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Fuzzy Parameters Analysis of Powder Snow Avalanches

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Abstract *This paper examines powder snow avalanches by introducing a pre–selected degree of variation, or fuzziness, in model parameters. Given a certain degree of uncertainty in the parameters, fuzzy set theory makes it possible to evaluate the uncertainty in the results thereby avoiding the need for a more complex stochastic analysis. Having assumed that model parameters are affected by a certain degree of variation, the response of the numerical model is calculated by solving the fuzzy equations. In this manner it is possible to evaluate how the results are affected by a given change in model parameters.*

1 Introduction

The goal of this investigation was to study powder snow avalanches by introducing a pre–determined degree of variation (*fuzziness*) in model parameters. It was motivated by the consideration that model parameters are usually affected by a degree of *uncertainty*, mainly due to measuring imprecision and the great variability of snow properties in both space and time. This approach could be particularly effective when dealing with snow avalanches, where it often proves difficult, or outright impossible, to gain sufficient knowledge of the characteristics of the avalanche itself.

Fuzzy set theory is a mathematical theory for the representation of uncertainty (see Zadeh (1968) and Zadeh (1988)). Given a certain degree of uncertainty in the parameters, fuzzy set theory makes it possible to evaluate the uncertainty in the results thereby avoiding the difficulties associated with stochastic analysis, since this method does not require a knowledge of probability distribution functions.

The model used in the numerical simulations is described in Clement Rastello (2002); it is based on earlier works by Hopfinger and Tochon-Danguy (1977), Hopfinger (1983), Beghin and Brugnot (1983) and Beghin and Olagne (1991).

It is assumed that some model parameters are affected by a certain amount of uncertainty (defined by the so-called *membership functions*) and the response of the numerical model is calculated by solving the fuzzy equations for different shapes of the membership functions. In this manner it is possible to work out a *quantitative* estimate of the influence of a given change in model parameters and hence to identify, and take into due account, the parameters having the most adverse effect on the model response.

2 Avalanche model

Compared to the model described by Beghin and Olagne (1991), the model used in this study is somewhat simplified, in as much as avalanche width is assumed to be constant (see Clement Rastello (2002)). An avalanche is defined as the turbulent motion of a dense fluid cloud (suspension of snow particles) moving downhill through the air under the effects of gravity. The shape of the avalanche is assumed to remain the same throughout its motion, i.e., the ratio between its height and length is assumed to be constant.

Snow entrainment is taken into account, while the sedimentation is neglected. This model has proved effective to describe the artificial avalanche of the Vallée de la Sionne (see Dufour, Gruber, and Ammann (2001)).

The equations of the model are:

- mass balance:

$$\frac{\mathrm{d}(\Delta\rho\, A)}{\mathrm{d}t} = \underbrace{\Delta\rho_N\, \beta h_N\, U}_{\text{Snow entrainment}} \tag{1}$$

- momentum balance:

$$\frac{\mathrm{d}\left[(\rho + k_v\, \rho_a)\, AU\right]}{\mathrm{d}t} = \underbrace{\Delta\rho\, A\, g \sin\theta}_{\text{Gravity}} - \underbrace{C_d\, \rho\, L\, U^2}_{\text{Friction}} \tag{2}$$

where $A = S_1 HL$ is the so-called "bidimensional volume", S_1 is a shape factor ($\pi/4$ for a semi-elliptical shaped avalanche), U the velocity of the gravity center, ρ avalanche density, ρ_a the density of the fluid medium (air), ρ_N density of ground snow, βh_N the height of the snow entrained, g gravity acceleration, θ the slope angle, k_v the added mass coefficient and C_d the friction coefficient between the avalanche and the ground. $\Delta\rho = \rho - \rho_a$ (or $\Delta\rho_N = \rho_N - \rho_a$) is the difference between snow (or ground snow) density and air density.

If $\beta = 0$ or $h_N = 0$, no snow is entrained during the avalanche motion, otherwise the quantity of snow entrained is assumed constant throughout the avalanche motion. It must be pointed out that this representation could be not very realistic (see the recent measurements of entrainment and deposition presented by Sovilla and Bartelt (2002)).

2.1 Analytical solution

If the contribution of friction in Eq. 2 is neglected ($C_d = 0$) the solution in terms of velocity U (or U_f), A and ρ as a function of distance x or (x_f) can be calculated analytically. Otherwise, a numerical integration is required. Neglecting the contribution of friction, velocity U turns out to be (if $U|_{x=0} = 0$):

$$U = \frac{\sqrt{2KMx + (KB + LM)x^2 + \frac{2}{3}(KN + LB)x^3 + \frac{1}{2}LNx^4}}{M + Bx + Nx^2} \tag{3}$$

where:

$$B = \beta h_N \Delta\rho_N + (1 + k_v)\rho_a E\sqrt{A_0} \tag{4a}$$

$$M = (1 + k_v)\rho_a A_0 + \Delta\rho_0 A_0 \tag{4b}$$

$$N = \frac{1}{4}(1 + k_v)\rho_a E^2 \tag{4c}$$

$$K = \Delta\rho_0 A_0\, g \sin\theta, \quad L = \beta h_N \Delta\rho_N\, g \sin\theta \tag{4d}$$

It is possible to link front velocity U_f with the velocity of the gravity center U (from Eq. 3) assuming the mass center of the avalanche to be situated approximately in the middle of its length L so that $x_f = x + \frac{L}{2}$:

$$U_f = \frac{U}{1 - \frac{1}{2k}\frac{\partial H}{\partial x_f}} \tag{5}$$

In the original work by Clement Rastello, coefficient E (volume growth rate) is introduced by differentiating $A = S_1 HL$ so that it is possible to write the evolution equation for A:

$$\frac{\mathrm{d}A}{\mathrm{d}x} = E\sqrt{A} \quad \text{with} \quad E = \frac{2\sqrt{\frac{S_1}{k}}\,\frac{\partial H}{\partial x_f}}{1 - \frac{1}{2k}\frac{\partial H}{\partial x_f}} \tag{6}$$

