ON BALANCING AND ORDER REDUCTION OF UNSTABLE PERIODIC SYSTEMS

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Abstract: We consider the direct application of balancing techniques to unstable periodic systems by extending the balancing concepts to arbitrary periodic systems. We extend first the balancing concepts to unstable discrete-time systems by defining the reachability and observability grammians from appropriate right and left coprime factorizations with inner denominators. Further, we extend this new balancing method to unstable linear time-varying discrete-time systems with periodically varying coefficient matrices. The new balancing approach serves as basis to develop balancing related order reduction methods for unstable periodic systems using accuracy enhancing square-root and balancing-free algorithms.

Keywords: Periodic systems, time-varying systems, discrete-time systems, balanced truncation, model reduction, numerical methods

1. INTRODUCTION

For a stable periodic system, numerical procedures for balancing (Varga, 1999) and model reduction (Longhi and Orlando, 1999; Varga, 2000) have been developed recently. These methods extend the well-known balancing and model reduction techniques for standard systems to the periodic case. To reduce unstable periodic systems, two embedding approaches extending similar techniques for standard systems, can be used in conjunction with balancing techniques.

In the first approach, the unstable periodic system is additively decomposed as the sum of its stable and an unstable parts. Then the order reduction is performed only on the stable part using appropriate balancing related methods. The reduced model is formed as the sum of the reduced stable part and the unstable part. This approach has the disadvantage that the unstable part is copied unmodified back into the reduced model, although sometimes a lower order approximation would be possible if this part is also reduced. In particular, if the unstable part is non-minimal, the reduced model results non-minimal too.

The second approach relies on coprime factorization techniques and therefore implicitly involves the reduction of both stable and unstable parts. The unstable periodic system can be expressed in a coprime factorized representation, where the factors are stable periodic systems. Using balancing related techniques, the compound system formed by appending the two factors is reduced and the reduced factors are recovered. Finally, the reduced periodic system is constructed from the coprime factorization of the reduced factors.

In this paper we consider a third approach which addresses the direct application of balancing techniques to unstable periodic systems by extending the balancing concepts to arbitrary periodic systems. Such an approach has been proposed recently for standard continuous-time systems by Zhou *et al.* (1999). We extend first this approach to unstable standard discrete-time systems. Further, we extend this new balancing method to unstable time-varying linear discrete-time systems with periodically varying coefficient matrices.

The new methods rely essentially on computing the controllability and observability grammians from appropriate right and left coprime factorizations with inner denominators. Transformation techniques allow to reduce the computational burden for computing these factorizations by solving reduced order Lyapunov equations instead full order Riccati equations. By using recursive factorization techniques, the factorizations can be determined directly with the state matrices in quasi-upper triangular forms which allows an efficient computation of the grammians.

The new balancing approach can serve as basis to develop balancing related order reduction methods for unstable periodic systems using accuracy enhancing square-root and balancing-free algorithms. The grammians can be computed directly in Cholesky factorized forms which can be employed to determine appropriate truncation matrices to perform model reduction of unstable periodic systems, analogously to methods developed for stable periodic systems in (Varga, 1999; Varga, 2000).

2. THE STANDARD CASE

Let $G(z)$ be a given discrete-time *transfer-function matrix* (TFM) without poles on the unit circle and let (A, B, C, D) be a stabilizable and detectable state-space representation satisfying

$$G(z) = C(zI - A)^{-1}B + D$$

In analogy to the continuous-time case (Zhou *et al.*, 1999) we define the controllability grammian P and the observability grammian Q as

$$P = \frac{1}{2\pi}\int_0^{2\pi} (e^{j\theta}I - A)^{-1}BB^T(e^{-j\theta}I - A^T)^{-1}\mathrm{d}\theta \quad (1)$$

$$Q = \frac{1}{2\pi}\int_0^{2\pi} (e^{-j\theta}I - A^T)^{-1}C^TC(e^{j\theta}I - A)^{-1}\mathrm{d}\theta \quad (2)$$

In the case of a stable system, P and Q are the usual positive semidefinite grammians satisfying the discrete-time matrix Lyapunov equations

$$\begin{aligned} APA^T + BB^T &= P \\ A^TQA + C^TC &= Q \end{aligned} \quad (3)$$

