if it belongs to a saturation region that satisfies the following properties:

$$\begin{aligned} y_{kl} = h^h_k h^v_l \quad &\text{with}\ \ h^h_k,\, h^v_l \in \{-1, 1\}\ \ (\forall k,\, l) \\ h^h_{k+T_k} = h^h_k \ \ &(1 \le k \le N - T_k) \\ h^v_{l+T_l} = h^v_l \ \ &(1 \le l \le M - T_l) \end{aligned} \tag{13}$$

The corresponding saturation region is also said to be *simple*.

Example 1: In order to make clear the difference between a generic saturation region of period (T_k, T_l) and a simple region of the same periodicity, let us consider the following 9×10 saturation regions:

$$\boldsymbol{S}_p = \begin{bmatrix} -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & 1 \\ 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & 1 \end{bmatrix} \tag{14}$$

$$\boldsymbol{S}_s = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & +1 & -1 & +1 & -1 & +1 & -1 & +1 & -1 & +1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & +1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & +1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \end{bmatrix} \tag{15}$$

It is worth noting that both the above region $\boldsymbol{S}_p$ and $\boldsymbol{S}_s$ exhibit a periodicity $(T_k, T_l) = (3, 2)$. However the output y_{kl} of region $\boldsymbol{S}_p$ cannot be expressed in form (13): hence region $\boldsymbol{S}_p$ is not simple. On the other hand we observe that the output of region $\boldsymbol{S}_s$ admits of expression (13), by choosing for h^h_k and h^v_k the following two sequences with period $T_k = 3$ and $T_l = 2$ respectively:

$$\begin{aligned} &(h^h_1, h^h_2, h^h_3, h^h_4, h^h_5, h^h_6, h^h_7, h^h_8, h^h_9) = \\ &\quad = (+1, +1, -1, +1, +1, -1, +1, +1, -1) \\ &(h^v_1, h^v_2, h^v_3, h^v_4, h^v_5, h^v_6, h^v_7, h^v_8, h^v_9, h^v_{10}) = \\ &\quad = (+1, -1, +1, -1, +1, -1, +1, -1, +1, -1) \end{aligned} \tag{16}$$

According to (6) the function g_{kl} corresponding to a *simple* saturation region can also be written as $g_{kl} = y_{kl} = h^h_k h^v_l$. The space variant template (8) admits of the following expression:

$$\begin{aligned} P_{kl,nm} &= g_{kl}\, A_{nm}\, g_{k+n,l+m} \\ &= h^h_k h^v_l\, A_{nm}\, h^h_{k+n} h^v_{l+m} \end{aligned} \tag{17}$$

The above transformation can be seen as the result of the applications of two operators, that act separately on the columns and on the rows of the template $\boldsymbol{A}$. These two operators, called horizontal and vertical operators, will be denoted with $\mathcal{H}^k$ and $\mathcal{V}^l$ respectively. They are defined as follows:

$$\begin{aligned} \mathcal{H}^k[\boldsymbol{A}] &= \begin{bmatrix} p^h_k\, A_{-1,-1} & p^h_k\, A_{-1,0} & p^h_k\, A_{-1,1} \\ A_{0,-1} & A_{0,0} & A_{0,1} \\ q^h_k\, A_{1,-1} & q^h_k\, A_{1,0} & q^h_k\, A_{1,1} \end{bmatrix} \\ \mathcal{V}^l[\boldsymbol{A}] &= \begin{bmatrix} p^v_l\, A_{-1,-1} & A_{-1,0} & q^v_l\, A_{-1,1} \\ p^v_l\, A_{0,-1} & A_{0,0} & q^v_l\, A_{0,1} \\ p^v_l\, A_{1,-1} & A_{1,0} & q^v_l\, A_{1,1} \end{bmatrix} \end{aligned} \tag{18}$$

where

$$\begin{aligned} p^h_k &= h^h_{k-1} h^h_k \qquad q^h_k = h^h_k h^h_{k+1} \\ p^v_l &= h^v_{l-1} h^v_l \qquad q^v_l = h^v_l h^v_{l+1} \end{aligned} \tag{19}$$

By use of (17) and (18) the following expression for the space-variant template $\boldsymbol{P}_{kl}$ is obtained:

$$\begin{aligned} \boldsymbol{P}_{kl} &= \mathcal{V}^l\{\mathcal{H}^k[\boldsymbol{A}]\} = \mathcal{H}^k\{\mathcal{V}^l[\boldsymbol{A}]\} = \\ &= \begin{bmatrix} p^h_k\, p^v_l\, A_{-1,-1} & p^h_k\, A_{-1,0} & p^h_k\, q^v_l\, A_{-1,1} \\ p^v_l\, A_{0,-1} & A_{0,0} & q^v_l\, A_{0,1} \\ q^h_k\, p^v_l\, A_{1,-1} & q^h_k\, A_{1,0} & q^h_k\, q^v_l\, A_{11} \end{bmatrix} \end{aligned} \tag{20}$$

Due to the fact that $h^h_k,\, h^v_l \in \{-1,1\}$ and hence $p^h_k,\, p^v_k,\, q^h_k,\, q^v_k \in \{-1,1\}$ only four different forms are admissible for the operators $\mathcal{H}^k$ and $\mathcal{V}^l$. For the sake of simplicity, such forms are denoted by removing from $\mathcal{H}^k$ and $\mathcal{V}^l$ superscripts k and l and by adding two indexes, corresponding to the values of $p^h_k,\, q^h_k$ and $p^v_k,\, q^v_k$ respectively. We have:

$$\begin{aligned} \mathcal{H}_{11}[\boldsymbol{A}] &= \begin{bmatrix} A_{-1,-1} & A_{-1,0} & A_{-1,1} \\ A_{0,-1} & A_{0,0} & A_{0,1} \\ A_{1,-1} & A_{1,0} & A_{1,1} \end{bmatrix} \\ \mathcal{H}_{1,-1}[\boldsymbol{A}] &= \begin{bmatrix} A_{-1,-1} & A_{-1,0} & A_{-1,1} \\ A_{0,-1} & A_{0,0} & A_{0,1} \\ -A_{1,-1} & -A_{1,0} & -A_{1,1} \end{bmatrix} \\ \mathcal{H}_{-1,1}[\boldsymbol{A}] &= \begin{bmatrix} -A_{-1,-1} & -A_{-1,0} & -A_{-1,1} \\ A_{0,-1} & A_{0,0} & A_{0,1} \\ A_{1,-1} & A_{1,0} & A_{1,1} \end{bmatrix} \\ \mathcal{H}_{-1,-1}[\boldsymbol{A}] &= \begin{bmatrix} -A_{-1,-1} & -A_{-1,0} & -A_{-1,1} \\ A_{0,-1} & A_{0,0} & A_{0,1} \\ -A_{1,-1} & -A_{1,0} & -A_{1,1} \end{bmatrix} \end{aligned} \tag{21}$$

$$\begin{aligned} \mathcal{V}_{11}[\boldsymbol{A}] &= \begin{bmatrix} A_{-1,-1} & A_{-1,0} & A_{-1,1} \\ A_{0,-1} & A_{0,0} & A_{0,1} \\ A_{1,-1} & A_{1,0} & A_{1,1} \end{bmatrix} \\ \mathcal{V}_{1,-1}[\boldsymbol{A}] &= \begin{bmatrix} A_{-1,-1} & A_{-1,0} & -A_{-1,1} \\ A_{0,-1} & A_{0,0} & -A_{0,1} \\ A_{1,-1} & A_{1,0} & -A_{1,1} \end{bmatrix} \\ \mathcal{V}_{-1,1}[\boldsymbol{A}] &= \begin{bmatrix} -A_{-1,-1} & A_{-1,0} & A_{-1,1} \\ -A_{0,-1} & A_{0,0} & A_{0,1} \\ -A_{1,-1} & A_{1,0} & A_{1,1} \end{bmatrix} \\ \mathcal{V}_{-1,-1}[\boldsymbol{A}] &= \begin{bmatrix} -A_{-1,-1} & A_{-1,0} & -A_{-1,1} \\ -A_{0,-1} & A_{0,0} & -A_{0,1} \\ -A_{1,-1} & A_{1,0} & -A_{1,1} \end{bmatrix} \end{aligned} \tag{22}$$

Example 2: With reference to the simple saturation region $\boldsymbol{S}_s$ shown in (15), we can readily compute the space-variant template $\boldsymbol{P}_{kl}$ for each k and l. As an example we consider the case $k = 5$ and $l = 5$. By using the values of $h^h_{4,5,6}$ and $h^v_{4,5,6}$ given in (16), we derive that $p^h_5 = 1$, $p^v_5 = -1$, $q^h_5 = -1$ and $q^v_5 = -1$. Then by substituting such coefficients in (20) we obtain the following expression for $\boldsymbol{P}_{55}$

$$\boldsymbol{P}_{55} = \begin{bmatrix} -A_{-1,-1} & A_{-1,0} & -A_{-1,1} \\ -A_{0,-1} & A_{0,0} & -A_{0,1} \\ A_{1,-1} & -A_{1,0} & A_{1,1} \end{bmatrix} \tag{23}$$

By exploiting the horizontal and the vertical operators defined in (21) and (22) the space-variant template $\boldsymbol{P}_{55}$ can be expressed as:

$$\boldsymbol{P}_{55} = \mathcal{H}_{1,-1}\, \{\mathcal{V}_{-1,-1}[\boldsymbol{A}]\} \tag{24}$$

We observe that, by following a similar procedure, the explicit expression of the other space-variant templates $\boldsymbol{P}_{kl}$, with $(k,l) \neq (5,5)$, can be easily computed.

Owing to (18) it is derived that two consecutive operators $\mathcal{H}^k = \mathcal{H}_{ab}$ and $\mathcal{H}^{k+1} = \mathcal{H}_{cd}$ must satisfy the constraint $b = c$; the same property holds for the operators $\mathcal{V}^l$. In order to give a compact characterization of such sequences we will introduce the following definition.

Definition 4: Given an oriented (connected) graph, containing n nodes, $a_1,\, \ldots a_n$, such that $a_1 \rightarrow a_2 \rightarrow a_3 \rightarrow \ldots \rightarrow a_{n-1} \rightarrow a_n \rightarrow a_1$, the corresponding closed sequence is denoted by $\mathcal{C}(a_1,\, a_2,\, \ldots a_n)$.

Owing to the above definition, it is easily derived that a closed sequence is not altered if the argument is shifted, i.e. $\mathcal{C}(a_1,\, a_2,\, \ldots a_n) = \mathcal{C}(a_i,\, a_{i+1},\, \ldots a_n,\, a_1,\, a_2,\, \ldots a_{i-1})$. With the notation $[\cdot]^p$ we denote the sequence obtained by iterating p times a generic operator; $p = 0$ denotes the null sequence.

According to the above notations, the set of all the admissible closed sequences for the horizontal and vertical operators can be expressed as:

$$\begin{aligned}&\mathcal{H} \text{ operators :}\\ &\qquad \mathcal{C}([\mathcal{H}_{-1,1}]^m, [\mathcal{H}_{1,1}]^p, [\mathcal{H}_{1,-1}]^m, [\mathcal{H}_{-1,-1}]^q)\\ &\mathcal{V} \text{ operators :}\\ &\qquad \mathcal{C}([\mathcal{V}_{-1,1}]^m, [\mathcal{V}_{1,1}]^p, [\mathcal{V}_{1,-1}]^m, [\mathcal{V}_{-1,-1}]^q)\end{aligned} \tag{25}$$

with

$$\begin{array}{ll} & (m=0,\ q=0,\ p>0)\\ \text{or} & (m=0,\ p=0,\ q>0)\\ \text{or} & (m=1,\ p\geq 0,\ q\geq 0)\end{array} \tag{26}$$

We are now ready to give the main result that can be used to test the existence of a stable equilibrium point in the original CNN.

Proposition 8: Let $\mathcal{C}(\mathcal{H}^1,\, \mathcal{H}^2,\, \dots \mathcal{H}^p)$ and $\mathcal{C}(\mathcal{V}^1,\, \mathcal{V}^2,\, \dots \mathcal{V}^q)$ be two admissible closed sequences of horizontal and vertical operators, respectively. Let a CNN be described by template $\boldsymbol{A}$. If there exist s, r, t, and u such that the set of conditions reported in Table III are satisfied, then there exist N and M such that the CNN exhibits at least one stable equilibrium point.