Height growth rate $\frac{\partial H}{\partial x_f}$, ratio $k = \frac{H}{L}$, E and coefficient S_1 are obtained experimentally as a function of slope angle θ only. For instance, for $\theta = 25°$ we get:

$$\frac{\partial H}{\partial x_f} \approx 0.085, \quad k \approx 0.3, \quad E \approx 0.2, \quad S_1 \approx 0.8 \tag{7}$$

3 Application of fuzzy set analysis to avalanches

In this study the influence of the *uncertainty* or *imprecision* affecting three parameters contained in Eq. 1 and 2 is assessed through a *fuzzy approach*. Parameters A_0, $\Delta\rho_0$ and E are considered to be fuzzy, i.e., associated with a certain degree of uncertainty, as defined through the so-called membership functions μ. Such functions represent the level of confidence (or imprecision) of a variable; a typical shape could be triangular or trapezoidal. Compared to stochastic analysis, this approach is a simpler and more intuitive way to allow for the uncertainty in some parameters, since it does not call for the determination of probability distribution functions.

The methods used to determine membership functions include: subjective evaluation or expert statement (by an expert on the subject), converted probabilities (using histograms or other probability diagrams), physical measures (often difficult), learning and adaptation (see, for instance, Schulz and Huwe (1997)).

Given a value of the membership function it is possible to obtain a *closed interval* within which lies the variable of interest. The projections of α-cut (cut at level α) on the axis of the variable define the left and right boundaries of this interval. For instance, α-cut $= 0.5$ in Figure 7 defines interval 160m^2 to 280m^2 for A_0, 120kg/m^3 to 210kg/m^3 for $\Delta\rho_0$ and 0.16 to 0.28 for E. As a results, the variables of interest (velocity $U(x)$ and average pressure $p(x) = \frac{1}{2}\,\rho\,U^2$) are now expressed by fuzzy numbers, Eq. 1 and 2 being fuzzy equations. It means that $U(x)$ and $p(x)$ are associated to a certain membership function to be determined by finding, for each level $0 \leq \alpha \leq 1$:

$$\text{Min/Max}\, U(x, A_0^\alpha, \Delta\rho_0^\alpha, E^\alpha) \quad \alpha \in [0,1] \tag{8a}$$

$$\text{Min/Max}\, p(x, A_0^\alpha, \Delta\rho_0^\alpha, E^\alpha) \quad \alpha \in [0,1] \tag{8b}$$

with:

$$\underline{A_0} \leq A_0 \leq \overline{A_0} \tag{9a}$$

$$\underline{\Delta\rho_0} \leq \Delta\rho_0 \leq \overline{\Delta\rho_0} \tag{9b}$$

$$\underline{E} \leq E \leq \overline{E} \tag{9c}$$

where the underline indicates the left boundary of a variable and the overline the right boundary of the same variable, corresponding to a certain (given) value of α-level.

The solution is found by minimizing and maximizing the fuzzy variables (subjected to the constraints in Eqs. 9a-9c) and calculating velocity and pressure for each level of α. Their range of variation represents their level of imprecision due to the imprecision in the input data. The determination of the membership function of U and p, i.e., their level of imprecision, is achieved by repeating the calculation for different α's. This problem can be viewed as an *optimization problem*, as explained in Dubois and Prade (1980).

4 Numerical simulations

The properties listed in Table 1 are used in the numerical simulations together with a friction coefficient $C_d = 0$. Such properties have proved useful to describe the avalanche

Table 1: Reference values of the avalanche parameters.

A_0	$\Delta\rho_0$	β	h_N	k_v	ρ_a	E	θ
m^2	kg/m^3	-	m	-	kg/m^3	-	$^\circ$
200	150	1	1.5	0.5	1.0	0.2	25

of the Vallée de la Sionne (see Dufour, Gruber, and Ammann (2001)); in this way the results are directly comparable with those found in the sensitivity analysis presented in Clement Rastello (2002).

It must be pointed out that the purpose of this work is not to propose a new model but to present the results of a method designed to take into account imprecision in the data. For this reason, the model has been conceived to be as simple as possible: the results presented below have been obtained by solving Eq. 3 analytically. Hence, the contribution is not in the model but in the way in which the model is used.

4.1 Discussion of the results

Six cases are represented by the membership functions (with triangular and trapezoidal shape, symmetrical and non symmetrical with respect to the *reference values* given in Table 1) as presented in Figure 1 (case 1), Figure 3 (case 2), Figure 5 (case 3), Figure 7 (case 4), Figure 9 (case 5) and Figure 11 (case 6). The results in terms of velocity U, average dynamic pressure p and velocity at $x = x_{max} = 4000$m ($U|_{x=L}$) are shown in Figure 2 (case 1), Figure 4 (case 2), Figure 6 (case 3), Figure 8 (case 4), Figure 10 (case 5) and Figure 12 (case 6).

Each diagram shows the value obtained for a different α-cut and makes it possible to investigate the effects of parameters imprecision on velocity and pressure and to identify the parameters with the most adverse influence on the response. For instance, in Figure 2 (left) for α-cut $= 0$, the velocity range at $x = 4000$m is ≈ 72m/s to 110m/s, i.e., a range -21.7% to $+19.5\%$ with respect to the reference value ≈ 92m/s. The same values can be found in the figure on the right, where the membership function for the same data is plotted. This range corresponds to a range -80.0% to $+80.0\%$ for the fuzzy input data in Figure 1 relating to the same value of α-cut $= 0$. It should be noted that, despite the linear membership in Figure 1, the resulting membership function is non linear.

As for case 2, it is obvious that the range for the same results is smaller in Figure 4 than in Figure 2 because in the former the data are much more precise (the membership functions are "crisper").

Moreover, Figures 5 and 6 make it possible to evaluate the influence of the slope of the membership functions. It can be seen that the increase in the membership function of Figure 5 (40m^2 to 200m^2 for A_0 30kg/m^3 to 150kg/m^3 for $\Delta\rho_0$ and 0.04 to 0.20 for E) is reflected in an increase 81.2m/s to 92.0m/s for $U|_{x=L}$ (left branch of Figure 6, right). The steep decrease in the Figure 5 (200m^2 to 280m^2 for A_0 150kg/m^3 to 210kg/m^3 for $\Delta\rho_0$ and 0.20 to 0.28 for E) is reflected in a less steep decrease 92.0m/s to 107.6m/s for $U|_{x=L}$ (right branch of Figure 6, right). Similar considerations apply to Figure 7 and Figure 8.