In the case of an unstable system consider the right coprime factorization

$$(zI - A)^{-1}B = N(z)M^{-1}(z)$$

with $M(z)$ an inner TFM. The factors can be computed according to (Zhou *et al.*, 1996) in the form

$$\begin{bmatrix} N(z) \\ M(z) \end{bmatrix} = \left[\begin{array}{c|c} A+BF & BW \\ \hline I & 0 \\ F & W \end{array}\right] := \left[\begin{array}{c|c} A_r & B_r \\ \hline I & 0 \\ F & W \end{array}\right]$$

where

$$\begin{aligned} F &= -WW^TB^TXA \\ W^T(I + B^TXB)W &= I \end{aligned} \quad (4)$$

and X is the stabilizing symmetric positive semidefinite solution of the Riccati equation

$$A^TX(I + BB^TX)^{-1}A - X = 0 \quad (5)$$

It follows from (1) that

$$\begin{aligned} P &= \frac{1}{2\pi}\int_0^{2\pi} N(e^{j\theta})N^T(e^{-j\theta})\mathrm{d}\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} (e^{j\theta}I - A_r)^{-1}B_rB_r^T(e^{-j\theta}I - A_r^T)^{-1}\mathrm{d}\theta \end{aligned}$$

and thus, for an unstable system, P fulfills the Lyapunov equation

$$(A + BF)P(A + BF)^T + BWW^TB^T = P \quad (6)$$

Similarly we compute the observability grammian Q by first determining the left coprime factorization

$$C(zI - A)^{-1} = \widetilde{M}^{-1}(z)\widetilde{N}(z)$$

where $\widetilde{M}(z)$ is an inner TFM. The factors can be computed in the form (Zhou *et al.*, 1996)

$$[\, N(z)\ M(z)\,] = \left[\begin{array}{c|cc} A+LC & I & L \\ \hline VC & 0 & V \end{array}\right]$$

where

$$\begin{aligned} L &= -A\widetilde{X}C^TV^TV \\ V(I + C\widetilde{X}C^T)V^T &= I \end{aligned} \quad (7)$$

and $\widetilde{X}$ is the stabilizing symmetric positive semidefinite solution of the Riccati equation

$$A(I + \widetilde{X}C^TC)^{-1}\widetilde{X}A^T - \widetilde{X} = 0 \quad (8)$$

The observability grammian Q thus satisfies

$$(A + LC)^TQ(A + LC) + C^TV^TVC = Q \quad (9)$$

For a minimal system, the grammians P and Q are positive definite (i.e., nonsingular), and they can be used to perform a system balancing by determining a coordinate transformation such that both grammians in the new coordinate system are equal and diagonal. Alternatively, P and Q can be employed to determine left and right truncation matrices L and T, respectively, to obtain a minimal or a reduced order system G_r with state space representation (LAT, LB, CT, D). Note that for a stable system $X = 0$, $\widetilde{X} = 0$, $W = I$ and $V = I$, thus P and Q are the standard grammians for a stable discrete-time system.

The emphasis on improving the accuracy of computations has led to so-called model reduction algorithms with *enhanced accuracy*. The grammians can be always determined directly in Cholesky factorized forms $P = SS^T$ and $Q = R^T R$, where S and R are upper-triangular matrices (Hammarling, 1982). The computation of L and T can be done from the singular value decomposition

$$RS = \left[\, U_1 \;\; U_2 \,\right] \mathrm{diag}(\Sigma_1, \Sigma_2) \left[\, V_1 \;\; V_2 \,\right]^T,$$

where

$$\begin{aligned} \Sigma_1 &= \mathrm{diag}(\sigma_1, \ldots, \sigma_r), \\ \Sigma_2 &= \mathrm{diag}(\sigma_{r+1}, \ldots, \sigma_n) \end{aligned}$$

and $\sigma_1 \geq \ldots \geq \sigma_r > \sigma_{r+1} \geq \ldots \geq \sigma_n \geq 0$.

The so-called *square-root* (**SR**) methods determine L and T as (Tombs and Postlethwaite, 1987)

$$L = \Sigma_1^{-1/2} U_1^T R, \qquad T = S V_1 \Sigma_1^{-1/2}.$$

This approach is usually numerically very accurate for well-equilibrated systems. However if the original system is highly unbalanced, potential accuracy losses can be induced in the reduced model if either of the truncation matrices L or T is ill-conditioned (i.e., nearly rank deficient).