Proof: We assume

$$N = \begin{cases} t-s+1+np & \text{if} \quad t\geq s\\ t-s+1+(n+1)p & \text{if} \quad t<s\end{cases}$$
$$M = \begin{cases} u-r+1+nq & \text{if} \quad r\geq u\\ u-r+1+(n+1)q & \text{if} \quad r<u\end{cases} \tag{27}$$

with $n = 0,\, 1,\, 2\dots$. We denote with $\mathrm{mod}(a,b)$ the rest of the division between two integers a and b. If the conditions of the Table III are fulfilled, then the space-variant template

$$\boldsymbol{P}_{kl} = \mathcal{H}^{\alpha}\{\mathcal{V}^{\beta}[\boldsymbol{A}]\} \tag{28}$$

where

$$\begin{cases} \alpha = \mathrm{mod}(s+k-1,p)\\ \beta = \mathrm{mod}(r+l-1,q)\end{cases} \tag{29}$$

satisfies all the conditions $\mathcal{E}$ reported in Proposition 2, for $1\leq k\leq N, 1\leq l\leq M$; hence, according to Proposition 6 and 7, the $N\times M$ CNN admits of at least one stable equilibrium point. ■

In the following section we propose a suitable algorithm based on Proposition 8 for checking the sufficient condition provided.

IV. Algorithm

The application of Proposition 8 to the $\mathcal{H}$ and the $\mathcal{V}$ admissible closed sequences is explained in Tables IV(a) and IV(b). The second column of Table IV(a) contains the six classes of admissible closed sequences for both $\mathcal{H}$ and $\mathcal{V}$ operators. Such classes are readily derived from expressions (25) and (26) and are denoted with $\mathcal{S}_{Hi}$ and $\mathcal{S}_{Vi}$ respectively ($i = 1, \dots 6$). Table IV(b) contains a set of conditions, equivalent to Proposition 8, that involves the horizontal operators $\mathcal{H}_I$, $\mathcal{H}_K$, $\mathcal{H}_L$, and the vertical operators $\mathcal{V}_J$, $\mathcal{V}_G$, $\mathcal{V}_M$, where $I,\, K,\, L,\, J,\, G,\, M, \in \{(1,1),\, (1,-1),\, (-1,1),\, (-1,-1)\}$. Finally, the third column of Table IV(a) shows the values of the parameters I, K, L and J, G, M for which the conditions given in Table IV(b) have to be verified.

It is seen that the total number of possible choices for the parameters I, L is 40; then each choice corresponds to one or more values of K. By considering that some cases are incorporated into others, the actual number of possible cases can be reduced to 16: they are listed in Table V(a) and numbered from $H1$ to $H16$. The same considerations are valid for the six $\mathcal{S}_{Vi}$ sequences, with respect to the parameters J, M and G. Table V(b) reports all the possible choices for the parameters J, M and G: they are numbered from $V1$ to $V16$.

The total number of cases reported in Table V can be further reduced by examining in more detail the set of conditions C, shown in Table IV(b).

To this end, let us consider the operators $\mathcal{H}_I$, $\mathcal{H}_L$, and $\mathcal{H}_K$ reported in Table IV(b) ($I,\ L,\ K \in \{(i,j)|i,j = \pm 1\}$). It is seen that $\boldsymbol{A}$ templates transformed by operator $\mathcal{H}_I$ must satisfy conditions C_{NW}, C_N, and C_{NE}. Such conditions involve only the second and the third row of $\mathcal{H}_I[\mathcal{V}_{J,G,M}[\boldsymbol{A}]]$ ($J,\ G,\ M \in \{(k,l)|k,l = \pm 1\}$), i.e. the rows that, according to (4), are labelled with $n = 0$ and $n = 1$ respectively. Since $\mathcal{H}_I = \mathcal{H}_{i,h}\quad (i,\ h \in \{\pm 1\})$ operates on the first and on the third row through indices i and h respectively, the first index i does not affect conditions C_{NW}, C_N, and C_{NE}. Hence the following statement holds:

Statement 1: The set of conditions C_{NW}, C_N, C_{NE} is satisfied by template $\mathcal{H}_{+1,h}[\mathcal{V}_{J,G,M}[\boldsymbol{A}]]$ ($J,\ G,\ M \in \{(k,l)|k,l = \pm 1\}$), if and only if it is also satisfied by template $\mathcal{H}_{-1,h}[\mathcal{V}_{J,G,M}[\boldsymbol{A}]]$.

The same considerations apply to operator $\mathcal{H}_L$ with reference to conditions C_{SW}, C_S, and C_{SE}, as stated below.

Statement 2: The set of conditions C_{SW}, C_S, C_{SE} is satisfied by template $\mathcal{H}_{h,+1}[\mathcal{V}_{J,G,M}[\boldsymbol{A}]]$ ($J,\ G,\ M \in \{(k,l)|k,l = \pm 1\}$), if and only if it is also satisfied by template $\mathcal{H}_{h,-1}[\mathcal{V}_{J,G,M}[\boldsymbol{A}]]$.

We point out that a similar statement does not hold for operator $\mathcal{H}_K$, since the involved conditions (i.e. C_W, C_0, C_E) concern all the template rows.

Using the above statements, the set of choices reported in Table V(a) may be reduced to a minimum. As an example, let us consider cases $H8$ and $H9$. They are identical for what concerns parameter K and are equivalent owing to the statements above. Hence, to prove the existence of at least one stable equilibrium point, only one case between

$H8$ and $H9$ should be used.

Now, let us consider cases $H8$ and $H5$. According to the above statements they are identical for the possible choices of parameters I and L, whereas $H5$ is less restrictive than $H8$ for what concerns parameter K. Hence, since it is sufficient that only one of the three cases $H5$, $H8$, and $H9$ is verified, only the less severe case $H5$ enters as an element of the minimum set of choices used for proving the existence of at least one stable equilibrium point.

Following a similar strategy, it turns out that only the first six choices of Table V(a), namely $H1$, $H2$, $H3$, $H4$, $H5$, and $H6$, form a minimum, independent set for what concerns operator $\mathcal{H}$.