The fact that, for α-cut $= 1$, the membership function of $U|_{x=L}$ turns out to be the same in terms of both range (3.9m/s) and position (Figures 10 and 12 on the right) is due to the fact that for α-cut $= 1$ the membership functions of A_0, $\Delta\rho_0$ and E are also the same, having the same range (30m^2 for A_0, 22.5kg/m^3 for $\Delta\rho_0$ and 0.03 for E) and position. In other words, the membership functions of A_0, $\Delta\rho_0$ and E in Figures 9 and 11 differ solely for the slant of the two branches. Again, the membership functions for $U|_{x=L}$

are non linear.

It can be also noticed that the imprecision in the data is reflected in a much larger imprecision in the value of pressure p.

These results can be very useful when the data are used, for instance, to design structural members, to protect buildings or town areas, or for planning purposes (hazard maps). In all cases it is possible to quantitatively evaluate how the imprecision in model parameter is reflected in the results.

Clement Rastello proposed a sensitivity analysis to study the influence of A_0, $\Delta\rho_0$ and E on the response in terms of velocity U. This approach examines the effect of one parameter at a time, given its range of variation. It means that the effect of the variation in each parameter is taken into account *separately*. Compared to a sensitivity analysis, the method presented here makes it possible to take into account an entire set of imprecise parameters and to determine the level of imprecision in the results. A different level of imprecision in the data is obtained by a change in the membership functions.

5 Conclusions

The main points discussed in the paper are briefly summarised below.

- The *fuzzy approach*, applied to a well-known model described in the literature, makes it possible to take into account a degree of *imprecision* in model parameters. This is very useful for avalanches, where only *approximate knowledge* is available.
- Given a set of membership functions, the influence of the *imprecision* affecting model parameters is evaluated quantitatively.
- A membership function can be determined by using an *expert judgment*. In this way, the contribution of an expert on the field is easily included in the model.
- A complex *statistical* analysis can be *avoided*.
- The set of parameters with the *greatest influence* on the response can be identified.

6 Acknowledgments

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7 Appendix - Some fuzzy set definitions

This Section presents a brief overview of some fuzzy sets definitions and operations. For a complete treatment, the reader is referred to Dubois and Prade (1980) and Kaufmann and Gupta (1991).

1. Let X be a set of elements. A is called a fuzzy (sub)set of X if A is a set of *ordered pairs*:

$$X = \{(x, \mu_A(x)), \quad x \in X, \quad \mu_A(x) \in [0, 1]\}. \tag{10}$$

where $\mu_A(x)$ represents the *grade of membership* of x in A. The closer $\mu_A(x)$ is to 1, the more x belongs to A, and conversely, the closer it is to 0, the less x belongs to A.

2. The *support* is the area where the membership function is greater than zero (A being a fuzzy set):

$$s_A = \{x : \mu_A(x) \geq 0\} \tag{11}$$

For instance, the support of the membership function in Figure 11 (left) is the interval $90 - 360\text{m}^2$.

3. The *core* is the area that contains elements having the maximum degree of membership to the fuzzy set A:

$$c_A = \{x : \mu_A(x) = 1\} \tag{12}$$

The core of the membership function in Figure 11 (left) is the interval $170 - 200\text{m}^2$.

4. The α-cut is the cut through the membership function of A at height α:

$$X_\alpha = \{x : \mu_A(x) = \alpha\} \tag{13}$$

For instance, the α-cut $= 0.5$ of the membership function in Figure 11 (left) is defined by points 130m^2 and 280m^2.

5. *Height* is the maximum value of the membership function of A:

$$h_A = \max_x \{\mu_A(x)\} \tag{14}$$

The height of the membership functions presented in the paper is 1.

Operations on fuzzy numbers are performed using the *extension principle*: the classical operators (addition, multiplication…) are thus extended to their fuzzy counterparts. For a binary operator $\otimes$, with $y, z \in \mathbb{R}$:

$$\mu_{A\otimes B}(x) = \max\{\min\{\mu_A(y), \mu_B(z)\} \mid\ y \otimes z = x\} \tag{15}$$

The degree of membership of x is the maximum of $\min\{\mu_A(y), \mu_B(z)\}$ over all the possible pairs of (y, z) for which $y \otimes z = x$ holds.

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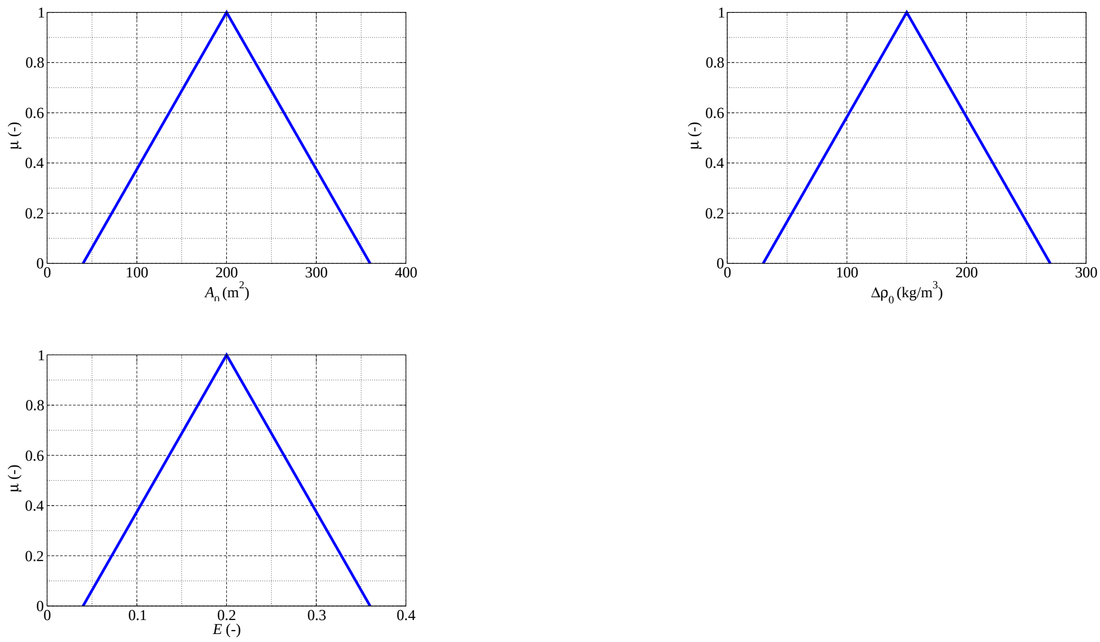

Figure 1: Membership functions for A_0, $\Delta\rho_0$ and E for case 1.

