ω-Limit Sets of Discrete Dynamical Systems

by

Andrew David Barwell

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School of Mathematics
College of Engineering and Physical Sciences
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Abstract

ω-limit sets are interesting and important objects in the study of discrete dynamical systems. Using a variety of methods, we present and extend existing results in this area of research. Of particular interest is the property of internal chain transitivity, and we present several characterizations of ω-limit sets in terms of this property. In so doing, we will often focus our attention on the property of pseudo-orbit tracing (shadowing), which plays a central role in many of the characterizations, and about which we prove several new results. We also make extensive use of symbolic dynamics, and prove new results relating to this method of analysis.

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List of Figures

Contents

Chapter 1

Introduction

ω-limit sets, as we will show, are an important and interesting phenomenon in the subject of dynamical systems, and there have been many treatments of their structure in recent literature (see for example [1], [4], [3], [9], [10], [11], [21], [22], [23], [26], [44]). Despite having a simple topological definition, ω-limit sets are complex objects, and take many different forms. Characterizations exist, but the insight gained from such studies can sometimes be limited by the complexity of the required technical definitions, and different characterizations may appear to bear no resemblance to one another. In this report we give an account of the existing material in this area, attempting to draw together many of the most commonly observed characteristics, and offer a new perspective using symbolic dynamics. We also characterize ω-limit sets for particular systems using simple and well-known properties, in particular that of internal chain transitivity, a property which has relevance in many areas of applied dynamical systems as well as topological dynamics [15], [26], [38], [53].

We will begin in this chapter by reviewing some of the central notions in discrete dynamical systems, giving examples to highlight certain key aspects of the theory. In Chapter 2 we will introduce some of the more specific definitions and terminology related to topological dynamics, in particular to ω-limit sets, and give some important results in this area, together with their proofs. In so doing, we will begin to see what restrictions can be put on the topological structure of ω-limit sets when we restrict our attention to interval maps. In Chapter 3 we will restrict our focus further to three specific properties of ω-limit sets, one being internal chain transitivity, which will be a focus of the remainder of this report, and which we show is equivalent to a property introduced independently by Šarkovs′kiĭ.

A lot of work has been done on ω-limit sets of interval maps (see [1], [4], [8], [9], [10]), but in Chapter 4 we take a fresh perspective using symbolic dynamics and kneading theory, developed primarily by Milnor and Thurston [35], and obtain new results which appear unaccessible using conventional analytic methods. We also prove several new results in the area of kneading theory which help us obtain our first dynamical characterization of ω-limit sets of interval maps in terms of internal chain transitivity.

The notion of pseudo-orbit tracing, or shadowing, has close links to that of internal chain transitivity, and was utilized by Bowen in [11] to examine the ω-limit sets of Axiom A diffeomorphisms; it has since become a popular area of research in its own right [16], [18], [28], [40], [52]. In Chapter 5 we introduce a number of different forms of this property, prove equivalences between them, and show how certain expansive properties of maps imply shadowing in its various forms. Finally in Chapter 6 we use much of what we develop in earlier chapters, particularly expansivity and shadowing from Chapter 5, to prove a number of characterizations of ω-limit sets in various spaces.

1.1 Dynamical Systems

In this work, we will focus on discrete dynamical systems, which are defined below. Whilst many of the definitions are consistent with general topological spaces, some of the more technical definitions require a metric, thus we define everything in terms of compact metric spaces unless otherwise stated, with particular emphasis on interval maps. For similar reasons maps on the space will be continuous in general unless otherwise stated (some instances where these conditions can be relaxed will be addressed in forthcoming work). These restrictions are consistent with previous work in this area (see [1], [4], [9], [10], [11], [12], [17], [44], [50] and many more). The terminology and symbols we use are also consistent with work in this area.

For X a compact metric space with metric d and $f : X \to X$ a continuous map, pick any $x \in X$ and any $A \subset X$. We shall denote by $f\restriction_A$ the function defined on the set A with image set $f(A)$. We use the standard definition of distance defined by $d(x, A) = \inf\{d(x, y) \ : \ y \in A\}$. For any $\epsilon > 0$ we denote the ball of radius ϵ centered at x by $B_\epsilon(x)$, and the set $\{x \in X \ : \ d(x, A) < \epsilon\}$ by $B_\epsilon(A)$. For any $n \in \mathbb{N}$, $f^{-n}(A)$ denotes the set $\{x \in X \ : \ f^n(x) \in A\}$; the n^{th} inverse image of A. For a point $y \in \mathbb{R}$ and a function $f : \mathbb{R} \to \mathbb{R}$, denote by $\lim_{x \downarrow y} f(x)$ and $\lim_{x \uparrow y} f(x)$ the limit of $f(x)$ as x tends to y from above and below respectively.

Definition 1.1.1 (Dynamical System, Orbit)**.** A *dynamical system* is a pair (X, f), where X is a compact metric space with metric d and f is a continuous mapping of X into itself. For a point x in the space X the *forward orbit* of x is the set

$$\text{orb}^+(x) = \{f^i(x) \; : \; i \in \mathbb{Z}, \; i \geq 0\},$$

where for $n \in \mathbb{Z}$, $f^{n+1} = f \circ f^n$, and f^0 is the identity on X. Where f is one-to-one, the *backward orbit* of $x = x_0$, $\text{orb}^-(x_0) = \{x_{-i}\}_{i \geq 0} \subset X$, is well-defined, where $f(x_{-i}) = x_{-i+1}$ for every $i > 0$. In our general discussion a map f will not necessarily be one-to-one, but in the case where f is a surjective map we can generate (possible several) backward orbits of a point x_0 by sequentially choosing $x_{i-1} \in f^{-1}(x_i)$ for each $i \leq 0$. Where a backward and forward orbit of a point x_0 are defined, we define the *full orbit* of x_0 as the set $\text{orb}(x_0) = \{x_i\}_{i \in \mathbb{Z}}$, where $f(x_i) = x_{i+1}$ for every $i \in \mathbb{Z}$. We will indicate explicitly when we are refering to a *full* orbit, and as we will usually be discussing only forward dynamics we will refer to the forward orbit of a point as simply the *orbit* of that point, unless there is any ambiguity.

Definition 1.1.2 (Periodic Point, Cycle)**.** If for some positive $n \in \mathbb{Z}$ we have that $f^n(x) = x$ we will say that x is a *periodic point* for f, and furthermore if $f^j(x) \neq x$ for any $j < n$ we will say that x has period n. The orbit of x will be referred to as a *periodic orbit* (of period n) or (n-)*cycle*. A point x is called *pre-periodic* if x is not periodic and there are integers $k, n > 0$ such that $f^{in+k}(x) = f^k(x)$ for every $i \geq 0$. A point x is said to be *asymptotically periodic* if there is a periodic point $z \in X$ such that $\lim_{n\to\infty} d(f^n(x), f^n(z)) = 0$.

It is easy to see that all periodic and pre-periodic points are asymptotically periodic.

We will denote the interior and closure of a set A by A° and $\overline{A}$ respectively, and for any $n \in \mathbb{N}$ the n^{th} derivative of a map f by $D^n f$.

Definition 1.1.3 (Regularly Closed Set)**.** A set $C \subset X$ is said to be *regularly closed* if $C = \overline{C^\circ}$.

The following are topological properties of all dynamical systems (X, f) and will appear regularly in the sequel.

Definition 1.1.4 (Topologically Transitive Map)**.** A map $f : X \to X$ is *topologically transitive* if for any pair of non-empty open sets U and V there is an integer $k > 0$ for which $f^k(U) \cap V \neq \emptyset$.

Definition 1.1.5 (Topologically Exact Map)**.** A map $f : X \to X$ is *topologically exact* if for any non-empty open set $U \subset X$ there is an integer $k > 0$ for which $f^k(U) = X$.

We will often refer to topologically transitive maps as simply *transitive*. Some authors refer to maps which are topologically exact as being *locally eventually onto*. It is easy to see that topologically exact maps are also transitive.

Remark 1.1.6. It is easy to see that any topologically exact map is also transitive, however the converse is not true in general. As an example, consider an irrational rotation of the circle. Then any open set will certainly meet any other open set, so the map is transitive, however there is no expansion of segments, so the map cannot be topologically exact.

Definition 1.1.7 (Sensitive Dependence to Initial Conditions (SDIC))**.** A map $f : X \to X$ has *sensitive dependence to initial conditions* if there exists a $\delta > 0$ such that for every $x \in X$ and neighbourhood N of x, there exists a $y \in N$ and an $n \geq 0$ such that $d(f^n(y), f^n(x)) > \delta$. The δ in this definition is known as the *sensitivity constant*.

The following definition was used by Devaney in [20] to define the behaviour of a function which has unpredictability whilst displaying regularity also.

Definition 1.1.8 (Chaotic Map)**.** A map $f : X \to X$ is called *chaotic* if it is topologically transitive, has a dense set of periodic points and has SDIC.

Example 1.1.9. We give two examples of maps which are chaotic, according to Devaney's definition. We make use of the fact that on certain spaces, some of the elements in the definition of chaos are redundant [5, 51].

1. Consider the unit circle $S = \{\mathbf{x} \in \mathbb{R}^2 \ : \ |\mathbf{x}| = 1\}$. If we denote by θ a point $(\cos\theta, \sin\theta) \in S$, for $0 \leq \theta < 2\pi$, we can define a map $g : S \to S$ by $g(\theta) = 2\theta \mod 2\pi$ (see Figure 1.1). So g simply doubles the angle a point in S makes with the positive x-axis. Then g is chaotic. Indeed, open arcs in S are stretched to double their length repeatedly under g, so after enough iterations any open arc will cover S, so g is certainly topologically transitive. Also, the periodic points of period n for g are precisely the $(2^n - 1)^{th}$ roots of unity, so periodic points are dense in S. Continuous transitive maps on metric spaces which have a dense set of periodic points are chaotic (see [5] and Appendix A), thus g is chaotic.

2. Let I be the compact interval $[0,1]$, let $s \in (1,2]$ and define a map $f_s : I \to I$ known as the *tent map* as

$$f_s(x) = \begin{cases} sx & x \in [0,1/2] \\ s(1-x) & x \in [1/2,1] \end{cases}$$

(see also Examples 1.2.1 and 4.4.3). The set $J = [f_s^2(1/2), f_s(1/2)]$ is invariant under f_s, so we lose no generality by considering $f_s \restriction_J$, which we call the *tent map core*. For $s \in (\sqrt{2}, 2]$, the tent map core is locally eventually onto [14], thus f_s is certainly transitive. Continuous transitive maps on an interval are chaotic [51], so f_s is chaotic.

$\square$

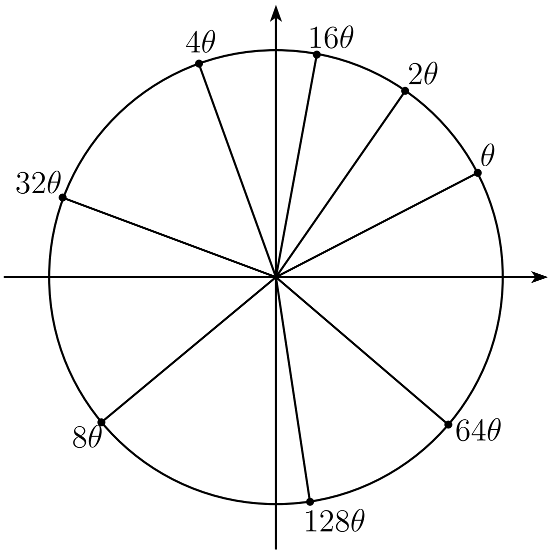

Figure 1.1: The doubling map is chaotic on S.

We will see later that topological transitivity is a property that has some interesting consequences with regards to ω-limit sets.

The following is a consequence of the intermediate value theorem which we will only quote here, but the proof can be found in many introductory courses in real analysis.

Lemma 1.1.10. *Suppose that $f : I \to I$ is a continuous function on a compact interval I. Then f has a fixed point in I.*

1.2 ω-Limit Sets

As we saw in the last section, even very simple functions can have complicated dynamics (this has also been explored in [29], [31] and [33]), so we look for ways to gain an overall understanding of how the system is behaving, particularly in the long term. ω-limit sets give us a way of doing this, providing an asymptotic description of the dynamics. For a sequence of points $\{x_n\}_{n\in\mathbb{N}}$, the *ω-limit set* of $\{x_n\}_{n\in\mathbb{N}}$ is given as the set of accumulation points of the sequence. It is formally defined as

$$\omega(\{x_n\}) = \bigcap_{n=0}^{\infty} \overline{\{x_k \ : \ k \geq n\}}$$

In particular, if $\{x_n\}_{n\in\mathbb{N}}$ is the orbit of a point $x \in X$ for a map $f : X \to X$, we define the ω-limit set of x (under f) as

$$\omega(x, f) = \bigcap_{n=0}^{\infty} \overline{\{f^k(x) \ : \ k \geq n\}}$$

(where we may drop the f if there is no ambiguity.)

In some cases the ω-limit set of a point may be a finite set – in particular if the point is asymptotically periodic (as is shown in Lemma 2.1.7). Furthermore, if the point is periodic or pre-periodic, the ω-limit set of that point will be precisely the periodic portion of the orbit.

It will often be the case that $\omega(x)$ is an infinite set, in which case it may take the form of a Cantor set (see Definition 2.2.9), a closed countable set or a set with non-empty interior. It may also be that an ω-limit set is a combination of these types of set, but it should be noted that for two ω-limit sets A and B, it is not necessarily the case that $A \cup B$ is an ω-limit set, even if the two intersect, as Example 1.2.1 shows (see also [10]).

Example 1.2.1. Consider the tent map

$$f_2(x) = \begin{cases} 2x & x \in [0, 1/2] \\ 2(1-x) & x \in [1/2, 1] \end{cases}$$

as defined in Example 1.1.9, and let us extend this to a map $g : [-1, 1] \to [-1, 1]$ defined by

$$g(x) = \begin{cases} -f_2(-x) & x \in [-1, 0] \\ f_2(x) & x \in [0, 1]. \end{cases}$$

The dynamics for the two halves of this map are disjoint about 0; i.e. no point in $(0, 1]$ is mapped to $[-1, 0)$ and vice-versa. Indeed the point $x = 1/2$ is mapped to the maximum 1 and then to 0, which is fixed. Consider the sets

$$H_1 = \{0\} \cup \bigcup_{n=0}^{\infty} \left\{ \frac{1}{2^n} \right\}$$

and

$$H_2 = \{0\} \cup \bigcup_{n=0}^{\infty} \left\{ -\frac{1}{2^n} \right\}.$$

For $x = 1/2^i \in H_1$ with $i \geq 0$, it is easy to see that $g^k(x) = 1/2^{i-k}$ for $k \leq i$, so H_1 is a backwards orbit of the point 0, similarly for H_2. We show in Chapter 4 that H_1 and H_2 are both ω-limit sets for g, but it is easy to see that $H = H_1 \cup H_2$ is not an ω-limit set for g since the dynamics are disjoint about 0 (points in H_1 will remain in H_1, similarly for H_2), even though the two sets are not disjoint (see Figure 1.2).

This example can be found also in [4], and we use it extensively throughout the sequel to demonstrate various dynamical properties.

$\square$

Example 1.2.2. A simpler example of an ω-limit set is a periodic orbit. Any periodic orbit of a map is an ω-limit set of any of the points in the orbit for that map. This is an example of when an ω-limit set is a *minimal set*, in other words a closed invariant set which contains no proper closed invariant set (see Chapter 2). We will see that an ω-limit set is either a minimal set, or properly contains one (neither of the sets H_1 nor H_2 in Example 1.2.1 are minimal, since $\{0\}$ is closed and invariant).

$\square$

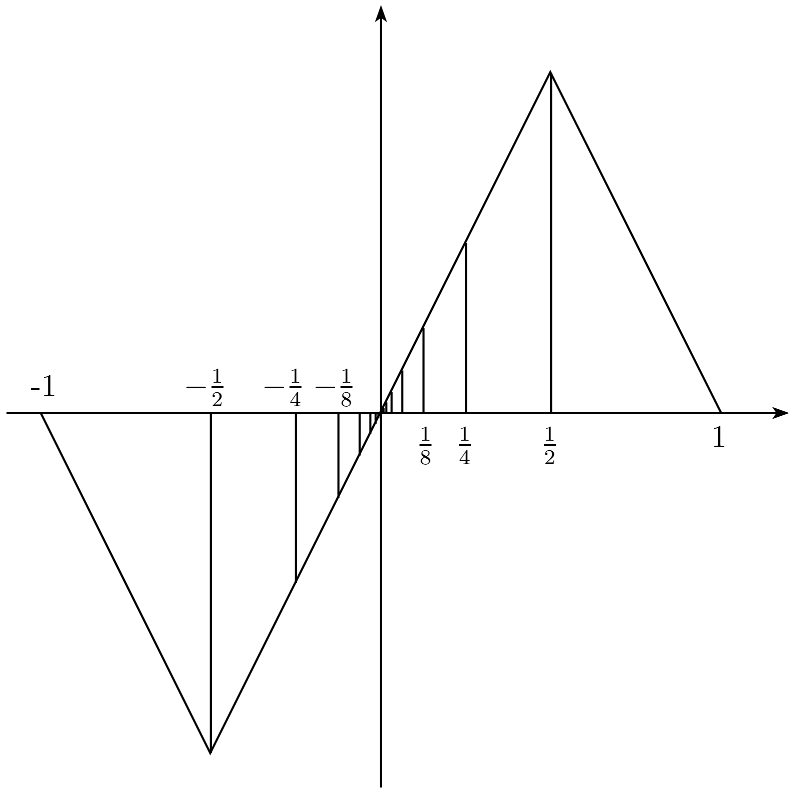

Figure 1.2: The dynamics of the double tent map are disjoint about 0.

Chapter 2

Topological Dynamics

In this chapter we investigate certain topological properties which regularly occur in the study of dynamical systems, and which will help us to improve our understanding of how ω-limit sets behave. The results stated are for dynamical systems (X, f) (a compact metric space X and a continuous function $f : X \to X$), unless stated otherwise. Our treatment is taken from [9], but examples can also be found in [1], [10] and [44].

2.1 General Properties of ω-Limit Sets

Definition 2.1.1 (Invariant Set)**.** We say that a set $A \subset X$ is *invariant* if $f(A) \subset A$. If $f(A) = A$ we say that A is *strongly invariant* (or *s-invariant* for short).

Central to the subject of ω-limit sets is the concept of an *attractor*. The following definition is taken from [26] but there are other definitions which are sometimes used and which may not be equivalent to this one (see for example [9], [19], [20]).

Definition 2.1.2 (Attractor)**.** A subset A of X is said to be an *attractor* for $f : X \to X$ if A is nonempty, compact, s-invariant and there is an open neighbourhood U of A such that

$$\lim_{n\to\infty} \sup\{d(f^n(x), A) \;:\; x \in U\} = 0$$

Definition 2.1.3 (Attracting/stable periodic orbit)**.** A periodic orbit A of a point under a map $f : X \to X$ is said to be *attracting (stable)* if A is an attractor for f.

Thus we see that $\omega(x, f)$ is *the* attractor for f restricted to $\mathrm{orb}(x, f)$ (but may not be an attractor for f). An example of a map f and an ω-limit set which is also an attractor for f is a stable periodic orbit of an interval map $f : [0, 1] \to [0, 1]$ (see [17], [33] for examples and exposition on stable orbits of interval maps). The set H_1 as defined in Example 1.2.1 is an example of an ω-limit set which is *not* an attractor for the map on the whole space.

Lemma 2.1.4. *For $x_0 \in X$ and $f : X \to X$ the set $A = \omega(x_0, f)$ is closed, non-empty and s-invariant.*

Proof. By definition, $A = \omega(x_0, f)$ is the intersection of countably many closed non-empty sets, so is itself closed. Since these sets are nested, by compactness we get that A is non-empty. To show that $f(A) \subseteq A$, suppose that $x \in A \setminus \mathrm{orb}^+(x_0)$ (if $x \in A \cap \mathrm{orb}^+(x_0)$ the claim is immediate), then x is a limit point of $\mathrm{orb}^+(x_0)$, so by continuity $f(x)$ is also a limit point of $\mathrm{orb}^+(x_0)$. Now suppose that $y \in A$, then there is a sequence $\{n_k\}_{k\geq 0} \subset \mathbb{N}$

such that $\lim_{k\to\infty} f^{n_k}(x_0) = y$. Since A is compact the sequence of points $f^{n_k-1}(x_0)$ for $k \geq 0$ has a convergent subsequence $f^{m_k}(x_0)$ with limit $z \in A$. So for every $k \in \mathbb{N}$, $m_k + 1 = n_{k'} \in \{n_k\}_{k\geq 0}$, so

$$f(z) = f\left(\lim_{k\to\infty} f^{m_k}(x_0)\right) = \lim_{k\to\infty} f^{m_k+1}(x_0) = y,$$

and hence $f(A) = A$. $\square$

The following property was first observed by Šarkovs′kiĭ, who proved in [45] that it is an inherent property of ω-limit sets. It was originally stated as a property of invariant sets, but we modify the definition slightly to remove the necessity of invariance. The term *weak incompressibility* seems to have appeared first in [4] and we adopt this term in this text.

Definition 2.1.5 (Weak Incompressibility)**.** A set $A \subset X$ is said to have *weak incompressibility* if for any proper non-empty subset $U \subset A$ which is open in A, $\overline{f(U)} \cap (A \setminus U) \neq \emptyset$. Equivalently we can say that for any non-empty closed subset $D \subsetneq A$ we have that $D \cap \overline{f(A \setminus D)} \neq \emptyset$.

Lemma 2.1.6. *If $A = \omega(x_0, f)$ for $f : X \to X$ and $x_0 \in X$ then A has weak incompressibility.*

Proof. Assume that for some closed $M \subsetneq A$ we have that $M \cap \overline{f(A \setminus M)} = \emptyset$, then by normality there are open sets U and V such that $\overline{U} \cap \overline{V} = \emptyset$, $M \subset U$ and $\overline{f(A \setminus M)} \subset V$. Thus $(A \setminus M) \subset f^{-1}(V) = W$, where W is open by continuity. $f\left(\overline{W}\right) = \overline{f(W)} = \overline{V}$, so $f\left(\overline{W}\right) \cap \overline{U} = \emptyset$.

Since $A = \omega(x_0, f) \subset (W \cup U)$ there is an integer $k_0 > 0$ such that $f^n(x_0) \in W \cup U$ for every $n \geq k_0$. Moreover, $f^n(x_0) \in W$ for infinitely many

$n \geq k_0$ and $f^m(x_0) \in U$ for infinitely many $m \geq k_0$, so for infinitely many $n \geq k_0$, $f^n(x_0) \in W$ and $f^{n+1}(x_0) \in U$. Thus there is a sequence $\{n_i\}_{i\in\mathbb{N}}$ such that $f^{n_i}(x_0) \in W$, $f^{n_i+1}(x_0) \in U$ and $y = \lim_{i\to\infty} f^{n_i}(x_0) \in \overline{W}$. Then $f(y) = f(\lim_{i\to\infty} f^{n_i}(x_0)) = \lim_{i\to\infty} f^{n_i+1}(x_0) \in \overline{U}$; i.e. $f(y) \in f\left(\overline{W}\right) \cap \overline{U}$ which contradicts the fact that $f\left(\overline{W}\right) \cap \overline{U} = \emptyset$. Hence $M \cap \overline{f(A \setminus M)} \neq \emptyset$. $\square$

We mentioned above that an asymptotically periodic point will have a finite ω-limit set; Lemma 2.1.7 formalizes this idea.

Lemma 2.1.7. *For $f : X \to X$ and $x_0 \in X$, $\omega(x_0, f)$ has only finitely many points if and only if x_0 is asymptotically periodic. Furthermore, if $\omega(x_0, f)$ contains infinitely many points then no isolated point of $\omega(x_0, f)$ is periodic.*

Proof. If x_0 is asymptotically periodic, then $L = \omega(x_0, f)$ is clearly a cycle, so is finite. Conversely, if L is finite then since L is invariant f must permute its elements, so L must contain a cycle. Suppose then that L is finite and properly contains a cycle C, then since every point in L is isolated, $L \setminus C$ is closed. Then by Lemma 2.1.6, $\overline{f(C)} \cap (L \setminus C) = C \cap (L \setminus C) \neq \emptyset$ – a clear contradiction. So if $L = \omega(x_0, f)$ is finite it must itself be a cycle, meaning that x_0 is asymptotically periodic. This proves the first part of the Lemma.

Suppose that $C \subseteq L$ is an m-cycle, and that $y \in C$ is isolated. Then y is fixed by $g = f^m$. Let $\omega_j = \omega(f^j(x_0), g)$, and notice that

$$\omega(x_0, f) = \bigcup_{j=0}^{m-1} \omega_j.$$

Indeed, if $y \in \cup_{j=0}^{m-1} \omega_j$ then clearly $y \in \omega(x_0, f)$. Conversely, if $y \in \omega(x_0, f)$ then $y = \lim_{k\to\infty} f^{n_k}(x_0)$ for some sequence $\{n_k\}_{k\in\mathbb{N}}$. Counting in base m

we see that for some $j < m$, $n_k \equiv j \mod m$ for infinitely many k, so $y \in \omega_j$. Moreover y is isolated, so suppose that $\omega_j \neq \{y\}$. Then $\omega_j \setminus \{y\}$ is closed and non-empty, which means that by Lemma 2.1.6, $(\omega_j \setminus \{y\}) \cap \overline{g(\{y\})} = (\omega_j \setminus \{y\}) \cap \{y\} \neq \emptyset$, which is impossible, so $\omega_j = \{y\}$. Hence $\lim_{k\to\infty} f^{km+j}(x_0) = y$, so letting $z = f^{m-j}(y)$ we see that $\lim_{k\to\infty} f^{km+i}(x_0) = f^i(z)$ for every $0 \leq i < m$. Thus $\lim_{n\to\infty} d(f^n(x_0), C) = 0$, so x_0 is asymptotically periodic, and L is finite; i.e. if L is infinite and has a periodic point then it can't be isolated. □

Lemma 2.1.8 tells us what happens to a connected subset under the action of the map f. This will help us deduce what basic forms an ω-limit set can take.

Lemma 2.1.8. *Let H be a connected subset of X and let $E = \cup_{k\geq 0} f^k(H)$. Then either the connected components of E are the sets $f^k(H)$ for every $k \geq 0$, or there are integers $m \geq 0$ and $p > 0$ such that the connected components of E are the sets $f^k(H)$ for $0 \leq k < m$ and $E_j = \cup_{k\geq 0} f^{m+j+kp}(H)$ for $0 \leq j < p$.*

Proof. By the continuity of f, connected subsets of X are mapped to connected subsets. Let S denote the set of non-negative integers m for which $f^m(H)$ and $f^{m+i}(H)$ are in the same component of E for some $i \in \mathbb{N}$. If $S = \emptyset$ then the connected components of E are the sets $f^k(H)$ for every $k \geq 0$.

So assume that $S \neq \emptyset$, and let m denote the smallest element of S. Then clearly $f^k(H)$ are disjoint components of E for every $0 \leq k < m$, so we look to determine the components of $E' = \cup_{k\geq m} f^k(H)$. Let p denote the smallest integer such that $f^m(H)$ and $f^{m+p}(H)$ are in the same component of E'. Since components are mapped into components, $f^{m+i}(H)$ and $f^{m+p+i}(H)$ are both in the same component for every $i \geq 0$; call this observation **(1)**.

In particular, for any fixed $j < p$ we have that the sets $f^{m+j+kp}(H)$ are all in the same component for every $k \geq 0$; i.e. for each $0 \leq j < p$, $E_j = \cup_{k \geq 0} f^{m+j+kp}(H)$ is contained in a component of E'. So we deduce that E' has r components, where $1 \leq r \leq p$. We claim that for any $i \geq m$ the sets $f^i(H), \ldots, f^{i+r-1}(H)$ are contained in distinct components of E'. Suppose not; i.e. that there are integers j, l where $i \leq j < l \leq i + r - 1$ for which $f^j(H)$ and $f^l(H)$ are in the same component. Then the sets $f^k(H)$ for all $k \geq j$ are contained in at most $l - j < r$ components, so some components of E' contain only finitely many of the sets $f^k(H)$. But observation (1) tells us that each of the r components in E' contain infinitely many of the sets $f^k(H)$, so we have a contradiction, and the claim is proved.

From this we can deduce that not only are $f^m(H), \ldots, f^{m+r-1}(H)$ contained in distinct components of E', so are $f^{m+1}(H), \ldots, f^{m+r}(H)$. Thus both $f^m(H)$ and $f^{m+r}(H)$ are in a different component to the sets $f^{m+1}(H), \ldots, f^{m+r-1}(H)$, which are all in distinct components. We have only r distinct components, so we must have that $f^m(H)$ and $f^{m+r}(H)$ are in the same component. By the definition of p we get that $r = p$. Finally, since $E' = \cup_{j=0}^{p-1} E_j$, and each E_j is contained in one of the p distinct components of E', we must have that the E_j are precisely the components of E'. $\square$

We can now prove our first significant result about the structure of ω-

limit sets, which for X a compact interval, tells us that if an ω-limit set has non-empty interior it must be a cycle of intervals.

Proposition 2.1.9. *Suppose that $f : I \to I$ is continuous for a compact interval I. If $L = \omega(x_0, f)$ contains an interval, then L is the union of finitely many disjoint closed intervals $J_1, \ldots, J_p$ such that $f(J_i) = J_{i+1}$ for $1 \leq i \leq p-1$ and $f(J_p) = J_1$.*

Proof. Let J_1 be a maximal subinterval of L, then J_1 is closed since L is closed. Since J_1 contains more than one point of $\mathrm{orb}^+(x_0)$, we have that $f^k(J_1) \cap J_1 \neq \emptyset$ for some $k > 0$. Thus by Lemma 2.1.8 there is an integer $p > 0$ such that the sets $J_k = f^{k-1}(J_1)$ for $1 \leq k \leq p$ are disjoint, by continuity of f are all closed intervals, and moreover $f(J_p) \subset J_1$. If $f^m(x) \in J_1$ for some $m \geq 0$, then $f^{m+i}(x) \in \cup_{k=1}^p J_k$ for every $i \geq 0$, thus $L \subset \cup_{k=1}^p J_k$. So since $\cup_{k=1}^p J_k \subset L$ we get that $\cup_{k=1}^p J_k = L$. Finally since L is invariant we must have that $f(J_p) = J_1$. $\square$

Example 2.1.10. An example of a map for which an ω-limit set is a cycle of disjoint intervals is a tent map f_s (see Example 1.1.9) with slope $s \in (\sqrt[4]{2}, \sqrt{2})$ and critical point $c = 1/2$. As mentioned in Example 1.1.9, the tent map is locally eventually onto provided $s \in (\sqrt{2}, 2]$, which means that any subinterval would eventually map onto the maximal invariant set $[f_s^2(c), f_s(c)]$ (called the *core*), so we could clearly not have a cycle of n disjoint intervals for $n \geq 2$. However for $s \in (1, \sqrt{2})$ we will get a cycle of at least two disjoint intervals, and if we further restrict the slope to the interval $s \in (\sqrt[4]{2}, \sqrt{2})$ we also get that these intervals form an ω-limit set.

To see this, consider the points $\{f_s^2(c), f_s^4(c), f_s^3(c), f_s(c)\}$. It can easily be shown that for $s \in (\sqrt[4]{2}, \sqrt{2})$,

1. $f_s^2(c) < f_s^4(c) < f_s^3(c) < f_s(c)$,

2. $c \in (f_s^2(c), f_s^4(c))$, and

3. $|f_s^2(c) - c| > |f_s^4(c) - c|$.

Thus we get that the intervals $[f_s^2(c), f_s^4(c)]$ and $[f_s^3(c), f_s(c)]$ are interchanged by f_s. Now consider the map $g_s := f_s^2$; this map has three turning points, one at each of the two pre-images of c under f_s, which we denote p_- and p_+, and one at c itself. Furthermore, it can be shown that $p_- < g_s(c) < g_s^2(c) < p_+$, so $g_s\restriction_{[g_s(c), g_s^2(c)]}$ is an upside-down tent map core, with slope $s^2 \in (\sqrt{2}, 2)$. Thus $g_s\restriction_{[g_s(c), g_s^2(c)]}$ is locally eventually onto, so is certainly transitive.