A *balancing-free square-root* (**BFSR**) algorithm proposed in (Varga, 1991) combines the advantages of a *balancing-free* (**BF**) approach (Safonov and Chiang, 1989) and of the **SR** approach. L and T are determined as

$$L = (Y^T X)^{-1} Y^T, \qquad T = X,$$

where X and Y are $n \times r$ matrices with orthogonal columns computed from two QR decompositions

$$S V_1 = X W, \quad R^T U_1 = Y Z$$

with W and Z non-singular and upper-triangular. The accuracy of the **BFSR** algorithm is usually better than either of **SR** or **BF** approaches.

We have the following analogous result to Theorem 4 of Zhou *et al.* (1999):

Theorem 1. Suppose $G(z)$ has no poles on the unit circle and let $G_r(z)$ be the TFM of the reduced order model with state-space realization (LAT, LB, CT, D), where L and T are the truncation matrices computed above. Then $G_r(z)$ has no poles on the unit circle and

$$\|G(z) - G_r(z)\|_\infty \leq 2 \sum_{r+1}^{n} \sigma_i$$

3. FURTHER NUMERICAL ASPECTS

The computation of grammians involves apparently the solution of two Riccati equations (5) and (8) of particular types. This can be however avoided easily using the technique developed in (Varga, 1993) to compute coprime factorizations with inner denominators. Let U_1 be an orthogonal transformation matrix which reduces A to an ordered *real Schur form* (RSF)

$$U_1^T A U_1 = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad U_1^T B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

where A_{11} contains the stable eigenvalues of A (i.e., those lying inside the unit circle), A_{22} contains the unstable eigenvalues of A, and the transformed $U_1^T B$ is partitioned in accordance with $U_1^T A U_1$. Using the transformed forms of the system matrices, the stabilizing feedback in (4) can be determined as

$$F = \left[\, 0 \;\; F_2 \,\right] U_1^T$$

where F_2 and W_2 are computed as

$$F_2 = -W_2 W_2^T B_2^T X_2 A_{22}$$

$$W_2^T (I + B_2^T X_2 B_2) W_2 = I$$

with X_2 being the stabilizing solution of the reduced order Riccati equation

$$A_{22}^T X_2 (I + B_2 B_2^T X_2)^{-1} A_{22} - X_2 = 0$$

Because A_{22} is anti-stable, X_2 is positive definite. Thus, since both A_{22} and X_2 are invertible, we can rewrite the above Riccati equation as a Lyapunov equation in the variable $Y = X_2^{-1}$

$$A_{22}^T Y A_{22} - B_2 B_2^T = Y$$

and F_2 can be alternatively computed as

$$F_2 = -B_2^T (Y + B_2 B_2^T)^{-1} A_{22}$$

To reduce the overall computational costs, it is possible to determine F_2 using the recursive approach of (Varga, 1993; Varga, 1998). With this method a second orthogonal transformation matrix U_2 is determined such that $U_2^T (A_{22} + B_2 F_2) U_2$ is further in a RSF. Thus, with

$$U = U_1 \begin{bmatrix} I & 0 \\ 0 & U_2 \end{bmatrix}$$

we have

$$\begin{bmatrix} N(z) \\ M(z) \end{bmatrix} = \left[\begin{array}{c|c} U^T(A+BF)U & U^TBW_2 \\ \hline U & 0 \\ FU & W_2 \end{array}\right]$$

and $U^T(A+BF)U$ is in RSF. The controllability grammian $\widehat{P}$ corresponding to the transformed matrices $\widehat{A} = U^T(A+BF)U$, $\widehat{B} = U^TB$ satisfies the Lyapunov equation

$$\widehat{A}\widehat{P}\widehat{A}^T + \widehat{B}W_2W_2^T\widehat{B}^T = \widehat{P}$$

and the controllability grammian in the original coordinates is given by $P = U^T\widehat{P}U$. If $\widehat{P}$ is determined in a Cholesky-factorized form $\widehat{P} = \widehat{S}\widehat{S}^T$ (e.g., by using the algorithm of Hammarling (1982)), then P can be easily computed in a similar form $P = SS^T$, where S is the upper triangular factor in the QR-decomposition of $U^T\widehat{S}$.