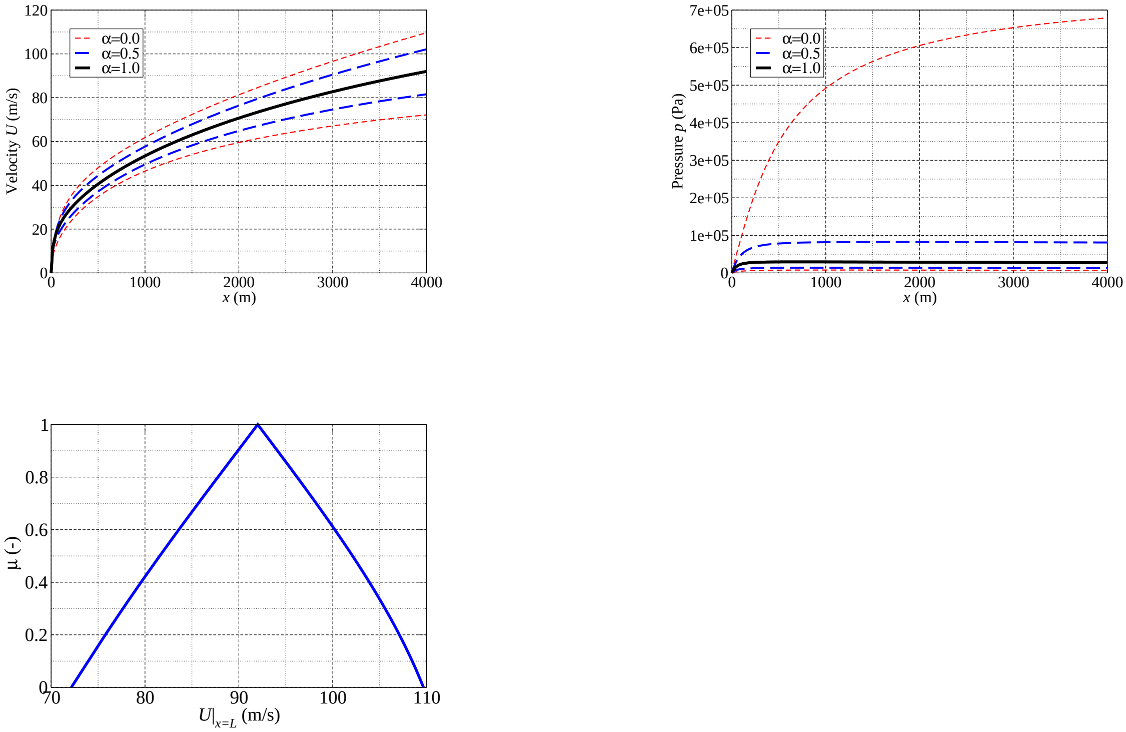

Figure 2: Velocity U and pressure p vs. distance x (left and center) and membership function for $U|_{x=L}$ (right) for case 1.

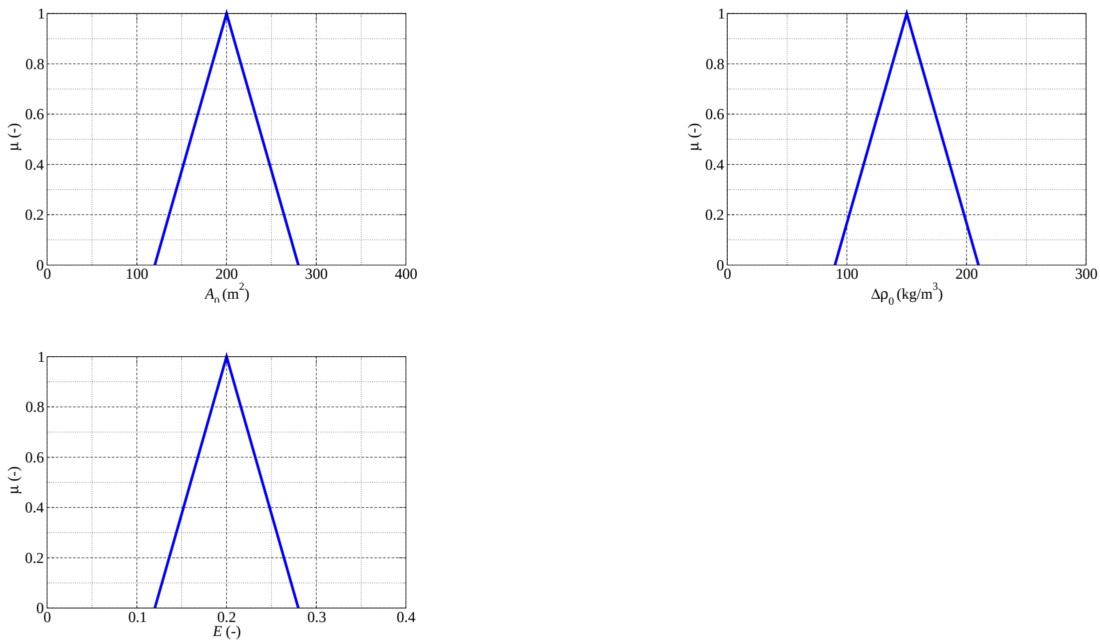

Figure 3: Membership functions for A_0, $\Delta\rho_0$ and E for case 2.