In Chapter 3, we show that a map $g : X \to X$ is transitive if and only if $X = \omega(x, g)$ for some $x \in X$ (Theorem 3.1.3), hence for the map g_s, $[g_s(c), g_s^2(c)] = \omega(x, g_s)$ for some $x \in [g_s(c), g_s^2(c)]$. In other words $[f_s^2(c), f_s^4(c)] = \bigcap_{n\in\mathbb{N}} \overline{\{f_s^{2k}(x) \ : \ k > n\}}$. Also $[f_s^3(c), f_s(c)] = f_s\big([f_s^2(c), f_s^4(c)]\big)$, and it can be shown that

$$f_s\left(\bigcap_{n\in\mathbb{N}} \overline{\{f_s^{2k}(x) \ : \ k > n\}}\right) = \bigcap_{n\in\mathbb{N}} \overline{\{f_s^{2k+1}(x) \ : \ k > n\}}.$$

Thus we have that

$$\omega(x, f_s) = \bigcap_{n\in\mathbb{N}} \overline{\{f_s^{k}(x) \ : \ k > n\}} = \big[f_s^2(c), f_s^4(c)\big] \cup \big[f_s^3(c), f_s(c)\big].$$

$\square$

So for the case where $f : I \to I$ we know that an ω-limit set is either a nowhere dense set (finite or infinite) or a cycle of closed disjoint intervals. In a later chapter we will make further observations about nowhere dense ω-limit sets; in the next section we will see that they can take a variety of different forms.

2.2 Minimal Sets

Definition 2.2.1 (CINE Set)**.** A set $A \subset X$ is said to be *CINE* if it is closed, invariant and non-empty.

Definition 2.2.2 (Minimal Set)**.** A set $M \subset X$ is a *minimal set* if it is CINE, and no proper subset of M is CINE.

The following is really an alternative definition of a minimal set, but we present it as a lemma.

Lemma 2.2.3. *For a CINE set $M \subset X$, the following are equivalent:*

1. *M is minimal;*
2. *M is the orbit closure of every one of its points;*
3. *M is the ω-limit set of every one of its points.*

Proof. $1 \Rightarrow 2$: Suppose that M is minimal and pick $x \in M$. Then $\overline{\mathrm{orb}^+(x)}$ is a CINE subset of M, hence $M = \overline{\mathrm{orb}^+(x)}$.

$2 \Rightarrow 3$: Now suppose that $M = \overline{\mathrm{orb}^+(x)}$ for every $x \in M$. Pick some $x \in M$; we want to show that $M = \omega(x)$. Then for any $z \in M$, $z \in \overline{\mathrm{orb}^+(x)}$

and $x \in \overline{\mathrm{orb}^+(z)}$. If $z \in \omega(x)$ we are done, so assume that $z \notin \omega(x)$, then $z \in \mathrm{orb}^+(x)$. Since $x \in \overline{\mathrm{orb}^+(z)}$, either $x \in \mathrm{orb}^+(z)$, in which case x is periodic and $M = \omega(x)$ and we're done, or $x \in \omega(z) \subset \omega(x)$. But then $z \in \omega(x)$, which is a contradiction. Hence for every $z \in M$, $z \in \omega(x)$, so $M \subset \omega(x) \subset \overline{\mathrm{orb}^+(x)} = M$, and so $M = \omega(x)$.

$3 \Rightarrow 1$: Finally suppose that the CINE set M is such that $M = \omega(x)$ for every $x \in M$. Let P be a CINE subset of M and pick $p \in P$. Then $\omega(p)$ is a CINE subset of P. But $p \in M$ so $M = \omega(p) \subset P$. Hence $P = M$ and we have that M is minimal. □

Corollary 2.2.4. *Every minimal set is s-invariant.*

Theorem 2.2.5. *Any two minimal sets must have empty intersection.*

Proof. Suppose that M_1 and M_2 are two distinct minimal sets, and that $A = M_1 \cap M_2 \neq \emptyset$. Then A is closed, and for every $a \in A$ and every $n \in \mathbb{N}$, $f^n(a) \in M_1 \cap M_2$, so A is invariant. But then A is a proper subset of both M_1 and M_2 which is CINE, contradicting the fact that M_1 and M_2 are minimal. □

So minimal sets share an intimate connection with ω-limit sets. In fact, by Lemma 2.1.7, finite ω-limit sets are precisely the periodic orbits of f, which must also be minimal sets. In other words, the finite minimal sets of a dynamical system coincide exactly with the finite ω-limit sets. Moreover, any finite subset of a compact interval I is a minimal set for some map.

Example 2.2.6. Consider the set $M = \{x_0, x_1, \ldots, x_n\} \subset I$, where $x_i < x_{i+1}$ for $i = 0, 1, \ldots, n-1$, and let $f : [x_0, x_n] \to [x_0, x_n]$ be the piecewise linear function defined by the rule $f(x_i) = x_{i+1}$ for $i = 0, 1, \ldots, n-1$ and $f(x_n) = x_0$ (see Figure 2.1). Then M is a periodic orbit for f and is thus minimal.

□

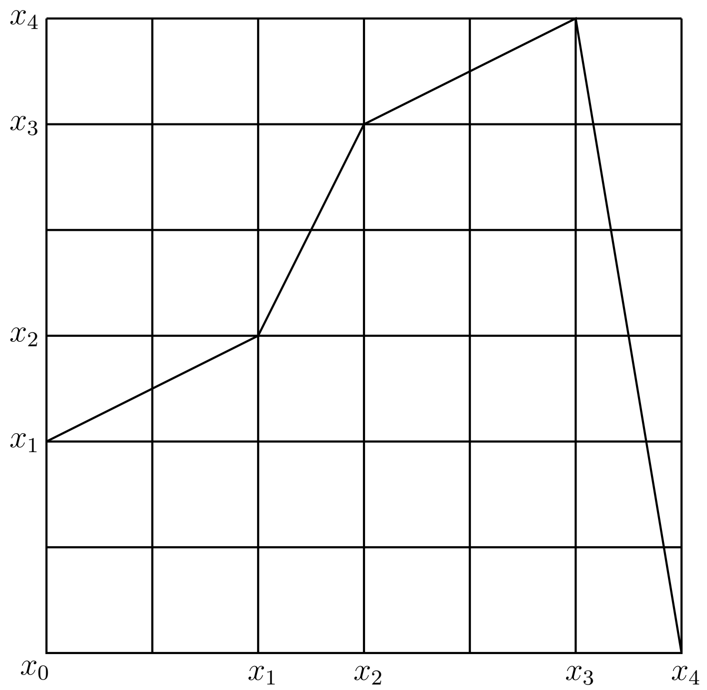

Figure 2.1: The set $\{x_0, x_1, x_2, x_3, x_4\}$ is a periodic orbit for the piecewise linear map shown.

Lemma 2.2.7. *Every CINE set $F \subset X$ contains a minimal set.*

Proof. Suppose that F is not minimal (else we are done). It then has at least one proper CINE subset. The set of all proper CINE subsets of F is partially ordered under inclusion, so by Hausdorff's Maximal Principle F contains a maximal chain, S (see Appendix B for notes on partial orders). Consider the intersection M of all elements of S. If $x \in M$ then $x \in T$ for all $T \in S$, so $f(x) \in T$ for all $T \in S$ and hence $f(x) \in M$, thus M is invariant. Also, M is the intersection of a collection of closed sets, so is itself closed, and since these sets are non-empty and nested, by compactness of X their intersection M is non-empty. Thus M is a CINE set, so let P be any CINE subset of M. P must be in S since S is maximal, and if it were not it would not be a subset of M. But then M is a subset of P, so we must have that $M = P$, hence M is minimal. □

So in particular, any ω-limit set is either a minimal set, or properly contains one. If the set is finite we have seen that it must be a minimal set, but if it is infinite there is no necessity for it being a minimal set. In fact the infinite minimal sets take a very special form.

Definition 2.2.8 (Totally Disconnected Set)**.** A closed set $C \subset X$ is said to be *totally disconnected* if the only connected subsets are singleton sets.

Definition 2.2.9 (Cantor Set)**.** A closed set $C \subset X$ is said to be a *Cantor set* if it has no isolated point and is totally disconnected.

Proposition 2.2.10. *Every infinite minimal set for a continuous map on a compact interval is a Cantor set.*

Proof. Let $M \subset I$ be an infinite minimal set. Since a subset of the real line is totally disconnected if it contains no interval, it is enough to show that M has no isolated point and contains no interval. If M had a periodic point p then $\mathrm{orb}^+(p)$ would be a proper CINE subset of M, contradicting the fact that M is minimal, so M has no periodic points. Also, since $M = \omega(x)$ for every $x \in M$, x is a limit point of M, and since x is not periodic it is not isolated. Moreover, if M contained an interval, by Proposition 2.1.9 it is a cycle of intervals $J_1, \ldots, J_m$, where $f^m(J_1) = J_1$, so by Lemma 1.1.10 there is a periodic point of period m in J_1 – a contradiction. $\square$

This result does not hold for general compact metric spaces. As a counter example, consider an irrational rotation of the unit circle S. It can be shown (see [9]) that every orbit for such a map is dense in S, so the whole space is an infinite minimal set by Lemma 2.2.3, but is certainly not a Cantor set.

So we are now in a position to state that an ω-limit set can either be finite, in which case it is a periodic orbit, or it can be infinite, in which case it is either minimal or properly contains a minimal set. Moreover, an infinite ω-limit set on a compact interval is either a Cantor set, a nowhere dense set which properly contains either a Cantor set or a periodic orbit, or it is a cycle of a finite number of closed disjoint intervals. This tells us quite a lot about the general structure of ω-limit sets, particularly on the interval. In the following chapters, we aim to make a more probing analysis of the types of set that can be ω-limit sets, and introduce some technical definitions which will help to characterize these sets.

Chapter 3

Transitivity, Internal Chain Transitivity and Expansivity

In this chapter we investigate further the property of transitivity, introduce a property inherent in ω-limit sets called *internal chain transitivity* and also look at certain expanding properties shared by different sets; these concepts will form the basis of our first characterizations of ω-limit sets. This is also where we begin to introduce results into the material presented. Unless stated otherwise, (X, f) is a dynamical system X (see Definition 1.1.1).

3.1 Transitivity

Recall that a map $f : X \to X$ is (topologically) transitive on X if for every pair of non-empty open sets $U, V \subset X$ there is an integer $k > 0$ for which $f^k(U) \cap V \neq \emptyset$ (we may also say that the set X is transitive with respect to the map f; the two descriptions are equivalent and we will use whichever is the more appropriate). Results in this section are from [9].

Lemma 3.1.1. *For a map $f : X \to X$ the following statements are equivalent:*

1. *f is transitive;*

2. *for every non-empty open set $W \subset X$, $\cup_{n\in\mathbb{N}} f^n(W)$ is dense in X;*

3. *for every pair of non-empty open sets $U, V \subset X$ there is an integer $k \geq 0$ for which $f^{-k}(U) \cap V \neq \emptyset$;*

4. *for every non-empty open set $W \subset X$, $\cup_{n\in\mathbb{N}} f^{-n}(W)$ is dense in X;*

5. *every closed, invariant, proper subset of X has empty interior.*

Proof. (1) $\Rightarrow$ (2): For any open $U, V \subset X$ there is a $k \in \mathbb{N}$ such that $f^k(U) \cap V \neq \emptyset$. Thus $\cup_{n\in\mathbb{N}} f^n(U)$ is dense in X.

(2) $\Rightarrow$ (3): Let $W \subset X$ be an open set, then we have that $\cup_{n\in\mathbb{N}} f^n(W)$ is dense in X. Thus for any open $V \subset X$ there is a $k \in \mathbb{N}$ such that $f^k(W) \cap V \neq \emptyset$. Thus there is an $x \in W$ such that $f^k(x) \in V$ i.e. $f^{-k}(V) \cap W \neq \emptyset$.

(3) $\Rightarrow$ (4): Analogous to (1) $\Rightarrow$ (2).

(4) $\Rightarrow$ (5): Suppose that for every open $W \subset X$, $\cup_{n\in\mathbb{N}} f^{-n}(W)$ is dense in X. Let $C \subsetneq X$ be CINE (see Definition 2.2.1), and suppose that C has non-empty interior. Then there is an open set $U \subset C$ and a $k \in \mathbb{N}$ for which $f^{-k}(X \setminus C) \cap U \neq \emptyset$, which contradicts the fact that C is invariant. Thus C has empty interior.

(5) $\Rightarrow$ (1): Suppose that every proper closed invariant subset of X has empty interior. Let U and V be non-empty open subsets of X such that for

every $k \in \mathbb{N}$, $f^k(U) \cap V = \emptyset$. Then $\overline{\cup_{k\in\mathbb{N}} f^k(U)}$ is a proper CINE subset of X with non-empty interior, which contradicts (5). Hence f is transitive. $\square$

Lemma 3.1.2. *If $f : X \to X$ is transitive, then $f(X) = X$.*

Proof. By Lemma 3.1.1 part (2), for every $x \in X$ there is a sequence of points $\{f^{n_k}(w_k)\}_{k\in\mathbb{N}}$ such that $f^{n_k}(w_k) \to x$ as $k \to \infty$. Then there is a subsequence $\{w_{k_j}\}_{j\in\mathbb{N}} \subset \{w_k\}_{k\in\mathbb{N}}$ such that $f^{n_{k_j}-1}(w_{k_j})$ converges to some $y \in X$. Set $x_j = f^{n_{k_j}-1}(w_{k_j})$. Then

$$x = \lim_{j\to\infty} f(x_j) = f\left(\lim_{j\to\infty} x_j\right) = f(y) \in f(X).$$

$\square$

We can now prove a result linking transitive sets to ω-limit sets.

Theorem 3.1.3. *A map $f : X \to X$ is transitive if and only if $X = \omega(x, f)$ for some $x \in X$.*

Proof. Suppose first that $X = \omega(x, f)$ for some $x \in X$. Then for every pair of non-empty, open $U, V \subset X$ there are integers $n > m > 0$ such that $f^m(x) \in U$ and $f^n(x) \in V$. Hence $f^{n-m}(U) \cap V \neq \emptyset$ and f is transitive.

Now suppose that f is transitive. For every $n \in \mathbb{N}$, by compactness the space X is covered by finitely many open balls of radius $1/n$. We enumerate the collection of such balls over all n as $\{U_k\}_{k\in\mathbb{N}} \subset 2^X$. Then by Lemma 3.1.1 part (4) we have that for every $k \in \mathbb{N}$ the set $G_k = \cup_{n\in\mathbb{N}} f^{-n}(U_k)$ is open and dense in X. X is compact and thus complete, so is a Baire space by the Baire Category Theorem (see for example [49]), hence the intersection

G of the G_k is also dense, so certainly there is a point $x \in X$ such that $x \in G$. Thus $\mathrm{orb}^+(x) \cap U_k \neq \emptyset$ for every k, so $\overline{\mathrm{orb}^+(x)} = X$. By Lemma 3.1.2, there is a $y \in X$ such that $f(y) = x$. If $y \in \mathrm{orb}^+(x)$ then x is periodic, so $\omega(x) = \overline{\mathrm{orb}^+(x)} = X$. If $y \notin \mathrm{orb}^+(x)$ then since $y \in X = \overline{\mathrm{orb}^+(x)}$ we must have that $y \in \omega(x)$, and since $f(y) = x$ we must have that $x \in \omega(x)$, so $\overline{\mathrm{orb}^+(x)} \subset \omega(x)$ by the fact that $\omega(x)$ is closed and invariant. Thus $\omega(x) = \overline{\mathrm{orb}^+(x)} = X$. $\square$

This result will have some use for us when we consider maps which are transitive on a subset of the whole space, allowing us to deduce that such a subset is an ω-limit set of some point in the subset itself.

3.2 Internal Chain Transitivity

We saw in the last section that transitivity on a set is not only strong enough to ensure the set is an ω-limit set, but ensures it is an ω-limit set of one of its points. There are many sets which are ω-limit sets of points outside the set (see Example 4.4.3), so transitivity is too strong a property to fully characterize these sets. In this section we introduce a weaker property which we will show implies ω-limit sets in certain spaces, and will form the basis of much of the work in following chapters.

Definition 3.2.1 (ϵ-Pseudo-Orbit)**.** For $\epsilon > 0$, the (finite or infinite) sequence of points $\{x_0, x_1, \ldots\} \subset X$ is an *ϵ-pseudo-orbit* if $d(f(x_i), x_{i+1}) < \epsilon$ for every $i \geq 0$. Where the parameter ϵ is not specified, we may simply refer to such a sequence of points as a *pseudo-orbit*.

The following lemma, which is well-known, makes use of uniform continuity to make deductions about the behaviour of maps near pseudo-orbits.

Lemma 3.2.2. *Let (X, f) be a dynamical system. For any $\epsilon > 0$ and $n \in \mathbb{N}$ there is a $\delta = \delta(n, \epsilon) > 0$ such that if $\{x_0, \ldots, x_n\}$ is a δ-pseudo-orbit and $y \in X$ is such that $d(y, x_0) < \delta$ then $d(f^k(y), x_k) < \epsilon$ for $k = 1, \ldots, n$.*

Proof. First notice that by uniform continuity of f, there is a $\delta < \frac{\epsilon}{2n}$ such that whenever $d(x, y) < \delta$ we have that $d(f^i(x), f^i(y)) < \frac{\epsilon}{2n}$ for $0 \leq i \leq n$. Thus for any δ-pseudo-orbit $\{x_0, \ldots, x_n\}$ of f we have that $d(f^j(x_0), x_j) < \frac{\epsilon}{2}$ for $j = 1, \ldots, n$. Indeed for any $j \leq n$ we have

$$\begin{aligned} d(f^j(x_0), x_j) &\leq d(f^j(x_0), f^{j-1}(x_1)) + \ldots + d(f(x_{j-1}), x_j) \\ &< \frac{j\epsilon}{2n} \leq \frac{\epsilon}{2}. \end{aligned}$$

Pick y such that $d(x_0, y) < \delta$, then $d(f^j(y), f^j(x_0)) < \frac{\epsilon}{2}$, so

$$d(f^j(y), x_j) \leq d(f^j(y), f^j(x_0)) + d(f^j(x_0), x_j) < \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

which completes the proof. □

Definition 3.2.3 (Chain Transitive Set)**.** A set $A \subset X$ is said to be *chain transitive* if for every $\epsilon > 0$ and any pair of points x and y in A there is an ϵ-pseudo-orbit $\{x_0 = x, x_1, \ldots, x_n = y\}$.

Definition 3.2.4 (Internally Chain Transitive Set)**.** A set $A \subset X$ is said to be *internally chain transitive* (or has *internal chain transitivity*) if for every $\epsilon > 0$ and any pair of points x and y in A there is an ϵ-pseudo-orbit

$\{x_0 = x, x_1, \ldots, x_m = y\} \subset A$, where $m \geq 1$. We will use the abbreviation ICT to mean whichever of the two terms is contextually appropriate.

Clearly every internally chain transitive set is chain transitive. The following is from [8] and demonstrates the link between ICT and invariance.

Proposition 3.2.5. *Suppose that A is a closed subset of X. If A is ICT then A is strongly-invariant.*

Proof. Let $x \in A$; we are going to show that $f(x) \in A$ and $f^{-1}(\{x\}) \cap A \neq \emptyset$. If x is a fixed point then we are done, so assume that $f(x) \neq x$. For every n there is a $1/n$-pseudo-orbit of points in A from x to x, via some distinct point. But then, for every n, we can have points $y_n, z_n \in A$ such that $d(f(y_n), x) < 1/n$ and $d(f(x), z_n) < 1/n$. By compactness of A, without loss of generality we may assume that $y_n \to y \in A$ and $z_n \to z \in A$. Then by continuity, $x = f(y)$ for $y \in A$. Moreover,

$$d(f(x), z) \leq d(f(x), z_n) + d(z_n, z) \to 0 \text{ as } n \to \infty.$$

So $f(x) = z \in A$, and $f(A) = A$. $\square$

Example 3.2.6. In this example we explore some of the properties mentioned above.

Consider the set $H = H_1 \cup H_2$ from Example 1.2.1; this set is ICT for the map g. Indeed, for any two points $x, y \in H$ and any $\epsilon > 0$ we can map x forward so that it is mapped onto either y or 0, whichever comes first. If it is y we are done, if it is 0 then notice that there is an $n \in \mathbb{N}$ such that

$1/2^n < \min\{|y|, \epsilon\}$, so by jumping to $z = \pm 1/2^n$ (whichever is closer to y), we can map z forward onto y.

Notice that the set $H \setminus \{0\}$ is chain transitive but not *internally* chain transitive, since we cannot get from any point in H_1 to any point in H_2 (or vice-versa) without mapping onto 0. H is also closed, since $0 \in H$ is the only limit point, so by Proposition 3.2.5 H is strongly-invariant.

It is also true that in general, not every subset of an ICT set is itself ICT, even if it is closed. To see this consider the set H_1 which is ICT. No proper subset of H_1 with more than one element but which does not contain the point 1 is ICT. Indeed such sets are not even invariant.

$\square$

Hirsch et al [26] investigated the link between ICT sets and ω-limit sets. Lemmas 3.2.7 and 3.2.10 and Theorem 3.2.9 are due to these authors; we include the proofs which will be helpful for what follows in later chapters.

Lemma 3.2.7. *For any dynamical system (X, f), the set $A = \omega(x_0, f)$ is ICT for any $x_0 \in X$.*

Proof. Let $\epsilon > 0$. By compactness of X, f is uniformly continuous, so there is a $\delta \in (0, \epsilon/3)$ such that for any $u, v \in X$ we have that $d(f(u), f(v)) < \epsilon/3$ whenever $d(u, v) < \delta$. Since $d(f^n(x_0), A) \to 0$ as $n \to \infty$ there is an $N \in \mathbb{N}$ such that $f^n(x_0)$ is in $B_\delta(A)$ for every $n \geq N$.

Let $a, b \in A$ be arbitrary, then by the previous observations there are integers $k > m \geq N$ such that $d(f^m(x_0), f(a)) < \epsilon/3$ and $d(f^k(x_0), b) < \epsilon/3$.

Then the set

$$Y = \{y_0 = a, y_1 = f^m(x_0), \ldots, y_{k-m} = f^{k-1}(x_0), y_{k-m+1} = b\} \subset B_\delta(A)$$

is an $\epsilon/3$-pseudo-orbit between a and b. So for every $y_i \in Y$ there is a $z_i \in A$ such that $d(y_i, z_i) < \delta < \epsilon/3$. Letting $z_0 = a$ and $z_{k-m+1} = b$ we have for every $i \leq k - m$

$$\begin{aligned} d(f(z_i), z_{i+1}) &\leq d(f(z_i), f(y_i)) + d(f(y_i), y_{i+1}) + d(y_{i+1}, z_{i+1}) \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 \\ &= \epsilon. \end{aligned}$$

Thus A is ICT. □

Definition 3.2.8 (Asymptotic Pseudo-Orbit)**.** A sequence of points $\{x_n\}_{n\in\mathbb{N}}$ in X is an *asymptotic pseudo-orbit* of f if

$$\lim_{n\to\infty} d(f(x_n), x_{n+1}) = 0.$$

Thus points in an asymptotic pseudo-orbit get progressively closer to the images of their predecessors in the sequence. The ω-limit set of an asymptotic pseudo-orbit $\{x_n\}_{n\in\mathbb{N}}$ is the set of limit points of subsequences of the set. Such ω-limit sets have some properties in common with ω-limit sets of regular orbits. They are certainly CINE sets, as is shown in [26], but they are also characterized by the property of ICT.

Theorem 3.2.9. *For a dynamical system (X, f), a closed set $A \subset X$ is ICT* *if and only if it is the ω-limit set of some asymptotic pseudo-orbit of f in A.*

Proof. The proof of sufficiency follows much the same course as the proof of Lemma 3.2.7. Suppose that A is the ω-limit set of some asymptotic pseudo-orbit $\{x_n\}_{n\in\mathbb{N}}$, and let $\epsilon > 0$. By compactness of X, f is uniformly continuous, so there is a $\delta \in (0, \epsilon/3)$ such that for any $u, v \in X$ we have that $d(f(u), f(v)) < \epsilon/3$ whenever $d(u, v) < \delta$. Again we note that there is an $N_1 \in \mathbb{N}$ such that x_n is in $B_\delta(A)$ for every $n \geq N_1$, but also observe that there is a second integer N_2 such that $d(f(x_n), x_{n+1}) < \epsilon/3$ for every $n \geq N_2$, by definition of asymptotic pseudo-orbits. We set $N = \max\{N_1, N_2\}$, and can now proceed exactly as before.

Let $a, b \in A$ be arbitrary, then by the previous observations there are integers $k > m \geq N$ such that $d(x_m, f(a)) < \epsilon/3$ and $d(x_k, b) < \epsilon/3$. Then the set

$$Y = \{y_0 = a, y_1 = x_m, \ldots, y_{k-m} = x_{k-1}, y_{k-m+1} = b\} \subset B_\delta(A)$$

is an $\epsilon/3$-pseudo-orbit between a and b. So for every $y_i \in Y$ there is a $z_i \in A$ such that $d(y_i, z_i) < \delta < \epsilon/3$. Letting $z_0 = a$ and $z_{k-m+1} = b$ we have for every $i \leq k - m$

$$\begin{aligned}
d(f(z_i), z_{i+1}) &\leq d(f(z_i), f(y_i)) + d(f(y_i), y_{i+1}) + d(y_{i+1}, z_{i+1}) \\
&< \epsilon/3 + \epsilon/3 + \epsilon/3 \\
&= \epsilon,
\end{aligned}$$

so A is ICT.

Now assume that A is ICT. Pick $x \in A$, then for any $\epsilon > 0$ by compactness there is a sequence of points $\{x = x_0, x_1, \ldots, x_m, x_{m+1} = x\} \subset A$ such that $A \subset \cup_{i \leq m} B_\epsilon(x_i)$. Since A is ICT, for each $i = 1, 2, \ldots, m$ there is an ϵ-pseudo-orbit $\{y_1^i = x_i, y_2^i, \ldots, y_{n_i}^i, y_{n_i+1}^i = x_{i+1}\} \subset A$ joining x_i and x_{i+1}. Thus the set

$$U_\epsilon = \{y_1^1, \ldots, y_{n_1}^1, y_1^2, \ldots, y_{n_2}^2, \ldots, y_1^m, \ldots, y_{n_m}^m, y_{n_m+1}^m\} \subset A$$

is an ϵ-pseudo-orbit connecting x to itself, and such that $A \subset \cup\{B_\epsilon(y) \,:\, y \in U_\epsilon\}$.

For every $k \in \mathbb{N}$, we have that $U_{1/k}$ is a $1/k$-pseudo-orbit of points in A joining x to itself and for which $A \subset \cup\{B_{1/k}(y) \;:\; y \in U_{1/k}\}$. Thus the infinite sequence of points $U = \cup_{k \in \mathbb{N}} U_{1/k}$ forms an asymptotic pseudo-orbit in A, so we need to show that its ω-limit set $\omega(U)$ is actually the set A.

Take $y \in \omega(U)$, then $y = \lim_{j \to \infty} x_{n_j}$ for a subsequence $\{x_{n_j}\}_{j \in \mathbb{N}} \subset U$. Since $U \subset A$ and A is closed we must have that $y \in A$. Now suppose that $y \in A$, then for every $k \in \mathbb{N}$ there is a $z_k \in U$ for which $y \in B_{1/k}(z_k)$ Thus $y = \lim_{k \to \infty} z_k$ and so $y \in \omega(U)$. □

So CINE sets which are ICT are precisely those which are ω-limit sets of some asymptotic pseudo-orbit. In light of Lemma 3.2.7 we would like to say the same for ω-limit sets of regular orbits, however due to the complexity of these sets it is not quite so simple. As we will see in Chapter 4, we will need to make further assumptions about either the map itself or the contents of an ICT set before we can ensure it is the ω-limit set of some point. However Lemma 3.2.7 will play a role in determining certain cases when ICT does imply an ω-limit set of a regular orbit (see Chapter 5).

Some maps have the property that their dynamics may be decomposed into disjoint sets, and if this is so it is not hard to see that no ω-limit set will intersect two such sets (see Example 1.2.1 and Remark 4.4.9). If a set $A \subset X$ is ICT this tells us that the dynamics of f cannot be decomposed over separate sets, albeit in a weaker sense than for those sets which are transitive.

Lemma 3.2.10. *Suppose that the set $A \subset X$ is ICT. Then whenever A is composed of two disjoint non-empty closed sets M_1 and M_2 we have that $f(M_1) \cap M_2 \neq \emptyset$.*

Proof. Suppose that $A = M_1 \cup M_2$ for disjoint non-empty closed sets M_1 and M_2. Pick $m \in M_1$ and $n \in M_2$ and set $\delta = \inf\{d(x, y) \ : \ x \in M_1, y \in M_2\}$, which is positive as M_1 and M_2 are closed and disjoint. Then since A is ICT there is a $\delta/2$-pseudo-orbit joining m and n in A. A is closed, so is invariant by Proposition 3.2.5, so to move from M_1 to M_2 in a $\delta/2$ jump there must exist some $x \in M_1$ such that $f(x) \in M_2$. $\square$

Notice that in Example 1.2.1, no pair of disjoint closed sets exist whose union is H such that either set is invariant with respect to g; H_1, H_2 and $\{0\}$ are the only closed invariant sets and they all intersect at 0.

Proposition 3.2.11 shows us that ICT is equivalent to weak incompressibility (see Definition 2.1.5) in dynamical systems. The result is due to Good and Raines and can be found in [8].

Proposition 3.2.11. *Let A be a closed subset of X. A is ICT if and only if it has weak incompressibility.*

Proof. Let A be weakly incompressible. If U is a proper nonempty open subset of A, let $F(U) = \overline{f(U)} \setminus U \neq \emptyset$.

Suppose that x and y are in A and that $\epsilon > 0$. Let $\mathcal{C}$ be a finite cover of A by $\epsilon/2$-neighbourhoods of points in A with no proper sub-cover, and let $\mathcal{B} = \{C \cap A \ : \ C \in \mathcal{C}\}$.

Suppose that $B_1 \in \mathcal{B}$. By weak incompressibility, $\overline{f(B_1)} \cap (A \setminus B_1) \neq \emptyset$, so unless $B_1 = A$ we have that $F(B_1) \neq \emptyset$, and there is some $B_2 \in \mathcal{B}$ such that $B_2 \cap \overline{f(B_1)} \neq \emptyset$, hence $B_2 \cap f(B_1) \neq \emptyset$. Suppose that we have chosen $B_j \in \mathcal{B}$, $j \leq k$, so that for each j there is some $i \leq j$ such that $B_j \cap f(B_i) \neq \emptyset$. Unless $B_1 \cup \ldots \cup B_k = A$, $F(B_1 \cup \ldots \cup B_k) \neq \emptyset$, so there is some $B_{k+1} \in \mathcal{B}$ such that $B_{k+1} \cap f(B_1 \cup \ldots \cup B_k) \neq \emptyset$, from which it follows that $B_{k+1} \cap f(B_j) \neq \emptyset$ for some $j < k+1$. Since $\mathcal{B}$ is a minimal finite cover, there is no $B_r \in \mathcal{B}$ for which $f(B_s) \cap B_r = \emptyset$ for every $B_s \in \mathcal{B}$, $B_s \neq B_r$. It follows then that for any $B, B' \in \mathcal{B}$ we can construct a sequence $B = B_1, B_2, \ldots, B_n = B'$ such that $B_{j+1} \cap f(B_j) \neq \emptyset$ for each $j < n$.

Now suppose that $x = x_0$, $f(x) \in B$ and $y \in B'$ for some $B,\ B' \in \mathcal{B}$. Then we can construct a sequence $B_1 = B, \ldots, B_n = B'$ as above. For $j = 1, \ldots, n-1$ choose any $x_j \in B_j \cap f^{-1}(B_{j+1})$, and put $x_n = y$. Then $x_0, \ldots, x_n$ is an ϵ-pseudo-orbit from x to y.

Conversely assume that A is ICT, then A is strongly-invariant by Proposition 3.2.5 and suppose that D is a proper, non-empty closed subset of A. Pick $y \in D$ and $x \in A \setminus D$. For each $n \in \mathbb{N}$, there is a $1/2^n$-pseudo-orbit from x to y. Let z_n be the last point in the pseudo-orbit that is not in D. Then z_n is such that $f(z_n)$ is within $1/2^n$ of D. Since A is compact we may assume that $z_n \to z$ for some z, and $f(z) \in D \cap \overline{f(A \setminus D)}$. □

3.3 Expansivity (I)

As we have noted in several of the proofs of various results, an ω-limit set $\omega(x_0, f)$ will act as an attractor for the orbit of the point x_0, so it may be surprising that these sets also have certain expansive properties. In [4], Balibrea and La Paz specify necessary and sufficient conditions for an infinite, nowhere dense subset of a compact interval I to be the ω-limit set of a point in I, including an expansive condition which we call *weakly-expansive* (see Definition 3.3.7). Here we will present some of their work regarding expansive properties, interspersed with original theory, and recast somewhat to fit our discussion (results are original unless stated otherwise). We do not present all of their material as it is outside the scope of this work. All results in this section are for a dynamical system (I, f), where I is a compact interval, unless otherwise stated.