An entirely similar computational approach can be devised to determine the observability grammian Q in a Cholesky factorized form $Q = R^TR$. As before, we can avoid the solution of a Riccati equation by solving instead a reduced order Lyapunov equation. All computational details follow by duality formulas.

4. THE PERIODIC CASE

In this section we extend the previous results for standard discrete-time systems to periodic systems of the form

$$\begin{aligned} x_{k+1} &= A_kx_k + B_ku_k \\ y_k &= C_kx_k + D_ku_k \end{aligned} \qquad (10)$$

where the matrices $A_k \in \mathrm{R}^{n_{k+1}\times n_k}$, $B_k \in \mathrm{R}^{n_{k+1}\times m}$, $C_k \in \mathrm{R}^{p\times n_k}$, $D_k \in \mathrm{R}^{p\times m}$ are periodic with period $K \geq 1$.

To simplify the presentation we introduce first some notation. For a K-periodic matrix X_k we use alternatively the *script* notation

$$\mathcal{X} := \operatorname{diag}(X_0, X_1, \ldots, X_{K-1}),$$

which associates the block-diagonal matrix $\mathcal{X}$ to the cyclic matrix sequence X_k, $k = 0, \ldots, K-1$. This notation is consistent with the standard matrix operations as for instance addition, multiplication, inversion as well as with several standard matrix decompositions (Cholesky, SVD). We denote with $\sigma\mathcal{X}$ the K-cyclic shift

$$\sigma\mathcal{X} = \operatorname{diag}(X_1, \ldots, X_{K-1}, X_0)$$

of the cyclic sequence X_k, $k = 0, \ldots, K-1$. By using the script notation, the periodic system (10) will be alternatively denoted by the quadruple $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})_{\mathbf{n}}$, where the time-varying state vector dimensions are denoted compactly by $\mathbf{n} = (n_0, \ldots, n_{K-1})$. We denote with $\mathcal{I}_{\mathbf{n}}$ the K-periodic identity matrix I_{n_k} with time-varying dimensions. The transition matrix of the system (10) is defined by the $n_j \times n_i$ matrix $\Phi_A(j,i) = A_{j-1}A_{j-2}\cdots A_i$, where $\Phi_A(i,i) := I_{n_i}$. The state transition matrix over one period $\Phi_A(j+K, j) \in \mathrm{R}^{n_j\times n_j}$ is called the *monodromy matrix* of system (10) at time j and its eigenvalues are called *characteristic multipliers* at time j.

In what follows, we assume that the monodromy matrix $\Phi_A(\tau+K, \tau)$ has no eigenvalues on the unit circle. Using a similar approach as for the standard case, we define the controllability grammian of a possibly unstable periodic system $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})_{\mathbf{n}}$ as the periodic semipositive definite matrix $\mathcal{P}$ satisfying the periodic Lyapunov equation analogous to (6)

$$\sigma\mathcal{P} = (\mathcal{A}+\mathcal{B}\mathcal{F})\mathcal{P}(\mathcal{A}+\mathcal{B}\mathcal{F})^T + \mathcal{B}\mathcal{W}\mathcal{W}^T\mathcal{B}^T \quad (11)$$

where

$$\begin{gathered} \mathcal{F} = -\mathcal{W}\mathcal{W}^T\mathcal{B}^T\mathcal{A}\sigma\mathcal{X} \\ \mathcal{W}^T(\mathcal{I} + \mathcal{B}^T\sigma\mathcal{X}\mathcal{B})\mathcal{W} = \mathcal{I} \end{gathered} \qquad (12)$$

and $\mathcal{X}$ is the stabilizing symmetric positive semidefinite solution of the periodic Riccati equation

$$\mathcal{A}^T\sigma\mathcal{X}(\sigma\mathcal{I}_{\mathbf{n}} + \mathcal{B}\mathcal{B}^T\sigma\mathcal{X})^{-1}\mathcal{A} - \mathcal{X} = 0 \quad (13)$$

These equations can be deduced from the conditions characterizing an all-pass periodic system established by Xie *et al.* (1996, Theorem 4.2).