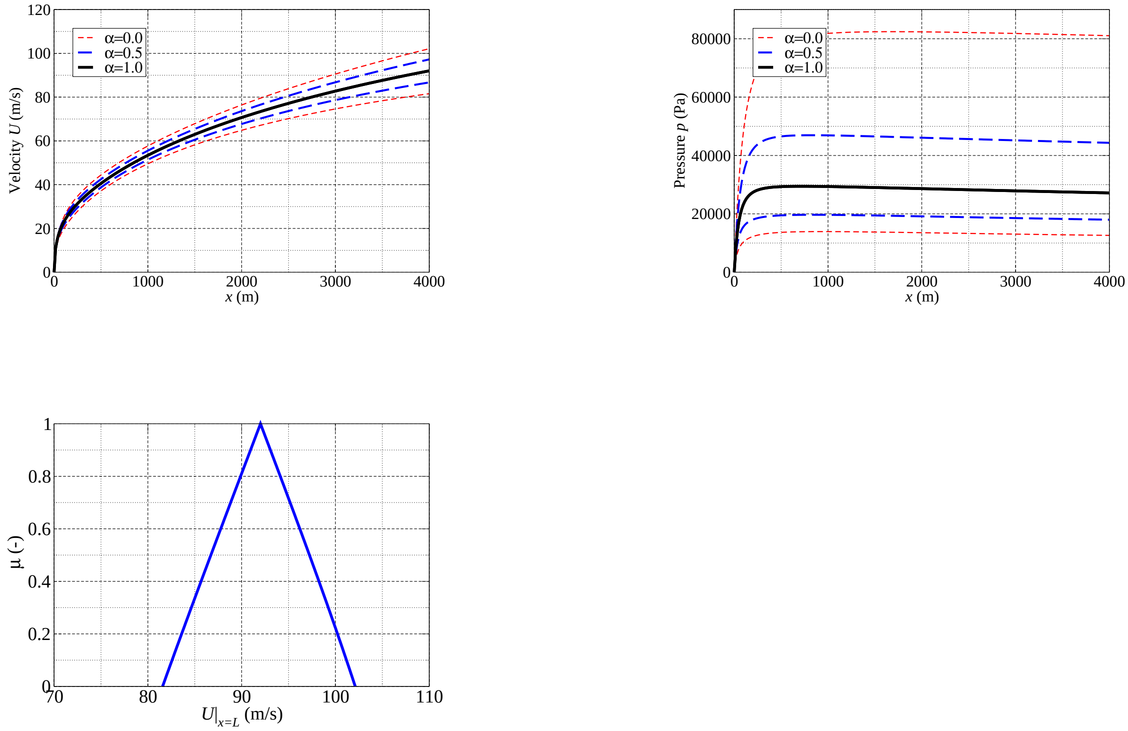

Figure 4: Velocity U and pressure p vs. distance x (left and center) and membership function for $U|_{x=L}$ (right) for case 2.

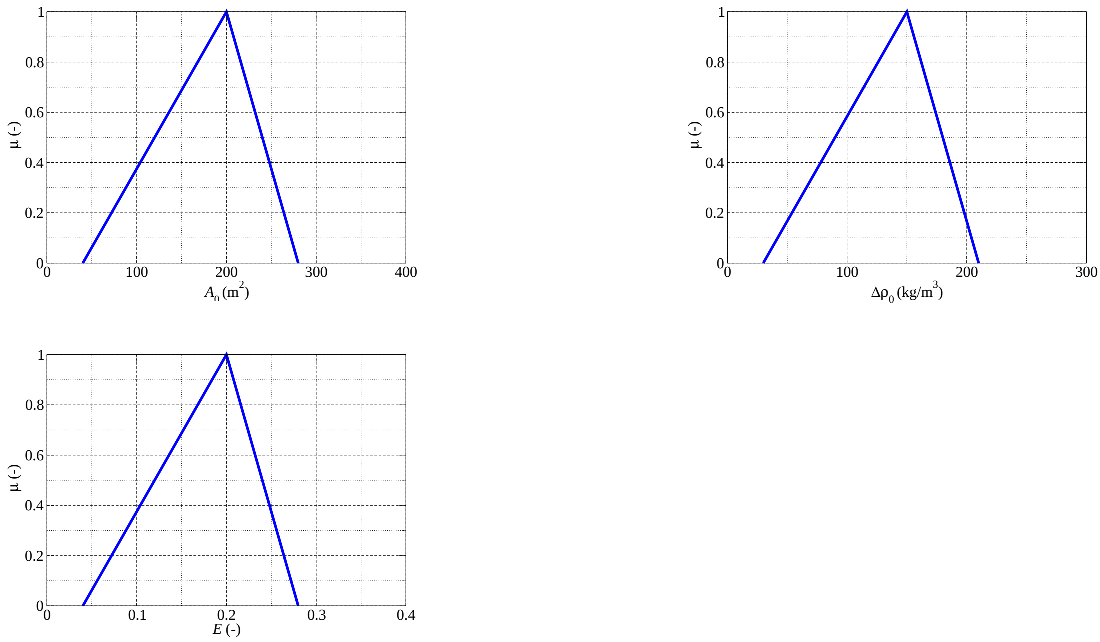

Figure 5: Membership functions for A_0, $\Delta\rho_0$ and E for case 3.