First let us recall that an ω-limit set $A \subset I$ is either nowhere dense or is a cycle of finitely many compact intervals. Moreover, if A is nowhere dense then it is either finite and thus a periodic orbit, or infinite. It is the infinite case that we consider here, and we make use of the fact that an infinite nowhere dense ω-limit set is either minimal (and thus a Cantor set) or properly contains a minimal set (see Lemma 2.2.7).

We saw in Lemma 3.2.10 that an ICT set cannot be decomposed into two closed disjoint sets such that either is invariant. Proposition 3.3.1 is from [4] and tells us what happens if one such set *contains* an invariant set.

Proposition 3.3.1. *Suppose that $A \subset I$ is an infinite, nowhere dense CINE set which is ICT, and suppose that it is decomposed into two disjoint, closed* *sets M_1 and M_2 such that M_2 contains an invariant set. Then there exists a point $x_1 \in M_1$ such that $\omega(x_1, f) \subset M_2$.*

Proof. Consider the set $A_0 = \{x \in M_1 \ : \ f(x) \in M_2\}$, which is non-empty by Lemma 3.2.10 and compact by continuity. Since A is nowhere dense it contains no intervals, hence we can choose $U_0 \supset M_1$ clopen in A with $U_0 \cap M_2 = \emptyset$.

Now set $A_i := f(A_{i-1}) \cap M_2$ for every $i \geq 1$. Thus A_n is the set of points $y \in M_2$ such that there is an $x_y \in M_1$ for which $f^n(x_y) = y$ and $f^i(x_y) \in M_2$ for every $0 < i \leq n$. Now choose clopen sets U_i such that $A_i \subset U_i$ and $f(U_i) \subset (U_{i+1} \cup U_0)$ for every $i \geq 1$; i.e. U_{i+1} is chosen so that it contains the part of U_i not mapped into M_1.

Suppose that the set $\Omega = \{n \in \mathbb{N} \ : \ f^i(x) \in M_2 \, \forall \, i \leq n, \text{ for some } x \in A_0\}$ were bounded above; i.e. there exists an $n_0 \in \mathbb{N}$ such that for every $x \in A_0$, $f^j(x) \in M_1$ for some $1 < j \leq n_0 + 1$. Then we would have that $f(A_{n_0}) \subset U_0$, and we can choose U_{n_0} so that $f(U_{n_0}) \subset U_0$. So

$$A = \left(A \cap \bigcup_{i=0}^{n_0} U_i \right) \cup \left(A \setminus \bigcup_{i=0}^{n_0} U_i \right),$$

and since M_2 contains an invariant set, $\cup_{i=0}^{n_0} U_i \neq A$ so the above is a decomposition of A into two non-empty, disjoint, closed sets for which

$$f\left(A \cap \bigcup_{i=0}^{n_0} U_i \right) \subset A \cap \bigcup_{i=0}^{n_0} U_i,$$

which is impossible by Lemma 3.2.10. Hence Ω has no upper bound.

If A_0 were finite, we would have some $x \in M_1$ for which $f^n(x) \in M_2$ for every $n \in \mathbb{N}$, hence $\omega(x, f) \subset M_2$. So suppose that A_0 is infinite, and that there is no $x \in A_0$ for which $f^n(x) \in M_2$ for every $n \in \mathbb{N}$ (if there is we are done). Call this condition **(*)**. Since A_0 is infinite and Ω is unbounded, there is a sequence $\{y_n\}_{n\in\mathbb{N}} \subset A_0$ and an increasing sequence $\{m_n\}_{n\in\mathbb{N}} \subset \mathbb{N}$ such that for every $n \in \mathbb{N}$, $f^i(y_n) \in M_2$ for all $0 < i \leq m_n$. A_0 is compact, so $\{y_n\}_{n\in\mathbb{N}}$ has a convergent subsequence, which without loss of generality we take to be $\{y_n\}_{n\in\mathbb{N}}$ itself. Then $x_1 = \lim_{n\to\infty} y_n \in A_0$. Also, for any $k \in \mathbb{N}$, f^k is continuous, so $f^k(x_1) = \lim_{n\to\infty} f^k(y_n)$. For any such k, choose $N_k \in \mathbb{N}$ such that $m_{N_k} \geq k$, then since $f^k(y_n) \in M_2$ for every $n > N_k$, we have $f^k(x_1) \in M_2$ since M_2 is closed. But this contradicts (*) since $k \in \mathbb{N}$ was arbitrary, so there must be some $x \in M_1$ for which $f^n(x) \in M_2$ for every $n \in \mathbb{N}$, and so $\omega(x, f) \in M_2$. □

Lemmas 3.3.2 and 3.3.5 are useful observations about ω-limit sets which will help us to isolate the required expansive property of infinite, nowhere dense subsets of the interval.

Lemma 3.3.2. *Suppose that $A = \omega(x_0, f)$ for some $x_0 \in I$. Then for every $a, b \in A$ and any neighbourhood U of a there is an increasing sequence of positive integers $\{k_i\}_{i\in\mathbb{N}}$ such that $b \in \overline{\cup_{i\in\mathbb{N}} f^{k_i}(U)}$.*

Proof. Since $a, b \in \omega(x_0, f)$, for any open neighbourhood V of b we have increasing sequences of positive integers $\{m_i\}_{i\in\mathbb{N}}$ and $\{n_i\}_{i\in\mathbb{N}}$, with $n_i > m_i$ for every i, such that $\lim_{i\to\infty} f^{m_i}(x_0) = a$ and $\lim_{i\to\infty} f^{n_i}(x_0) = b$, so there is some $N \in \mathbb{N}$ such that $f^{m_i}(x_0) \in U$ and $f^{n_i}(x_0) \in V$ for every $i \geq N$. For every $i \in \mathbb{N}$ let $z_i := f^{m_i}(x_0)$ and $k_i := n_i - m_i$. Then clearly $\lim_{i\to\infty} z_i = a$ and $\lim_{i\to\infty} f^{k_i}(z_i) = b$, and since $z_i \in U$ for infinitely many i, we must have that $b \in \overline{\cup_{i \in \mathbb{N}} f^{k_i}(U)}$. $\square$

We now identify two properties of ω-limit sets, which we use to prove an expansive property for certain ω-limit sets.

Definition 3.3.3 (Property α)**.** For any $A \subset I$ closed and nowhere dense, suppose $a \in A$ and $M \subsetneq A$ is CINE. We say that A has *property* α if for any open $V \supset M$ for which $A \setminus V \neq \emptyset$ and $A \cap V$ is closed, and for any neighbourhood U of a, there is some $k_0 \in \mathbb{N}$ such that for every $n > k_0$, $f^n(U) \cap V \neq \emptyset$.

Definition 3.3.4 (Property β)**.** We say that the set $A \subset I$ has *property* β if for every $a, b \in A$ (where we do not exclude the case $a = b$) and any neighbourhoods U of a and V of b, $f^n(U) \cap V \neq \emptyset$ for infinitely many $n \in \mathbb{N}$.

Note that sets such as those labeled V in Definition 3.3.3 exist since A is closed and nowhere dense. The definition of property α is quite restrictive, but we will show that both it and property β are propeties of ω-limit sets, and together the two properties imply a type of expansivity (Definition 3.3.7).

Note also that properties α and β are independent: the set of natural numbers n for which $f^n(U) \cap V \neq \emptyset$ in property β need not be cofinite, where as in property α it is cofinite. Moreover, there is a set (labelled V) in property α which necessarily does not contain A, where as neither set U nor V in property β are so restricted. However both properties are implied by (but do not imply) topological mixing (Definition 4.1.10) and by topological exactness (Definition 1.1.5).

Lemma 3.3.5. *Suppose that for some $x_0 \in I$, $A = \omega(x_0, f)$ is infinite and nowhere dense. Then A has property α.*

Proof. Pick $a \in A$ and let U be any neighbourhood of a. Consider the set $J = \cup_{j\in\mathbb{N}} f^j(U)$. Certainly $\overline{J} \cap A$ is CINE. If $M \subsetneq A$ is CINE, given an open set $V \supset M$ for which $A \setminus V \neq \emptyset$ and $A \cap V$ is closed, by Proposition 3.3.1 there is a $b \in A \setminus V$ for which $\omega(b, f) \subset V$. By Lemma 3.3.2, $b \in \overline{J}$, so $f^k(b) \in \overline{J}$ for all $k \in \mathbb{N}$ and since $\omega(b, f) \subset V$, for some $m \in \mathbb{N}$, $f^n(b) \in V$ for every $n > m$. There are then two possible cases:

1. $f^{m+r}(b) \in f^{k_0}(U)$ for some $k_0, r \in \mathbb{N}$. Then $f^n(U) \cap V \neq \emptyset$ for every $n > k_0$.

2. For every n, $f^n(b) \notin f^k(U)$ for any $k \in \mathbb{N}$. Then $f^{m+i}(b)$ is a limit point of J for every $i \in \mathbb{N}$. But $f^{m+i}(b) \in V$, so $V \cap f^{k_0}(U) \neq \emptyset$ for some $k_0 \in \mathbb{N}$ since V is open in I. Take $z \in V \cap f^{k_0}(U)$, then $f^i(z) \in V \cap f^{k_0+i}(U)$ for every $i \in \mathbb{N}$. Hence $f^n(U) \cap V \neq \emptyset$ for all $n > k_0$.

$\square$

Remark 3.3.6. It is easy to see that any ω-limit set $A = \omega(x_0)$ has property β, since each point in A is approached to within arbitrarily close distances by the orbit of the point x_0. Thus the neighbourhood U of a contains an iterate of x_0 whose orbit must hit the neighbourhood V of b infinitely often.

Definition 3.3.7 (Weakly-Expansive Set)**.** Suppose that $A \subset I$ is an infinite, nowhere dense CINE set. Then A is *weakly-expansive* if there exists $r > 0$ such that for any $a \in A$ and any neighbourhood U of a, the diameter of $f^n(U)$ is greater than r for some $n \in \mathbb{N}$.

Proposition 3.3.8 and Corollary 3.3.9 are extracted from [4], but the proofs are reworked to fit with the material in this chapter.

Proposition 3.3.8. *Suppose that $A \subset I$ is an infinite, nowhere dense CINE set, that A has properties α and β, and that A contains a proper minimal subset. Then A is weakly-expansive.*

Proof. Suppose there is a proper, closed, invariant set $B \subsetneq A$ which contains all the proper minimal subsets of A. For any $x \in A \setminus B$ set $r = \frac{1}{2}d(x, B)$. Take $y \in A$ and any neighbourhood U of y. If $y \in B$ then $f^n(U) \cap B \neq \emptyset$ for every $n \geq 0$ and by property β, $f^k(U) \cap B_r(x) \neq \emptyset$ for infinitely many $k \in \mathbb{N}$. Thus the diameter of $f^k(U)$ is greater than r for infinitely many k. If $y \notin B$ let $V \supset B$ be open such that $A \cap V$ is closed and $A \setminus V \neq \emptyset$, and let V be of distance at least r from $B_{r/2}(x)$ (notice that in this case we may have $y = x$). Then by property α there is a $k_0 \in \mathbb{N}$ for which $f^n(U) \cap V \neq \emptyset$ for every $n > k_0$, and again we have $f^n(U) \cap B_{r/2}(x) \neq \emptyset$ for infinitely many $n \in \mathbb{N}$ by property β. So for infinitely many $n > k_0$ we have that $f^n(U)$ has diameter greater than r.

Now suppose that there is no such $B \subsetneq A$ containing all the proper minimal subsets of A. If there was only one proper minimal subset then this would be such a B, hence there are at least two proper minimal subsets of A. We claim that there is an $r > 0$ such that for any $x \in A$, $d(x, M) > 2r$ for some minimal set M. If not, given a decreasing sequence $\{r_n\}_{n \in \mathbb{N}}$ of positive real numbers, where $\lim_{n \to \infty} r_n = 0$, for any of these r_n we could find $x_n \in A$ for which $d(x_n, M) < r_n$ for all minimal subsets M. Then $x = \lim_{n \to \infty} x_n \in A$ is such that $x \in M$ for every minimal set M, which is impossible by Theorem 2.2.5.

So take $y \in A$ and let B be a minimal set with $d(y, B) > 2r$. Take any neighbourhood U of y with diameter $r_0 < r$. Let $V \supset B$ be open such that

$A \cap V$ is closed and $A \setminus V \neq \emptyset$, and let V be of distance at least r from U. Then by property α there is a $k_0 \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$ for every $n > k_0$. Also by property β, for infinitely many $n \in \mathbb{N}$, $f^n(U) \cap U \neq \emptyset$. So for infinitely many $n > k_0$, $f^n(U)$ has diameter greater than r. □

Corollary 3.3.9. *Suppose that $A = \omega(x_0, f)$ is an infinite and nowhere dense ω-limit set for some $x_0 \in I$, and that A contains a proper minimal set. Then A is weakly-expansive.*

Note that this case is not vacuous — the ω-limit sets H_1 and H_2 in Example 1.2.1 both have proper minimal subset $\{0\}$. It is easy to see that both H_1 and H_2 are weakly-expansive, since any open set lying to either side of 0 will eventually cover either $[0, 1]$ or $[-1, 0]$; this is due to the fact that all tent maps with gradient in the region $(\sqrt{2}, 2]$ are locally eventually onto [14].

We will return to the subject of expansivity in Chapter 5, where we define a sequence of expansive properties, satisfied by a class of interval map, and which imply properties allowing us to draw further conclusions between ICT sets and ω-limit sets.

Chapter 4

Symbolic Dynamics and Kneading Theory

Symbolic dynamics is the process of representing the behaviour of mathematical systems via a sequence of symbols from a (usually finite) alphabet. Marston Morse was one of the first to use symbols in this way, in his 1921 paper *Recurrent geodesics on a surface of negative curvature* [36]. Then in 1938, Hedlund and Morse [25] wrote a paper entitled *Symbolic Dynamics* in which they build on this theory, proving a number of results about the space of infinite sequences of symbols. Further contributions to the theory have since been made by Smale [48], Gottschank and Hedlund [24] and Metropolis, Stein and Stein [34]. Similar techniques were also used by Shannon as early as 1948 when developing his theory of communication [46].

In 1988 (with pre-prints dating back to 1977), John Milnor and William Thurston introduced an ingenious and original way of analyzing piecewise monotone maps of a compact interval, based on the concept of symbolic dynamics, in their paper *On Iterated Maps of the Interval* [35]. Their idea was to record symbolically where in relation to the various local extrema of the map each iterate of a specific local extremum fell, and associate with each such point its complete symbolic itinerary, known as a *kneading invariant*. They were then able to make some remarkable observations about the map in question using only the information stored in these kneading invariants. This process has become known as *kneading theory*, which together with symbolic dynamics provides a convenient method for analyzing piecewise monotone maps, and has also become a popular area of research in its own right (see for example [27], [32], [42]).

Kneading theory has been updated many times since the original paper of Milnor and Thurston (see [17], [19], [20] for examples), and in this chapter we present a version of the theory most relevant to our discussion, which has evolved from the work of de Melo and van Strien [19] on piecewise monotone maps, along with some preliminary material on shift spaces. We then show how kneading theory can be used to make useful observations about the behaviour of certain maps which would otherwise have been very difficult to deduce, and in particular we present a characterization of a class of ω-limit sets for maps of the interval, which includes the widely studied post-critical ω-limit sets of critically non-recurrent tent maps (see [21], [22], and [41]).

4.1 Shift Spaces

In this section we introduce the idea of symbol spaces and the shift map, which together form what are known as *shift spaces* – one of the basic precepts in symbolic dynamics. The ideas in this section form the basis of much of what follows, but it is worth noting that they also have many other applications, particularly in coding theory.

Consider an alphabet (usually finite) of symbols $\mathcal{A}$ and define spaces $X = \mathcal{A}^{\mathbb{N}}$ to be all infinite sequences $(x_0 x_1 \ldots)$ over $\mathcal{A}$ (where we allow 0 to be a natural number in this case, for the sake of easy notation) and $Z = \mathcal{A}^{\mathbb{Z}}$ all bi-infinite sequences $(\ldots x_{-1} \cdot x_0 x_1 \ldots)$ over $\mathcal{A}$, where the '$\cdot$' defines where the sequence is read from i.e. its central point. Finite sequences $(x_1 x_2 \ldots x_n),\ x_i \in \mathcal{A}$ will be called finite words, or just words. The shift map σ acts on X and Z as follows. For $(x_0 x_1 x_2 \ldots) \in X$ and $(\ldots x_{-1} \cdot x_0 x_1 \ldots) \in Z$ we have

$$\sigma((x_0 x_1 x_2 \ldots)) = (x_1 x_2 x_3 \ldots)$$

and

$$\sigma((\ldots x_{-1} \cdot x_0 x_1 \ldots)) = (\ldots x_{-1} x_0 \cdot x_1 \ldots).$$

We can define a metric d on X and Z as follows. For $x = (x_0 x_1 \ldots), y = (y_0 y_1 \ldots) \in X$, let $d(x, y) = 1/2^k$, where k is the smallest positive integer for which $x_k \neq y_k$. Similarly, for $x = (\ldots x_{-1} \cdot x_0 x_1 \ldots), y = (\ldots y_{-1} \cdot y_0 y_1 \ldots) \in Z$, let $d(x, y) = 1/2^k$, where k is the smallest positive integer for which either $x_k \neq y_k$ or $x_{-k} \neq y_{-k}$. So elements are close in X or Z if they agree on a large initial or central segment respectively. Open balls $B_{1/2^k}(x)$ and $B_{1/2^k}(z)$ with radius $1/2^k$ about the points $x = (\ldots x_{-1} \cdot x_0 x_1 \ldots) \in Z$ and $z = (z_0 z_1 z_2 \ldots) \in X$ are then given by

$$B_{1/2^k}(x) = \big\{(\ldots y_{-1} \cdot y_0 y_1 \ldots) \in Z \ : \ y_i = x_i \text{ for every } -k < i < k\big\}$$

and

$$B_{1/2^k}(z) = \big\{(y_0 y_1 y_2 \ldots) \in X \ : \ y_i = z_i \text{ for every } i < k\big\}.$$

This metric gives rise to the Tychonoff product topologies $\mathcal{T}_{\mathbb{N}}$ and $\mathcal{T}_{\mathbb{Z}}$ on X and Z respectively. Indeed if we furnish the alphabet $\mathcal{A}$ with the discrete topology, then open sets in $\mathcal{T}_{\mathbb{N}}$ and $\mathcal{T}_{\mathbb{Z}}$ correspond to countable unions of open balls from X and Z with respect the the metric d. $\mathcal{A}$ is finite so is compact, hence X and Z are both compact by Tychonoff's Theorem (see [47], for example). The shift map can easily be shown to be continuous with respect to d on both X and Z, thus (X, σ) and (Z, σ) are dynamical systems.

Definition 4.1.1 (Shift Space)**.** Subsets $W \subset X$ and $Y \subset Z$ are called *shift spaces* if they are invariant with respect to σ, and closed with respect to the topologies $\mathcal{T}_{\mathbb{N}}$ and $\mathcal{T}_{\mathbb{Z}}$ respectively.

In particular, the spaces X and Z are shift spaces, known as *full shifts* over the alphabet $\mathcal{A}$.

Definition 4.1.2 (Sub-Shifts of Finite Type)**.** For a set of words $\mathcal{F}$, define the sets $X_{\mathcal{F}} \subset X$ and $Z_{\mathcal{F}} \subset Z$ as

$$X_{\mathcal{F}} = \{x \in X \ : \ x \text{ does not contain any word from } \mathcal{F}\}$$

and

$$Z_{\mathcal{F}} = \{z \in Z \ : \ z \text{ does not contain any word from } \mathcal{F}\}.$$

Then $X_{\mathcal{F}}$ and $Z_{\mathcal{F}}$ are called *sub-shifts* of the full shifts X and Z. In particular, $X_{\mathcal{F}}$ and $Z_{\mathcal{F}}$ are called *sub-shifts of finite type* (or just *shifts of finite type*) if $\mathcal{F}$ is a finite set.

Note that the terms *shift of finite type* and *sub-shift* are also used by many authors to refer to the shift map σ acting upon the relevant shift space. In what follows we will use the terms to mean either the space or the map, whichever is contextually appropriate, denoting the space in the form $X_{\mathcal{F}}$ and the map as σ.

Lemma 4.1.3. *Shifts of finite type are shift spaces.*

Proof. Let $X_{\mathcal{F}}$ be a shift of finite type. Clearly $X_{\mathcal{F}}$ is invariant since if $x \in X_{\mathcal{F}}$ contains no word from $\mathcal{F}$ then neither does $\sigma(x)$. Moreover, suppose that $\lim_{n\to\infty} x_n = x$ for a sequence $\{x_n\}_{n\in\mathbb{N}} \subset X_{\mathcal{F}}$, and suppose that $x \notin X_{\mathcal{F}}$. Then at some point we will see that a word in $\mathcal{F}$ appears in x. But for any $k \in \mathbb{Z}$ there is an $n \in \mathbb{N}$ such that $d(x_n, x) < 1/2^k$. Thus for any integer k we can find a point x_n which agrees with x up to the first k symbols. Thus if x contains a word from $\mathcal{F}$ then so must an infinite number of the x_n; a contradiction that $\{x_n\}_{n\in\mathbb{N}} \subset X_{\mathcal{F}}$. Thus $x \in X_{\mathcal{F}}$ and so $X_{\mathcal{F}}$ is closed. $\square$

Definition 4.1.4 (Sofic Shifts)**.** Let G be a finite directed graph with edges E_G. For each $e \in E_G$, let e^- denote the initial point of e and e^+ the final point. Let $\mathcal{A}$ be a finite set of labels, let $L : E_G \to \mathcal{A}$ and let $\mathcal{G} = (G, L)$. A bi-infinite path on G is a bi-infinite sequence of edges $\pi = \ldots e_{-1} \cdot e_0 e_1 \ldots$ such that e_n^+ and e_{n+1}^- meet at a vertex. We denote the shift space of all paths on G by Z_G. L can be extended to paths around G in the natural way: $L(\pi) = \ldots L(e_{-1}) \cdot L(e_0) L(e_1) \ldots$.

A shift space is *sofic* if it takes the form

$$Z_{\mathcal{G}} = \big\{L(\pi) : \pi \in Z_G\big\},$$

for some $\mathcal{G}$.

We will return to sofic shifts in Chapter 5. For now we simply note that all shifts of finite type are sofic shifts, and all sofic shifts are shift spaces, the proofs of which can be found in [30].

The following Lemma about ω-limit sets of sequences in shift spaces is well-known.

Lemma 4.1.5. *Suppose that X is a shift space, and that $\boldsymbol{s} \in X$. Then $\boldsymbol{t} \in \omega(\boldsymbol{s}, \sigma)$ if and only if every finite initial segment of $\boldsymbol{t}$ occurs infinitely often in $\boldsymbol{s}$.*

Proof. Suppose that every finite initial segment of $\mathbf{t} = (t_0 t_1 \ldots)$ occurs infinitely often in $\mathbf{s}$. Pick $\epsilon > 0$, then there is an $n \in \mathbb{N}$ for which $1/2^n < \epsilon$, and we have that $(t_0 \ldots t_n)$ occurs in $\mathbf{s}$. So by the metric on X, $\mathrm{orb}(\mathbf{s}, \sigma)$ gets within $1/2^n$ (and thus ϵ) of $\mathbf{t}$, hence $\mathbf{t} \in \omega(\mathbf{s}, \sigma)$.

Now suppose that $\mathbf{t} \in \omega(\mathbf{s}, \sigma)$, and pick $n \in \mathbb{N}$. Then there is an $\epsilon > 0$ for which $\epsilon < 1/2^n$, and $\mathrm{orb}(\mathbf{s}, \sigma)$ gets within ϵ (and also within $1/2^n$) of $\mathbf{t}$ infinitely often. So by the metric on X, $(t_0 \ldots t_n)$ occurs infinitely often in $\mathbf{s}$. $\square$

In Lemma 4.1.8 we take advantage of internal chain transitivity (ICT) and the structure of the shift space to construct a sequence s containing all finite words from elements in a set Λ, such that $\omega(s, \sigma) = \Lambda$. This is a reworking of similar theory from [7], which is originally due to Good, Knight and Raines. First we show how pseudo-orbits appear in shift spaces.

Lemma 4.1.6. *Suppose that Ω is an alphabet and Λ is a subset of $\Omega^{\mathbb{N}}$. For $\epsilon > 0$, if $\{s_0, s_1, \ldots\}$ is an ϵ-pseudo-orbit, then for $n \in \mathbb{N}$ with $1/2^n < \epsilon \leq$*

$1/2^{n-1}$, *for each* $i \geq 0$ *the first* $n-1$ *symbols of* $\sigma(s_i)$ *agree with the first* $n-1$ *symbols of* s_{i+1}.

Proof. We have that for each $i \geq 0$, there is a positive integer n_i for which $d\big(\sigma(s_i), s_{i+1}\big) = 1/2^{n_i} < \epsilon$. Thus the first $n_i - 1$ symbols of $\sigma(s_i)$ and s_{i+1} coincide. Suppose that $n \in \mathbb{N}$ is such that $1/2^n < \epsilon \leq 1/2^{n-1}$. If $n > \min\{n_i \; : \; i \geq 0\}$ then $n - 1 \geq \min\{n_i \; : \; i \geq 0\}$, in which case $\epsilon \leq 1/2^{n-1} \leq 1/2^{n_i}$ for some i, which contradicts the fact that $\{s_0, s_1, \ldots\}$ is an ϵ-pseudo-orbit. Thus $n \leq \min\{n_i \; : \; i \geq 0\}$ and so for each $i \geq 0$ the first $n-1$ symbols of $\sigma(s_i)$ agree with the first $n-1$ symbols of s_{i+1}. $\square$

We now proceed with the construction of a sequence s for which $\omega(s, \sigma)$ is a specific ICT set Λ.

Suppose that Ω is an alphabet, N is a positive integer and Λ is a non-empty, ICT subset of $\Omega^{\mathbb{N}}$. Let $\mathcal{A}$ be the collection of all finite words of length greater than N which appear in elements of Λ, and enumerate $\mathcal{A}$ as $\{\theta_n\}$. For every $n \in \mathbb{N}$ pick $q_n, q_{n+1} \in \Lambda$ such that θ_n is the initial segment of q_n and θ_{n+1} is the initial segment of q_{n+1}. Also for each $n \in \mathbb{N}$ pick $m_n > \max\{|\theta_n|, |\theta_{n+1}|\}$ and let $\epsilon = 1/2^{m_n+1}$. Then since Λ is ICT there is an ϵ-pseudo-orbit $\{q_{n,0} = q_n, q_{n,1}, \ldots, q_{n,k_n} = q_{n+1}\} \subset \Lambda$ joining q_n and q_{n+1}. By Lemma 4.1.6, for each $n \in \mathbb{N}$ the points $q_{n,0} = q_n, q_{n,1}, \ldots, q_{n,k_n} = q_{n+1}$ are such that for $1 \leq i \leq k_n$, the first $m_n - 1$ symbols of $\sigma(q_{n,i-1})$ agree with the first $m_n - 1$ symbols of $q_{n,i}$.

We construct an element $s(\Lambda, N) \in \Omega^{\mathbb{N}}$ as follows: For every $n \in \mathbb{N}$ we make a new word ϕ_n from θ_n, θ_{n+1} and the ϵ-pseudo-orbit joining the corresponding q_n, q_{n+1}, by taking the first symbol of each $q_{n,i}$ and construct-

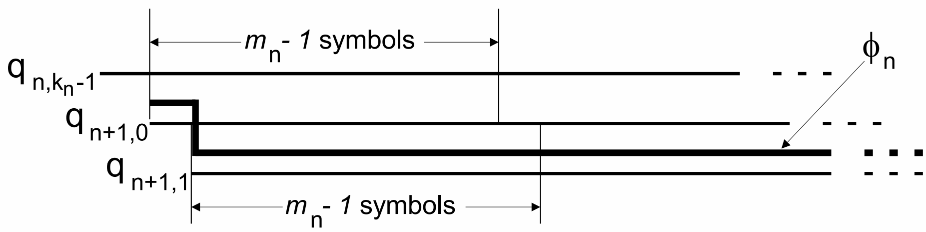

Figure 4.1: The relationship between the various sequences in the construction of $s(\Lambda, N)$.

ing ϕ_n sequentially from these. So ϕ_n begins with an initial segment of $q_{n-1,k_{n-1}} = q_{n,0}$ (the first section of which is θ_n) and ends with the first symbol of q_{n,k_n-1}, then ϕ_{n+1} begins with an initial segment of $\sigma(q_{n,k_n-1}) = q_{n+1,0}$ (see Figure 4.1). The sequence $s(\Lambda, N)$ is then the concatenation of all the ϕ_n. Notice that for each i, the agreement between $\sigma(q_{n,i})$ and $q_{n,i+1}$ is *at least as long* as each of θ_n and θ_{n+1}.

Definition 4.1.7 (Arbitrary Length, Infinite Repetition Sequence)**.** We call the element $s(\Lambda, N)$ in the above construction an *arbitrary length, infinite repetition sequence* for the ICT set Λ.

Lemma 4.1.8. *Suppose that Ω is an alphabet and $\Lambda \subset \Omega^{\mathbb{N}}$ is a shift space which is ICT. If for some $N \in \mathbb{N}$, $s = s(\Lambda, N)$ is an arbitrary length, infinite repetition sequence for Λ, then $\Lambda = \omega(s, \sigma)$.*

Proof. Let $\mathcal{A}$ be the set of all finite words of length greater than N which occur in elements of Λ. Then since Λ is invariant, $\mathcal{A}$ is also the set of all finite words of length greater than N which occur as initial segments of elements of Λ. We first show that $\mathcal{A}$ is the set of all infinitely repeating words of length greater than N occurring in s.

Let $V \in \mathcal{A}$ be a word of length greater than N. Then V occurs as a sub-word infinitely often in $\mathcal{A}$, and hence by construction infinitely often in s. Now suppose that the finite word V has length greater than N and occurs infinitely often in s. By the construction of s, there is a $n \in \mathbb{N}$ for which all of the words from $\mathcal{A}$ occurring in $\sigma^n(s)$ are of length greater than that of V, so pick an occurrence of V in $\sigma^n(s)$. Since for every $k \in \mathbb{N}$, $\sigma^k(s)$ begins with a segment of some point in Λ (see Figure 4.1), this V must be the initial segment of some point in Λ, so must also be the initial segment of some word from $\mathcal{A}$, and since $\mathcal{A}$ is invariant under taking subwords of length greater than N, we must have that $V \in \mathcal{A}$.

Pick $t \in \Lambda$. Then every finite initial segment of t of length greater than N is in $\mathcal{A}$, so occurs infinitely often in s, and hence by Lemma 4.1.5, $t \in \omega(s, \sigma)$. Now pick $t \in \omega(s, \sigma)$. Then by Lemma 4.1.5 every finite initial segment of t of length greater than N occurs infinitely often in s, and so is in $\mathcal{A}$. Λ is closed, thus $t \in \Lambda$ (since t is the limit of points in Λ with initial segments of increasing length which agree with those of t). Hence $\Lambda = \omega(s, \sigma)$ as required. $\square$

The next result is also from [7] and is a characterization of ω-limit sets for shifts of finite type.

Theorem 4.1.9. *Let $\mathcal{F}$ be a finite collection of words, and let $\Lambda \subset X_{\mathcal{F}}$ be a closed set. Then $\Lambda = \omega(x, \sigma)$ for some $x \in X_{\mathcal{F}}$ if and only if Λ is ICT.*

Proof. If $\Lambda = \omega(x, \sigma)$ for some $x \in X_{\mathcal{F}}$ then by Lemma 3.2.7 we have that Λ is ICT, so necessity is dealt with. To prove sufficiency, assume that $\Lambda \subset X_{\mathcal{F}}$ is a closed set which is ICT.

Let $N = \max\{|F| \ : \ F \in \mathcal{F}\}$ and form the arbitrary length, infinite repetition sequence $s = s(\Lambda, N)$. By Proposition 3.2.5, Λ is invariant so is a shift space, and by Lemma 4.1.8, $\Lambda = \omega(s, \sigma)$; we need to show that $s \in X_{\mathcal{F}}$.

Suppose not i.e. that there is some $F \in \mathcal{F}$ that appears in s. Thus by the construction of s, there is some $k_0 \in \mathbb{N}$ for which $\sigma^{k_0}(s)$ begins with F. $|F| \leq N$, so since $\sigma^k(s)$ begins with an initial segment of some point in Λ of length at least N for every $k \in \mathbb{N}$, we have that F must be the initial segment of some point in Λ. This is a contradiction since $\Lambda \subset X_{\mathcal{F}}$, hence $s \in X_{\mathcal{F}}$. $\square$

The following property is related to (although stronger than) that of transitivity, and we introduce it here as it is a property of certain shift spaces.