Similarly, we define the periodic observability grammian $\mathcal{Q}$ satisfying the periodic Lyapunov equation analogous to (9)

$$\mathcal{Q} = (\mathcal{A}+\mathcal{L}\mathcal{C})^T\sigma\mathcal{Q}(\mathcal{A}+\mathcal{L}\mathcal{C}) + \mathcal{C}^T\mathcal{V}^T\mathcal{V}\mathcal{C} \quad (14)$$

where

$$\begin{gathered} \mathcal{L} = -\mathcal{A}\widetilde{\mathcal{X}}\mathcal{C}^T\mathcal{V}^T\mathcal{V} \\ \mathcal{V}(\mathcal{I} + \mathcal{C}\widetilde{\mathcal{X}}\mathcal{C}^T)\mathcal{V}^T = \mathcal{I} \end{gathered} \qquad (15)$$

and $\widetilde{\mathcal{X}}$ is the stabilizing symmetric semipositive definite solution of the periodic Riccati equation

$$\mathcal{A}(\mathcal{I}_{\mathbf{n}} + \widetilde{\mathcal{X}}\mathcal{C}^T\mathcal{C})^{-1}\widetilde{\mathcal{X}}\mathcal{A}^T - \sigma\widetilde{\mathcal{X}} = 0 \qquad (16)$$

Let $\mathcal{P} = \mathcal{S}^T\mathcal{S}$ and $\mathcal{Q} = \mathcal{R}^T\mathcal{R}$ be the Cholesky factorizations of grammians. For a minimal system, in analogy with the standard stable case (Tombs and Postlethwaite, 1987), we can use the singular value decomposition

$$\mathcal{R}\mathcal{S} = \mathcal{U}\Sigma\mathcal{V}^T, \qquad (17)$$

to compute the balancing transformation matrix $\mathcal{T}$ and its inverse $\mathcal{T}^{-1}$ as

$$\mathcal{T} = \mathcal{S}\mathcal{V}\Sigma^{-1/2}, \qquad \mathcal{T}^{-1} = \Sigma^{-1/2}\mathcal{U}^T\mathcal{R}.$$

It is easy to show that the Lyapunov-transformed system $(\sigma\mathcal{T}^{-1}\mathcal{A}\mathcal{T}, \sigma\mathcal{T}^{-1}\mathcal{B}, \mathcal{C}\mathcal{T}, \mathcal{D})_{\mathbf{n}}$ has the controllability and observability grammians equal and diagonal. We call such a realization of the possibly unstable periodic system (10) a *balanced* realization.

Algorithms with enhanced accuracy for periodic model reduction have been developed by Varga (2000). The truncation formulas to determine directly the matrices of the reduced system $(\mathcal{A}_r, \mathcal{B}_r, \mathcal{C}_r, \mathcal{D}_r)$ generalize those in the standard case. Let us write the singular value decomposition (17) at each time instant k in the partitioned form

$$R_kS_k = [\,U_{k,1}\ U_{k,2}\,]\begin{bmatrix}\Sigma_{k,1} & 0\\ 0 & \Sigma_{k,2}\end{bmatrix}[\,V_{k,1}\ V_{k,2}\,]^T \quad (18)$$

where $\Sigma_{k,1} \in \mathrm{R}^{r_k\times r_k}$, $U_{k,1} \in \mathrm{R}^{n_k\times r_k}$, $V_{k,1} \in \mathrm{R}^{n_k\times r_k}$ and $\Sigma_{k,1} > 0$. From the above decomposition define, with $\widetilde{\Sigma}_1 = \mathrm{diag}\,(\Sigma_{0,1}, \ldots, \Sigma_{K-1,1})$, the *truncation* matrices

$$\mathcal{L} = \widetilde{\Sigma}_1^{-\frac{1}{2}}\mathcal{U}_1^T\mathcal{R}, \qquad \mathcal{T} = \mathcal{S}\mathcal{V}_1\widetilde{\Sigma}_1^{-\frac{1}{2}}. \quad (19)$$

Then the reduced system can be computed as

$$(\mathcal{A}_r, \mathcal{B}_r, \mathcal{C}_r, \mathcal{D}_r)_{\mathbf{r}} = (\sigma\mathcal{L}\mathcal{A}\mathcal{T}, \sigma\mathcal{L}\mathcal{B}, \mathcal{C}\mathcal{T}, \mathcal{D})_{\mathbf{r}} \quad (20)$$

Since the computation of the reduced model relies exclusively on square-root information (the Cholesky factors of grammians), this model reduction method is called the *square-root* approach. This approach leads to a guaranteed enhancement of the overall numerical accuracy of computations. The key computation in determining the truncation matrices $\mathcal{L}$ and $\mathcal{T}$ is the solution of the two periodic Lyapunov equations (11) and (14) with time-varying dimensions directly for the Cholesky factors of the grammians. Numerically stable algorithms for these computations have been developed recently by Varga (1999).