Finally, if operator $\mathcal{V}$ and Table V(b) are considered, a similar reasoning reduces the effectively independent choices to $V1$, $V2$, $V3$, $V4$, $V5$, and $V6$.

The final minimum set of possible choices, extracted from Table V together with the set of constrained to be verified, is shown in Table VI.

The above considerations allow us to reformulate the sufficient conditions provided by Proposition 8 according to the following $\boldsymbol{Algorithm}$:

- I - consider each one of the possible $6\times6=36$ cases obtained by combining a case Hm, $(1\leq m\leq6)$ of Table VI(a) with a case Vn, $(1\leq n\leq6)$ of Table VI(b);
- II - check the constraints reported in Table VI(c), for the prescribed values of the parameters I, L, K and J, M, G;
- III - if such constraints are verified for at least one of the 36 considered cases, then the CNN exhibits at least one stable equilibrium point.

We remark that the above procedure simply requires to check some sets of inequalities, expressed in term of the template elements; hence it exploits both the local connectivity and the CNN space-invariant structure.

We will show in the next Section that the above algorithm considerably extends the class of CNNs for which a rigorous proof of the existence of a stable equilibrium point is available.

V. Comparison with Previous Results

In this section we report all the main classes of CNNs for which the existence of at least one stable equilibrium point has been rigorously proved. We will denote such classes by $\boldsymbol{C_A}$, $\boldsymbol{C_B}$, $\boldsymbol{C_C}$, $\boldsymbol{C_D}$, and $\boldsymbol{C_E}$. Then we compare such classes with the class of CNN (hereafter denoted by $\boldsymbol{C}$) that satisfies the sufficient conditions provided through Proposition 8 and the corresponding Algorithm presented in the previous Section. As pointed out in Section III we assume that the input and the bias terms be null.

Class $\boldsymbol{C_A}$ [8]: A CNN described by equation (3) presents a stable equilibrium point if the comparison matrix of $\hat{\boldsymbol{A}}-\boldsymbol{U}$ is a non-singular M-matrix.

Class $\boldsymbol{C_B}$ (Theorem 2 of [10]): A CNN described by equation (3) presents a stable equilibrium point if there exists a permutation $\{1,\,2,\ldots n\}\ \rightarrow\ \{\lambda_1,\,\lambda_2,\lambda_n\}$ such that:

$$\hat{A}_{\lambda_i\lambda_i}-1>\sum_{j=i+1}^{n}|\hat{A}_{\lambda_i\lambda_j}|\ \ \forall\, i \tag{30}$$

Class $\boldsymbol{C_C}$ (Theorem 4 of [10]): A space-invariant CNN described by equations (5) exhibits at least one stable equilibrium point if the template elements satisfy at least one of the following inequalities:

$$\begin{aligned}
&A_{0,0}-1>|A_{0,1}|+|A_{1,1}|+|A_{1,0}|+|A_{1,-1}|\\
&A_{0,0}-1>|A_{-1,1}|+|A_{0,1}|+|A_{1,1}|+|A_{1,0}|\\
&A_{0,0}-1>|A_{-1,0}|+|A_{-1,1}|+|A_{0,1}|+|A_{1,1}|\\
&A_{0,0}-1>|A_{-1,-1}|+|A_{-1,0}|+|A_{-1,1}|+|A_{0,1}|\\
&A_{0,0}-1>|A_{0,-1}|+|A_{-1,-1}|+|A_{-1,0}|+|A_{-1,1}|\\
&A_{0,0}-1>|A_{1,-1}|+|A_{0,-1}|+|A_{-1,-1}|+|A_{-1,0}|\\
&A_{0,0}-1>|A_{1,0}|+|A_{1,-1}|+|A_{0,-1}|+|A_{-1,-1}|\\
&A_{0,0}-1>|A_{1,1}|+|A_{1,0}|+|A_{1,-1}|+|A_{0,-1}|
\end{aligned} \tag{31}$$

Remark 1: Class $\boldsymbol{C_C}$ coincides with Class $\boldsymbol{C_B}$ in case of CNNs described by space-invariant templates (see Theorem 4 of [10]).

Remark 2: Class $\boldsymbol{C_A}$ is included in Class $\boldsymbol{C_B}$ and therefore in Class $\boldsymbol{C_C}$ for CNNs described by space-invariant templates (see Theorem 1 of [10]).

Since our results explicitly refer to space-invariant CNNs, according to the above Remarks 1 and 2 it is sufficient to compare class $\boldsymbol{C}$ (i.e. the class defined through Proposition 8 and the corresponding Algorithm) with Class $\boldsymbol{C_C}$.

Comparison of class $\boldsymbol{C_C}$ with class $\boldsymbol{C}$: The following Propositions holds:

Proposition 9: Class $\boldsymbol{C}$ is not included in Class $\boldsymbol{C_C}$.

Proof: Let us consider the following template:

$$\boldsymbol{A}=\begin{bmatrix}-a & a & a\\ -a & A_{0,0} & a\\ a & a & -a\end{bmatrix}\qquad 1\leq A_{0,0}\leq a \tag{32}$$

It belongs to Class $\boldsymbol{C}$, because it satisfies the constraints of Table IV(b) for the case $H1-V6$. On the other hand it is seen that it does not satisfy anyone of the inequalities (31) and therefore does not belong to Class $\boldsymbol{C_C}$.

This implies that Class $\boldsymbol{C}$ is not included in Class $\boldsymbol{C_C}$. ■

Proposition 10: Class $\boldsymbol{C_C}$ is not included in Class $\boldsymbol{C}$.