Definition 4.1.10 (Topological Mixing/Weak Mixing)**.** For a dynamical system (X, f), the map f is *topologically mixing* if for every pair of open sets U and V there is an $N \in \mathbb{N}$ for which $f^n(U) \cap V \neq \emptyset$ for every $n \geq N$. f is said to be *(topologically) weakly mixing* if the map $(f \times f) : (X \times X) \to (X \times X)$, defined by $(f \times f)(x, y) = (f(x), f(y))$, is transitive.

Lemma 4.1.11. *Suppose for a dynamical system (X, f) that $f : X \to X$ is mixing, then $(f \times f) : (X \times X) \to (X \times X)$ has mixing.*

Proof. Suppose that $f : X \to X$ is mixing, let $U = U_1 \times U_2$ and $V = V_1 \times V_2$ be open sets in $X \times X$.

By mixing there are positive integers N_1 and N_2 such that for every $n \geq N_1$ we have that $f^n(U_1) \cap V_1 \neq \emptyset$ and for every $n \geq N_2$ we have that $f^n(U_2) \cap V_2 \neq \emptyset$. Let $N = \max\{N_1, N_2\}$, then for every $n \geq N$ we have that $f^n(U_1) \cap V_1 \neq \emptyset$ and $f^n(U_2) \cap V_2 \neq \emptyset$, so $(f \times f)^n(U_1 \times U_2) \cap (V_1 \times V_2) \neq \emptyset$ for every such n. $\square$

Corollary 4.1.12. *In a dynamical system (X, f), if f is mixing it is also weak mixing.*

So mixing is a stronger property than weakly mixing. These properties will be useful in our analysis in Chapter 5.

Proposition 4.1.13. *For a finite alphabet Ω, the full shift $\Omega^{\mathbb{N}}$ is mixing.*

Proof. Suppose that $X = \Omega^{\mathbb{N}}$ is the full shift, and that U, V are basic open sets in X (we lose no generality in assuming that U and V are basic open, since if not we can "shrink" to basic open sets to get the same result). Then there are words $(t_0 t_1 \ldots t_p)$ and $(s_0 s_1 \ldots s_m)$ such that $U = \{(r_0 r_1 \ldots) : r_i = t_i \text{ for every } 0 \leq i \leq p\}$ and $V = \{(r_0 r_1 \ldots) : r_i = s_i \text{ for every } 0 \leq i \leq m\}$. Since we are in the full shift, for every $n \geq 0$ there is an $x_n \in X$ for which $\sigma^n(x_n)$ begins with $(s_0 s_1 \ldots s_m)$. In other words $\sigma^n(x_n) \in V$, so for every $n \geq p$, $\sigma^{n+1}(t_0 t_1 \ldots t_p x_{n-p}) \in V$. Since $(t_0 t_1 \ldots t_p x_{n-p})$ is an element of U for every $n \geq p$ we have that $\sigma^{n+1}(U) \cap V \neq \emptyset$ for every $n \geq p$. $\square$

4.2 The Kneading Theory

One common use of symbolic dynamics is to assign infinite sequences to points in a compact interval. There is not, in general, a one-to-one relationship between points in the interval and the set of infinite sequences over an alphabet, so it is often necessary to determine whether or not a given sequence relates to a point in the interval.

The main precept in kneading theory is the assignment of the kneading invariants, which are strongly related to the critical points of the map. By examining the kneading invariants of a map and comparing them to a specific sequence of symbols, we can determine whether or not that sequence corresponds to a point in the interval. Thus kneading theory is often used together with symbolic dynamics.

We will be looking at maps with a finite number of monotone pieces on a compact interval I, and the first part of this section is dedicated to defining the basic terminology. We then focus on some results which follow from the definitions, where arguments are quite often symmetric in terms of an order relation, in which case we will focus only on one half of the argument; the other being analogous.

The dynamics of a continuous map $f : I \to I$ are completely dependent upon the position of the map with respect to the diagonal $f(x) = x$, so we lose no generality in letting I be the interval $[0, 1]$. The map f is said to be l-modal if it is continuous and there exist l critical points (local extrema) $c_1 < c_2 < \ldots < c_l$ such that f is strictly monotone on each of the $l + 1$ sub-intervals I_i, where $I_{i+1} = (c_i, c_{i+1})$ for $i = 1, \ldots, l - 1$, $I_1 = [0, c_1)$ and

$I_{l+1} = (c_l, 1]$. We define the *symbolic dynamics* for l-modal maps as follows.

- Define the *alphabet* $\Omega = \{I_1, \ldots, I_{l+1}, C_1, \ldots, C_l\}$.
- The full shift space is denoted $\Omega^{\mathbb{N}}$, where

$$\Omega^{\mathbb{N}} = \{(s_0 s_1 \ldots) | s_i \in \Omega\}.$$

- The *polarity* of an element s_i is determined by the map $\rho : \Omega \to \{-1, +1\}$, where

$$\rho(s_i) = \begin{cases} +1 & \text{if either } D(f) > 0 \text{ on } s_i, \text{ or } s_i = C_j \text{ for some } j \\ -1 & \text{if } D(f) < 0 \text{ on } s_i. \end{cases}$$

- We say that a finite word $(s_0 \ldots s_n)$ is *even* if $\Pi_{i=0}^n \rho(s_i) = 1$ and *odd* if $\Pi_{i=0}^n \rho(s_i) = -1$.
- We say that two sequences $(s_0 s_1 \ldots)$ and $(t_0 t_1 \ldots)$ in $\Omega^{\mathbb{N}}$ have *discrepancy* m if $s_i = t_i$ for every $i \leq m - 1$ and $s_m \neq t_m$.
- The *address* of a point $x \in I$ is given by the map $A : I \to \Omega$, where

$$A(x) = \begin{cases} I_i & x \in I_i \\ C_i & x = c_i. \end{cases}$$

- The *itinerary* of a point x is the sequence $It(x) = (x_0 x_1 \ldots)$ where $It_i(x) = x_i = A(f^i(x))$, and define $It(x)\restriction_N = (x_0 x_1 \ldots x_N)$.
- For $x \in I$ and $N \in \mathbb{N}$, let $I_N(x) = \{y \in I \ : \ It(y)\restriction_N = It(x)\restriction_N\}$.

We denote the set of all itineraries of a map f by Σ_f, where we will drop the subscript f when there is no ambiguity. If f is an l-modal map, then usually $\Sigma_f \subsetneq \Omega^{\mathbb{N}}$. We furnish $\Omega^{\mathbb{N}}$ with the metric d and the shift map σ as defined in Section 4.1. Notice that $(\sigma \circ It)(x) = (It \circ f)(x)$ for every $x \in I$. Σ_f is not in general a shift space, but we can put certain conditions upon subsets $\Lambda \subset I$ which make $It(\Lambda)$ a shift space (see Lemma 4.2.5 and Theorem 4.3.12).

Definition 4.2.1 (Parity Lexicographic Ordering)**.** Order the elements of Ω as follows: $I_i < C_i < I_{i+1}$ for $i = 1, \ldots, l$. For two sequences $\mathbf{s} = (s_0 s_1 \ldots)$ and $\mathbf{t} = (t_0 t_1 \ldots)$ with discrepancy m, we say that $\mathbf{s} \prec \mathbf{t}$ if either $(s_0 \ldots s_{m-1})$ is even, *and* $s_m < t_m$, or $(s_0 \ldots s_{m-1})$ is odd, *and* $s_m > t_m$.

By defining the lexicographic ordering $\prec$ on $\Omega^{\mathbb{N}}$ as above, we ensure that the ordering of the sequence space preserves that on the real line. Results 4.2.2, 4.2.3 and 4.2.4 are known, and follow on from the definition of the order $\prec$.

Proposition 4.2.2. *Let $f : I \to I$ be l-modal, $x, y \in I$. If $It(x) \prec It(y)$ then $x < y$.*

Proof. Let the discrepancy of $It(x) = (x_0 x_1 x_2 \ldots)$ and $It(y) = (y_0 y_1 y_2 \ldots)$ be m, with $It(x) \prec It(y)$.

If $m = 0$ then the result is clear, since then $x_0 < y_0$ so $x < y$. Thus we assume (for an inductive argument) that the result holds for sequences with discrepancy up to $m - 1$ for $m > 0$, and prove the result for m.

Consider $It(f(x)) = (x_1 x_2 \ldots)$ and $It(f(y)) = (y_1 y_2 \ldots)$. If $\rho(x_0) = 1$, then $It(f(x)) \prec It(f(y))$, since we have not changed the parity of

$(x_0 \dots x_{m-1})$, thus by our inductive hypothesis, $f(x) < f(y)$. Since $x_0 = y_0$, $\rho(x_0) = 1$ and $f(x) < f(y)$, we have that $x < y$.

If $\rho(x_0) = -1$, then $It(f(x)) \succ It(f(y))$ since the parity of $(x_0 \dots x_{m-1})$ has changed, thus $f(x) > f(y)$ by inductive hypothesis. Since $x_0 = y_0$, $\rho(x_0) = -1$ and $f(x) > f(y)$, we have that $x < y$.

If $x_0 = y_0 = C_i$ for some i, then $x = y$ and so $It(x) = It(y)$, contrary to our hypothesis. $\square$

Lemma 4.2.3. *Let $w \in I$ and $N \in \mathbb{N}$, then the set $I_N(w)$ is an interval in I. Moreover, if $f^n(w) = c_k$ for some $n \leq N$ and some $1 \leq k \leq l$, then $I_N(w) = \{w\}$, otherwise $I_N(w)$ is an open interval.*

Proof. Suppose that for some $N \in \mathbb{N}$ there is an $n \leq N$ for which $f^n(w) = c_k$ for some $1 \leq k \leq l$. Notice that if a point $y \in I$ has more than one immediate pre-image, they are all in different intervals of monotonicity. So there is only one pre-image of each point in the backwards orbit of c_k which falls into the same interval of monotonicity as that of the point in the orbit of w. Thus if $It_i(y) = It_i(w)$ for every $i < n$ and $It_n(y) = It_n(w) = C_k$ then $y = w$ and so $I_N(w) = \{w\}$, since w is the only point in $It_0(w)$ which maps onto c_k after n iterations.

Now suppose that $f^n(w) \neq c_k$ for any $n \leq N$, and $x \in I_N(w)$. Then there is an $\epsilon > 0$ such that $B_\epsilon(x) \in I_N(w)$, so $I_N(w)$ is open. Suppose that $x, y, z \in I$ s.t. $x < y < z$ and $It(x)\restriction_N = It(z)\restriction_N = It(w)\restriction_N$. We want to show that $It(y)\restriction_N = It(x)\restriction_N$. Suppose not; i.e. $It(y)$ and $It(x)$ have discrepancy $j < N$. Suppose $It(y) \succ It(x)$. Then $It(y) \succ It(z)$ since certainly $It(x)\restriction_j = It(z)\restriction_j$. But then $y > z$ – a contradiction. So suppose

$It(y) \prec It(x)$. But then $y < x$ which is another contradiction.

We have shown that if $x < y < z$ and $x, z \in I_N(w)$ for some $w \in I$, then $y \in I_N(w)$. So if $I_N(w)$ is not a single point it is an open interval. $\square$

In particular the ordering on the sequence space gives us that for any $\mathbf{t} \in \Omega^{\mathbb{N}}$ the set $A = \{x \in I \ : \ It(x) \prec \mathbf{t}\}$ is an interval in I.

We show now that the itinerary map is continuous at points which are not in the backwards orbit of any critical point, and use this to shown that images under the itinerary map of CINE subsets of the interval not containing any critical point are shift spaces. This does not hold in general if we allow the set to contain critical points, as we see later in Theorem 4.3.12.

Lemma 4.2.4. *Suppose that $x \in I$ and that $C_k \notin It(x)$ for any $1 \leq k \leq l$. Then the itinerary map It is continuous at x.*

Proof. Pick $\epsilon > 0$. For every $i \geq 0$ define $\eta_i = \min\{|f^i(x) - c_k| \ : \ 1 \leq k \leq l\} > 0$. Choose $N \in \mathbb{N}$ such that $1/2^N < \epsilon$. Then for every $y \in U_i = f^{-i}\big(B_{\eta_i}(f^i(x))\big)$, $A(f^i(y)) = A(f^i(x))$ for every i, where f^{-i} is the inverse image of f^i. Set $U = \bigcap_{i \leq N} U_i$, then for every $y \in U$, $It(y)\restriction_N = It(x)\restriction_N$. The U_i are open so U is open, and $x \in U$ so $U \neq \emptyset$. Hence there is a $\delta > 0$ such that $B_\delta(x) \subset U$. So for every $y \in B_\delta(x)$ we have that $It(y)\restriction_N = It(x)\restriction_N$ and so $d(It(x), It(y)) \leq 1/2^N < \epsilon$. $\square$

Lemma 4.2.5. *Suppose that $f : I \to I$ is an l-modal map with critical points $c_1, \ldots, c_l$. If $\Lambda \subset I$ is a CINE set such that $c_j \notin \Lambda$ for any $1 \leq j \leq l$ then $It(\Lambda)$ is a shift space.*

Proof. We have that

$$\sigma(It(\Lambda)) = (\sigma \circ It)(\Lambda) = (It \circ f)(\Lambda) \subset It(\Lambda),$$

so $It(\Lambda)$ is invariant. Moreover $It : \Lambda \to It(\Lambda)$ is continuous by Lemma 4.2.4, so $It(\Lambda)$ is compact since Λ is compact. Thus $It(\Lambda)$ is closed since $\Omega^{\mathbb{N}}$ is a Hausdorff space. $\square$

Definition 4.2.6 (Limit Itineraries)**.** Define the *upper-* and *lower-limit itinerary* of a point $x \in I$ as $It(x^+) = \lim_{y \downarrow x} It(y)$ and $It(x^-) = \lim_{y \uparrow x} It(y)$ respectively.

Lemma 4.2.7 shows that the limits $It(x^+)$ and $It(x^-)$ exist and are consistent with the ordering on $\Omega^{\mathbb{N}}$, and is based on notes in [19].

Lemma 4.2.7. *Let $f : I \to I$ be an l-modal map.*

1. *If $f^i(x) \neq c_k$ for any i and for any $1 \leq k \leq l$ then $It(x) = It(x^+) = It(x^-)$.*

2. *If $f^n(x) = c_k$ for some $n \in \mathbb{N}$ and some $1 \leq k \leq l$, then for small enough δ*

   (a) $It_n(y) = I_{k+1}$ for all $y \in (x, x+\delta)$ and $It_n(y) = I_k$ for all $y \in (x-\delta, x)$ if $It(y)\restriction_{n-1}$ is even, and

   (b) $It_n(y) = I_k$ for all $y \in (x, x+\delta)$ and $It_n(y) = I_{k+1}$ for all $y \in (x-\delta, x)$ if $It(y)\restriction_{n-1}$ is odd.

   For the least such n we have that $It(x)\restriction_{n-1} = It(x^+)\restriction_{n-1} = It(x^-)\restriction_{n-1}$.

3. *For all $x \in I$, C_k does not appear in $It(x^+)$ or $It(x^-)$ for any $1 \leq k \leq l$.*

Proof. Let $x \in I$ be fixed. Notice that by the continuity of f, for any $n \in \mathbb{N}$ there is a $\delta > 0$ for which It_n is constant on $(x, x+\delta)$ and $(x-\delta, x)$, although if $It_n(x) = C_k$ for some $1 \leq k \leq l$ then if $It(x)\lceil_{n-1}$ is even, $It_n(y) = I_{k+1}$ for all $y \in (x, x+\delta)$ and $It_n(y) = I_k$ for all $y \in (x-\delta, x)$ by the definition of the ordering $\prec$ (and vice-versa if $It(x)\lceil_{n-1}$ is odd). This is because by Lemma 4.2.3, when $It_n(x) = C_k$, $I_N(x) = \{x\}$ for every $N \geq n$. Thus C_k does not appear in $It(x^+)$ or $It(x^-)$ for any $1 \leq k \leq l$.

Now suppose that $f^n(x) = c_k$ for some $n \in \mathbb{N}$ and some $1 \leq k \leq l$, and that n is minimal in this respect. Then by the continuity of f there is a $\delta > 0$ such that for any $y \in (x-\delta, x+\delta)$, $f^i(y) \neq c_j$ for every $i < n$ and for all $1 \leq j \leq l$. So for all $i < n$ there is a $\delta > 0$ for which $It_i(y) = It_i(y^+)$ for every $y \in (x, x+\delta)$ and $It_i(y) = It_i(y^-)$ for every $y \in (x-\delta, x)$, and in particular $It(x)\lceil_{n-1} = It(x^+)\lceil_{n-1} = It(x^-)\lceil_{n-1}$. Now suppose that $f^i(x) \neq c_k$ for all $i \in \mathbb{N}$ and all $1 \leq k \leq l$; this must mean that $It(x) = It(x^+) = It(x^-)$ by the continuity of It. □

Lemma 4.2.8 is not stated in [19], and is stated implicitly but not proved in [35]. We prove it here as we need it for Theorem 4.2.13.

Lemma 4.2.8. *For every $1 \leq j \leq l$, $\sigma(It(c_j^+)) = \sigma(It(c_j^-))$.*

Proof. By the continuity of f and the fact that c_j is an extremum, for $1 \leq j \leq l$ there is an open neighbourhood U of c_j such that for every $y \in U \cap [0, c_j)$ there is an $x \in U \cap (c_j, 1]$ for which $f(x) = f(y)$; i.e. for every itinerary $\mathbf{s}$ of a point in $U \cap [0, c_j)$ there is an itinerary $\mathbf{t}$ of a point in $U \cap (c_j, 1]$ such that $\sigma(\mathbf{s}) = \sigma(\mathbf{t})$. Call this observation (1).

Suppose that $\sigma(It(c_j^+)) \neq \sigma(It(c_j^-))$, and their discrepancy is $p > 0$. Thus $\sigma(It(c_j^+))\lceil_n = \sigma(It(c_j^-))\lceil_n$ for every $n < p$ but $\sigma(It(c_j^+))\lceil_p \neq \sigma(It(c_j^-))\lceil_p$ (so in fact $\sigma(It(c_j^-))\lceil_p \prec \sigma(It(c_j^+))\lceil_p$). Certainly there is a point $x < c_j$ such that $\sigma(It(x))\lceil_p = \sigma(It(c_j^-))\lceil_p$, and by observation (1) there is a $y > c_j$ such that $\sigma(It(y))\lceil_p = \sigma(It(x))\lceil_p$. So we have $y > c_j$ with

$$\sigma(It(y))\lceil_p = \sigma(It(c_j^-))\lceil_p \prec \sigma(It(c_j^+))\lceil_p,$$

and so for some $z \in (c_j, y)$ we have that $It(y) \prec It(z)$, which contradicts Proposition 4.2.2. Hence $\sigma(It(c_j^+)) = \sigma(It(c_j^-))$. □

Definition 4.2.9 (Kneading Invariants)**.** For an l-modal map f, the kneading invariants K_i for $1 \leq i \leq l$ are defined as $K_i = \sigma(It(c_i^+)) = \sigma(It(c_i^-))$.

It is worth mentioning that some authors, such as Collet and Eckmann in [17], use alternative kneading invariants defined as the itineraries of the images of the critical points. The differences in the definitions are explored in [19] but we do not address them here.

Definition 4.2.10 (Concatenation of Sequences)**.** For two sequences of symbols A and B (which may be either finite or infinite), we use the notation $A \sqsubset B$ to denote the concatenation of A with B.

Lemma 4.2.11. *If $f^n(x) = c_k$ for some $n \in \mathbb{N}$ and some $1 \leq k \leq l$ then*

- $\sigma^n(It(x^+)) = It(c_k^+)$ *and* $\sigma^n(It(x^-)) = It(c_k^-)$ *if* $It(x)\lceil_{n-1}$ *is even;*
- $\sigma^n(It(x^+)) = It(c_k^-)$ *and* $\sigma^n(It(x^-)) = It(c_k^+)$ *if* $It(x)\lceil_{n-1}$ *is odd.*

Proof. Suppose that $It(x)\lceil_{n-1}$ is even and let $It(x) = (x_0 \ldots x_{n-1} C_k \ldots)$. Then by Lemma 4.2.7, $It(x^+) = (x_0 \ldots x_{n-1} I_{k+1} \ldots)$, and as $y \downarrow x$, $f^n(y) \downarrow f^n(x) = c_k$. Thus

$$It(x^+) = \left(x_0 \ldots x_{n-1} \sqsubset \lim_{y \downarrow f^n(x)} It(y)\right),$$

and so

$$\sigma^n(It(x^+)) = \lim_{y \downarrow f^n(x)} It(y) = \lim_{y \downarrow c_k} It(y) = It(c_k^+).$$

Similarly, $It(x^-) = (x_0 \ldots x_{n-1} I_k \ldots)$, and as $y \uparrow x$, $f^n(y) \uparrow f^n(x) = c_k$. Thus

$$It(x^+) = \left(x_0 \ldots x_{n-1} \sqsubset \lim_{y \uparrow f^n(x)} It(y)\right),$$

and so

$$\sigma^n(It(x^+)) = \lim_{y \uparrow f^n(x)} It(y) = \lim_{y \uparrow c_k} It(y) = It(c_k^-).$$

The proof is analogous if $It(x)\lceil_{n-1}$ is odd. ☐

In the next section we will manipulate sequences in $\Omega^{\mathbb{N}}$ to produce results about maps of the interval. It will thus be important for us to know whether a certain sequence is actually the itinerary of a point in the interval. As has been suggested, the key to this is the set of kneading invariants $\{K_i\}_{1 \le i \le l}$. To see why this might be the case, consider the map as shown in Figure 4.2. In order to preserve the parity lexicographic ordering, itineraries of points should fall between $It(f(k))$ and $It(f(c))$.

Definition 4.2.12 and Theorem 4.2.13 are reworkings of similar material in [19].

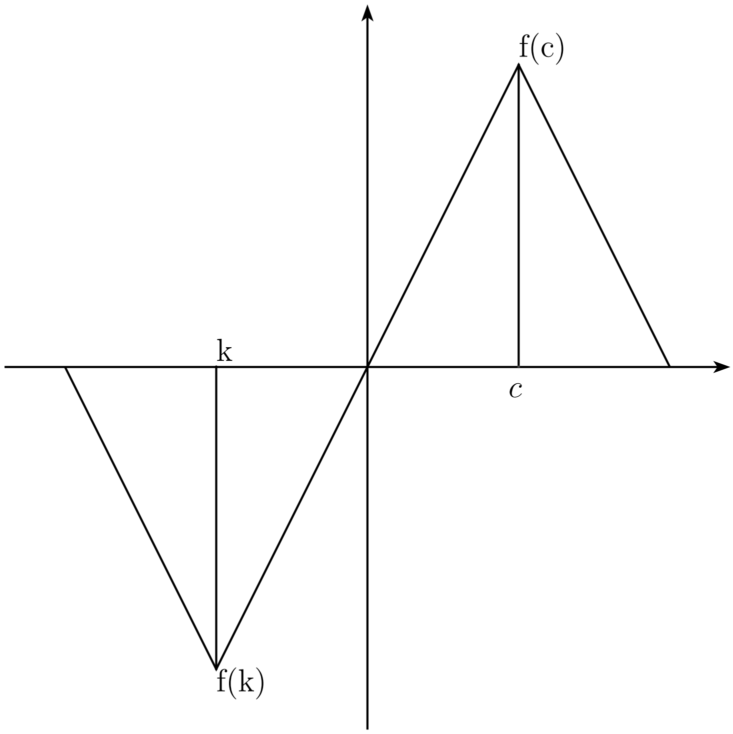

Figure 4.2: All itineraries fall between those of $f(k)$ and $f(c)$.

Definition 4.2.12 (Admissibility Conditions)**.** Let Σ_K denote the subset of $\Omega^{\mathbb{N}}$ which consists of sequences $\mathbf{s} = (s_0 s_1 \ldots)$ obeying the following conditions. For every $n \geq 0$ and for $1 \leq k \leq l$

$$\sigma^n(\mathbf{s}) = It(c_k) \text{ if } s_n = C_k \tag{4.1}$$

$$K_k \prec \sigma^{n+1}(\mathbf{s}) \prec K_{k+1} \text{ if } s_n = I_{k+1} \text{ and } \rho(s_n) = +1 \tag{4.2}$$

$$K_{k+1} \prec \sigma^{n+1}(\mathbf{s}) \prec K_k \text{ if } s_n = I_{k+1} \text{ and } \rho(s_n) = -1, \tag{4.3}$$

where we ignore the redundant invariants K_{l+1} if $s_n = I_{l+1}$, or K_0 if $s_n = I_1$. Sequences in Σ_K will be called *admissible*. Let $\hat{\Sigma}_K$ be defined as Σ_K but with $\prec$ replaced by $\preceq$ in (4.1) to (4.3).

Theorem 4.2.13. $\Sigma_K \subseteq It(I) \subseteq \hat{\Sigma}_K$.

Proof. Let $x \in I$ with $It(x) = \mathbf{s}$ and suppose that $f^n(x) \in I_{k+1} = (c_k, c_{k+1})$ for some $n \geq 0$ and some $1 \leq k \leq l-1$. Then $f^{n+1}(x) \in (f(c_k), f(c_{k+1}))$ if $\rho(I_{k+1}) = +1$ and $f^{n+1}(x) \in (f(c_{k+1}), f(c_k))$ if $\rho(I_{k+1}) = -1$. Hence by (the contra-positive of) Proposition 4.2.2, $K_k \preceq \sigma^{n+1}(\mathbf{s}) \preceq K_{k+1}$ if $\rho(I_{k+1}) = +1$, and $K_{k+1} \preceq \sigma^{n+1}(\mathbf{s}) \preceq K_k$ if $\rho(I_{k+1}) = -1$. Thus $It(x) \in \hat{\Sigma}_K$. Suppose that $f^n(x) \in I_1$ (the case where $f^n(x) \in I_{l+1}$ is analogous). Then $f^{n+1}(x) < f(c_1)$ if $\rho(I_1) = +1$ and $f^{n+1}(x) > f(c_1)$ if $\rho(I_1) = -1$. Hence by Proposition 4.2.2, $\sigma^{n+1}(\mathbf{s}) \preceq K_1$ if $\rho(I_1) = +1$, and $K_1 \preceq \sigma^{n+1}(\mathbf{s})$ if $\rho(I_1) = -1$. Thus $It(x) \in \hat{\Sigma}_K$.

Now let $\mathbf{t} = (t_0 t_1 \ldots) \in \Sigma_K$ and suppose that there is no $x \in I$ for which $It(x) = \mathbf{t}$. Then $I = A \cup B$, where $A = \{x \in I \ : \ It(x) \prec \mathbf{t}\}$ and $B = \{x \in I \ : \ It(x) \succ \mathbf{t}\}$. A and B are intervals by Lemma 4.2.3 with $0 \in A$ and $1 \in B$, and furthermore $A \cap B = \emptyset$. The real numbers $a = \sup A$ and $b = \inf B$ both exist, with $a \leq b$, so if $a \notin A$ and $b \notin B$ we have that $A \cup B$ is not connected and hence a contradiction. We show that $b \notin B$; the case $a \notin A$ is analogous, and is given in [19].

Suppose that $b \in B$, then for every $y < b$ we must have $y \notin B$ and hence $It(y) \prec \mathbf{t}$, so

$$It(b^-) \preceq \mathbf{t} \prec It(b). \tag{4.4}$$

By Lemma 4.2.4, any discontinuity of the itinerary map must occur at a point which is in the backward orbit of a turning point. By (4.4) this must be the case for the point b; i.e. there is a non-negative integer n for which

$f^n(b) = c_k$ for some $1 \leq k \leq l$. Let n be minimal in this respect. Thus by Lemma 4.2.7, $It(b)\restriction_{n-1} = \mathbf{t}\restriction_{n-1} = It(b^-)\restriction_{n-1}$ (which is vacuous if $n = 0$). For all $x < b$ with x sufficiently close to b we have that either $f^n(x) \in I_{k+1}$ or $f^n(x) \in I_k$. Suppose that $f^n(x) \in I_{k+1}$ for all such x; the other case is analogous. Thus $It_n(b^-) = I_{k+1}$, so since $It(b) \succ It(b^-)$ we must have that $\mathbf{t}\restriction_{n-1}$ is odd, and $c_k \leq t_n \leq I_{k+1}$.

If $t_n = c_k$ then since $\mathbf{t} \in \Sigma_K$ we would have that $\sigma^n(\mathbf{t}) = It(c_k)$ and then $It(b) = \mathbf{t}$, which contradicts the assumption that there is no such b. So $t_n = I_{k+1}$.

Now $\sigma^n(It(b^-)) = It(c_k^+)$ by Lemma 4.2.11, and thus $\sigma^{n+1}(It(b^-)) = \sigma(It(c_k^+)) = K_k$ by Lemma 4.2.8. Hence by (4.4) and Proposition 4.2.2, if $\rho(I_{k+1}) = +1$ then $\sigma^{n+1}(\mathbf{t}) \preceq \sigma^{n+1}(It(b^-)) = K_k$, and if $\rho(I_{k+1}) = -1$ then $\sigma^{n+1}(\mathbf{t}) \succeq \sigma^{n+1}(It(b^-)) = K_k$, both of which contradict the admissibility conditions (4.1) to (4.3), which is a contradiction to the fact that $\mathbf{t} \in \Sigma_K$. Hence $b \notin B$. $\square$

4.3 Locally Pre-Critical Maps

Consider a set $\Lambda \subset I$ and the corresponding set of itineraries $It(\Lambda)$; generally there is not a one-to-one correspondence between points in Λ and their itineraries in $It(\Lambda)$. This is due to the fact that distinct points in the basins of attraction of periodic orbits will often have identical itineraries. We wish to avoid this situation, so we place a condition on piecewise monotone maps which guarantees that the itinerary map is a bijection. The results in this section are original, unless otherwise stated.

Definition 4.3.1 (Locally Pre-Critical)**.** We say that the l-modal map $f : I \to I$ with critical points $c_1, \ldots, c_l$ is *locally pre-critical* if for every open interval $U \subset I$ there is an $n \in \mathbb{N}$ such that $c_k \in f^n(U)$ for some $1 \le k \le l$.

Locally pre-critical is a similar but weaker property than *topologically exact* (also referred to as *locally eventually onto* – see Definition 1.1.5), since locally pre-critical does not specify the need for expansivity.

Example 4.3.2. The map g in Example 1.2.1 is locally pre-critical, since any open interval will contain an open interval in either $(0, 1)$ or $(-1, 0)$, and both f_2 and $-f_2$ are locally eventually onto on $[0, 1]$ and $[-1, 0]$ respectively [13]. However g is not locally eventually onto, since $[0, 1]$ and $[-1, 0]$ are both invariant under g.

$\square$

Definition 4.3.3 (Wandering Interval)**.** For an l-modal map $f : I \to I$, an interval $J \subset I$ is called a *wandering interval* if $f^n(J) \cap f^m(J) = \emptyset$ for every $n > m \ge 0$ and the orbit $\{f^n(J)\}_{n \ge 0}$ does not tend towards a cycle.

The following definition was introduced implicitly by Milnor and Thurston in [35] and used in [17] and [19] in discussions about the distribution of itineraries of points.

Definition 4.3.4 (Homterval)**.** For an l-modal map $f : I \to I$, an interval $J \subset I$ is called a *homterval* if f^n is a homeomorphism on J for every $n \ge 0$.

Locally pre-critical is equivalent to saying that the map *has no homtervals*. De Melo and van Strien show that this is the case when the map has no wandering intervals and no attracting periodic orbit ([19], Lemma 3.1). We also have the following two Theorems, courtesy of Collet and Eckmann, which are variations of results from [17] and concern the logistic map $f_\mu(x) = \mu x(1-x)$ for $1 \leq \mu \leq 4$:

Theorem 4.3.5. *If f_μ has no attracting (stable) periodic orbit then it has no homterval.*

Theorem 4.3.6. *The kneading sequence of f_μ is periodic if and only if the map has a stable periodic orbit.*

Thus we get the following corollary:

Corollary 4.3.7. *For the family of logistic maps $f_\mu(x) = \mu x(1-x)$, $1 \leq \mu \leq 4$, if the kneading sequence is not periodic then the map has no homterval.*

We define and investigate various expansive properties of maps in Chapter 5, which allow us to make deductions on their behaviour similar to those which we make for locally pre-critical maps. Corollary 4.3.7 shows us however that there are many smooth, locally pre-critical maps, which is useful since it demonstrates that there are many maps other than expanding maps for which the theory in this and the following sections on locally pre-critical maps applies.

By ensuring that a map is locally pre-critical we can make several useful deductions about the behaviour of the associated itinerary map.