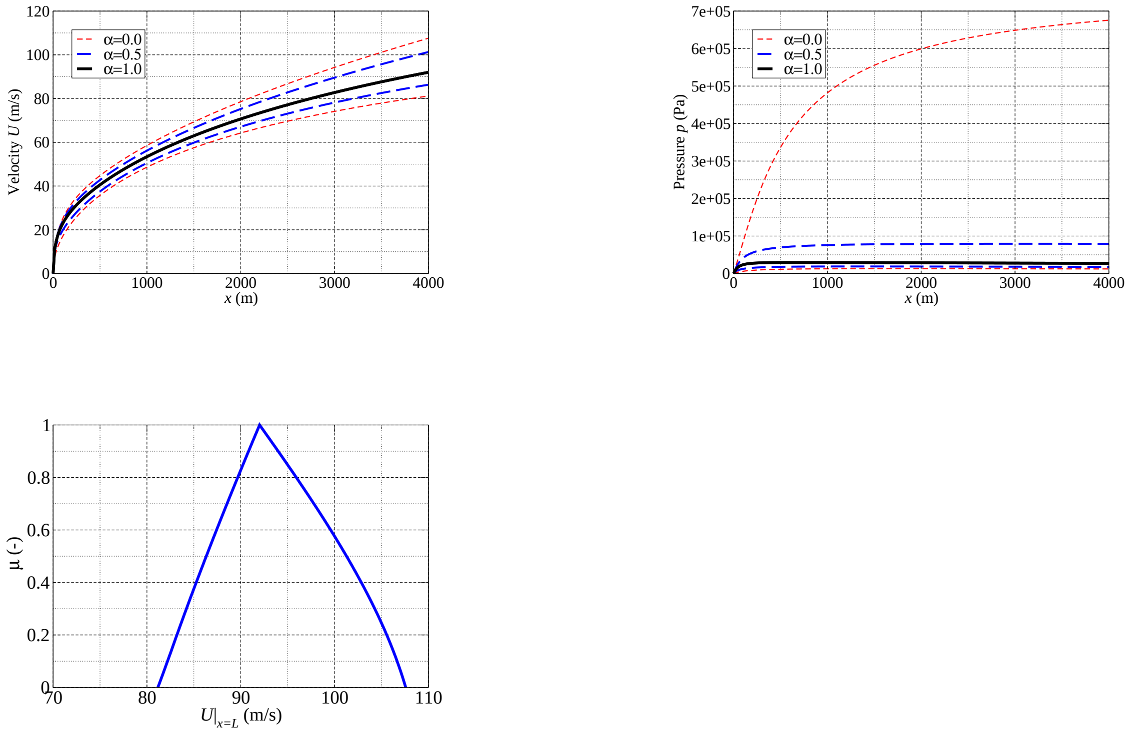

Figure 6: Velocity U and pressure p vs. distance x (left and center) and membership function for $U|_{x=L}$ (right) for case 3.

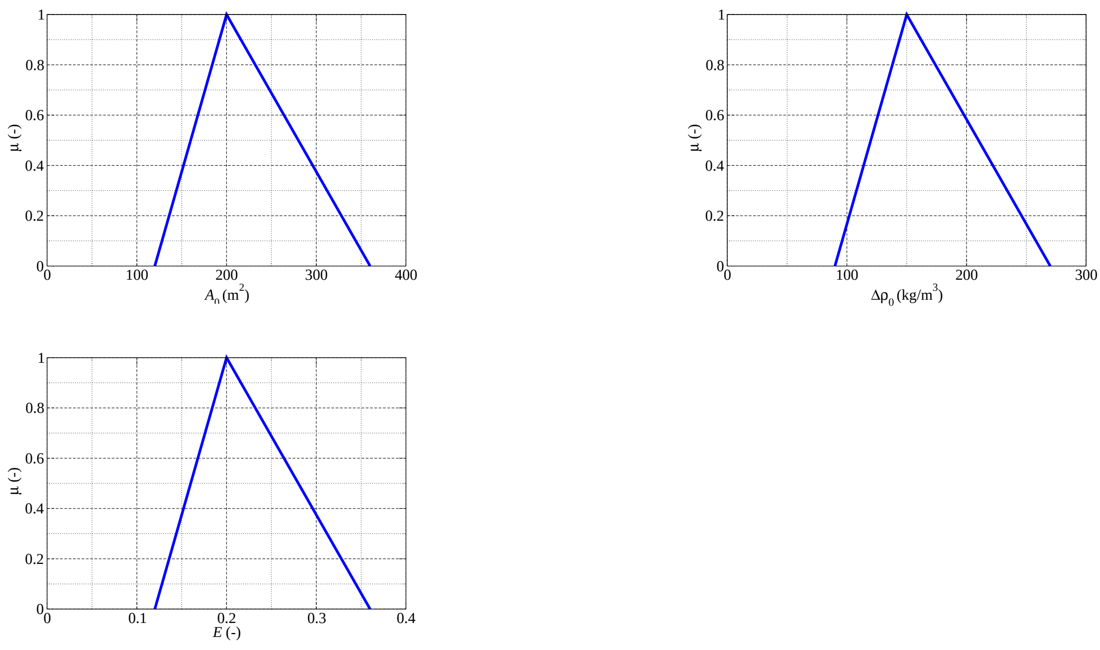

Figure 7: Membership functions for A_0, $\Delta\rho_0$ and E for case 4.