Proof: Let us consider the following template:

$$\boldsymbol{A}=\begin{bmatrix}a & -a & -5a\\ -a & A_{0,0} & A_{0,1}\\ A_{1,-1} & A_{1,0} & A_{1,1}\end{bmatrix} \tag{33}$$

with

$$\begin{cases} A_{0,0}-1 = |A_{0,1}|+|A_{1,-1}|+|A_{1,0}|+|A_{1,1}|+\varepsilon \\ \varepsilon > 0 \end{cases} \tag{34}$$

We observe that it belongs to $\boldsymbol{C_C}$ for any positive ε. It is easily verified that the only two saturation regions, that admit of a stable equilibrium, for any positive ε are $\boldsymbol{S}$ and $-\boldsymbol{S}$, where:

$$\boldsymbol{S} = \begin{bmatrix} 1 & -1 & 1 & -1 & \cdots & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & \cdots & 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & \cdots & 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 & \cdots & 1 & 1 & -1 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \tag{35}$$

We note that the above equilibrium point is not simple, according to Definition 3 and therefore cannot be detected through Proposition 8. This implies that Class $\boldsymbol{C_C}$ is not included in Class $\boldsymbol{C}$. ■

Remark 3: In a forthcoming paper [12], the authors have shown that there exist two other classes of matrices $\hat{\boldsymbol{A}} - \boldsymbol{U}$ that guarantee the existence of at least one stable equilibrium point (see Theorems 10 and 11 of [12]). The first class (denoted in [12] as $\mathcal{R}_0$) represents an extension of class $\boldsymbol{C_A}$: it is easily proved that for space-invariant templates class $\mathcal{R}_0$ is included in the class defined by (31) by substituting $\geq$ to $>$ in each inequality. The second class (denoted in [12] as $\mathcal{F}_0$) is in general different from $\mathcal{R}_0$, but in case of space invariant templates it is a subclass of $\mathcal{R}_0$. By exploiting the same arguments used in the proof of Propositions 9 and 10, it is easily derived that no one of the classes $\mathcal{R}_0$ and $\boldsymbol{C}$ is included in the other one.

Class $\boldsymbol{C_D}$ [11]: A CNN described by equation (3) presents a stable equilibrium point if the following condition holds:

$$\forall\, i:\ \sum_j \hat{\boldsymbol{A}}_{ij} - 1 > 0 \tag{36}$$

Comparison of class $\boldsymbol{C_D}$ with class $\boldsymbol{C}$: It is readily derived that Class $\boldsymbol{C_D}$ can be defined through the constraints of Table IV(b) obtained by combining case $H1$ of Table VI(a) (i.e. $I = L = K = (1,1)$) and case $V1$ of Table VI(b) (i.e. $J = M = G = (1,1)$). Hence Class $\boldsymbol{C_D}$ is included in Class $\boldsymbol{C}$

Class $\boldsymbol{C_E}$ [13]: A space-invariant CNN described by equations (5) is stable almost everywhere and therefore exhibits at least one stable equilibrium point if the signs of the template elements are arranged according to anyone of the following configurations:

$$\begin{bmatrix} + & + & + \\ + & A_{0,0} & + \\ + & + & + \end{bmatrix} \qquad \begin{bmatrix} - & + & - \\ - & A_{0,0} & - \\ - & + & - \end{bmatrix} \tag{37}$$

$$\begin{bmatrix} - & - & - \\ + & A_{0,0} & + \\ - & - & - \end{bmatrix} \qquad \begin{bmatrix} - & + & - \\ + & A_{0,0} & + \\ - & + & - \end{bmatrix} \tag{38}$$

Comparison of class $\boldsymbol{C_E}$ with class $\boldsymbol{C}$: Templates with any of the sign configurations shown in (37) are a particular case of Class $\boldsymbol{C}$, when cases $H1 - V1$ and $H1 - V2$ respectively are considered. Templates with the any of the sign configurations shown in (38) are a particular case of Class $\boldsymbol{C}$, when cases $H2 - V1$ and $H2 - V2$ respectively are considered. It is derived that Class $\boldsymbol{C_E}$ is included in Class $\boldsymbol{C}$.

Finally we give two examples of templates which describe CNN belonging to class $\boldsymbol{C}$ and not to the others.

Example 3: The first example is a space-invariant $N \times M$ CNN defined by the following template $\boldsymbol{A}_1$

$$\boldsymbol{A}_1 = \begin{bmatrix} -r & +s & +r \\ -s & +p & +s \\ -r & -s & -r \end{bmatrix} \qquad \begin{cases} s > 0 \\ p > 1 \\ r > 0 \end{cases} \tag{39}$$

This CNN exhibits at least one stable equilibrium point if the template elements satisfy the inequalities previously reported for Classes $\boldsymbol{C_C}$, $\boldsymbol{C_D}$, $\boldsymbol{C_E}$ and $\boldsymbol{C}$ and summarized in Table below. From this Table it is derived that Classes $\boldsymbol{C_C}$, $\boldsymbol{C_D}$ and $\boldsymbol{C_E}$ are included in Class $\boldsymbol{C}$.

Classes	Conditions
$\boldsymbol{C_C}$	$p-1 > 2(s+r)$
$\boldsymbol{C_D}$	$p-1 > \begin{cases} s+2r & \text{if} \quad r > s \\ 2s+r & \text{if} \quad r < s \end{cases}$
$\boldsymbol{C_E}$	N.A.
$\boldsymbol{C}$: $H1 - V2$	$p-1 > \begin{cases} r & \text{if} \quad r > s \\ 2s-r & \text{if} \quad r < s \end{cases}$

If we assume $s = 1$, $p = 4$ and $r = 2$ then only the conditions of Class $\boldsymbol{C}$ are satisfied. As an example a 5×5 CNN, described by such parameters, exhibit $16,012$ equilibrium points. One of them is defined by the following state:

$$\boldsymbol{x}_1 = \begin{bmatrix} +4 & -7 & +7 & -7 & +6 \\ +3 & -8 & +8 & -8 & +9 \\ +3 & -8 & +8 & -8 & +9 \\ +3 & -8 & +8 & -8 & +9 \\ +2 & -5 & +5 & -5 & +8 \end{bmatrix} \tag{40}$$

Example 4: The second example is again a space-invariant $N \times M$ CNN, defined by the following template $\boldsymbol{A}_2$

$$\boldsymbol{A}_2 = \begin{bmatrix} +r & +s & +r \\ -s & +p & -s \\ -r & -s & +r \end{bmatrix} \qquad \begin{cases} s > 0 \\ p > 1 \\ r > 0 \end{cases} \tag{41}$$

This CNN exhibits at least one stable equilibrium point if the template elements satisfy the inequalities reported in the following Table.