Lemma 4.3.8. *The itinerary map $It : I \to \Sigma$ is a bijection if and only if the associated map $f : I \to I$ is locally pre-critical.*

Proof. Suppose that f is locally pre-critical. By the definition of Σ, the itinerary map $It : I \to \Sigma$ is onto, so let $x, y \in I$ for $x < y$. Then there is an $n \in \mathbb{N}$ for which $c_k \in f^n((x, y))$ for some k, and let n be minimal in this respect. Then for every $i < n$, $f^i((x, y))$ is strictly contained in an interval of monotonicity, so $f^i(x) \neq f^i(y)$ for all such i. Assume that for every $i < n$, $f^i(x) \neq c_j$ and $f^i(y \neq c_j)$ for any $1 \leq j \leq l$. If this is not the case then $It_i(x) \neq It_i(y)$ for the smallest such i and we are done. Thus for every $i < n$, $f^i([x, y])$ is strictly contained in an interval of monotonicity, and so the endpoints of $f^i([x, y])$ are $f^i(x)$ and $f^i(y)$ for each $i < n$. Thus c_k lies between $f^n(x)$ and $f^n(y)$ and hence $It_n(x) \neq It_n(y)$. Thus It is one-to-one.

Now suppose that f is not locally pre-critical, then there is an open interval $U \subset I$ for which $c_k \notin f^n(U)$ for any critical point c_k and any $n \geq 0$. So for every $n \geq 0$ we have that for any $u, v \in U$ such that $u < v$, $f^n([u, v])$ is strictly contained in an interval of monotonicity, and hence $It(u) = It(v)$. $\square$

Lemma 4.3.9. *If a map $f : I \to I$ is locally pre-critical then the inverse itinerary map $It^{-1} : \Sigma \to I$ is continuous.*

Proof. Pick $\mathbf{s} \in \Sigma$, where $\mathbf{s} = It(x)$ for some $x \in I$ and let $\epsilon > 0$. Then for every $N \in \mathbb{N}$, $I_N(x)$ is an interval by Lemma 4.2.3. Since the itinerary map is bijective by Lemma 4.3.8, $\cap_{N \in \mathbb{N}} I_N(x) = \{x\}$, so there is an $N^* \in \mathbb{N}$ such that for every $y \in I_{N^*}(x)$ we have $|x - y| < \epsilon$. Set $\delta = 1/2^{N^*}$, then whenever $\mathbf{t} \in \Sigma$ such that $d(\mathbf{s}, \mathbf{t}) < \delta$, $It^{-1}(\mathbf{t}) \in I_{N^*}(x)$ and so $|It^{-1}(t) - It^{-1}(\mathbf{s})| < \epsilon$. $\square$

Lemma 4.3.10. *Suppose that the l-modal map $f : I \to I$ is locally pre-critical. Then $x > y$ implies that $It(x) \succeq It(x^-) \succ It(y^+) \succeq It(y)$.*

Proof. It is clear from the definition of the limit itineraries that $It(x) \succeq It(x^-)$ and $It(y^+) \succeq It(y)$. Moreover by Proposition 4.2.2 and the fact that the itinerary map is bijective, every $z \in (y, x)$ is such that $It(x^-) \succ It(z) \succ It(y^+)$. $\square$

In Lemma 4.2.5 we showed that CINE subsets of the interval which don't contain critical points map to shift spaces under the itinerary map. Theorem 4.3.12 shows precisely when the set of itineraries of a collection of points is closed. Corollary 4.3.13 then tells us when a CINE set Λ for a locally pre-critical map has an image $It(\Lambda)$ under the itinerary map which is a shift space. If it can be shown that such a set is ICT, by Theorem 4.1.9 there is an $s \in \Omega^{\mathbb{N}}$ for which $It(\Lambda) = \omega(s, \sigma)$. This fact will be exploited in future results.

Lemma 4.3.11. *Suppose that the l-modal map $f : I \to I$ with critical points $c_1, \ldots, c_l$ is locally pre-critical, and $y \in I$ is either critical or pre-critical. Then there is no $x \in I$ for which either $It(x) = It(y^+)$ or $It(x) = It(y^-)$.*

Proof. By Lemma 4.3.8 we have that the itinerary map is a bijection. Suppose that $It(x) = It(y^+)$ (the case for $It(x) = It(y^-)$ is similar). Then certainly $x \neq y$ since $C_j \notin It(x)$ by Lemma 4.2.7(3) for any $1 \leq j \leq l$, so $x > y$ by the definition of $It(y^+)$. But then for every $z \in (y, x)$, $It(z) = It(y^+) = It(x)$ by Lemma 4.2.3, which contradicts the fact that the itinerary map is a bijection. $\square$

Theorem 4.3.12. *Suppose that the l-modal map $f : I \to I$ with critical points $c_1, \ldots, c_l$ is locally pre-critical, and $\Lambda \subset I$. $It(\Lambda)$ is closed if and* *only if Λ is closed and there is no sequence of points $\{x_n\}_{n\in\mathbb{N}} \subset \Lambda$ for which $\lim_{n\to\infty} x_n = y$ for any critical or pre-critical point y.*

Proof. Suppose that $It(\Lambda)$ is closed. It^{-1} is continuous by Lemma 4.3.9, so since $It(\Lambda)$ is compact we have that Λ is compact and thus closed. If there were a sequence $\{x_n\}_{n\in\mathbb{N}} \subset \Lambda$ as in the statement of the theorem, there would be a subsequence $\{x_{n_j}\}_{j\in\mathbb{N}}$ for which either $\lim_{j\to\infty} It(x_{n_j}) = It(y^+)$ or $\lim_{j\to\infty} x_{n_j} = It(y^-)$. Thus the sequence $\{It(x_{n_j})\}_{j\in\mathbb{N}} \subset It(\Lambda)$ has no limit in $It(\Lambda)$ by Lemma 4.3.11, contradicting the fact that $It(\Lambda)$ is closed.

Now suppose that Λ is closed and contains no such sequence $\{x_n\}_{n\in\mathbb{N}}$ as in the statement of the theorem. Suppose for a contradiction that $It(\Lambda)$ is not closed, then there must be a sequence of points $\{s_j\}_{j\in\mathbb{N}} \subset It(\Lambda)$ for which $\lim_{j\to\infty} s_j = s \notin It(\Lambda)$. Write $It^{-1}(s_j) = x_j$ for every j, then $\{x_j\}_{j\in\mathbb{N}} \subset \Lambda$ and

$$s = \lim_{j\to\infty} It(x_j) \tag{4.5}$$

Λ is compact, so there is a subsequence $\{x_{j_n}\}_{n\in\mathbb{N}}$ which has a limit $x \in \Lambda$. By assumption, x is neither a critical nor a pre-critical point, so the itinerary map is continuous at x. If $\{x_j\}_{j\in\mathbb{N}}$ had a limit this would have to be x also, but then

$$s = \lim_{j\to\infty} It(x_j) = It\left(\lim_{j\to\infty} x_j\right) = It(x) \in It(\Lambda),$$

which is a contradiction. Thus there must be another subsequence $\{x_{j_m}\}_{m\in\mathbb{N}}$ which has a limit $z \neq x$, where $z \in \Lambda$.

Both $It(x)$ and $It(z)$ are in $It(\Lambda)$, and since f is locally pre-critical,

$It(x) \neq It(z)$; let $d(It(x), It(z)) = \epsilon$. The itinerary map is continuous at both x and z, so there are values $\delta_x, \delta_z > 0$ such that whenever $u \in B_{\delta_x}(x)$ we have that $It(u) \in B_{\epsilon/4}(It(x))$ and whenever $v \in B_{\delta_z}(z)$ we have that $It(v) \in B_{\epsilon/4}(It(z))$. Since x and z are limits of their corresponding sequences, there are integers $M, N > 0$ such that for every $n \geq N$, $x_{j_n} \in B_{\delta_x}(x)$ so $It(x_{j_n}) \in B_{\epsilon/4}(It(x))$ (and similarly for every $m \geq M$, $x_{j_m} \in B_{\delta_z}(z)$ so $It(x_{j_m}) \in B_{\epsilon/4}(It(z))$).

By (4.5), there is a $p \in \mathbb{N}$ such that for every $j \geq p$, $It(x_j) \in B_{\epsilon/4}(s)$. But infinitely many of these will be in $B_{\epsilon/4}(It(x))$ and infinitely many will be in $B_{\epsilon/4}(It(z))$. This is impossible by the definition of ϵ, so no such s exists and we conclude that $It(\Lambda)$ is closed. $\square$

Corollary 4.3.13. *Suppose that the l-modal map $f : I \to I$ with critical points $c_1, \ldots, c_l$ is locally pre-critical, and $\Lambda \subset I$. $It(\Lambda)$ is a shift space if and only if Λ is CINE and there is no sequence of points $\{x_n\}_{n \in \mathbb{N}} \subset \Lambda$ for which $\lim_{n\to\infty} x_n = y$ for any critical or pre-critical point y.*

Proof. Suppose that $It(\Lambda)$ is a shift space, in other words is CINE. Then Λ is closed by Theorem 4.3.12 and there is no such sequence $\{x_n\}$ as stated. Also since $It(\Lambda)$ is invariant we have

$$(It \circ f)(\Lambda) = (\sigma \circ It)(\Lambda) \subset It(\Lambda),$$

thus $f(\Lambda) \subset \Lambda$ since It is bijective by Lemma 4.3.8. Thus Λ is CINE.

Now suppose that Λ is CINE and there is no sequence of points $\{x_n\}_{n \in \mathbb{N}} \subset \Lambda$ for which $\lim_{n\to\infty} x_n = y$ for any critical or pre-critical point y. By Theorem 4.3.12 $It(\Lambda)$ is also closed, and since Λ is invariant,

$$(\sigma \circ It)(\Lambda) = (It \circ f)(\Lambda) \subset It(\Lambda),$$

so $It(\Lambda)$ is invariant. Thus $It(\Lambda)$ is a shift space. $\square$

4.4 Symbolic Dynamics and ω-Limit Sets

In [7], we show that for $X_{\mathcal{F}}$ a shift of finite type, closed, invariant, non-empty (CINE) subsets of $X_{\mathcal{F}}$ are internally chain transitive (ICT) if and only if they are the ω-limit set of some point $\mathbf{s} \in X_{\mathcal{F}}$. In other words, internal chain transitivity completely characterizes ω-limit sets of shifts of finite type. Example 1.2.1 tells us that this is not the case for maps of the interval in general, where the set H from that example is ICT but not an ω-limit set for the map g. However in [7], we show that for certain subsets of a compact interval this is the case for tent maps. The proof of this relies heavily upon the kneading theory, and in this section, which is original work unless stated otherwise, we generalize the result to l-modal maps. (It will be assumed that f is an l-modal map on the interval $I = [0, 1]$ with critical points $c_1, \ldots, c_l$, and that Σ is the set of itineraries of points in I.)

In order to use symbolic dynamics to analyze the dynamics of maps on their ω-limit sets, we would like to have an analogue of Lemma 4.1.5 to enable us to tell when a point $y \in I$ is in the ω-limit set of a point x, but it is clear that when moving between the interval and the sequence space, problems occur when a critical point is in the forward orbit of a point under consideration. Ensuring that the map is locally pre-critical allows us to avoid this problem, as is shown in Proposition 4.4.2, which requires the following Lemma.

Lemma 4.4.1. *Suppose that the l-modal map $f : I \to I$ with critical points $c_1, \ldots, c_l$ is locally pre-critical, and let $y \in I$. For every $\epsilon > 0$ there is a positive integer $N = N(\epsilon, y)$ such that for every $x \in I$ which satisfies $It(x)\restriction_N = It(y^+)\restriction_N$ or $It(x)\restriction_N = It(y^-)\restriction_N$ we have $x \in (y - \epsilon, y + \epsilon)$.*

Proof. Case 1: suppose that there is no $k \in \mathbb{N}$ such that $f^k(y) = c_j$ for any $1 \leq j \leq l$. Thus $It(y^+) = It(y) = It(y^-)$ and the itinerary map is continuous at y by Lemma 4.2.4. Assume the lemma is false; i.e. that there is an $\epsilon > 0$ such that for every $n \in \mathbb{N}$ there is a point $z_n \in I$ such that $It(z_n)$ and $It(y)$ agree up to their first n places, but $z_n \notin (y - \epsilon, y + \epsilon)$. Certainly $\lim_{n\to\infty} It(z_n) = It(y)$. Since I is compact, $\lim_{n\to\infty} z_n$ exists, and since It is continuous at y we have $It(\lim_{n\to\infty} z_n) = It(y)$. Because the itinerary map is bijective we get that $\lim_{n\to\infty} z_n = y$, which contradicts our assumption that the lemma is false.

Case 2: suppose that there is a $k \in \mathbb{N}$ for which $f^k(y) = c_j$ for some $1 \leq j \leq l$, and k is minimal in this respect. Thus by Lemma 4.2.7, the discrepancy between $It(y)$ and $It(y^+)$ is k, as it is between $It(y)$ and $It(y^-)$. Assume that the lemma is false for $It(y^+)$; i.e. that there is an $\epsilon > 0$ such that for every $n \in \mathbb{N}$ there is a point $z_n \in I$ such that $It(z_n)$ and $It(y^+)$ agree up to their first n places, but $z_n \notin (y - \epsilon, y + \epsilon)$ (the proof is analogous for $It(y^-)$). Pick $z \in (y, y+\epsilon)$, then since f is locally pre-critical, $It(z) \succ It(y^+)$ by Lemma 4.3.10; let their discrepancy be m. Let $m^* = \max\{m, k\}$, then by assumption there is a $z_{m^*} \in I$ such that $It(z_{m^*}) \restriction_{m^*} = It(y^+) \restriction_{m^*}$ and $|z_{m^*} - y| \geq \epsilon$. We must have that $z_{m^*} > y$ by the agreement of $It(z_{m^*})$ with $It(y^+)$ past the discrepancy with $It(y)$, hence $z_{m^*} > y + \epsilon > z$. So we have $y < z < z_{m^*}$, with

$$It(z)\restriction_{m^*} \succ It(y^+)\restriction_{m^*} = It(z_{m^*})\restriction_{m^*},$$

which contradicts Lemma 4.2.2. □

The next proposition exploits symbolic dynamics to give conditions under which one point is in the ω-limit set of another.

Proposition 4.4.2. *Suppose that the l-modal map $f : I \to I$ is locally pre-critical. For $x, y \in I$, either*

1. *x is periodic or pre-periodic, in which case $y \in \omega(x, f)$ if and only if arbitrarily long initial segments of $It(y)$ occur infinitely often in $It(x)$, or*

2. *x is neither periodic nor pre-periodic, in which case $y \in \omega(x, f)$ if and only if arbitrarily long initial segments of $It(y^+)$ or $It(y^-)$ (or possibly both) occur infinitely often in $It(x)$.*

Proof. Consider case 1. If x is periodic or pre-periodic, $\omega(x, f)$ is a subset of $orb(x, f)$ and is a cycle. So if $y \in \omega(x, f)$ then y occurs infinitely often in $orb(x, f)$ and thus arbitrarily long initial segments of $It(y)$ occur infinitely often in $It(x)$. Similarly, if arbitrarily long initial segments of $It(y)$ occur infinitely often in $It(x)$, then y must be in the periodic part of $orb(x, f)$. In other words, $y \in \omega(x, f)$.

For case 2, let $y \in \omega(x, f)$. Then since x is neither periodic nor pre-periodic, $y \in orb(x, f)$ at most once. Pick $n \in \mathbb{N}$; we want to show that either $It(y^+)\restriction_n$ or $It(y^-)\restriction_n$ occurs infinitely often in $It(x)$. By Lemma 4.2.7, for any $i \in \mathbb{N}$ there is a $\delta_i > 0$ such that $It_i(x)$ is constant on $(y, y + \delta_i)$ and $(y - \delta_i, y)$. Let $\delta = \min\{\delta_i \ : \ i \leq n\}$ (so $\delta > 0$), then $It(z)\restriction_n = It(y^+)\restriction_n$ for all $z \in (y, y + \delta)$ and $It(z)\restriction_n = It(y^-)\restriction_n$ for all $z \in (y - \delta, y)$. Since $y \in \omega(x, f)$, there are infinitely many $k \in \mathbb{N}$ for which $|f^k(x) - y| < \delta$, and since $y \in orb(x, f)$ at most once there are infinitely many k for which either $f^k(x) \in (y, y + \delta)$ or $f^k(x) \in (y - \delta, y)$. Thus for these infinitely many k, either $It(f^k(x))\restriction_n = It(y^+)\restriction_n$ or $It(f^k(x))\restriction_n = It(y^-)\restriction_n$.

Suppose now that for $x, y \in I$, arbitrarily long initial segments of $It(y^+)$ occur infinitely often in $It(x)$ (the proof for $It(y^-)$ is analogous), and let $\epsilon > 0$. By Lemma 4.4.1 there is an $N(\epsilon) \in \mathbb{N}$ such that for every $z \in I$ which satisfies $It(z)\restriction_{N(\epsilon)} = It(y^+)\restriction_{N(\epsilon)}$ we have $z \in (y - \epsilon, y + \epsilon)$. By assumption there is a $k \in \mathbb{N}$ for which $\sigma^k(It(x))\restriction_{N(\epsilon)} = It(y^+)\restriction_{N(\epsilon)}$; so $It(f^k(x))\restriction_{N(\epsilon)} = It(y^+)\restriction_{N(\epsilon)}$ and thus $|f^k(x) - y| < \epsilon$. Since ϵ was arbitrary, $y \in \omega(x, f)$. $\square$

We now return to Example 1.2.1 and show that the set H_1 is an ω-limit set for the map f_2 (and hence for g since the dynamics are disjoint about 0). Since the dynamics are identical on $[-1, 0]$ and disjoint from those on $[0, 1]$, we also get that H_2 is an ω-limit set for the map $-f_2$.

Example 4.4.3 (Using symbolic dynamics to identify an ω-limit set)**.** Recall the map f_2, the tent map as described in Example 1.1.9 with constant gradient $|s| = 2$ (the upper half of the map in Figure 4.3), which is locally pre-critical.

Consider the set

$$H_1 = \{0\} \cup \bigcup_{n=0}^{\infty} \left\{ \frac{1}{2^n} \right\}.$$

The symbolic dynamics can be defined over the set $\Omega = \{0, 1, C\}$ where $A(x) = 0$ if $x \in [0, 1/2)$, $A(x) = C$ if $x = 1/2$ and $A(x) = 1$ if $x \in (1/2, 1]$.

For a point $x = 1/(2^{i+1}) \in H_1$ for any $i \geq 0$, notice that x is mapped to $1/(2^i)$, then $1/(2^{i-1})$ and so on until it is mapped onto 1, then the fixed point 0. Hence for such an x we have $It(x) = (0^i C 10^\infty)$ and $It(x^+) = (0^i 110^\infty)$, where for example 10^i means 1 followed by i repetitions of a 0. The only points in H_1 not of this form are 1 which has itinerary (10^∞) and 0 which has itinerary (0^∞). Moreover, this map has kneading invariant $K = It(f(c)) = (10^\infty)$.

Consider the sequence $\mathbf{s} = (0110^2 110^3 110^4 11 \ldots)$, which is the itinerary of a point $y_0 \in [0, 1]$ by Theorem 4.2.13. Notice that for any $x \in H_1$, arbitrarily long segments of $It(x^+)$ occur infinitely often in $It(y_0) = \mathbf{s}$. Hence by Proposition 4.4.2 $H_1 \subseteq \omega(y_0, f_2)$. Now suppose that $x \notin H_1$. Then $It(x^+)$ and $It(x^-)$ will always contain a word of the form $10^n 1$, for some $n \in \mathbb{N}$, or 111, neither of which appears infinitely often in $It(y_0)$. So by Proposition 4.4.2 $x \notin \omega(y_0, f_2)$ for any such x. Hence $H_1 = \omega(y_0, f_2)$.

$\square$

So symbolic dynamics is a powerful tool for establishing whether a given set is the ω-limit set of some point in the interval. In the case of Example 4.4.3, it was relatively easy to write down the itinerary of a point whose ω-limit set would be the set H_1, since we knew exactly which points we needed

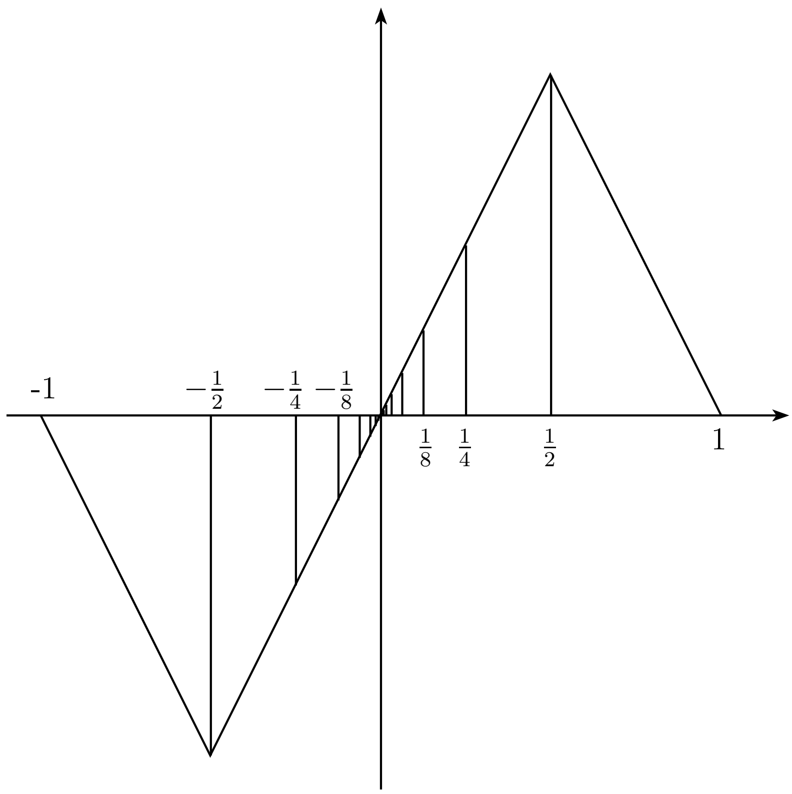

Figure 4.3: The interval $[0, 1]$ is invariant under f_2; the upper half of the map.

to include. However to develop a more general theory we need a better idea of how the itinerary map affects the structure of sets in the interval.

Theorem 4.4.4. *Suppose that $f : I \to I$ is a locally pre-critical l-modal map with critical points $c_1, \ldots, c_l$ and that $\Lambda \subset I$ is a set which does not contain c_k for any $1 \le k \le l$ and for which $f(\Lambda) \subset \Lambda$. Then $It\restriction_\Lambda$ is a homeomorphism.*

Proof. Since $c_k \notin \Lambda$ for any $1 \le k \le l$ and $f(\Lambda) \subset \Lambda$, no $x \in \Lambda$ is pre-critical, so It is continuous on Λ by Lemma 4.2.4. $It\restriction_\Lambda$ is bijective by Lemma 4.3.8, and It^{-1} is continuous by Lemma 4.3.9. $\square$

Lemma 4.4.5. *Suppose that $f : I \to I$ is locally pre-critical, $\Lambda \subset I$ is closed* *and $c_k \notin \Lambda$ for any $1 \leq k \leq l$. Then Λ is ICT if and only if $It(\Lambda)$ is ICT.*

Proof. Suppose that the closed set Λ is ICT. By Proposition 3.2.5, Λ is invariant, so no point $x \in \Lambda$ is pre-critical. To show that $It(\Lambda)$ is ICT, pick $r, s \in It(\Lambda)$, where $r = It(y)$ and $s = It(x)$ for some $x, y \in \Lambda$, and let $\epsilon > 0$. We need a sequence $\{s = s_0, s_1, \dots, s_n = r\} \subset It(\Lambda)$ for which $d(\sigma(s_{i-1}), s_i) < \epsilon$ for every $1 \leq i \leq n$. Now It is continuous on Λ by Lemma 4.2.4, so is also uniformly continuous since Λ is compact. So there is a $\delta > 0$ such that for every $x, y \in \Lambda$ for which $|x - y| < \delta$ we have $d(It(x), It(y)) < \epsilon$. Since Λ is ICT there exist $x_0 = x, x_1, \dots, x_n = y$ for which $|f(x_{i-1}) - x_i| < \delta$ for every $1 \leq i \leq n$. Hence $d\big(It(f(x_{i-1}), It(x_i)\big) < \epsilon$. Thus, setting $s_i = It(x_i)$ and noting that by conjugation $It(f(x_{i-1})) = \sigma(It(x_{i-1}))$, we get that $d(\sigma(s_{i-1}), s_i) < \epsilon$ for every $1 \leq i \leq n$.

Now suppose that $It(\Lambda)$ is ICT. Since by Theorem 4.4.4, $It^{-1} : It(\Lambda) \to \Lambda$ is a homeomorphism we have that $It(\Lambda)$ is closed, so is compact since the full shift is compact [30]. Thus we can show that Λ is ICT using an identical (but symmetric) argument to that given above. ☐

Proposition 4.4.6. *Suppose that $f : I \to I$ is a locally pre-critical, l-modal map with critical points $c_1, \dots, c_l$, and that $L \subset I$ is closed, invariant, non-empty and does not contain c_j for any $1 \leq j \leq l$. Suppose also that for $x_0 \in I$, $\{f^n(x_0) \ : \ n \in \mathbb{N}\}$ is bounded away from c_j for each $1 \leq j \leq l$. Then $It(L) = \omega(It(x_0), \sigma)$ if and only if $L = \omega(x_0, f)$.*

Proof. Since $\{f^n(x_0) \ : \ n \in \mathbb{N}\}$ is bounded away from c_j for $1 \leq j \leq l$, It is a homeomorphism on $\overline{\{f^n(x_0) \ : \ n \in \mathbb{N}\}}$ by Theorem 4.4.4. So

$$\begin{aligned}
\omega(It(x_0), \sigma) &= \bigcap_{n \geq 0} \overline{\{\sigma^k(It(x_0)) \ : \ k > n\}} \\
&= \bigcap_{n \geq 0} \overline{\{It(f^k(x_0)) \ : \ k > n\}} \\
&= \bigcap_{n \geq 0} It\left(\overline{\{f^k(x_0) \ : \ k > n\}}\right) \ \text{(since } It \text{ is continuous)} \\
&= It\left(\bigcap_{n \geq 0} \overline{\{f^k(x_0) \ : \ k > n\}}\right) \ \text{(since } It \text{ is injective)} \\
&= It(\omega(x_0, f))
\end{aligned}$$

Suppose that $It(L) = \omega(It(x_0), \sigma)$, then by the above calculation, $It(L) = It(\omega(x_0, f))$. It is a homeomorphism on L by Theorem 4.4.4, so certainly it is injective, hence we must have that $L = \omega(x_0, f)$.

Now suppose that $L = \omega(x_0, f)$, then clearly $It(L) = It(\omega(x_0, f))$; i.e. $It(L) = \omega(It(x_0), \sigma)$, again by the above calculation. □

By ensuring that a CINE set Λ contains the images of none of the critical points, we get some interesting structure in the associated space of itineraries, as was shown in Lemma 4.2.5 – a result extended by the following theorem.

Theorem 4.4.7. *Suppose that $f : I \to I$ is a locally pre-critical, l-modal map with critical points $c_1, \ldots, c_l$, and that $\Lambda \subset I$ is a CINE set which does not contain $f(c_j)$ for any $1 \leq j \leq l$.*

Then $It(\Lambda)$ is a shift space which is a subset of a shift of finite type $X_{\mathcal{F}}$ for some finite but non-empty set of words $\mathcal{F}$.

Proof. Λ is CINE so $It(\Lambda)$ is CINE by Theorem 4.3.12. Moreover, $c_j \notin \Lambda$ for

$1 \leq j \leq l$ since $f(c_j) \notin \Lambda$ and Λ is invariant, thus $It(\Lambda)$ is a shift space by Lemma 4.2.5.

Since $f(c_j) \notin \Lambda$ for $1 \leq j \leq l$ and Λ is closed, we must have that Λ is in fact bounded away from each $f(c_j)$. The inverse itinerary map is continuous by Lemma 4.3.9, so is uniformly continuous and $It(\Lambda)$ must therefore be bounded away from each K_j. In particular for every $1 \leq j \leq l$ there is an $n_j \in \mathbb{N}$ such that $d(K_j, \mathbf{s}) \geq 1/2^{n_j}$ for every $\mathbf{s} \in It(\Lambda)$. Hence the discrepancy between $\mathbf{s}$ and K_j is at most n_j for every $\mathbf{s} \in It(\Lambda)$. Set

$$\mathcal{F} = \{K_j\restriction_{n_j} : \ 1 \leq j \leq l\}.$$

We claim that $It(\Lambda) \subset X_{\mathcal{F}}$, where $X = \Omega^{\mathbb{N}}$ for $\Omega = \{I_1 \ldots I_{l+1}, C_1, \ldots, C_l\}$ as defined above. Indeed, for any $\mathbf{s} \in It(\Lambda)$, no initial segment of $\mathbf{s}$ can be a word from $\mathcal{F}$ by the discrepancy between $\mathbf{s}$ and K_j as noted above. So suppose that there is some $\mathbf{s} \in It(\Lambda)$ and some $F \in \mathcal{F}$ such that $\mathbf{s}$ contains the word F; i.e. F is the initial segment of $\sigma^k(\mathbf{s})$ for some $k \in \mathbb{N}$. But $It(\Lambda)$ is invariant, so $\sigma^k(\mathbf{s}) = \mathbf{t} \in \Lambda'$ has F as its initial segment, which we have said can't happen. Thus $It(\Lambda) \subset X_{\mathcal{F}}$. $\square$

The main result of this chapter is Theorem 4.4.8, which characterizes those ω-limit sets of locally pre-critical, piecewise monotone interval maps which do not contain the image of any critical point. Such ω-limit sets are well-studied, and include the post-critical ω-limit sets of tent maps whose critical point is non-recurrent (see [21], [22], [41] for examples).

Theorem 4.4.8. *Suppose that $f : I \to I$ is a locally pre-critical, l-modal map with critical points $c_1, \ldots, c_l$, and that $\Lambda \subset I$ is a closed, non-empty set* *which does not contain $f(c_j)$ for any $1 \leq j \leq l$.*

Then Λ is ICT if and only if $\Lambda = \omega(x_0, f)$ for some $x_0 \in I$.

Proof. Suppose that $\Lambda = \omega(x_0, f)$, then Λ is ICT by Lemma 3.2.7. Now suppose that $\Lambda \subset I$ is a non-empty, closed set which does not contain the image of any critical point of f, and is ICT. Thus Λ is invariant by Proposition 3.2.5, and so contains no critical or pre-critical points. Furthermore, since Λ is closed, there is no sequence of points in Λ which converges to a critical or pre-critical point. Let $\Gamma = It(\Lambda)$. Then by Lemmas 4.2.5 and 4.4.5, Γ is closed, invariant, non-empty and ICT. Since $f(c_j) \notin \Lambda$ for any $1 \leq j \leq l$ and Λ is closed it must be bounded away from each $f(c_j)$, so by uniform continuity of It^{-1}, Γ must be bounded away from each K_j. Thus for every $1 \leq j \leq l$ there is an $N_j \in \mathbb{N}$ such that the discrepancy between t and K_j is less than N_j for every $t \in \Gamma$; let $N = \max\{N_j \ : \ 1 \leq j \leq l\}$. Now construct the arbitrary length, infinite repetition sequence $s = s(\Gamma, N)$. By Lemma 4.1.8, $\Gamma = \omega(s, \sigma)$.

We want to have that $s = It(x_0)$ for some $x_0 \in [0, 1]$, so we show that the admissibility conditions (4.1) to (4.3) in Definition 4.2.12 are satisfied by s. By the construction of s, for any $k \in \mathbb{N}$, $\sigma^k(s)$ begins with at least the first N symbols of some $t \in \Gamma$, within which we can see a discrepancy with every kneading invariant K_j, as noted above. Since each such t is the itinerary of some point in Λ, this discrepancy tells us that $\sigma^k(s)$ satisfies conditions (4.1) to (4.3). Thus by Theorem 4.2.13, there is an $x_0 \in I$ for which $s = It(x_0)$. Hence $\Gamma = \omega(It(x_0), \sigma)$, where $f^k(x_0)$ is bounded away from c_j for $1 \leq j \leq l$. So by Proposition 4.4.6 we have that $\Lambda = \omega(x_0, f)$ as required. $\square$

Remark 4.4.9. It may appear that the (implied) condition of Λ not containing any critical point in Theorem 4.4.8 is simply an artifact of using symbolic dynamics. However we need only look as far as Examples 1.2.1 and 4.4.3 to see that not every CINE set which is ICT is an ω-limit set. Indeed the set $H = H_1 \cup H_2$ which contains both critical points (and their images) is ICT but is not an ω-limit set for g. We require that Λ does not contain the *image* of any critical point in order to ensure that the admissibility conditions are met when constructing the arbitrary length, infinite repetition sequence in the proof of the theorem, however it is not known whether this condition is actually required for the theorem to hold.

In Chapter 5 we identify a strong form of a property known as *shadowing*, which is a sufficient (but not necessary) condition of ICT sets to be ω-limit sets and which depends on the behaviour of the critical point of a map, implying that a property of greater subtlety than ICT is required to characterize ω-limit sets which contain critical points. In particular, we show that by omitting the critical points of a map from a set, the map has this strong form of shadowing on such sets.