The *square-root* method is essentially a balancing based truncation approach. To avoid accuracy losses potentially induced by balancing, an alternative is to use a *balancing-free* approach to determine the truncation matrices. A *square-root balancing-free* approach for the periodic case, which combines both these desirable features, has been proposed recently by Varga (2000). Consider the QR-decompositions

$$\mathcal{S}\mathcal{V}_1 = \widetilde{\mathcal{T}}\mathcal{X}, \quad \mathcal{R}^T\mathcal{U}_1 = \widetilde{\mathcal{Z}}\mathcal{Y}, \quad (21)$$

where $\mathcal{X}$ and $\mathcal{Y}$ are nonsingular matrices and $\widetilde{\mathcal{T}}$ and $\widetilde{\mathcal{Z}}$ are matrices with orthonormal columns. With the already computed $\widetilde{\mathcal{T}}$ we define the corresponding $\widetilde{\mathcal{L}}$ as

$$\widetilde{\mathcal{L}} = (\widetilde{\mathcal{Z}}^T\widetilde{\mathcal{T}})^{-1}\widetilde{\mathcal{Z}}^T. \quad (22)$$

It is easy to show that the periodic system $(\sigma\widetilde{\mathcal{L}}\mathcal{A}\widetilde{\mathcal{T}}, \sigma\widetilde{\mathcal{L}}\mathcal{B}, \mathcal{C}\widetilde{\mathcal{T}}, \mathcal{D})_{\mathbf{r}}$ with $\widetilde{\mathcal{L}}$ and $\widetilde{\mathcal{T}}$ defined in (21) and (22) is Lyapunov-similar to the reduced system $(\mathcal{A}_r, \mathcal{B}_r, \mathcal{C}_r, \mathcal{D}_r)_{\mathbf{r}}$ obtained with the *square-root* approach.

Similarly to the standard case we can avoid the solution of periodic Riccati equations by solving instead periodic Lyapunov equations of lower order. Consider the periodic orthogonal matrix $\mathcal{U}$ such that $\mathcal{A}$ is in an ordered EPRSF (Bojanczyk *et al.*, 1992; Varga, 1999)

$$\begin{aligned} U_{k+1,1}^T A_k U_{k,1} &= \begin{bmatrix} A_{k,11} & A_{k,12}\\ 0 & A_{k,22}\end{bmatrix}\\ U_{k+1,1}^T B_k &= \begin{bmatrix} B_{k,1}\\ B_{k,2}\end{bmatrix}\end{aligned} \quad (23)$$

where $\mathcal{A}_{11}$ contains the stable characteristic values of $\mathcal{A}$ (i.e., those lying in the unit circle), $\mathcal{A}_{22}$ contains the unstable characteristic values of $\mathcal{A}$, and the transformed $\sigma\mathcal{U}_1^T\mathcal{B}$ is partitioned in accordance with $\sigma\mathcal{U}_1^T AU_1$.

Using the transformed forms of the system matrices, the stabilizing periodic state feedback can be determined as

$$F_k = [\,0\ \ F_{k,2}\,]\,U_{k,1}^T$$

where $\mathcal{F}_2$ and $\mathcal{W}_2$ can be computed as

$$\mathcal{F}_2 = -\mathcal{W}_2\mathcal{W}_2^T\mathcal{B}_2^T\sigma\mathcal{X}_2\mathcal{A}_{22}$$

$$\mathcal{W}_2^T(\mathcal{I} + \mathcal{B}_2^T\sigma\mathcal{X}_2\mathcal{B}_2)\mathcal{W}_2 = \mathcal{I}$$

with $\mathcal{X}_2$ being the stabilizing solution of the reduced order periodic Riccati equation

$$\mathcal{A}_{22}^T\sigma\mathcal{X}_2(\mathcal{I} + \mathcal{B}_2\mathcal{B}_2^T\sigma\mathcal{X}_2)^{-1}\mathcal{A}_{22} - \mathcal{X}_2 = 0$$