Classes	Conditions
$\boldsymbol{C_C}$	$p-1>2(s+r)$
$\boldsymbol{C_D}$	$p-1>\begin{cases} 3s & if \quad r<s \\ r+2s & if \quad r>s \end{cases}$
$\boldsymbol{C_E}$	N.A.
$\boldsymbol{C}$: $H2-V2$	$p-1>\begin{cases} -r & if \quad r<s \\ r-2s & if \quad r>s \end{cases}$

Also in this case, Classes $\boldsymbol{C_C}$, $\boldsymbol{C_D}$ and $\boldsymbol{C_E}$ are subsets of Class $\boldsymbol{C}$. In particular, if we suppose that $s=3$, $p=2$ and $r=2$, then only the conditions of Class $\boldsymbol{C}$ are satisfied. A 5×5 CNN described by these parameters possesses 8 equilibrium points, one of which is the following:

$$\boldsymbol{x}_2 = \begin{bmatrix} +10 & -11 & +11 & -11 & +6 \\ -9 & +12 & -12 & +12 & -5 \\ +9 & -12 & +12 & -12 & +5 \\ -9 & +12 & -12 & +12 & -5 \\ +4 & -9 & +9 & -9 & +4 \end{bmatrix} \qquad (42)$$

VI. Conclusion

We have investigated the properties of stable equilibrium points in space-invariant CNNs. We have yielded a set of sufficient conditions (and a simple algorithm for checking them) ensuring the existence of at least one stable equilibrium point. Such conditions present two main characteristics: a) they exploit both the CNN local connectivity and the space-invariant structure and hence they are directly expressed in terms of the template elements; b) they are different from the results reported in the literature [8]-[13] and include some of them. In particular they considerably extend the class of CNN, for which the existence of a stable equilibrium point is rigorously proved.

We point out that the complete characterization of the class of CNN that exhibits at least one stable equilibrium point is a fundamental step for understanding CNN global dynamics.

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TABLE I
SET OF OPERATORS DENOTED BY C.

$C_{NW}[\boldsymbol{A}] = \sum_{n=0}^{n=1}\sum_{m=0}^{m=1} A_{nm}$	$C_{N}[\boldsymbol{A}] = \sum_{n=0}^{n=1}\sum_{m=-1}^{m=1} A_{nm}$	$C_{NE}[\boldsymbol{A}] = \sum_{n=0}^{n=1}\sum_{m=-1}^{m=0} A_{nm}$
$C_{W}[\boldsymbol{A}] = \sum_{n=-1}^{n=1}\sum_{m=0}^{m=1} A_{nm}$	$C_{0}[\boldsymbol{A}] = \sum_{n=-1}^{n=1}\sum_{m=-1}^{m=1} A_{nm}$	$C_{E}[\boldsymbol{A}] = \sum_{n=-1}^{n=1}\sum_{m=-1}^{m=0} A_{nm}$
$C_{SW}[\boldsymbol{A}] = \sum_{n=-1}^{n=0}\sum_{m=0}^{m=1} A_{nm}$	$C_{S}[\boldsymbol{A}] = \sum_{n=-1}^{n=0}\sum_{m=-1}^{m=1} A_{nm}$	$C_{SE}[\boldsymbol{A}] = \sum_{n=-1}^{n=0}\sum_{m=-1}^{m=0} A_{nm}$

TABLE II
SET OF CONSTRAINS (DENOTED BY $\mathcal{E}$) OF PROPOSITION 2.

$C_{NW}[\boldsymbol{P}_{11}] > 1$	$C_N[\boldsymbol{P}_{1l}] > 1 \;\; \forall\, l \neq \{1, M\}$	$C_{NE}[\boldsymbol{P}_{1M}] > 1$
$C_W[\boldsymbol{P}_{k1}] > 1 \quad \forall\, k \neq \{1, N\}$	$C_0[\boldsymbol{P}_{kl}] > 1 \quad \forall\, k \neq \{1, N\}, \forall\, l \neq \{1, M\}$	$C_E[\boldsymbol{P}_{kM}] > 1 \quad \forall\, k \neq \{1, N\}$
$C_{SW}[\boldsymbol{P}_{N1}] > 1$	$C_S[\boldsymbol{P}_{Nl}] > 1 \;\; \forall\, l \neq \{1, M\}$	$C_{SE}[\boldsymbol{P}_{NM}] > 1$

TABLE III
SET OF CONSTRAINS OF PROPOSITION 8.

$C_{NW}\left[\mathcal{H}^s[\mathcal{V}^r(\boldsymbol{A})]\right] > 1$	$C_N\left[\mathcal{H}^s[\mathcal{V}^j(\boldsymbol{A})]\right] > 1 \quad \forall\, i$	$C_{NE}\left[\mathcal{H}^s[\mathcal{V}^u(\boldsymbol{A})]\right] > 1$
$C_W\left[\mathcal{H}^i[\mathcal{V}^r(\boldsymbol{A})]\right] > 1 \quad \forall\, j$	$C_0\left[\mathcal{H}^i[\mathcal{V}^j(\boldsymbol{A})]\right] > 1 \quad \forall\, i, j$	$C_E\left[\mathcal{H}^i[\mathcal{V}^u(\boldsymbol{A})]\right] > 1 \quad \forall\, j$
$C_{SW}\left[\mathcal{H}^t[\mathcal{V}^r(\boldsymbol{A})]\right] > 1$	$C_S\left[\mathcal{H}^t[\mathcal{V}^j(\boldsymbol{A})]\right] > 1 \quad \forall\, i$	$C_{SE}\left[\mathcal{H}^t[\mathcal{V}^u(\boldsymbol{A})]\right] > 1$

TABLE IV

APPLICATION OF PROPOSITION 8 TO THE ADMISSIBLE $\mathcal{H}$ AND $\mathcal{V}$ SEQUENCES: (A) ADMISSIBLE SEQUENCES; (B) CONSTRAINED TO BE VERIFIED.