Chapter 5

Pseudo-Orbit Shadowing

The idea of a pseudo-orbit is one with many practical considerations. Indeed when we study orbits of points in an abstract setting, such as within an arbitrary topological space, we are assuming implicitly that we are able to calculate the *position*, or *value* of each point in the space to an infinite degree of accuracy. In practice of course this is not possible, and whether we are calculating the positions of protons in a particle accelerator or the movement of a low pressure system in the earth's atmosphere, small errors will inevitably enter our calculations and distort our findings and predictions. This is due to unavoidable rounding errors inherent in the calculating device we are using; be it a hand-held calculator or a powerful supercomputer, the value of a point in a system (and thus all iterates thereof) can only be found to a finite number of decimal places (assuming the use of base-10 arithmetic). What we have in fact is a δ-pseudo-orbit (see Chapter 3), where δ can be treated as the rounding error. Hence we should like to know that for each such pseudo-orbit there exists a real orbit in the system which approximates the pseudo-orbit to a degree of accuracy we can control; if this is the case we can conclude that our calculations, based upon the pseudo-orbit, contain a bounded level of inaccuracy. This property is known as *shadowing*, or *pseudo-orbit tracing*.

In [11], Bowen uses shadowing to study ω-limit sets of a certain class of diffeomorphism, which is where the definition seems to have originated. However since then many different versions of shadowing have been introduced to describe a similar type of behaviour in different classes of maps [28, 40].

We saw in Theorem 4.1.9 that for shifts of finite type, ω-limit sets are precisely the ICT subsets of the shift space, whilst in [52], Walters proves that a subshift has the shadowing property if and only if it is of finite type. Thus shadowing would appear to be a useful and important property for us to consider in our pursuit of understanding ω-limit sets in terms of internal chain transitivity. In the first half of this chapter we investigate the links between the various versions of shadowing, prove implications where they exist and give examples where appropriate.

In [18], Coven et al. identify a condition on tent maps which implies shadowing, which we state here without proof, and which relies upon the nature of the critical point of the map. They also give an expansivity condition for general maps which implies shadowing. In the second half of this chapter we investigate further how expansivity properties give rise to shadowing, basing one of our properties upon that given by Coven et al. and extending their result to show that maps possess a stronger form of shadowing in this case (Theorem 5.2.22).

Much of the material in this chapter is the result of work in collaboration with Chris Good, Piotr Oprocha and Brian Raines [8]. As in previous chapters, work is original unless stated otherwise, and we will indicate where a result or example is from [8].

Throughout this chapter, (X, f) is a dynamical system unless stated otherwise.

5.1 The Shadowing Property

Recall that a δ-pseudo-orbit is a sequence of points $\{x_0, x_1, \ldots\} \subset X$ such that $d(f(x_i), x_{i+1}) < \delta$ for $i \geq 0$.

Definition 5.1.1 (Shadowing/Pseudo-Orbit Tracing)**.** Let the set K be either $\{0\} \cup \mathbb{N}$ or $\{0, 1, \ldots, k-1\}$ for some $k \in \mathbb{N}$, and let $\epsilon > 0$ be a real number. The sequence $\{y_n\}_{n \in K}$ is said to *ϵ-shadow* the sequence $\{x_n\}_{n \in K}$ if for every $n \in K$ we have that $d(y_n, x_n) < \epsilon$, and is said to *asymptotically shadow* the sequence $\{x_n\}_{n \in \mathbb{N}}$ if $\lim_{n \to \infty} d(y_n, x_n) = 0$. If both properties hold we say that $\{y_n\}_{n \in \mathbb{N}}$ *asymptotically ϵ-shadows* $\{x_n\}_{n \in \mathbb{N}}$.

The map f is said to have the *pseudo-orbit tracing property* (or *shadowing*) if for every $\epsilon > 0$ there is a $\delta > 0$ such that for every δ-pseudo-orbit $\{x_0, x_1, \ldots\} \subset X$ there is a point $x \in X$ whose orbit $\{f^n(x) : n \in \mathbb{N}\}$ ϵ-shadows the pseudo-orbit.

Notice that the definition does not tell us the value of δ in relation to ϵ, so from a practical point of view it offers little help, except to say that the map is somehow "well-behaved" on pseudo-orbits. However from a mathematical point of view, we shall see that this property (and others like it) allow us to discern a great deal about the behaviour of "true" orbits near pseudo-orbits, and from this make further statements about the link between internal chain transitivity and ω-limit sets.

We begin with a number of results which will help us determine when a map has the shadowing property. The following fact is well-known.

Lemma 5.1.2. *A map $f : X \to X$ has shadowing if and only if for every $\epsilon > 0$ there is a $\delta > 0$ such that every finite δ-pseudo-orbit is ϵ-shadowed.*

Proof. Clearly if f has shadowing every finite pseudo-orbit is shadowed, so assume conversely that for every $\epsilon > 0$ there is a $\delta > 0$ such that every finite δ-pseudo-orbit is ϵ-shadowed. Let $\epsilon > 0$, let δ be the constant given by the statement for $\epsilon/2$, and let $\{x_n\}_{n\in\mathbb{N}}$ be an infinite δ-pseudo-orbit. For each $n \in \mathbb{N}$ there is a $y_n \in X$ which $\epsilon/2$-shadows $\{x_1, \dots, x_n\}$. Some subsequence $\{y_{n_k}\}_{k\in\mathbb{N}}$ has a limit $y \in X$, so for any $m \in \mathbb{N}$, there is a $n_k > m$, such that $d(f^m(y_{n_k}), f^m(y)) < \epsilon/2$. Then

$$\begin{aligned} d(f^m(y), x_m) &\leq d(f^m(y), f^m(y_{n_k})) + d(f^m(y_{n_k}), x_m) \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon. \end{aligned}$$

□

Results 5.1.3 and 5.1.4 are due to Coven, Kan and Yorke [18]. We provide no proof of either result here, but in Theorem 5.2.22 we show that the condition in Lemma 5.1.3, when satisfied, implies a stronger form of shadowing which implies the form given in Definition 5.1.1.

Lemma 5.1.3. *A map $f : X \to X$ has shadowing if for every $\epsilon > 0$ there is a $\delta > 0$ such that for every $x \in X$,*

$$\overline{B_{\epsilon+\delta}(f(x))} \subseteq f\left(\overline{B_\epsilon(x)}\right).$$

So provided that a slightly expanded neighbourhood of the image of a point is contained in the image of the neighbourhood of that point, the map f has shadowing on X. This is really an expansive condition, albeit different from locally eventually onto (which we know is satisfied by the tent map f_λ for $\lambda \in (\sqrt{2}, 2]$, see Example 1.1.9 and [14]). Indeed Coven et al. remark that while the condition of Lemma 5.1.3 is satisfied by the tent map for $\lambda = 2$, it is not satisfied by the tent map for any value of $\lambda \in (\sqrt{2}, 2)$ since it fails at the critical point. Lemma 5.1.4 addresses this issue, where the kneading invariant is given by $K = (K_0 K_1 \ldots)$, $c = 1/2$ is the critical point of the map, and the symbolic dynamics for the tent map are exactly as defined in Example 4.4.3.

Lemma 5.1.4. *For $\lambda \neq 2$, the tent map f_λ has the shadowing property if and only if for every $\epsilon > 0$ there is a positive integer n such that $|f_\lambda^n(c) - c| \leq \epsilon$, and either $K_n = C$ or $K\restriction_{n-1}$ is even if $K_n = 0$ and odd if $K_n = 1$.*

In other words the tent map f_λ with slope λ has the shadowing property if and only if the orbit of $f_\lambda(c)$ returns to within arbitrarily small neighbourhoods of c, and such that when it returns after n iterations, we have that $(K_0 \ldots K_{n-1} K_n) \preceq (K_0 \ldots K_{n-1} C)$.

The following theorem is due to Walters [52], and shows that shadowing is inherent in a class of subshift (see Definition 4.1.2).

Theorem 5.1.5. *A subshift has shadowing if and only if it is of finite type.*

In analyzing this result, we actually see that the same is true of a strictly stronger form of shadowing. This is the motivation for our next definition, which is new in the literature.

Definition 5.1.6 (Shadowing with Direct Hit/h-Shadowing)**.** For a subset $Y \subset X$ we say that the map $f : X \to X$ has *shadowing with direct hit on* Y (or simply *h-shadowing on* Y) if and only if for every $\epsilon > 0$ there is a $\delta > 0$ such that for every finite δ-pseudo-orbit $\{x_0, x_1, \ldots, x_m\} \subseteq Y$ there is $y \in X$ such that $d(f^i(y), x_i) < \epsilon$ for every $i < m$ and $f^m(y) = x_m$.

If $Y = X$ then we say simply that f has *h-shadowing*.

Lemma 5.1.2 shows that a map with h-shadowing also has shadowing; we will see below that the converse is not the case. First we relate h-shadowing to shift spaces.

Theorem 5.1.7. *A subshift has h-shadowing if and only if it is of finite type.*

Proof. Suppose that a subshift $\sigma : X_{\mathcal{F}} \to X_{\mathcal{F}}$ has h-shadowing, then by Lemma 5.1.2 it is clear that σ has shadowing, so by Theorem 5.1.5 it is a shift of finite type.

Now suppose that $\sigma : X_{\mathcal{F}} \to X_{\mathcal{F}}$ is a shift of finite type and let $\epsilon > 0$. Suppose that the maximum number of symbols in any word in $\mathcal{F}$ is M, and pick $k \in \mathbb{N}$ such that $k > M$ and $1/2^k \leq \epsilon$; let $\delta = 1/2^k$. Suppose that $\{s_0, s_1, \ldots, s_n\} \subset X_{\mathcal{F}}$ is a δ-pseudo-orbit, where $s_i = (s_i^0 s_i^1 s_i^2 \ldots)$, then by Lemma 4.1.6, the first k symbols of $\sigma(s_i)$ agree with those of s_{i+1}, for $i = 0, 1, \ldots, n-1$. Let $t = (t^0 t^1 \ldots)$ be the element formed by setting t^i to be the first symbol of s_i, for $i = 0, 1, \ldots, n-1$, and setting $(t^n t^{n+1} \ldots) = s_n$. Then $\sigma^i(t) = (s_i^0 s_{i+1}^0 s_{i+2}^0 \ldots)$ for $i < n$. Moreover for $j \leq \min\{n-i, k\}$, $s_{i+j}^0 = s_i^j$, so $\sigma^i(t) = (s_i^0 s_i^1 s_i^2 \ldots s_i^{k-1} \ldots)$ for $i < n$ and thus $d(\sigma^i(t), s_i) \leq 1/2^{k+1} < \epsilon$. Also by construction, $\sigma^n(t) = s_n$, so provided $t \in X_{\mathcal{F}}$ we have that σ has h-shadowing.

Suppose that $t \notin X_{\mathcal{F}}$, then for some $i \in \mathbb{N}$, $\sigma^i(t)$ begins with some word $F \in \mathcal{F}$. Thus by the construction of t and since $k > M$, we are forced to conclude that either s_i begins with F for some $i < n$, or $\sigma^j(s_n)$ begins with F for $j \geq 0$, both of which are contradictions so we are done. □

We stay with shift spaces for Proposition 5.1.8, which uses symbolic dynamics to show that an l-modal map $f : I \to I$ has h-shadowing on a CINE subset of a compact interval I provided the set does not contain the image of any critical point. This is precisely the hypothesis of Theorems 4.4.7 and 4.4.8, and demonstrates the connection between shadowing and our use of symbolic dynamics in Theorem 4.4.8. This connection will be explored further in the next chapter.

Proposition 5.1.8. *Suppose that $f : I \to I$ is a locally pre-critical l-modal map with critical points $c_1, \ldots, c_l$, and that $\Lambda \subset I$ is CINE. If $f(c_k) \notin \Lambda$ for any $1 \leq k \leq l$ then f has h-shadowing on Λ.*

Proof. Fix $\epsilon > 0$. The inverse itinerary map It^{-1} is continuous by Theorem 4.4.4, thus also uniformly continuous since $It(\Lambda)$ is closed by Theorem 4.3.12 and thus compact, so there is an $\eta_\epsilon > 0$ for which $|x - y| < \epsilon$ whenever $d(It(x), It(y)) < \eta_\epsilon$ for every $x, y \in \Lambda$. Similarly, the itinerary map It is uniformly continuous on the compact set Λ by Theorem 4.4.4, so for every

$\nu > 0$ there is a $\delta_\nu > 0$ such that $d(It(x), It(y)) < \nu$ whenever $|x - y| < \delta_\nu$. Since $f(c_k) \notin \Lambda$ for any $1 \leq k \leq l$ and $It(\Lambda)$ is closed it is bounded away from the kneading invariants K_k for every $1 \leq k \leq l$. So there is a $\chi \in \mathbb{N}$ such that $It(x)\restriction_\chi$ satisfies admissibility conditions (4.1) to (4.3) for every $x \in \Lambda$. Now set $\nu := \min\{1/2^\chi, \eta_\epsilon/2\}$, then there is an $m \in \mathbb{N}$ such that $\nu \leq 1/2^m < \eta_\epsilon$ and $1/2^m \leq 1/2^\chi$.

Consider a δ_ν-pseudo-orbit $\{x_0, x_1, \ldots, x_r\} \subset \Lambda$, which gives rise to a ν-pseudo-orbit $\{s_0, s_1, \ldots, s_r\} \subset It(\Lambda)$ by Lemma 4.4.5. Then by the metric on $\Omega^\mathbb{N}$ and the definition of ν, the first m symbols in $\sigma(s_i)$ correspond to the first m symbols in s_{i+1}, and in particular the first χ symbols in $\sigma(s_i)$ correspond to the first χ symbols in s_{i+1}. Let $\mathbf{s}$ be a sequence constructed from taking first element of s_i for $0 \leq i \leq r-1$ in turn and appending this with s_r as the tail of $\mathbf{s}$. We claim that $\mathbf{s} = It(z)$ for some $z \in I$. i.e. that $\mathbf{s}$ satisfies admissibility conditions (4.1) to (4.3). The condition will be violated either in the middle of s_r or across the join between s_i and s_{i+1} for some i. The former cannot occur, since $t\restriction_\chi$ satisfies admissibility conditions (4.1) to (4.3) for every $t \in It(\Lambda)$, and $It(\Lambda)$ is invariant so every iterate of s_r is in $It(\Lambda)$ also. Thus as soon as we see χ symbols of an iterate of s_r we know that it satisfies admissibility conditions (4.1) to (4.3). Notice also that a join incorporates at least χ symbols of the outgoing word, say s_i, so since $\sigma^i(\mathbf{s})$ begins with s_i we remain in s_i for more than χ symbols, so can conclude that $\sigma^i(\mathbf{s})$ still satisfies admissibility conditions (4.1) to (4.3). Hence $\mathbf{s} = It(z)$ for some $z \in I$.

To see that the δ_ν-pseudo-orbit is ϵ-shadowed, notice that since $\sigma(s_{i-1})$ agrees in the first m places with s_i for every $1 \leq i \leq r$, we have that for every $1 \leq i \leq r$, $d(\sigma(s_{i-1}), s_i) \leq \frac{1}{2^m} < \eta_\epsilon$, i.e. $d(\sigma^i(s), s_i) < \eta_\epsilon$, so $d(\sigma^i(It(z)), It(x_i)) < \eta_\epsilon$ and by conjugation $d(It(f^i(z)), It(x_i)) < \eta_\epsilon$. Thus $|f^i(z) - x_i| < \epsilon$ for every $1 \leq i \leq r$. $\square$

To see that h-shadowing is a strictly stronger property than shadowing we identify a class of maps which do not have h-shadowing. This requires the following lemma, which demonstrates how pseudo-orbits behave under homeomorphisms [8].

Lemma 5.1.9. *Let $f : X \to X$ be a homeomorphism.*

1. *Suppose that f has h-shadowing, and for any $\epsilon > 0$, let δ be as given by h-shadowing for ϵ. Then for any two δ-pseudo-orbits $\{x_i\}_{i=0}^n$ and $\{y_i\}_{i=0}^n$ with $x_n = y_n = z$ we have that $d(x_i, y_i) < 2\epsilon$ for $0 \leq i \leq n$.*
2. *Conversely, suppose that there are points x, y, z with $x \neq y$, and suppose that for every $\delta > 0$ there is an $n > 0$ and δ-pseudo-orbits $\{x_1 = f(x), x_2, \ldots, x_n = z\}$ and $\{y_1 = f(y), y_2, \ldots, y_n = z\}$. Then f cannot have h-shadowing.*

Proof. (1): Let z_0 be a point which ϵ-shadows $\{x_i\}_{i=0}^n$ with direct hit, and let z_1 be a point which ϵ-shadows $\{y_i\}_{i=0}^n$ with direct hit. Then $f^n(z_0) = f^n(z_1) = z$, so since f is one-to-one, we must have that $f^i(z_0) = f^i(z_1)$ for $0 \leq i \leq n$. In other words, both $\{x_i\}_{i=0}^n$ and $\{y_i\}_{i=0}^n$ are ϵ-shadowed by the same orbit, forcing the result.

(2): Suppose that f has h-shadowing, let $\epsilon = d(x, y)/2$, and let δ be as given by h-shadowing for ϵ. Consider the δ-pseudo-orbits $\{x_0 = x, x_1 = f(x), \ldots, x_n = z\}$ and $\{y_0 = y, y_1 = f(y), \ldots, y_n = z\}$. Then there are points z_0 and z_1 which ϵ-shadow $\{x_i\}_{i=0}^n$ and $\{y_i\}_{i=0}^n$ respectively, with direct hit. As in (1), since f is one-to-one we must have that $z_0 = z_1$. But then

$$\begin{aligned} d(x,y) &\leq d(x,z_0) + d(z_0,z_1) + d(z_1,y) \\ &< \epsilon + 0 + \epsilon \\ &= d(x,y). \end{aligned}$$

This contradiction proves the result. ☐

Theorem 5.1.10 [8] says that when homeomorphisms have weak mixing (see Definition 4.1.10) they cannot have h-shadowing.

Theorem 5.1.10. *Let $f : X \to X$ be a topologically weakly mixing homeomorphism. If X has more than one element then f does not have h-shadowing.*

Proof. Fix $x, y \in X$ such that $x \neq y$. By topological weak mixing, for any $\delta > 0$ there is an $n > 0$ such that $f^n(B_\delta(f(x))) \cap B_\delta(x) \neq \emptyset$, $f^n(B_\delta(f(y))) \cap B_\delta(x) \neq \emptyset$. In particular, for every $\delta > 0$ there are δ-pseudo-orbits

$$\{x, f(x), \ldots, f^{n-1}(x), x\}$$

and

$$\{y, f(y), \ldots, f^{n-1}(y), x\}.$$

Then by Lemma 5.1.9 (2), f cannot have h-shadowing. ☐

Example 5.1.11 (Maps with shadowing but not h-shadowing)**.** Bi-infinite sub-shifts are homeomorphisms, and as we know from [52] shifts of finite type have shadowing. By Corollary 4.1.12, a mixing shift of finite type acting on a non-singleton set is thus an example of a map which satisfies the conditions of Theorem 5.1.10, so does not have h-shadowing, but does have shadowing [8].

As a specific example, consider the full shift, $Z = \{0, 1\}^{\mathbb{Z}}$ on the alphabet $\{0, 1\}$. Z is a weakly mixing homeomorphism (see Proposition 4.1.13), and has shadowing as mentioned above. Suppose Z has h-shadowing, let $\epsilon = 1/4$ and let δ be the constant given by the definition of h-shadowing for ϵ. There is an $n \in \mathbb{N}$ such that $1/2^n < \delta$. Let $s = 0^{-\infty} \cdot 10^{\infty}$ and let $t = 1^{-\infty} \cdot 1^{\infty}$. Now let

$$\begin{aligned}
s_0 &= s, \\
s_1 &= 0^{-\infty} \cdot 10^{n-1}1^{\infty}, \\
s_{i+1} &= \sigma^i(s_1) \text{ for } 1 \leq i < 2n, \\
\text{and } s_{2n+1} &= t.
\end{aligned}$$

Then $\{s_0 = s, s_1, \ldots, s_{2n+1} = t\}$ is a δ-pseudo-orbit in Z, so there is some $r \in Z$ for which $d(\sigma^i(r), s_i) < \epsilon$ for $0 \leq i \leq 2n$ and $\sigma^{2n+1}(r) = s_{2n+1} = t$. But then $r = t$, and so

$$d(r, s_0) = d(r, t) = 1/2 > 1/4 = \epsilon,$$

so Z cannot have h-shadowing.

$\square$

With reference to Theorem 3.2.9, we introduce the following further variation of shadowing, which will allow us to apply a similar analysis to Theorem 3.2.9 for ω-limit sets of true orbits.

Definition 5.1.12 (Limit Shadowing)**.** The map $f : X \to X$ is said to have *limit shadowing* on $L \subset X$ if for every asymptotic pseudo-orbit $\{x_n\}_{n\in\mathbb{N}} \subset L$ there is a point $x \in X$ which asymptotically shadows $\{x_n\}_{n\in\mathbb{N}}$. If $L = X$ we simply say that f has *limit shadowing*.

Although this is, strictly speaking, not a stronger property than shadowing (see Example 5.1.13), for maps which have this property we will be able to demonstrate a clear link between ICT and ω-limit sets. The following example, found in [40], shows how a map can have limit shadowing but not shadowing.

Example 5.1.13. Consider a strictly increasing interval map $f : [0,1] \to [0,1]$, for which the points $0,\ 1/3,\ 2/3,\ 1$ are fixed, for which $f(x) > x$ on $(0,1/3)$ and on $(1/3,2/3)$ and for which $f(x) < x$ on $(2/3,1)$. So 0 and 1 are unstable fixed points, 2/3 is a stable fixed point, and 1/3 is stable from below and unstable from above.

For $5 < m \in \mathbb{N}$, let $V_s^m = B_{1/m}(s)$, for $s \in \{0, 1/3, 2/3, 1\}$. Furthermore, let $W_1^m = (0,1/3) \setminus (V_0^m \cup V_{1/3}^m)$ and let $W_2^m = (1/3,1) \setminus (V_{1/3}^m \cup V_{2/3}^m \cup V_1^m)$.

By the definition of f, we see that for every $m > 5$ there is an $a_m > 0$ such that for every $x \in W_1^m$

$$|f(x) - 1/3| \leq |x - 1/3| - 2a_m, \tag{5.1}$$

and for every $x \in W_2^m$

$$|f(x) - 2/3| \leq |x - 2/3| - 2a_m. \tag{5.2}$$

Consider a sequence $\{x_k\}_{k\geq 0} \subset [0,1]$ such that

$$d_k = |f(x_k) - x_{k+1}| \to 0 \text{ as } k \to \infty.$$

We will show that such orbits are always asymptotically shadowed.

There exist $m_0 \in \mathbb{N}$ and $b_1 > 0$ such that for every $m \geq m_0$ and for every $x \in f(V_s^m)$ we have that $d(x, V_u^m) \geq b_1$ provided $u \neq s$. Moreover, there exists $k_0 \in \mathbb{N}$ such that for every $k \geq k_0$ we have that $d_k < b_1$. This implies that for every $m \geq m_0$, for every $k \geq k_0$ and for every $x \in f(V_s^m)$, if $u \neq s$ we have that

$$|f(x_k) - x_{k+1}| < d(x, V_u^m). \tag{5.3}$$

In what follows, it will be assumed that we have chosen $m \geq m_0$ and $k \geq k_0$.

By (5.3), for $x_k \in V_s^m$ we have that $f(x_k) \in f(V_s^m)$ so $|f(x_k) - x_{k+1}| < d(f(x_k), V_u^m)$ for $u \neq s$. So since $V_s^m \cap V_u^m = \emptyset$ for $u \neq s$ we have

$$x_{k+1} \notin V_u^m \text{ provided } x_k \in V_s^m. \tag{5.4}$$

Let $r_1 = f(1/3 - 1/m) - (1/3 - 1/m)$, $r_2 = f(2/3 - 1/m) - (2/3 - 1/m)$ and $r_3 = (2/3 + 1/m) - f(2/3 + 1/m)$. For any $m \geq m_0$, by the definition of f

there is a $b_2(m) > 0$ for which

$$b_2(m) < \min\{r_1,\ r_2,\ r_3\}.$$

We claim that for every $m \geq m_0$ there is an $s \in \{0, 1/3, 2/3, 1\}$ and a $K(m) \geq k_0$ such that

$$x_k \in V_s^m \text{ for every } k \geq K(m). \tag{5.5}$$

It follows from (5.4) that if (5.5) holds we have established limit shadowing for f, since then we can let our shadowing orbit be that of s, whichever of the fixed points that is. In fact, as long as we can show that for some $m \geq m_0$, $x_k \in V_0^m \cup V_{1/3}^{\cup} V_{2/3}^m \cup V_1^m$ for every $k \geq k_0$ we are done by the same reasoning.

To prove claim (5.5), fix $m \geq m_0$ and let $l(m) \geq k_0$ be such that for $k \geq l(m)$,

$$d_k < \min\{a_m, b_2(m)\}. \tag{5.6}$$

Case 1: $x_{l(m)} \in W_2^m$.

Then for $k \geq l(m)$, provided $x_k \in W_2^m$ we have by (5.2) and (5.6) that

$$\begin{aligned} |x_{k+1} - 2/3| &\leq |x_{k+1} - f(x_k)| + |f(x_k) - 2/3| \\ &< a_m + |x_k - 2/3| - 2a_m \\ &= |x_k - 2/3| - a_m. \end{aligned} \tag{5.7}$$

Also, for $k \geq l(m)$ we have

$$\min\{r_2,\ r_3\} > |f(x_k) - x_{k+1}|$$

by the choice of $b_2(m)$ and $l(m)$. So, combining this with (5.7), if $x_{k+j} \in \{(2/3 - 1/m),\ (2/3 + 1/m)\}$ for some $j > 0$, $x_{k+j+1} \in V_{2/3}^m$ and we are done. If no such $x_{k+j} \in \{(2/3 - 1/m),\ (2/3 + 1/m)\}$ then $x_{k_i} \in V_{2/3}^m$ for some $i > 0$ by (5.7) and we are done.

Case 2: $x_{l(m)} \in W_1^m$.

Then for $k \geq l(m)$, provided $x_k \in W_1^m$ we have by (5.1) and (5.6)

$$\begin{aligned} |x_{k+1} - 1/3| &\leq |x_{k+1} - f(x_k)| + |f(x_k) - 1/3| \\ &< a_m + |x_k - 1/3| - 2a_m \\ &= |x_k - 1/3| - a_m. \end{aligned}$$

So, again using the choice of $b_2(m)$, there is a $j > 0$ for which either $1/3 > x_{k+j} \in V_{1/3}^m$, or $x_{k+j} \in (1/3, 2/3)$ in which case (after several iterations if needs be) we are reduced to Case 1.

This proves claim (5.5), and thus f has limit shadowing.

Let $\epsilon = 1/6$. To see that f does not have shadowing, for any $0 < \delta < 1/6$ consider the sequence $\{x_k\}_{k\geq 0}$ defined as follows:

- $x_0 = \delta/2$;
- $x_k = f^k(x_0)$ for every $0 < k < J$, where $|x_{J-1} - 1/3| \leq \delta/2$;

- $x_{J+i} = f^i(1/3 + \delta/2)$ for every $i \geq 0$.

This is a δ-pseudo-orbit for any $\delta > 0$ we choose, and it is not ϵ-shadowed by any $x \in [0, 1]$, so f cannot have shadowing.

□

Since Example 5.1.13 shows that limit shadowing does not imply shadowing, the property of *strong limit shadowing* (or *s-limit shadowing*) was introduced by Lee and Sakai [28], which does imply shadowing (indeed shadowing is part of the condition for a map to have this stronger form).

Definition 5.1.14 (s-Limit Shadowing)**.** A map $f : X \to X$ has *s-limit shadowing on* $Y \subseteq X$ if for every $\epsilon > 0$ there is $\delta > 0$ such that the following two conditions hold:

1. for every δ-pseudo-orbit $\{x_n\}_{n\in\mathbb{N}} \subseteq Y$ of f, there is $y \in X$ such that y ϵ-shadows $\{x_n\}_{n\in\mathbb{N}}$, and
2. for every asymptotic δ-pseudo-orbit $\{z_n\}_{n\in\mathbb{N}} \subseteq Y$ of f, there is $y \in X$ such that y asymptotically ϵ-shadows $\{z_n\}_{n\in\mathbb{N}}$.

In the special case $Y = X$ we say that f has *s-limit shadowing.*

We explore the links between limit shadowing, s-limit shadowing and h-shadowing in Lemma 5.1.15 and Theorem 5.1.16 [8].

Lemma 5.1.15. *If* $\Lambda \subseteq f(\Lambda) \subseteq X$ *and* f *has s-limit shadowing on* Λ *then* f *has limit shadowing on* Λ*. In particular, if* f *is surjective and has s-limit shadowing then* f *also has limit shadowing.*

Proof. Fix any asymptotic pseudo orbit $\{x_n\}_{n\in\mathbb{N}} \subseteq \Lambda$, fix any $\epsilon > 0$ and let δ be the constant given from the definition of s-limit shadowing for ϵ. There is N such that $d(f(x_n), x_{n+1}) < \delta$ for all $n \geq N$. There is also $y \in \Lambda$ such that $f^N(y) = x_N$. Then the sequence

$$y, f(y), \dots, f^N(y) = x_N, x_{N+1}, x_{N+2}, \dots \tag{5.8}$$

is an asymptotic δ-pseudo-orbit in Λ. Now, it is enough to apply the definition of s-limit shadowing and the proof is completed. □

Theorem 5.1.16. *Suppose that $\Lambda \subseteq X$ is closed.*

1. *If there is an open set U such that $\Lambda \subseteq U$ and f has h-shadowing on U, then f has s-limit shadowing on Λ.*
2. *If Λ is invariant and $f\restriction_\Lambda$ has h-shadowing then $f\restriction_\Lambda$ has s-limit shadowing.*
3. *If f has h-shadowing then f has s-limit shadowing.*

Proof. (1): Notice that by Lemma 5.1.2, f has shadowing already, so the first half of the definition of s-limit shadowing is satisfied trivially.

Fix $\epsilon > 0$ such that $B_{3\epsilon}(\Lambda) \subseteq U$ and denote $\epsilon_n = 2^{-n-1}\epsilon$. By the definition of h-shadowing there are $\{\delta_n\}_{n\in\mathbb{N}}$ such that every finite δ_n-pseudo-orbit in U is ϵ_n-shadowed with direct hit. Fix any infinite δ_0-pseudo-orbit $\{x_n\}_{n\in\mathbb{N}} \subseteq \Lambda$ such that $\lim_{n\to\infty} d(f(x_n), x_{n+1}) = 0$. There is an increasing sequence $\{k_i\}_{i\in\mathbb{N}}$ such that $\{x_n\}_{n=k_i}^{\infty}$ is an infinite δ_i-pseudo-orbit and obviously $k_0 = 0$. Note that if w is a point such that $f^{k_i}(w) = x_{k_i}$ then the sequence

$$w, f(w), \ldots, f^{k_i}(w), x_{k_i+1}, \ldots, x_{k_{i+1}}$$

is a δ_i-pseudo-orbit.

Let z_0 be a point which ϵ_0-shadows the δ_0-pseudo-orbit $x_0, \ldots, x_{k_1}$ with exact hit (i.e. such that $f^{k_1}(z_0) = x_{k_1}$). Notice that $f^j(z_0) \in U$ for $0 \leq j \leq k_1$.

For $i \in \mathbb{N}$, assume that z_i is a point which ϵ_i-shadows the δ_i-pseudo-orbit

$$z_{i-1}, f(z_{i-1}), \ldots, f^{k_i}(z_{i-1}), x_{k_i+1}, \ldots, x_{k_{i+1}} \subseteq U$$

with exact hit. Then by h-shadowing there is a point z_{i+1} which ϵ_{i+1}-shadows the δ_{i+1}-pseudo-orbit

$$z_i, f(z_i), \ldots, f^{k_{i+1}}(z_i), x_{k_{i+1}+1}, \ldots, x_{k_{i+2}} \subseteq U$$

with exact hit. Thus we can produce a sequence $\{z_i\}_{i=0}^{\infty}$ with the following properties:

1. $d(f^j(z_{i-1}), f^j(z_i)) < \epsilon_i$ for $j \leq k_i$ and $i \geq 1$,

2. $d(f^j(z_i), x_j) < \epsilon_i$ for $k_i < j \leq k_{i+1}$ and $i \geq 0$,

3. $f^{k_{i+1}}(z_i) = x_{k_{i+1}}$ for $i \geq 0$,

4. $d(f^j(z_i), \Lambda) < \epsilon$ for $j \leq k_{i+1}$,

There is an increasing sequence $\{s_i\}_{i\in\mathbb{N}}$ such that the limit $z = \lim_{i\to\infty} z_{s_i}$

exists.