Because $\mathcal{A}_{22}$ is anti-stable, $\mathcal{X}_2$ is positive definite. Thus, since both $\mathcal{A}_{22}$ and $\mathcal{X}_2$ are invertible, we can rewrite the above periodic Riccati equation as a periodic Lyapunov equation in the variable $\mathcal{Y} = \mathcal{X}_2^{-1}$

$$\mathcal{A}_{22}^T\mathcal{Y}\mathcal{A}_{22} - \mathcal{B}_2\mathcal{B}_2^T = \sigma\mathcal{Y} \quad (24)$$

and $\mathcal{F}_2$ can be computed as

$$\mathcal{F}_2 = -\mathcal{B}_2^T(\sigma\mathcal{Y} + \mathcal{B}_2\mathcal{B}_2^T)^{-1}\mathcal{A}_{22}$$

Since the periodic submatrix $A_{k,22}$ in the EPRSF (23) has constant dimension, the algorithm of Varga (1997) can be employed to solve (24).

It is also possible in the periodic case to reduce the overall computational costs, by determining $\mathcal{F}_2$ using a recursive approach similar to the approach for standard system presented in (Varga, 1993). For this purpose, the periodic Schur method for pole assignment (Sreedhar and Van Dooren, 1993) can be extended to compute recursively coprime factorizations for periodic systems. We will not enter into the details of such a method, but discuss only the outcome of it. The periodic state feedback $\mathcal{F}_2$ is determined simultaneously with a second orthogonal periodic transformation matrix $\mathcal{U}_2$ such that $\sigma\mathcal{U}_2^T(\mathcal{A}_{22} + \mathcal{B}_2\mathcal{F}_2)\mathcal{U}_2$ is in a PRSF. Thus, with

$$U_k = U_{k,1}\begin{bmatrix} I & 0 \\ 0 & U_{k,2} \end{bmatrix}$$

we have that the controllability grammian $\widehat{\mathcal{P}}$ corresponding to the transformed system matrices $\widehat{\mathcal{A}} = \sigma\mathcal{U}^T(\mathcal{A} + \mathcal{B}\mathcal{F})\mathcal{U}$ and $\widehat{\mathcal{B}} = \sigma\mathcal{U}^T\mathcal{B}$ satisfies the periodic Lyapunov equation

$$\widehat{\mathcal{A}}\widehat{\mathcal{P}}\widehat{\mathcal{A}}^T + \widehat{\mathcal{B}}\mathcal{W}_2\mathcal{W}_2^T\widehat{\mathcal{B}}^T = \sigma\widehat{\mathcal{P}}$$

The controllability grammian in the original coordinates is given by $\mathcal{P} = \mathcal{U}^T\widehat{\mathcal{P}}\mathcal{U}$. If $\widehat{\mathcal{P}}$ is determined in a Cholesky-factorized form $\widehat{\mathcal{P}} = \widehat{\mathcal{S}}\widehat{\mathcal{S}}^T$ using the algorithm developed by Varga (1999) for time-varying dimensions, then $\mathcal{P}$ can be easily computed in a similar form $\mathcal{P} = \mathcal{S}\mathcal{S}^T$, where $\mathcal{S}$ is the upper triangular factor in the QR-decomposition of $\mathcal{U}^T\widehat{\mathcal{S}}$.

An entirely similar computational approach can be devised to determine the periodic observability grammian $\mathcal{Q}$. As before, we can avoid the solution of a periodic Riccati equation by solving instead a reduced order periodic Lyapunov equation. All computational details follow by dual formulas.

5. CONCLUSIONS

We extended the balancing concepts for stable systems to unstable standard and periodic discrete-time systems. This allows the application of balancing related accuracy enhancing order reduction methods to unstable periodic systems. The main computational problems are the computation of coprime factorizations with inner denominators, the solution of sign definite periodic Lyapunov equations with constant and time-varying dimensions, and the computation of truncation matrices for model reduction using square-root and balancing-free techniques. The main computational ingredient to solve these problems is the computation of extended periodic real Schur form.

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