Cases	Admissible Sequences	Admissible Values for the Parameters I, K, L, J, G, M
S_{H1}	$\mathcal{C}([\mathcal{H}_{11}]^p)$ $(m=0,\ p>0,\ q=0)$	$I = (1,1)$ $L = (1,1)$ $K = (1,1)$
S_{H2}	$\mathcal{C}([\mathcal{H}_{-1,-1}]^q)$ $(m=0,\ p=0,\ q>0)$	$I = (-1,-1)$ $L = (-1,-1)$ $K = (-1,-1)$
S_{H3}	$\mathcal{C}(\mathcal{H}_{-1,1},\ \mathcal{H}_{1,-1})$ $(m=1,\ p=0,\ q=0)$	$I = (-1,1)$ or $(1,-1)$ $L = (-1,1)$ or $(1,-1)$ $K = (-1,1)$ and $(1,-1)$
S_{H4}	$\mathcal{C}(\mathcal{H}_{-1,1},[\mathcal{H}_{1,1}]^p,\mathcal{H}_{1,-1})$ $(m=1,\ p>0,\ q=0)$	$I = (1,-1)$ or $(-1,1)$ or $(1,1)$ $L = (1,-1)$ or $(-1,1)$ or $(1,1)$ $K = (1,-1)$ and $(-1,1)$ and $(1,1)$
S_{H5}	$\mathcal{C}(\mathcal{H}_{-1,1},\mathcal{H}_{1,-1},[\mathcal{H}_{-1,-1}]^q)$ $(m=1,\ p=0,\ q>0)$	$I = (1,-1)$ or $(-1,1)$ or $(-1,-1)$ $L = (1,-1)$ or $(-1,1)$ or $(-1,-1)$ $K = (1,-1)$ and $(-1,1)$ and $(-1,-1)$
S_{H6}	$\mathcal{C}(\mathcal{H}_{-1,1},[\mathcal{H}_{1,1}]^p,\mathcal{H}_{1,-1},[\mathcal{H}_{-1,-1}]^q)$ $(m=1,\ p>0,\ q>0)$	$I = (1,-1)$ or $(-1,1)$ or $(1,1)$ or $(-1,-1)$ $L = (1,-1)$ or $(-1,1)$ or $(1,1)$ or $(-1,-1)$ $K = (1,-1)$ and $(-1,1)$ and $(-1,-1)$ and $(-1,-1)$
S_{V1}	$\mathcal{C}([\mathcal{V}_{11}]^p)$ $(m=0,\ p>0,\ q=0)$	$J = (1,1)$ $M = (1,1)$ $G = (1,1)$
S_{V2}	$\mathcal{C}([\mathcal{V}_{-1,-1}]^q)$ $(m=0,\ p=0,\ q>0)$	$J = (-1,-1)$ $M = (-1,-1)$ $G = (-1,-1)$
S_{V3}	$\mathcal{C}(\mathcal{V}_{-1,1},\ \mathcal{V}_{1,-1})$ $(m=1,\ p=0,\ q=0)$	$J = (-1,1)$ or $(1,-1)$ $M = (-1,1)$ or $(1,-1)$ $G = (-1,1)$ and $(1,-1)$
S_{V4}	$\mathcal{C}(\mathcal{V}_{-1,1},[\mathcal{V}_{1,1}]^p,\mathcal{V}_{1,-1})$ $(m=1,\ p>0,\ q=0)$	$J = (1,-1)$ or $(-1,1)$ or $(1,1)$ $M = (1,-1)$ or $(-1,1)$ or $(1,1)$ $G = (1,-1)$ and $(-1,1)$ and $(1,1)$
S_{V5}	$\mathcal{C}(\mathcal{V}_{-1,1},\mathcal{V}_{1,-1},[\mathcal{V}_{-1,-1}]^q)$ $(m=1,\ p=0,\ q>0)$	$J = (1,-1)$ or $(-1,1)$ or $(-1,-1)$ $M = (1,-1)$ or $(-1,1)$ or $(-1,-1)$ $G = (1,-1)$ and $(-1,1)$ and $(-1,-1)$
S_{V6}	$\mathcal{C}(\mathcal{V}_{-1,1},[\mathcal{V}_{1,1}]^p,\mathcal{V}_{1,-1},[\mathcal{V}_{-1,-1}]^q)$ $(m=1,\ p>0,\ q>0)$	$J = (1,-1)$ or $(-1,1)$ or $(1,1)$ or $(-1,-1)$ $M = (1,-1)$ or $(-1,1)$ or $(1,1)$ or $(-1,-1)$ $G = (1,-1)$ and $(-1,1)$ and $(-1,-1)$ and $(-1,-1)$

(a)

$C_{NW}\,[\mathcal{H}_I\,[\mathcal{V}_J\,(\boldsymbol{A})]]>1$	$C_N\,[\mathcal{H}_I\,[\mathcal{V}_G\,(\boldsymbol{A})]]>1$	$C_{NE}\,[\mathcal{H}_I\,[\mathcal{V}_M\,(\boldsymbol{A})]]>1$
$C_W\,[\mathcal{H}_K\,[\mathcal{V}_J\,(\boldsymbol{A})]]>1$	$C_0\,[\mathcal{H}_K\,[\mathcal{V}_G\,(\boldsymbol{A})]]>1$	$C_E\,[\mathcal{H}_K\,[\mathcal{V}_M\,(\boldsymbol{A})]]>1$
$C_{SW}\,[\mathcal{H}_L\,[\mathcal{V}_J\,(\boldsymbol{A})]]>1$	$C_S\,[\mathcal{H}_L\,[\mathcal{V}_G\,(\boldsymbol{A})]]>1$	$C_{SE}\,[\mathcal{H}_L\,[\mathcal{V}_M\,(\boldsymbol{A})]]>1$

(b)