For any $j, n \in \mathbb{N}$ there exist $i_0 \geq 0$ and $m \geq i_0$ such that $k_{i_0} < j \leq k_{i_0+1}$ and $d(f^j(z), f^j(z_{s_m})) < \epsilon_{n+1}$. So we get

$$
\begin{aligned}
d(f^j(z), x_j) &\leq d(f^j(z), f^j(z_{s_m})) + d(f^j(z_{i_0}), x_j) + \sum_{i=i_0}^{s_m-1} d(f^j(z_i), f^j(z_{i+1})) \\
&\leq \epsilon_{n+1} + \epsilon_{i_0} + \sum_{i=i_0}^{s_m-1} \epsilon_{i+1} \\
&\leq \epsilon 2^{-n-2} + \sum_{i=i_0}^{\infty} 2^{-i-1}\epsilon \leq \epsilon(2^{-n-2} + 2^{-i_0}) \\
&\leq \epsilon(2^{-n-2} + 1).
\end{aligned}
$$

But we can fix n to be arbitrarily large in that case, which immediately implies that

$$
d(f^j(z), x_j) \leq \epsilon.
$$

Furthermore, for any n, let $j > k_{n+2}$. There is $i_1 \geq n+2$ such that $k_{i_1} < j \leq k_{i_1+1}$ and there is $m > i_1$ such that $d(f^j(z), f^j(z_{s_m})) < \epsilon_{n+1}$. Then as before we obtain

$$
\begin{aligned}
d(f^j(z), x_j) &\leq \epsilon(2^{-n-2} + 2^{-i_1}) \\
&\leq \epsilon(2^{-n-2} + 2^{-n-2}) = \epsilon_n.
\end{aligned}
$$

This immediately implies that $\limsup_{j\to\infty} d(f^j(z), x_j) \leq \epsilon_n$ which, since n was arbitrary, finally gives $\lim_{j\to\infty} d(f^j(z), x_j) = 0$. This shows that f has s-limit shadowing on Λ.

(2) follows from (1), since $U = \Lambda$ is open in Λ. (3) is a special case of (1) with $\Lambda = X$. $\square$

The next result follows immediately from Theorem 5.1.16 (1) and Lemma 5.1.15.

Corollary 5.1.17. *Suppose that there is an open set U such that f has h-shadowing on U. If $\Lambda \subseteq U$ is closed with $\Lambda \subseteq f(\Lambda)$ then f has limit shadowing on Λ.*

Despite this last result, it is not known whether shadowing and limit shadowing together imply s-limit shadowing.

We finish this section by proving a result from [8] which shows that provided we can find some iterate of a map which has h-shadowing, we can deduce that the map itself has h-shadowing.

Theorem 5.1.18. *If Λ is a closed set such that $f(\Lambda) \supset \Lambda$ then the following conditions are equivalent:*

1. *f has h-shadowing on Λ,*
2. *f^n has h-shadowing on Λ for some $n \in \mathbb{N}$,*
3. *f^n has h-shadowing on Λ for all $n \in \mathbb{N}$,*

Proof. Implication (3) implies (2) is trivial. Implication (1) implies (3) is also obvious, since for any $\delta > 0$ and $n > 0$ if $\{y_0, y_1, \ldots, y_m\}$ is δ-pseudo-orbit for f^n then the sequence

$$y_0, f(y_0), \ldots, f^{n-1}(y_0), y_1, f(y_1), \ldots, f^{n-1}(y_{m-1}), y_m$$

is δ-pseudo-orbit for f.

For the proof of implication (2) implies (1), fix $\epsilon > 0$ and suppose that f^n has h-shadowing on Λ for some $n \in \mathbb{N}$. Let X have metric d. By Lemma 3.2.2 there is an $\epsilon' > 0$ such that if $\{x_0, \ldots, x_n\} \subseteq \Lambda$ is an ϵ'-pseudo-orbit and $y \in X$ is such that $d(y, x_0) < \epsilon'$ then $d(f^k(y), x_k) < \epsilon$ for $k = 1, \ldots, n$.

By h-shadowing there is a $\delta > 0$ such that every δ-pseudo-orbit of f^n is ϵ'-shadowed by a point in X which hits the last element of the pseudo-orbit. Again by Lemma 3.2.2 (with $y = x_0$), there is a $\gamma < \frac{\delta}{n}$ such that whenever $\{x_0, \ldots, x_n\}$ is a γ-pseudo-orbit for f we have that $d(f^i(x_0), x_i) < \delta$ for $i = 1, \ldots, n$.

Let $\{x_0, \ldots, x_m\} \subseteq \Lambda$ be any γ-pseudo-orbit for f, and write $m = jn + r$ for some $j \geq 0$ and some $r < n$. Since f is surjective on Λ (i.e. $\Lambda \subset f(\Lambda)$) there is a point $z \in \Lambda$ such that $f^{n-r}(z) = x_0$. Then

$$\{z, f(z), \ldots, f^{n-r}(z), x_1, \ldots, x_m\} \subset \Lambda$$

is a γ-pseudo-orbit for f, which we re-enumerate to obtain the sequence $\{y_0, \ldots, y_{(j+1)n}\}$. We now claim that $\{y_0, y_n, y_{2n}, \ldots, y_{(j+1)n}\}$ is a δ-pseudo-orbit for f^n. Indeed, $\{y_0, \ldots, f^{n-r}(y_0) = y_{n-r}, \ldots, y_n\}$ is a γ-pseudo-orbit (of length $n+1$) for f and so $d(f^n(y_0), y_n) < \delta$. Similarly we have

$$d(f^n(y_{kn}), y_{(k+1)n}) < \delta \text{ for } 1 \leq k \leq j.$$

By h-shadowing of f^n there is u such that that $d(f^{kn}(u), y_{kn}) < \epsilon'$ for $k = 0, 1, \ldots, j+1$ and $f^{(j+1)n}(u) = y_{(j+1)n}$. Thus by the definition of

ϵ' we have that $d(f^{kn+i}(u), y_{kn+i}) < \epsilon$ for $k = 0, \ldots, j+1$ and for $i = 0, \ldots, n-1$. So the point u ϵ-shadows the γ-pseudo-orbit $\{y_0, \ldots, y_{(j+1)n}\} = \{z, f(z), \ldots, f^{n-r}(z) = x_0, x_1, \ldots, x_m\}$, and consequently the point $w = f^{n-r}(u)$ ϵ-shadows the γ-pseudo-orbit $\{x_0, \ldots, x_m\}$ with exact hit, since

$$f^m(w) = f^{(j+1)n}(u) = y_{(j+1)n} = x_m.$$

$\square$

5.2 Expansivity (II)

In our pursuit of properties which guarantee pseudo-orbit shadowing in continuous maps, we have isolated a type of expansivity which not only implies the standard form of shadowing, as is shown in [18], but also implies the stronger property of h-shadowing. However the expansive property is not easy to verify in most cases, so we prove a series of implications between this and other forms of expansivity, and link these to the various versions of shadowing. What we present here is a theory which highlights properties of maps which are easy to verify, are commonly observed, and which imply the existence of and links between shadowing, in its various forms.

Definition 5.2.1 (Positively Expansive)**.** The map $f : X \to X$ is said to be *positively expansive* if there is a *constant of expansivity* $b > 0$ such that for any $x, y \in X$ the condition

$$d(f^n(x), f^n(y)) < b \text{ for every } n \geq 0$$

implies that $x = y$.

Proposition 5.2.2 is from [8]; Proposition 5.2.3 is new.

Proposition 5.2.2. *Suppose that f is a positively expansive map with shadowing, then f has h-shadowing.*

Proof. Let $0 < \epsilon < b$ and let $\delta > 0$ be the constant as given in the definition of shadowing for ϵ. Fix any δ-pseudo orbit $\{x_0, x_1, \ldots, x_m\}$ and extend it to the infinite δ-pseudo-orbit $\{x_0, x_1, \ldots, x_m, f(x_m), f^2(x_m), \ldots\}$ Let z be a point which ϵ-shadows the above pseudo-orbit, then $d(f^{j+m}(z), f^j(x_m)) < b$ for all $j \geq 0$, which by positive expansivity implies that $f^m(z) = x_m$. $\square$

Proposition 5.2.3. *On a compact interval I, suppose that the map $f : I \to I$ is positively expansive. Then any infinite, nowhere dense, CINE set $A \subset I$ is weakly-expansive.*

Proof. Suppose $f : I \to I$ is positively expansive with constant b, let A be any infinite, nowhere dense, CINE subset of I and let $x \in A$ have a neighbourhood U. Choose $y \in U$ with $y \neq x$, then there is a $k \in \mathbb{N}$ such that $|f^k(x) - f^k(y)| \geq b$, so certainly $f^k(U)$ has diameter at least b and hence A is weakly-expansive. $\square$

Although the definition of weak expansivity refers to infinite, nowhere dense, CINE sets, it is easy to see from the above proof that for a positively expansive map on a compact interval I with expansive constant b, *any* subset $A \subset I$ has the property that for any $a \in A$ and any neighbourhood U of a there is some $k \in \mathbb{N}$ such that the diameter of $f^k(U)$ is greater than b.

Recall from the definition of a full orbit, that for a surjective map we may define a backward orbit of a point by choosing successive pre-images of a point. Definition 5.2.4 weakens Definition 5.2.1 by making use of the existence of full orbits for surjective maps. There are many similar expansive definitions in the literature (see [2], [28], [37] for examples), so we use the term c-expansive to distinguish from these.

Definition 5.2.4 (c-Expansive)**.** The surjective map $f : X \to X$ is said to be c*-expansive* if there is a *constant of expansivity* $c > 0$ such that for any $x, y \in X$ and any full orbits $\{f^n(x)\}_{n\in\mathbb{Z}}$ and $\{f^n(y)\}_{n\in\mathbb{Z}}$ through x and y respectively the condition

$$d(f^n(x), f^n(y)) < c \text{ for every } n \in \mathbb{Z}$$

implies that $x = y$.

It is easy to see that c-expansivity is equivalent to saying that there is a $c > 0$ such that for every $x, y \in X$ such that $x \neq y$ there is a $k \in \mathbb{Z}$ for which $d(f^k(x), f^k(y)) \geq c$. Thus any shift space X is c-expansive with constant $1/2$, since for any two elements $s, t \in X$ with $s \neq t$ we have that for some $k \in \mathbb{N}$, $d(\sigma^k(s), \sigma^k(t)) = 1 > 1/2$.

Theorem 5.2.5 [8] is due to Oprocha:

Theorem 5.2.5. *If f is c-expansive then the following conditions are equivalent:*

1. *f has the shadowing property,*

2. *f has the s-limit shadowing property.*

Proof. We have to prove (1) $\implies$ (2) since the converse implication is trivial. Fix $\epsilon > 0$ and assume that $\epsilon < b/2$ where b is the expansive constant. Let $\delta > 0$ be a constant provided by the shadowing property for ϵ. Shadowing implies that Definition 5.1.14 (1) holds. To prove 5.1.14 (2), let $\{x_n\}_{n\in\mathbb{N}}$ be an asymptotic δ-pseudo-orbit that is ϵ-shadowed by the point z.

Suppose, for a contradiction, that $d(f^n(z), x_n)$ does not converge to 0 as $n \to \infty$. Since X is compact (so that every sequence has a convergent subsequence), there are points p_0 and q_0 in X and an infinite subset N_0 of $\mathbb{N}$ such that

1. $\lim\limits_{n\to\infty, n\in N_0} f^n(z) = p_0$, and
2. $\lim\limits_{n\to\infty, n\in N_0} x_n = q_0$.

Let $d(p_0, q_0) = \eta > 0$. By continuity,

$$\lim_{n\to\infty, n\in N_0} f^{n+k}(z) = p_k = f^k(p_0)$$

for all $k \geq 0$. By continuity, the fact that $\{x_n\}$ is an asymptotic pseudo-orbit and that

$$d\big(x_{n+1}, f(q_0)\big) \leq d\big(x_{n+1}, f(x_n)\big) + d\big(f(x_n), f(q_0)\big),$$

we have that

$$\lim_{n\to\infty, n\in N_0} x_{n+1} = q_1 = f(q_0).$$

Hence $\lim_{n\to\infty, n\in N_0} x_{n+k} = q_k = f^k(q_0)$ for all $k \geq 0$.

Since X is compact, there is an infinite subset N_{-1} of N_0 such that

$$\lim_{n\to\infty, n\in N_{-1}} f^{n-1}(z) = p_{-1} \text{ and } \lim_{n\to\infty, n\in N_{-1}} x_{n-1} = q_{-1},$$

for some p_{-1} and q_{-1} in X. Again, continuity and the fact that $\{x_n\}$ is an asymptotic pseudo-orbit imply that $f(p_{-1}) = p_0$ and $f(q_{-1}) = q_0$. Repeating this argument we can find a nested sequence of infinite sets $\{N_{-k}\}_{k\in\mathbb{N}}$, and sequences of points $\{p_{-i}\}_{i\in\mathbb{N}}$ and $\{q_{-i}\}_{i\in\mathbb{N}}$, such that for all $0 < k \in \mathbb{N}$

1. $0 \leq n - k$ for all $n \in N_{-k}$,
2. $\lim_{n\to\infty, n\in N_{-k}} f^{n-k}(z) = p_{-k}$ and $f(p_{-k}) = p_{-k+1}$,
3. $\lim_{n\to\infty, n\in N_{-k}} x_{n-k} = q_{-k}$ and $f(q_{-k}) = q_{-k+1}$.

Thus the sequences $\{p_k\}_{k\in\mathbb{Z}}$ and $\{q_k\}_{k\in\mathbb{Z}}$ are the specific full orbits passing through p_0 and q_0 respectively, which satisfy the above conditions. Moreover

$$d(p_k, q_k) \leq \begin{cases} \sup_{n\in N_0} d\big(f^{n+k}(z), x_{n+k}\big), & \text{if } k \geq 0, \\ \sup_{n\in N_k} d\big(f^{n+k}(z), x_{n+k}\big), & \text{if } k < 0. \end{cases}$$

Since $\epsilon < b/2$ and z ϵ-shadows $\{x_n\}$, $d(p_k, q_k) < b/2$ for all $k \in \mathbb{Z}$. It follows by c-expansivity that

$$0 = d(p_0, q_0) = \lim_{n\to\infty, n\in N_0} d(f^n(z), x_n) = \eta > 0,$$

which is a contradiction, and the result follows. $\square$

Definition 5.2.6 (Topologically Hyperbolic)**.** The surjective map $f : X \to X$ is said to be *topologically hyperbolic* if it is c-expansive and has the shadowing property.

Thus a topologically hyperbolic map automatically has s-limit shadowing by Theorem 5.2.5.

Many authors refer to hyperbolic properties in the context of expanding maps (see for example [11], [28], [37], [40]) but is not clear where the definition of topologically hyperbolic originated. An important class of topologically hyperbolic maps are the shifts of finite type (one or two-sided), which as noted above have c-expansivity and have shadowing [52].

As we will see in Theorem 6.1.1 and Corollary 6.1.2, for topologically hyperbolic maps ω-limit sets are completely characterized by internal chain transitivity.

The term *uniformly expanding* is usually used to describe maps which increase the distance between any pair of points by at least some amount $\mu > 1$ (see for example [2], [39]). We introduce four variations of this property which will enable us to demonstrate the existence of shadowing properties in various maps. The first three properties can be observed in many classes of maps, the widely studied class of interval maps being one example. All results in the remainder of this section are from [8].

Definition 5.2.7 (Uniformly Expanding)**.** If there are $\delta > 0$, $\mu > 1$ such that $d(f(x), f(y)) \geq \mu d(x, y)$ provided that $x, y \in A \subseteq X$ and $d(x, y) < \delta$ then we say that f is *uniformly expanding on A.*

It is a consequence that if f is uniformly expanding on A then for each

$x \in A$ there is an open set $U \ni x$ such that $f\restriction_{U\cap A}$ is one-to-one. Furthermore we have the following.

Lemma 5.2.8. *If f is uniformly expanding on the invariant set $\Lambda \subseteq X$ then f is positively expansive on Λ.*

Proof. Let $\delta > 0$ and $\mu > 1$ be as given in the definition of uniformly expanding, and put $b = \delta/2$. Pick $x \neq y \in \Lambda$ and assume that $d(x, y) < b$ (else we are done). Then

$$d(f^i(x), f^i(y)) \geq \mu d(f^{i-1}(x), f^{i-1}(y))$$

whenever $d(f^{i-1}(x), f^{i-1}(y)) \leq b$. Then certainly there is a $k \in \mathbb{N}$ such that $d(f^{k-1}(x), f^{k-1}(y)) \leq b < d(f^k(x), f^k(y))$, which completes the proof. $\square$

Definition 5.2.9 (Strongly Uniformly Expanding)**.** If there are $\delta > 0$, $\mu > 1$ such that $d(f(x), f(y)) \geq \mu d(x, y)$ provided that $x \in A \subseteq X$ and $d(x, y) < \delta$ then we say that f is *strongly uniformly expanding on A.*

We note that if f is strongly uniformly expanding on A, then it is trivially uniformly expanding on A.

Definition 5.2.10 (Piecewise Expanding)**.** For $\Lambda \subseteq X$ closed, we say that f is *piecewise expanding on* Λ if there exist $\mu > 1$ and a finite number of open sets $U_1, \ldots, U_n$ such that $\Lambda \subset \bigcup_{j=1}^n U_j$ and $d(f(x), f(y)) \geq \mu d(x, y)$ for every $x, y \in U_j$, $j = 1, \ldots, n$.

In the sequel, we will regularly use Λ to denote the closed set on which a map is piecewise expanding.

Remark 5.2.11. Notice that by normality, if f is piecewise expanding on Λ, then there is an open set $U \supset \Lambda$ whose closure is also contained in $\bigcup_{j=1}^{n} U_j$. Therefore f is piecewise expanding on the closure of U, for any such U.

Definition 5.2.12 (Open/Locally One-to-One Maps)**.** For a subset $A \subseteq X$, we say that f is *open on* A if for every $x \in A$ and every neighbourhood U of x, $f(x) \in f(U)^{\circ}$, and that f is *locally one-to-one* on A if for every $x \in A$ there is an open set $U \ni x$ such that $f\restriction_U$ is injective.

Definition 5.2.13 (Lebesgue Number)**.** For any open cover U of a set $A \subset X$, the *Lebesgue number* of this cover is the constant δ such that for any $x \in A$, the open δ-neighbourhood around x is contained in some member of the cover.

Piecewise expanding is a stronger property than that of both uniformly expanding and locally uniformly expanding, since it requires f to be expansive on an open cover of the set Λ, rather than just the set itself. The following lemma demonstrates this fact.

Lemma 5.2.14. *If f is piecewise expanding on the closed set $\Lambda \subseteq X$ then f is strongly uniformly expanding on Λ.*

Proof. Let $\mu > 1$ be the constant and $U_1, \ldots, U_n$ be the open cover of Λ as given in the definition of piecewise expanding, and let η be the Lebesgue number of this cover. Define $\delta = \eta/2$. Then for every $x \in \Lambda$, $y \in X$, if $d(x, y) < \delta$ we have that $y \in B_\eta(x)$ and so there is some U_i for which $x, y \in U_i$. Thus $d(f(x), f(y)) \geq \mu d(x, y)$ and we get that f is strongly uniformly expanding on Λ. $\square$

The similarities between Definitions 5.2.9 and 5.2.10 allow us to prove many of the following results in “tandem”, considering first the case of a strongly uniformly expanding map, and then that of a piecewise expanding map. Much of the detail in these separate cases is very similar, but we include full proofs for completeness.

The last of the four uniformly expanding properties is a derivation of a property introduced implicitly in Coven *et al* [18], which they use to prove shadowing, and from it we get a strengthening of this result. Note that our property bears no resemblance to the property of *locally expanding* introduced in [10].

Definition 5.2.15 (Locally Uniformly Expanding)**.** For a subset $\Lambda \subseteq X$ we say that a continuous map $f \colon X \to X$ is *locally uniformly expanding on* Λ if there is a $\mu > 1$ and a $\nu > 0$ such that for every $x \in \Lambda$ and every $\epsilon < \nu$ we have that $B_{\mu\epsilon}(f(x)) \subseteq f\left(B_\epsilon(x)\right)$.

Definition 5.2.15 depends on the structure of open neighbourhoods and so there may be some difficulty to verify it in practice. As it happens, local uniform expanding is a property of piecewise expanding maps which are open on a specific set. To prove this we need the following lemma.

Lemma 5.2.16. *Let $\Lambda \subseteq X$ be closed and suppose that one of the following hold:*

(i) f is strongly uniformly expanding on Λ with respect to the constants $\mu > 1$ and $\delta > 0$, and for $x, q \in \Lambda$ and $\eta > 0$, we have that $B_{\mu\eta}(f(x)) \subset f(U)^\circ$ for some open set U, where $x \in U \subseteq B_{\delta/2}(q)$,

(ii) f *is piecewise expanding on* Λ *with respect to the constant* $\mu > 1$ *and the open cover* $U_1, \ldots, U_n$ *of* Λ*, and for some* $i \leq n$*,* $\eta > 0$ *we have that* $B_{\mu\eta}(f(x)) \subset f(U)^\circ$ *for some open set* U*, where* $x \in U \subseteq U_i$*.*

Then for any $\rho < \eta$*,*

$$B_{\mu\rho}(f(x)) \subseteq f(B_\rho(x)).$$

Proof. Denote $V = U \cap f^{-1}\big(B_{\mu\rho}(f(x))\big)$ and notice that $f(V) = B_{\mu\rho}(f(x))$ from our assumption. We consider the two cases from the assumptions of the lemma independently.

(i): First note that $V \subset B_{\delta/2}(q)$. Suppose that $V \not\subseteq B_\rho(x)$. Then there is $y \in V \setminus B_\rho(x)$, so $\rho \leq d(x,y) < \delta$, thus $d(f(x), f(y)) \geq \mu\rho$ and we get that $f(y) \notin B_{\mu\rho}(f(x))$ which is impossible. Thus $V \subseteq B_\rho(x)$ and

$$f(V) = B_{\mu\rho}(f(x)) \subseteq f(B_\rho(x)).$$

(ii): Suppose that $V \not\subseteq B_\rho(x)$. Then there is $y \in V \setminus B_\rho(x)$, so $d(x,y) \geq \rho$ and $x, y \in U \subset U_i$, thus $d(f(x), f(y)) \geq \mu\rho$ and we get that $f(y) \notin B_{\mu\rho}(f(x))$ which is impossible. Thus

$$f(V) = B_{\mu\rho}(f(x)) \subseteq f(B_\rho(x)).$$

□

Lemma 5.2.17. *Let* $\Lambda \subseteq X$ *be closed.*

(i) If f *is strongly uniformly expanding and open on* Λ *then* f *is locally uniformly expanding on* Λ*.*

(ii) If f is piecewise expanding on Λ and open on a neighbourhood Q of Λ then f is locally uniformly expanding on a neighbourhood $W \subseteq Q$ of Λ.

Proof. (i): Let $\mu > 1$, $\delta > 0$ be the constants from Definition 5.2.9 . Fix $x \in \Lambda$ and let U be an open neighbourhood of x such that $U \subseteq B_{\delta/2}(x)$. There is an $\eta = \eta(x) > 0$ such that $B_{\mu\eta}(f(x)) \subset f(U)^\circ$ since f is open at x. Fix any $\rho < \eta$, then by Lemma 5.2.16(i) we have that $B_{\mu\rho}(f(x)) \subseteq f(B_\rho(x))$.

Let $U' = f^{-1}\big(B_{\mu\eta/2}(f(x))\big) \cap U$. Then $x \in U'$ and we can take $\nu = \nu(x) < \eta/2$ so that $B_\nu(x) \subseteq U'$.

Take any $z \in B_\nu(x)$ and $\epsilon \leq \nu$ so that $B_\epsilon(z) \subseteq B_\nu(x)$, then $f(z) \in f(U') = B_{\mu\eta/2}(f(x))$ and so

$$B_{\mu\epsilon}(f(z)) \subseteq B_{\mu\eta}(f(x)) \subseteq f(U)^\circ.$$

Since $z \in U \subseteq B_{\delta/2}(x)$, by Lemma 5.2.16(i) we get that $B_{\mu\epsilon}(f(z)) \subseteq f(B_\epsilon(z))$. In other words, for $x \in \Lambda$ and $z \in X$ we have that

$$B_\epsilon(z) \subseteq B_{\nu(x)}(x) \;\Rightarrow\; B_{\mu\epsilon}(f(z)) \subseteq f(B_\epsilon(z)). \tag{5.9}$$

Λ is compact and $\nu(x)$ is well defined for every $x \in \Lambda$, so there are $x_1, \ldots, x_s$ such that

$$\Lambda \subseteq \bigcup_{i=1}^{s} B_{\nu(x_i)/2}(x_i).$$

Denote $\xi = \min_i \nu(x_i)/2$, fix any $\epsilon < \xi$ and any $x \in \Lambda$. There is i such that $x \in B_{\nu(x_i)/2}(x_i)$ and so $B_\epsilon(x) \subseteq B_{\nu(x_i)}(x_i)$. Hence by (5.9), $B_{\mu\epsilon}(f(x)) \subseteq f(B_\epsilon(x))$.

(ii): Let $\mu > 1$ be the constant and $U_1, \ldots, U_n$ be the open cover of Λ

as given in Definition 5.2.10. Thus $Q \cap \bigcup_{i \leq n} U_i$ is an open neighbourhood of Λ, and moreover there is a neighbourhood $W \supset \Lambda$ such that f is piecewise expanding on $\overline{W} \subset \left(Q \cap \bigcup_{i \leq n} U_i\right)$ (see Remark 5.2.11). We will show that f is locally uniformly expanding on $\overline{W}$.

Fix $x \in \overline{W}$ and let U be an open neighbourhood of x such that $U \subseteq U_i$ for some i. There is an $\eta = \eta(x) > 0$ such that $B_{\mu\eta}(f(x)) \subset f(U)^\circ$ since f is open at x. Fix any $\rho < \eta$, then by Lemma 5.2.16(ii) we have that $B_{\mu\rho}(f(x)) \subseteq f(B_\rho(x))$.

Let $U' = f^{-1}\big(B_{\mu\eta/2}(f(x))\big) \cap U$. Then $x \in U'$ and we can take $\nu = \nu(x) < \eta/2$ so that $B_\nu(x) \subseteq U'$.

Take any $z \in B_\nu(x)$ and $\epsilon \leq \nu$ so that $B_\epsilon(z) \subseteq B_\nu(x)$, then $f(z) \in f(U') = B_{\mu\eta/2}(f(x))$ and so

$$B_{\mu\epsilon}(f(z)) \subseteq B_{\mu\eta}(f(x)) \subseteq f(U^\circ).$$

Since $z \in U \subseteq U_i$, by Lemma 5.2.16(ii) we get that $B_{\mu\epsilon}(f(z)) \subseteq f(B_\epsilon(z))$. In other words, for $x \in \overline{W}$ and $z \in X$ we have that

$$B_\epsilon(z) \subseteq B_{\nu(x)}(x) \;\Rightarrow\; B_{\mu\epsilon}(f(z)) \subseteq f(B_\epsilon(z)). \tag{5.10}$$

Note that $\overline{W}$ is compact and $\nu(x)$ is well defined for every $x \in \overline{W}$, so there are $x_1, \ldots, x_s$ such that

$$\overline{W} \subseteq \bigcup_{i=1}^{s} B_{\nu(x_i)/2}(x_i).$$

Denote $\xi = \min_i \nu(x_i)/2$, fix any $\epsilon < \xi$ and any $x \in \overline{W}$. There is i such that

$x \in B_{\nu(x_i)/2}(x_i)$ and so $B_\epsilon(x) \subseteq B_{\nu(x_i)}(x_i)$. Hence by (5.10), $B_{\mu\epsilon}(f(x)) \subseteq f(B_\epsilon(x))$. $\square$

When the map is locally one-to-one we can reverse the implication in Lemma 5.2.17.

Lemma 5.2.18. *Let $\Lambda \subseteq X$ be closed.*

(i) If f is locally uniformly expanding and locally one-to-one on Λ then f is strongly uniformly expanding on Λ.

(ii) If f is locally uniformly expanding on a neighbourhood of Λ and locally one-to-one on Λ then f is piecewise expanding on Λ.

Proof. Let μ, ν be as given in Definition 5.2.15. Certainly for every $x \in \Lambda$ there is a $\delta(x)$ such that f is one-to-one on $B_{\delta(x)}(x)$, and the collection of such neighbourhoods cover Λ. Take β to be their Lebesgue number and let $\delta = \min\{\beta, \nu\}$, then f is one-to-one on $B_\delta(x)$ for every $x \in \Lambda$. We consider the two cases independently.

(i): Let $x \in \Lambda$ and suppose that $d(x, y) = \eta < \delta$. Suppose also that $d(f(x), f(y)) < \mu d(x, y) = \mu\eta$, then

$$f(y) \in B_{\mu\eta}(f(x)) \subseteq f(B_\eta(x))$$

since f is locally uniformly expanding at x, and $\eta < \nu$. But $y \notin B_\eta(x)$, so there is a $z \in B_\eta(x)$ for which $f(z) = f(y)$, contradicting the fact that f is one-to-one on $B_\delta(x)$. Thus $d(f(x), f(y)) \geq \mu d(x, y)$ for every $x \in \Lambda$ with $d(x, y) < \delta$, and hence f is strongly uniformly expanding on Λ.

(ii): Let W be the neighbourhood of Λ on which f is locally uniformly expanding. Now consider a cover of Λ consisting of $\delta/3$-neighbourhoods of points in Λ, take a finite sub-cover $\{B_{\delta/3}(x_i) \ : \ 1 \leq i \leq n\}$, and let $U_i = W \cap B_{\delta/3}(x_i)$. Fix any $i \leq n$ and $x, y \in U_i$. Then $\eta = d(x, y) < 2\delta/3$ and $d(x, x_i) < \delta/3$, $d(y, x_i) < \delta/3$, thus since $y \notin B_\eta(x)$ and $B_\eta(x) \subseteq B_\delta(x_i)$, we get that f is one-to-one on $B_\eta(x)$.

Suppose that $d(f(x), f(y)) < \mu d(x, y) = \mu\eta$. Then we have that

$$f(y) \in B_{\mu\eta}(f(x)) \subseteq f(B_\eta(x)),$$

since f is locally uniformly expanding at x, and $\eta < \nu$. But $y \notin B_\eta(x)$, so there is a $z \in B_\eta(x)$ for which $f(z) = f(y)$, contradicting the fact that f is one-to-one on $B_\delta(x)$. Thus $d(f(x), f(y)) \geq \mu d(x, y)$ for every $x, y \in U_i$, for any i, and hence f is piecewise expanding on Λ. $\square$

The assumption of Lemma 5.2.18 that f is locally one-to-one on Λ is essential, as shown by the following example, which makes use of some specific properties of the interval to find a map which is locally uniformly expanding but neither locally one-to-one nor strongly uniformly or piecewise expanding.

Example 5.2.19. For each $n \in \mathbb{N}$, let $a_n = 1/2^n$ and $b_n = 3/2^{n+2}$, so that b_n is the midpoint of the line segment $[a_{n+1}, a_n]$.

Consider the continuous function $f \colon [0, 1] \to [0, 1]$, taking the value $f(0) = 0$, whose graph (see Figure 5.1) is the piecewise linear curve passing consecutively through the points

$$(1, 1),\ (a_1, 0),\ (b_1, 1),\ (a_2, 0),\ (b_2, 1/2),\ (a_3, 0),\ \ldots, (a_n, 0),\ (b_n, 1/2^{n-1}), \ldots$$

Note that the gradient on the interval (a_{n+1}, b_n) is 8 and on (b_n, a_n) is -8.

Let $\mu = 4/3$, $\nu = 1/4$ and let $\Lambda = \{0\} \cup \{a_n : n = 1, 2, 3, \ldots\}$. For any $0 < \epsilon < 1/4$, the interval $[0, \epsilon)$ is an open neighbourhood of points Λ and there is $n \in \mathbb{N}$ such that $a_{n+1} < \epsilon \leq a_n$.

We show will show that f is locally uniformly expanding on Λ. However $f\restriction_U$ is not one-to-one for any open neighbourhood U of zero, and f is clearly not uniformly or piecewise expanding on Λ.

First, we claim that

$$B_{\mu\epsilon}(f(0)) \subseteq f\big(B_\epsilon(0)\big).$$

Case 1: $3/2^{n+2} = b_n < \epsilon \leq a_n$. Then $[1/2^{n+1}, 3/2^{n+2}] \subseteq [0, \epsilon) \subseteq B_{1/2^n}(0)$ so that

$$f\big(B_\epsilon(0)\big) \supseteq f\big([0, 3/2^{n+2}]\big) \supseteq [0, 4/2^{n+1}] \supseteq B_{\mu/2^n}(0) \supseteq B_{\mu\epsilon}(0).$$

Case 2: $a_{n+1} < \epsilon \leq b_n$. Then $[3/2^{n+3}, 1/2^{n+1}] \subseteq [0, \epsilon) \subseteq [0, 3/2^{n+2})$ so $f\big(B_\epsilon(0)\big) \supseteq f\big([0, 3/2^{n+3}]\big) = [0, 4/2^{n+2}] \supseteq B_{2\mu/2^{n+2}}(0) \supseteq B_{\mu\epsilon}(0)$.