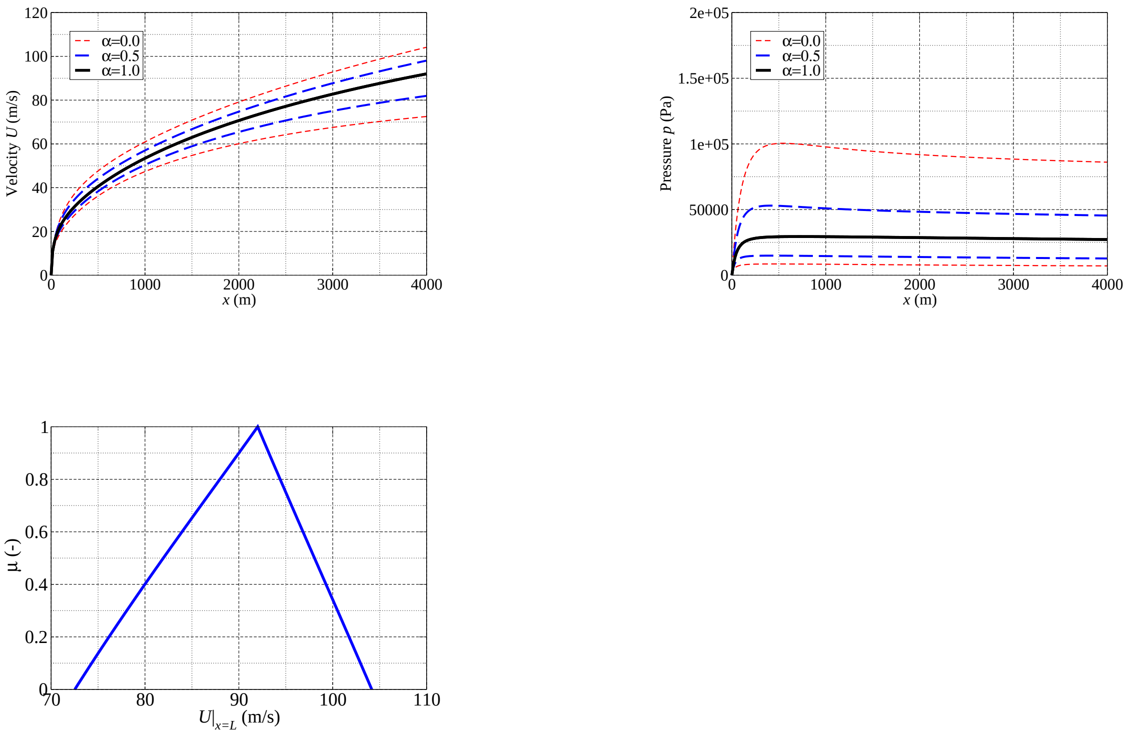

Figure 8: Velocity U and pressure p vs. distance x (left and center) and membership function for $U|_{x=L}$ (right) for case 4.

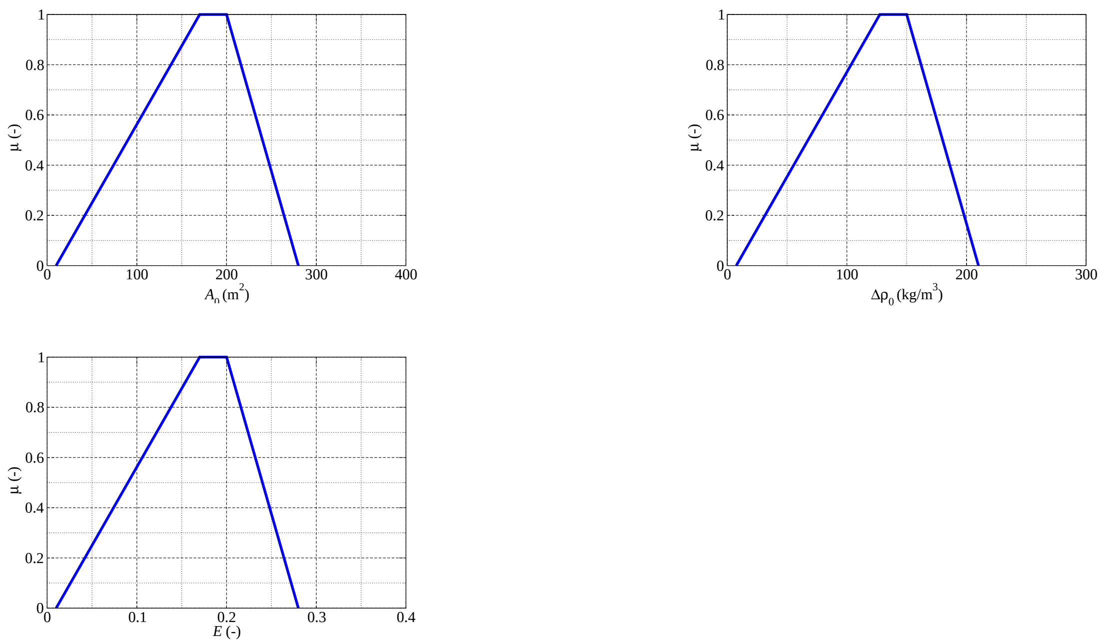

Figure 9: Membership functions for A_0, $\Delta\rho_0$ and E for case 5.

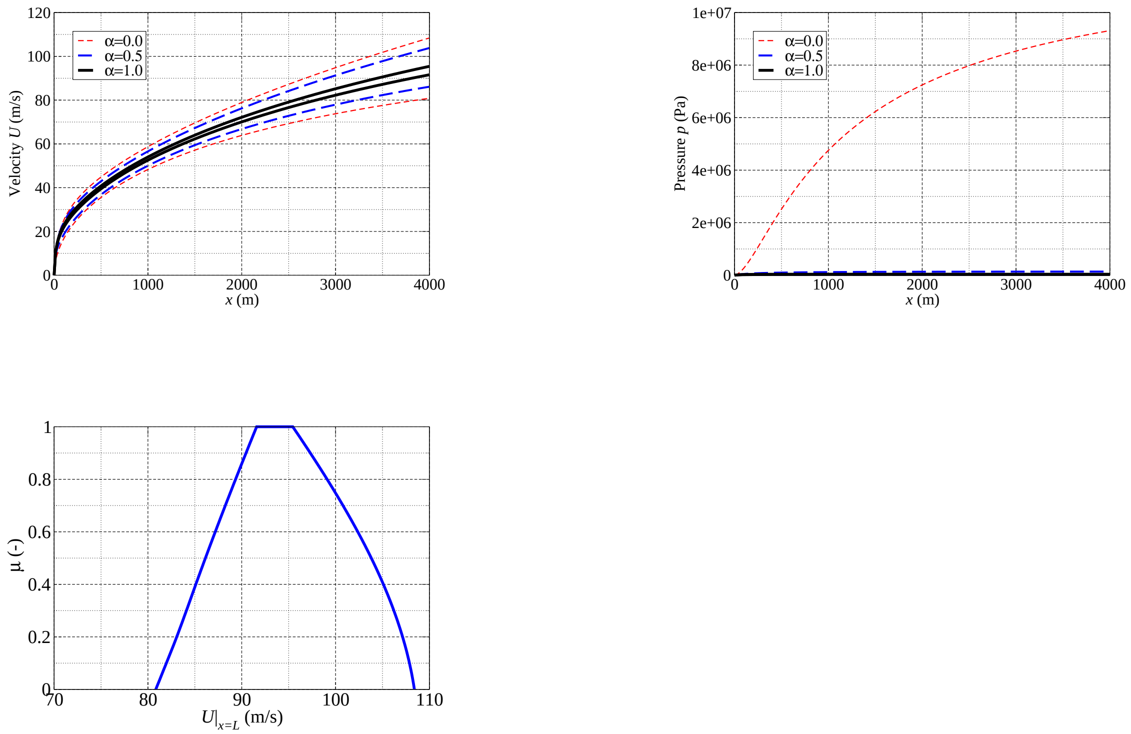

Figure 10: Velocity U and pressure p vs. distance x (left and center) and membership function for $U|_{x=L}$ (right) for case 5.