TABLE V

TOTAL NUMBER OF POSSIBLE CHOICES: (A) FOR THE PARAMETERS I, L AND K; (B) FOR THE PARAMETERS J, M AND G

I	L	K	Case
$(1,1)$	$(1,1)$	$(1,1)$	$H1$
$(-1,-1)$	$(-1,-1)$	$(-1,-1)$	$H2$
$(1,-1)$	$(1,-1)$	$(1,-1)$ and $(-1,1)$	$H3$
$(1,-1)$	$(-1,1)$	$(1,-1)$ and $(-1,1)$	$H4$
$(-1,1)$	$(1,-1)$	$(1,-1)$ and $(-1,1)$	$H5$
$(-1,1)$	$(-1,1)$	$(1,-1)$ and $(-1,1)$	$H6$
$(1,-1)$	$(1,1)$	$(1,-1)$ and $(-1,1)$ and $(1,1)$	$H7$
$(-1,1)$	$(1,1)$	$(1,-1)$ and $(-1,1)$ and $(1,1)$	$H8$
$(1,1)$	$(1,-1)$	$(1,-1)$ and $(-1,1)$ and $(1,1)$	$H9$
$(1,1)$	$(-1,1)$	$(1,-1)$ and $(-1,1)$ and $(1,1)$	$H10$
$(1,-1)$	$(-1,-1)$	$(1,-1)$ and $(-1,1)$ and $(-1,-1)$	$H11$
$(-1,1)$	$(-1,-1)$	$(1,-1)$ and $(-1,1)$ and $(-1,-1)$	$H12$
$(-1,-1)$	$(1,-1)$	$(1,-1)$ and $(-1,1)$ and $(-1,-1)$	$H13$
$(-1,-1)$	$(-1,1)$	$(1,-1)$ and $(-1,1)$ and $(-1,-1)$	$H14$
$(1,1)$	$(-1,-1)$	$(1,-1)$ and $(-1,1)$ and $(1,1)$ and $(-1,-1)$	$H15$
$(-1,-1)$	$(1,1)$	$(1,-1)$ and $(-1,1)$ and $(1,1)$ and $(-1,-1)$	$H16$

(a)

J	M	G	Case
$(1,1)$	$(1,1)$	$(1,1)$	$V1$
$(-1,-1)$	$(-1,-1)$	$(-1,-1)$	$V2$
$(1,-1)$	$(1,-1)$	$(1,-1)$ and $(-1,1)$	$V3$
$(1,-1)$	$(-1,1)$	$(1,-1)$ and $(-1,1)$	$V4$
$(-1,1)$	$(1,-1)$	$(1,-1)$ and $(-1,1)$	$V5$
$(-1,1)$	$(-1,1)$	$(1,-1)$ and $(-1,1)$	$V6$
$(1,-1)$	$(1,1)$	$(1,-1)$ and $(-1,1)$ and $(1,1)$	$V7$
$(-1,1)$	$(1,1)$	$(1,-1)$ and $(-1,1)$ and $(1,1)$	$V8$
$(1,1)$	$(1,-1)$	$(1,-1)$ and $(-1,1)$ and $(1,1)$	$V9$
$(1,1)$	$(-1,1)$	$(1,-1)$ and $(-1,1)$ and $(1,1)$	$V10$
$(1,-1)$	$(-1,-1)$	$(1,-1)$ and $(-1,1)$ and $(-1,-1)$	$V11$
$(-1,1)$	$(-1,-1)$	$(1,-1)$ and $(-1,1)$ and $(-1,-1)$	$V12$
$(-1,-1)$	$(1,-1)$	$(1,-1)$ and $(-1,1)$ and $(-1,-1)$	$V13$
$(-1,-1)$	$(-1,1)$	$(1,-1)$ and $(-1,1)$ and $(-1,-1)$	$V14$
$(1,1)$	$(-1,-1)$	$(1,-1)$ and $(-1,1)$ and $(1,1)$ and $(-1,-1)$	$V15$
$(-1,-1)$	$(1,1)$	$(1,-1)$ and $(-1,1)$ and $(1,1)$ and $(-1,-1)$	$V16$

(b)

TABLE VI

MINIMUM SET OF POSSIBLE CHOICES EXPLOITED BY THE PROPOSED ALGORITHM: (A) FOR THE PARAMETERS I, L AND K; (B) FOR THE PARAMETERS J, M AND G; (C) CONSTRAINED TO BE VERIFIED

I	L	K	*Case*
$(1,1)$	$(1,1)$	$(1,1)$	$H1$
$(-1,-1)$	$(-1,-1)$	$(-1,-1)$	$H2$
$(1,-1)$	$(1,-1)$	$(1,-1)$ and $(-1,1)$	$H3$
$(1,-1)$	$(-1,1)$	$(1,-1)$ and $(-1,1)$	$H4$
$(-1,1)$	$(1,-1)$	$(1,-1)$ and $(-1,1)$	$H5$
$(-1,1)$	$(-1,1)$	$(1,-1)$ and $(-1,1)$	$H6$

(a)

J	M	G	*Case*
$(1,1)$	$(1,1)$	$(1,1)$	$V1$
$(-1,-1)$	$(-1,-1)$	$(-1,-1)$	$V2$
$(1,-1)$	$(1,-1)$	$(1,-1)$ and $(-1,1)$	$V3$
$(1,-1)$	$(-1,1)$	$(1,-1)$ and $(-1,1)$	$V4$
$(-1,1)$	$(1,-1)$	$(1,-1)$ and $(-1,1)$	$V5$
$(-1,1)$	$(-1,1)$	$(1,-1)$ and $(-1,1)$	$V6$

(b)

$C_{NW}\left[\mathcal{H}_I\left[\mathcal{V}_J\left(\boldsymbol{A}\right)\right]\right]>1$	$C_{N}\left[\mathcal{H}_I\left[\mathcal{V}_G\left(\boldsymbol{A}\right)\right]\right]>1$	$C_{NE}\left[\mathcal{H}_I\left[\mathcal{V}_M\left(\boldsymbol{A}\right)\right]\right]>1$
$C_{W}\left[\mathcal{H}_K\left[\mathcal{V}_J\left(\boldsymbol{A}\right)\right]\right]>1$	$C_{0}\left[\mathcal{H}_K\left[\mathcal{V}_G\left(\boldsymbol{A}\right)\right]\right]>1$	$C_{E}\left[\mathcal{H}_K\left[\mathcal{V}_M\left(\boldsymbol{A}\right)\right]\right]>1$
$C_{SW}\left[\mathcal{H}_L\left[\mathcal{V}_J\left(\boldsymbol{A}\right)\right]\right]>1$	$C_{S}\left[\mathcal{H}_L\left[\mathcal{V}_G\left(\boldsymbol{A}\right)\right]\right]>1$	$C_{SE}\left[\mathcal{H}_L\left[\mathcal{V}_M\left(\boldsymbol{A}\right)\right]\right]>1$

(c)