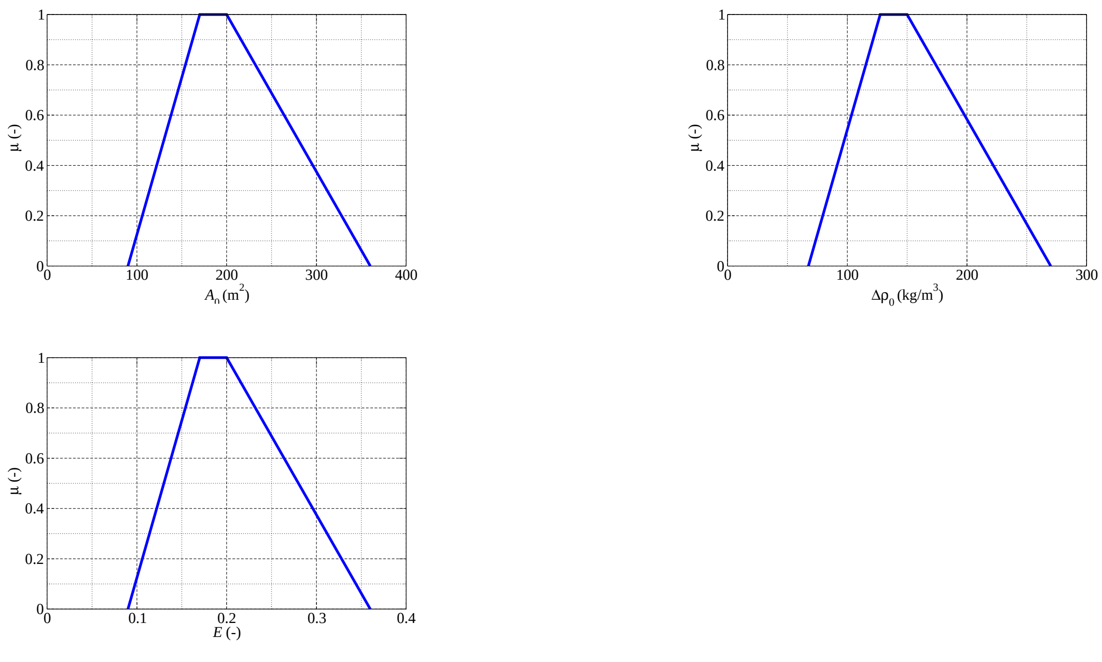

Figure 11: Membership functions for A_0, $\Delta\rho_0$ and E for case 6.

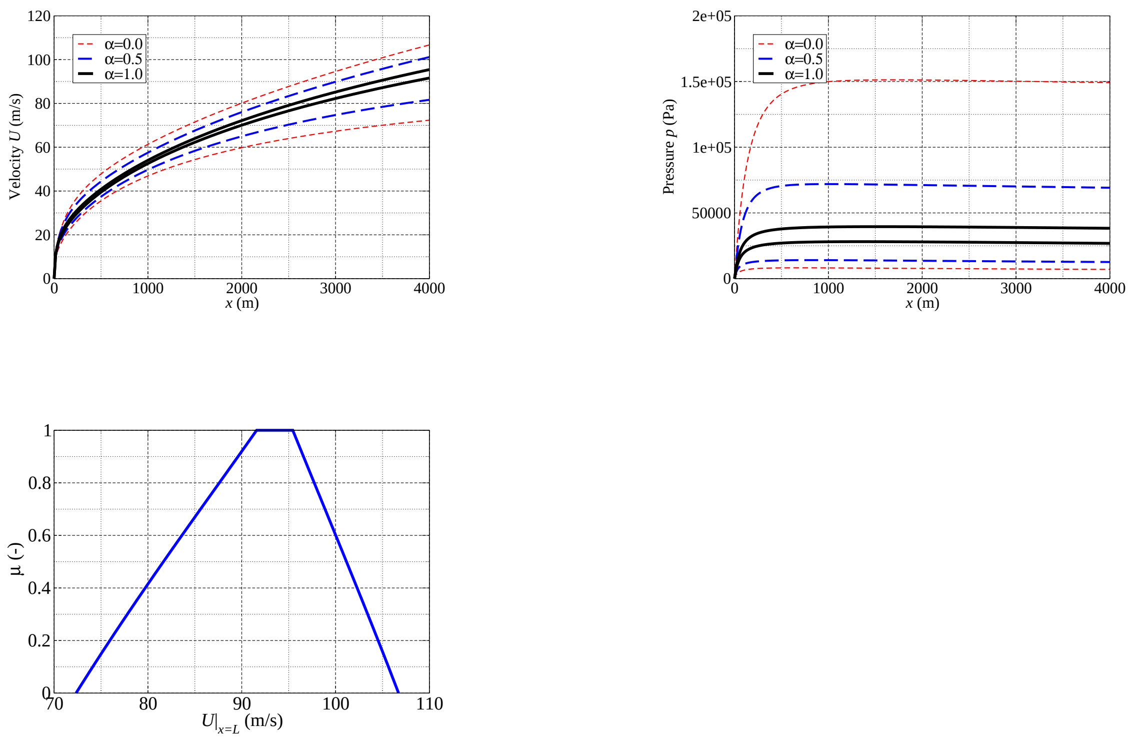

Figure 12: Velocity U and pressure p vs. distance x (left and center) and membership function for $U|_{x=L}$ (right) for case 6.