The proof of the claim is finished. Now fix any $a_n \in \Lambda \setminus \{0\}$ and consider $B_\epsilon(a_n) = (a_n - \epsilon, a_n + \epsilon)$. If $b_{n+1} \notin B_\epsilon(a_n)$, then $B_\epsilon(a_n) \subseteq (b_{n+1}, b_{n-1})$ and $f\big(B_\epsilon(a_n)\big) = [0, 8\epsilon) \supseteq B_{\mu\epsilon}(f(a_n))$. If $b_{n+1} \in (a_n - \epsilon, a_n + \epsilon)$, then $f\big(B_\epsilon(a_n)\big) = f\big([0, a_n + \epsilon)\big)$. The argument now follows by the argument at 0, since

$$B_{\mu\epsilon}(f(a_n)) = B_{\mu\epsilon}(0) \subset f(B_\epsilon(0)) \subset f(B_{\epsilon + a_n}(0)) = f([0, \epsilon + a_n)).$$

□

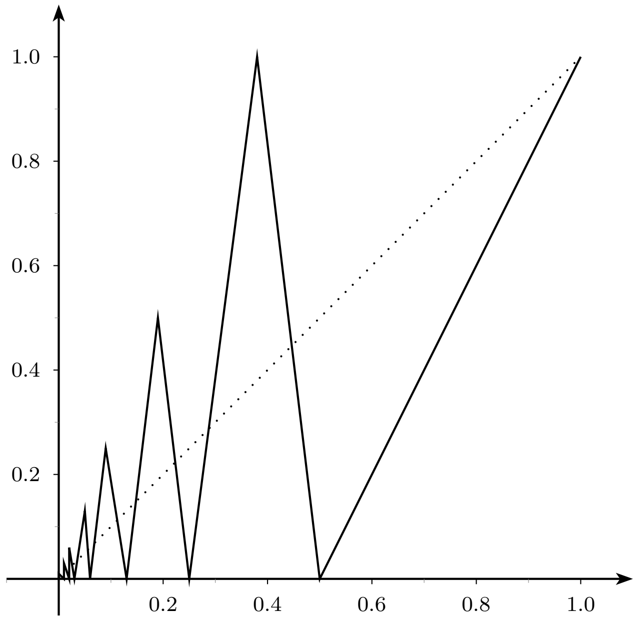

Figure 5.1: The graph of the function f from Example 5.2.19

The following results illustrate the similarities and differences between piecewise or strongly uniformly expanding maps and locally uniformly expanding maps.

Corollary 5.2.20. *Let $\Lambda \subseteq X$ closed and suppose that $f : X \to X$ is open and locally one-to-one on Λ. Then f is strongly uniformly expanding on Λ if and only if f is locally uniformly expanding on Λ.*

Proof. Follows directly from Lemmas 5.2.17 and 5.2.18. □

Theorem 5.2.21. *Let $\Lambda \subseteq X$ closed, then the following conditions are equivalent:*

(i) f is piecewise expanding on Λ and open on an open neighborhood Q of Λ.

(ii) f locally one-to-one on Λ and locally uniformly expanding on an open neighborhood W of Λ.

Proof. To prove the first implication, suppose that f is piecewise expanding on Λ and open on Q. Let μ be the constant and $U_1, \ldots, U_n$ be the open cover given in Definition 5.2.10, then by Lemma 5.2.17(ii) we get that f is locally uniformly expanding on $\overline{W}$, where $\overline{W} \subset Q \cap \bigcup_i U_i$ and W is an open neighborhood of Λ. To see that f is locally one-to-one on $\overline{W}$, let δ be the Lebesgue number for the open cover $U_1, \ldots, U_n$ of $\overline{W}$. Fix $x \in \overline{W}$ and $y, z \in B_\delta(x)$, then $y, z \in U_i$ for some i, and so $d(f(y), f(z)) \geq \mu d(y, z)$. Thus $f(z) \neq f(y)$ for any $y, z \in B_\delta(x)$, so f is locally one-to-one on $\overline{W}$.

Now suppose that f is locally uniformly expanding on W and locally one-to-one on Λ, and let μ, ν be as given in Definition 5.2.15. By Lemma 5.2.18 we get that f is piecewise expanding on Λ. To see that f is open on W, take any $x \in W$ and any $0 < \epsilon < \nu$, then $B_{\mu\epsilon}(f(x)) \subseteq f(B_\epsilon(x))$, which implies that $f(x) \in f(B_\epsilon(x))^\circ$. It is enough to put $Q = W$ and the proof is finished. □

Finally we present a result which is a strengthening of a result from [18], and relates expansivity to shadowing in a direct sense.

Theorem 5.2.22. *For $\Lambda \subseteq X$, if $f : X \to X$ is locally uniformly expanding on Λ then f has h-shadowing on Λ.*

Proof. Let $\epsilon > 0$, let μ, ν be as given in Definition 5.2.15, let $\epsilon' = \min\{\epsilon, \nu\}$ and let $\delta = (\mu - 1)\epsilon'$. Then for every $x \in \Lambda$

$$B_{\epsilon'+\delta}(f(x)) \subseteq B_{\mu\epsilon'}(f(x)) \subset f\left(B_{\epsilon'}(x)\right). \tag{5.11}$$

Suppose that $\{x_0, \ldots, x_m\} \subset \Lambda$ is a δ-pseudo-orbit. Notice that by (5.11) we have $B_{\epsilon'+\delta}(f(x_i)) \subset f(B_{\epsilon'}(x_i))$ for $i = 0, 1, \ldots, m-1$, so

$$B_{\epsilon'}(x_{i+1}) \subseteq f(B_{\epsilon'}(x_i)) \qquad \text{for } i = 0, 1, \ldots, m-1. \tag{5.12}$$

Let $J_0 = B_{\epsilon'}(x_0)$ and then define inductively $J_i = f^{-i}\big(B_{\epsilon'}(x_i)\big) \cap J_{i-1}$.

Clearly the J_i are nested, and by (5.12) we can prove by induction that $f^i(J_i) = B_{\epsilon'}(x_i)$, since

$$\begin{aligned} B_{\epsilon'}(x_i) \supset f^i(J_i) &\supset B_{\epsilon'}(x_i) \cap f^i(J_{i-1}) \supset B_{\epsilon'}(x_i) \cap f(B_{\epsilon'}(x_{i-1})) \\ &\supset B_{\epsilon'}(x_i) \cap B_{\epsilon'}(x_i) = B_{\epsilon'}(x_i). \end{aligned}$$

In particular, $f^i(J_m) \subset B_{\epsilon'}(x_i)$, for $i = 0, 1, \ldots, m$ and $f^m(J_m) = B_{\epsilon'}(x_m)$, thus there is a point $y \in J_m$ such that $f^i(y) \in B_{\epsilon'}(x_i)$ and for which $f^m(y) = x_m$. □

Figure 5.2 shows what we call the "family tree" of implications between shadowing and expansivity, illustrating how all the various properties of shadowing and expansivity relate to one another. Arrows represent an implication between two properties, with any extra conditions in parentheses where required.

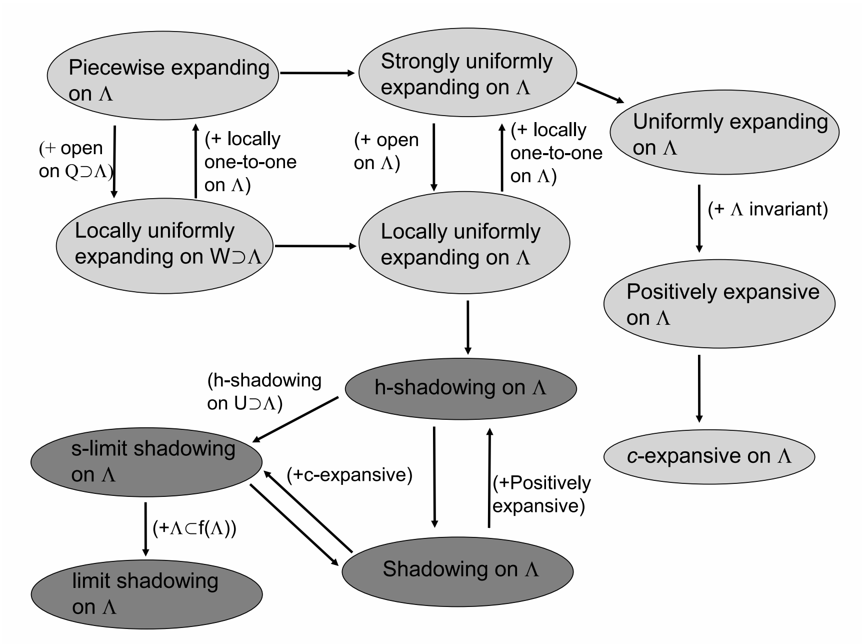

Figure 5.2: The "family tree", relating the various types of expansivity and shadowing.

Chapter 6

ω-Limit Sets in Various Spaces

In Chapter 4 we proved our first main result linking internal chain transitivity (ICT) to ω-limit sets (Theorem 4.4.8). This seems a natural characterization, particularly in view of Theorem 4.1.9, and we pursue this link throughout this chapter, proving many results showing when this characterization applies. We also introduce another dynamical property which seems more closely linked to ω-limit sets (Definition 6.1.4), which is based upon a similar property introduced by Balibrea and La Paz in [4].

The material in this chapter draws upon that in the previous chapter, often relying upon a form of pseudo-orbit shadowing to attain the required characterization. Some of the material is again the result of collaboration with Good, Oprocha and Raines [8], and as before, where a result is due to collaboration or another author we will indicate this clearly.

6.1 ω-Limit Sets of Maps on Compact Metric Spaces

We begin this section with a result that uses a version of shadowing to show when ω-limit sets are characterized by ICT. This result, from [8], is the first in a sequence of results which provide different conditions for when this characterization applies.

Theorem 6.1.1. *Let (X, f) be a dynamical system. If f has limit shadowing on a closed set $\Lambda \subseteq X$, then for any closed subset $A \subset \Lambda$ the following conditions are equivalent:*

1. *A has weak incompressibility,*
2. *A is ICT,*
3. *there is a point $x_A \in X$ such that $A = \omega(x_A, f)$.*

In particular, if f has limit shadowing then properties (1–3) are equivalent properties of any closed invariant subset of X.

Proof. If A is an ω-limit set then it necessarily has weak incompressibility as was shown in [44]. This gives (3) $\implies$ (1). By Theorem 3.2.11 we have (1) $\iff$ (2). The last implication (2) $\implies$ (3) follows by the fact that every closed ICT set A is the ω-limit set of an asymptotic pseudo-orbit $\{x_n\}_{n\in\mathbb{N}}$ by Lemma 3.2.9, thus there is a point $x_A \in X$ whose orbits asymptotically shadows $\{x_n\}_{n\in\mathbb{N}}$, giving $A = \omega(x_A, f)$. $\square$

In light of Lemma 5.1.15 and Theorem 5.2.5 we get the following corollary [8] from Theorem 6.1.1.

Corollary 6.1.2. *Let (X, f) be a dynamical system, $f : X \to X$ be surjective and $\Lambda \subseteq X$ be closed. If f is topologically hyperbolic, then the following conditions are equivalent on any closed subset $A \subset \Lambda$:*

1. *A has weak incompressibility,*
2. *A is ICT,*
3. *there is a point $x_A \in X$ such that $A = \omega(x_A, f)$.*

Example 6.1.3. In Remark 4.4.9 we give an example of a set which is ICT and CINE but which is not an ω-limit set. Thus the condition of topological hyperbolicity cannot be dropped from Corollary 6.1.2.

We also note that the implied condition of shadowing cannot be dropped on its own. As a specific example, in [7] we give an example of a subset A of a sofic shift S which is CINE and ICT but not an ω-limit set of any point in S (although it will be the ω-limit set of some point in the full shift by Theorem 4.1.9). We will show that the shift map $\sigma : S \to S$ is c-expansive but not shadowing on S.

The sofic shift in question is generated by the graph G with vertices a and b with directed edges $[a, a]$ labelled 0, $[a, a]$ labelled 1, $[a, b]$ labelled 2 and $[b, b]$ labelled 0 (see Figure 6.1).

So sequences in the space S are all those which can be formed by following bi-infinite paths around the graph G. For example a sequence can have any combination of 1's and 0's, but once it has a 2 it must then be followed by an infinite string of '0's. The metric d and shift map σ are as defined in Section 4.1. S is a compact metric space and σ is continuous on S, so this example is relevant to the discussion (see [30]).

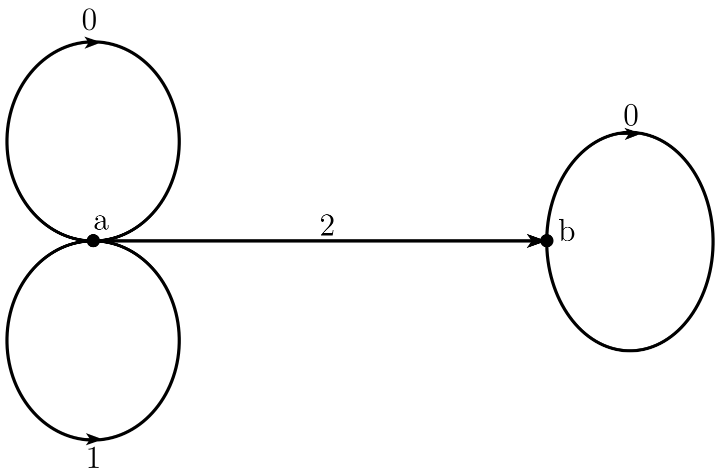

Figure 6.1: The sofic space S is generated by the graph G.

The set A is given as all forward and backward shifts of the points $\mathbf{0} = (0^{-\infty}.0^{\infty})$, $\mathbf{1} = (1^{-\infty}.1^{\infty})$, $\mathbf{s} = (0^{-\infty}.1^{\infty})$, $\mathbf{t} = (1^{-\infty}.20^{\infty})$, where the decimal point lies to the immediate left of the 0^{th} element. A is CINE since $\mathbf{0}$ and $\mathbf{1}$ are fixed, shifts of $\mathbf{s}$ tend towards $\mathbf{1}$ and shifts of $\mathbf{t}$ tend towards $\mathbf{0}$. Moreover A is ICT since for any $n \in \mathbb{N}$ there are forward shifts of $\mathbf{s}$ and $\mathbf{t}$ which agree with central segments of $\mathbf{1}$ and $\mathbf{0}$ of length n respectively, and backward shifts of $\mathbf{s}$ and $\mathbf{t}$ which agree with central segments of $\mathbf{0}$ and $\mathbf{1}$ of length n respectively, so there is a $1/2^n$-pseudo-orbit joining any two points of A.

To see that σ is c-expansive, notice that for any two distinct sequences in the shift space there will be a point where the two sequences have different symbols, thus if we shift to this point these two shifted sequences will be a distance 1 apart, thus σ is c-expansive with constant 1. To see that σ does not have the shadowing property on S, consider a pseudo-orbit which contains at least two instances of the sequence $\mathbf{t}$. No such orbit can be ϵ-shadowed by a real orbit for any $0 < \epsilon < 1$, since such an orbit would be a sequence containing more than one central segment of $\mathbf{t}$ i.e more than one occurrence of a 2, which can't happen for any sequence in S. This is related to the reason why A cannot be the ω-limit set of any point in S; any such point must contain arbitrarily long central segments of $\mathbf{t}$ infinitely often, which as mentioned above cannot be the case.

$\square$

In [4], Balibrea and La Paz define a property set property for interval maps which seems closely linked to shadowing, but better approximates the dynamics of maps on their ω-limit sets. To this end we define the following, more general set property for maps on compact metric spaces which has similarities to that in [4].

Definition 6.1.4 (Dynamically Indecomposable)**.** For a dynamical system (X, f), we say that a set $\Lambda \subseteq X$ is *dynamically indecomposable* if for every $\epsilon > 0$, every pair of points $x, y \in \Lambda$ and every pair of open sets U, V such that $x \in U$ and $y \in V$ there is $m > 0$ and a sequence of regularly closed sets $J_0, J_1, \ldots, J_m$ (see Definition 1.1.3) for which

1. $x \in (J_0)^\circ,\ J_0 \subseteq U$,
2. $J_{i+1} \subseteq f(J_i)$ for $i = 0, 1, \ldots, m-1$,
3. $J_i \subseteq B_\epsilon(\Lambda)$ for $i = 0, 1, \ldots, m$,
4. $y \in (J_m)^\circ,\ J_m \subseteq V$.

Next we present Theorem 6.1.5 and Theorem 6.1.6 (the latter being similar to a result in [4]). The two results, together with Theorem 6.1.7, relate dynamical indecomposability to shadowing and ω-limit sets. They appear in [8] but are due primarily to this author.

Theorem 6.1.5. *Let (X, f) be a dynamical system. If $\Lambda \subseteq X$ is ICT, f has h-shadowing on Λ and is open on a neighbourhood of Λ, then Λ is dynamically indecomposable.*

Proof. Let $\epsilon > 0$, pick $x, y \in \Lambda$ and let U and V be open with $x \in U$ and $y \in V$. Certainly there is an $\eta > 0$ for which $B_\eta(x) \subseteq U$ and $B_\eta(y) \subseteq V$. There is also ξ such that f is open on $B_\xi(\Lambda)$. Let $\epsilon' = \min\{\eta, \epsilon, \xi/2\}$. Let δ be provided for $\epsilon'/2$ by h-shadowing. By the assumptions Λ is ICT, so there is a δ-pseudo-orbit $\{x_0 = x, x_1, \ldots, x_m = y\} \subset \Lambda$. Thus there is a $z \in X$ for which $d(f^i(z), x_i) < \epsilon'/2$ and $f^m(z) = x_m = y$.

So let $J_0 = \overline{B_{\epsilon'/2}(x_0)}$ and for $i = 0, 1, \ldots, m-1$, let

$$J_{i+1} = \overline{f(J_i) \cap B_{\epsilon'/2}(x_{i+1})}.$$

The J_i are regularly closed by the openness and continuity of f. We claim that

- $x \in (J_0)^\circ,\ J_0 \subseteq U$;
- $J_{i+1} \subseteq f(J_i)$ for $i = 0, 1, \ldots, m-1$;
- $J_i \subseteq B_\epsilon(\Lambda)$ for $i = 0, 1, \ldots, m$;
- $y \in (J_m)^\circ,\ J_m \subseteq V$.

Since f is open at $f^i(z) \in B_\xi(\Lambda)$, $f^i(z) \in (J_i)^\circ \neq \emptyset$, and the first three statements hold immediately. Furthermore, there is $0 < r < \epsilon'/2$ such that

$$f((J_i)^\circ) \supset B_r(f^{i+1}(z))$$

and in particular,

$$y = f^m(z) \in f((J_{m-1})^\circ) \cap B_r(x_m) \subset (J_m)^\circ.$$

This proves that the claim holds, and as an immediate consequence we see that Λ is dynamically indecomposable. $\square$

Theorem 6.1.6. *Let (X, f) be a dynamical system, and suppose that $\Lambda \subseteq X$ is a closed set which is dynamically indecomposable. Then $\Lambda = \omega(x_\Lambda, f)$ for some $x_\Lambda \in X$.*

Proof. Since Λ is compact, there is a sequence of points $\{z_n \ : \ n \in \mathbb{N}\}$ in Λ such that $\Lambda = \overline{\{z_n\}_{n\in\mathbb{N}}}$. Enumerate the collection $\{B_{1/p}(z_n) \ : \ n, p \in \mathbb{N}\}$ as $\{B_k \ : \ k \in \mathbb{N}\}$, then for every k there is an $n_k \in \mathbb{N}$ such that $z_{n_k} \in B_k$. We define a sequence of natural numbers $\{m_n \ : \ n \in \mathbb{N}\}$ and a sequence of regularly closed sets $\{J_k \ : \ k \in \mathbb{N}\}$ as follows.

1. For $m_1 = 1$, let J_{m_1} be the closure of any basic open subset of B_1 such that $z_{n_1} \in (J_{m_1})^\circ$

2. Given J_{m_i} such that $z_{n_i} \in (J_{m_i})^\circ$ consider the point $z_{n_{i+1}} \in B_{i+1}$. Since Λ is dynamically indecomposable, we can define basic open sets $I^0_{m_i}$ and $\{I_j \ : \ m_i + 1 \leq j \leq m_{i+1}\}$ whose closures $J^0_{m_i}$ and $\{J_j \ :$

$m_i + 1 \leq j \leq m_{i+1}\}$ respectively are contained in $B_{1/i}(\Lambda)$, and for which $z_{n_i} \in (J^0_{m_i})^\circ \subseteq (J_{m_i})^\circ$, $J_{m_i+1} \subseteq f(J^0_{m_i})$, $J_{j+1} \subseteq f(J_j)$ for $j = m_i + 1, \ldots, m_{i+1} - 1$, and $z_{n_{i+1}} \in (J_{m_{i+1}})^\circ \subseteq B_{i+1}$.

By the construction of the J_k's, for every $k \in \mathbb{N}$ there is a closed set $D \subseteq J_{k-1}$ such that $f(D) = J_k$. Hence, for every $k \in \mathbb{N}$ there is a closed set $J^{(k)} \subseteq J_0$ such that $f^k(J^{(k)}) = J_k$. The $J^{(k)}$ are nested, so by compactness $K = \bigcap_{k \in \mathbb{N}} J^{(k)} \neq \emptyset$.

For $x_\Lambda \in K$, $f^i(x_\Lambda) \in J_i$ for every $i \in \mathbb{N}$, so certainly $\Lambda \subset \omega(x_\Lambda, f)$. Suppose that $z \in X \setminus \Lambda$, then there are disjoint open sets U and V for which $z \in U$ and $\Lambda \subseteq V$. Since $\bigcup\{J_j : m_i \leq j \leq m_{i+1}\} \subseteq B_{1/i}(\Lambda)$ there is an $N \in \mathbb{N}$ for which $f^n(x_\Lambda) \in V$ for every $n \geq N$, hence $z \notin \omega(x_\Lambda, f)$. Thus $\Lambda = \omega(x_\Lambda, f)$. □

Thus using dynamical indecomposability, we can make a link between internally chain transitive sets and ω-limit sets for maps which are strongly uniformly expanding (and thus also h-shadowing) on the given set. This is made precise in Theorem 6.1.7.

Theorem 6.1.7. *Let (X, f) be a dynamical system. If f is strongly uniformly expanding on the closed set Λ and open on a neighbourhood of $\Lambda \subseteq X$ then for any subset $A \subset \Lambda$ the following are equivalent:*

1. *A has weak incompressibility;*

2. *A is ICT;*

3. *$A = \omega(x_A, f)$ for some $x_A \in X$.*

Proof. By the same arguments as in the proof of Theorem 6.1.1 it suffices to prove (2) $\Longrightarrow$ (3). By the fact that f is strongly uniformly expanding on A and open on a neighbourhood of A, Lemma 5.2.17 implies that f is locally uniformly expanding on A. This in turn gives us that f has h-shadowing on A by Theorem 5.2.22. Then $A = \omega(x_A, f)$ for some $x_A \in X$ follows by Theorems 6.1.5 and 6.1.6. $\square$

Since strongly uniformly expanding is a weaker property than both piecewise uniformly expanding and locally uniformly expanding, we see that Theorem 6.1.7 applies to a number of maps on various spaces. One example is the class of piecewise linear interval maps, including tent maps, which are among the maps explored in the next section.

6.2 ω-Limit Sets of Interval Maps

We begin this section by analyzing piecewise expanding maps on a compact interval I. The structure of the interval allows us to make certain useful observations about such maps, which will be useful in proving some of the results which follow.

There is a large class of piecewise expanding interval maps, namely the class of piecewise linear maps, where each interval of linearity has gradient greater than 1. If f is such a map and Λ is a closed set which contains no local extremum, then it is easy to see that f is piecewise expanding on Λ.

Remark 6.2.1. Notice that a continuous interval map $f : I \to I$ cannot be piecewise expanding on any set $\Lambda \subseteq I$ which contains a local extremum, since there must be infinitely many pairs of points on either side of the extremum which map to the same point. Furthermore, such a map is trivially open on any set Λ which does not contain any of the local extrema. Thus if an interval map $f : I \to I$ is piecewise expanding on the closed set $\Lambda \subseteq I$, it is also open on a neighbourhood of Λ.

Proposition 6.2.2 and Theorem 6.2.3 are both in [8] but are again due to this author.

Proposition 6.2.2. *If $f : I \to I$ is a map which is piecewise expanding on a closed set Λ, then there is an open set $U \supset \Lambda$ such that f is locally uniformly expanding on $\overline{U}$.*

Proof. Remark 6.2.1 gives us that f is open on a neighbourhood of Λ, and by Lemma 5.2.17 we have that f is locally uniformly expanding on a neighbourhood V of Λ. Then by normality there is a set U such that $\Lambda \subseteq U \subset \overline{U} \subseteq V$. □

The next result gives us a test for when ω-limit sets of interval maps are characterized by ICT. The result is similar to that of Theorem 6.1.7, but we use different results to prove it, thus demonstrating the robustness of the theory.

Theorem 6.2.3. *Suppose that $\Lambda \subseteq I$ is closed and the continuous map $f : I \to I$ is piecewise expanding on Λ. Then for any closed subset $A \subset \Lambda$ the following are equivalent:*

1. *A has weak incompressibility;*

2. *A is ICT;*

3. $A = \omega(x_A, f)$ *for some* $x_A \in I$.

Proof. As previously, we only have to prove (2) $\implies$ (3). By Proposition 6.2.2, there is an open set $U \supset A$ such that f is locally uniformly expanding on $\overline{U}$, and by Theorem 5.2.22, f has h-shadowing on $\overline{U}$, indeed on U itself. Furthermore, by Proposition 3.2.5 A is invariant. Thus by Corollary 5.1.17 f has limit shadowing on A, and finally by Theorem 6.1.1 we obtain that $A = \omega(x_A, f)$ for some $x_A \in I$. □

Theorem 6.2.3 also has similarities to Theorem 4.4.8, and taking these two results together it would be reasonable to conjecture that a locally pre-critical interval map has some shadowing or expansive properties on a set which contains no critical points. However as we have already mentioned, a map which is locally pre-critical need not be uniformly expanding, even on a set which contains none of the critical points (Corollary 4.3.7), nor can we prove whether such a map has any form of shadowing. These remain unanswered questions.

Throughout this chapter (and previous chapters) we have proved cases of when an internally chain transitive set is an ω-limit set. We have seen that there is a direct link in the case of shifts of finite type (Theorem 4.1.9), but when we move to general compact metric spaces we need to place further restrictions on the behaviour of the map on the set in question (Theorems 6.1.1, 6.1.7 and Corollary 6.1.2), of which the interval is a specific case (Theorems 4.4.8 and 6.2.3).

However, there are cases of interval maps where these extra conditions are satisfied trivially, meaning that the characterization of ω-limit sets by

ICT applies for the whole space; Corollary 6.2.5 [8] is such a case.

Theorem 6.2.4. *The full tent map* $f_2 : [0,1] \to [0,1]$ *has h-shadowing.*

Proof. Follows trivially from Theorem 5.2.22, since the full tent map is locally uniformly expanding (on the whole interval). □

Corollary 6.2.5. *For the full tent map* $f_2 : [0,1] \to [0,1]$ *and any closed subset* $A \subset [0,1]$, *the following are equivalent:*

1. *A has weak incompressibility;*

2. *A is ICT;*

3. $A = \omega(x_A, f_2)$ *for some* $x_A \in [0,1]$.

Proof. To show that (2) $\implies$ (3), notice that by surjectivity, f_2 is open on any neighbourhood of A, so by Theorem 6.1.5 A is dynamically indecomposable. Furthermore A is closed, so by Theorem 6.1.6 there is some $x_A \in [0,1]$ for which $A = \omega(x_A, f_2)$. The other implications are dealt with as in Theorem 6.1.1. □

Remark 6.2.6. Coven, Kan and Yorke show in [18] that in the family of tent maps $\{f_s\}$, f_s has shadowing precisely when the critical point c is recurrent and obeys certain parity rules with respect to its orbit (a result which is extended to a larger class of piecewise linear maps in [16]). Although the full tent map f_2 has h-shadowing, this is not true for other tent maps, regardless of the behaviour of the critical point. The reason for this is that the standard form of the tent map f_s is not surjective for $1 < s < 2$, so for any $0 < \delta \leq 1 - f_s(c)$ the point $f_s(c) + \delta$ has no pre-image but can be the final point in a δ-pseudo-orbit, hence the map cannot have h-shadowing.

By Theorem 4.4.8, we see that for a tent map f_s with critical point c, an ICT subset of the interval is an ω-limit set provided it does not contain $f_s(c)$. To provide proof that this is not a redundant condition, even in tent maps, in a forthcoming paper we will give an example of a tent map f which has an ICT set containing $f(c)$ which is not an ω-limit set.

Example 6.2.7. In this example we use the kneading theory to identify an ICT set for a tent map f, containing $f(c)$, which does not have the shadowing property, yet which is an ω-limit set. We define the symbolic dynamics for the family of tent maps $\{f_s\}$ with slope s and critical point c as in Example 4.4.3.

Let W be the word 100, and consider the symbolic sequence $K = W1^{n_1}W1^{n_2}W1^{n_3}...$, where $\{n_i\}$ is an increasing sequence of positive, odd integers; then by [17, Lemma III.1.6], K is the kneading sequence of some tent map f. Consider the set Λ' defined as follows:

$$\Lambda' = \{\sigma^j(1^k K),\ \sigma^j(1^k W 1^\infty)\ :\ j,k \in \mathbb{N}\}.$$

It can be shown that Λ' is a set of limit-itineraries of points in a set $\Lambda \subset [0,1]$. Indeed 1^∞ is the itinerary of the unstable fixed point, sequences of the form $1^k W 1^\infty$ are itineraries of pre-images of the fixed point along a certain pre-image branch, and sequences of the form $1^k K$ are limit itineraries of pre-images of c which come from the fixed point. It is not hard to show that Λ' is closed and ICT (see Section 4). By [6, Theorem 5.4] we also see that

$\Lambda = \omega(z)$, where z is the point with itinerary

$$It(z) = W1^{k_1}W1^{n_1}W1^{k_2}W1^{n_1}W1^{n_2}W1^{k_3}W1^{n_1}W1^{n_2}W1^{n_3}...,$$

and where for each $j \in \mathbb{N}$, $k_j > n_j$ is an odd integer.
However, since K is not recurrent we get that c is not recurrent and so f does not have the shadowing property by [18, Theorem 4.2].

$\square$

We end this chapter by noting that whilst we have proved many cases where ICT is equivalent to being an ω-limit set, it is still far from clear exactly what other conditions we need to place upon a set to ensure this is the case in general. This question remains open.

Chapter 7

Concluding Remarks

7.1 Internal Chain Transitivity

Whilst there are many dynamical properties which can be observed in ω-limit sets, many of which we have investigated here, it seems that the most useful is that of internal chain transitivity (ICT). We have seen that in terms of shifts of finite type, ICT gives us a full description of maps on their ω-limit sets (Chapter 4). Furthermore we have found that the same is true of a certain class of interval maps, when we restrict our attention to sets which contain no pre-critical points (Chapters 4 and 6). Whilst we acknowledge that ICT gives us a picture of the behaviour of maps on their ω-limit sets which is far from complete, and that there exist other characterizations in the literature, we believe that characterizations in terms of ICT are helpful since they allow us to make comparisons about the behaviour of different maps with respect to a single property. For instance, the fact that ω-limit sets are fully described in terms of ICT for shifts of finite type but not interval maps, or even other shift maps such as sofic shifts (Chapter 5), indicate a fundamental structural difference in these spaces, and this may help us to isolate further properties which will provide a more general characterization. Furthermore, ICT has many applications in the applied sciences, such as computer science (as was hinted at in Chapter 5) and the biological sciences [15], [26], [53], and is a topic of research in its own right [8], [26], [38]. Thus we feel that a full characterization of ω-limit sets which incorporates this property would be useful and instructive, and we will pursue such a description in future work.

7.2 Shadowing

In pursuing an understanding of how and when ICT, which describes the dynamics on a set in terms of pseudo-orbits, characterizes ω-limit sets, it was necessary that we concentrate our focus on maps which have a element of pseudo-orbit shadowing (Chapters 5 and 6), at least on their ω-limit sets (which are intrinsically ICT). This is somewhat misleading, as it is clear from Example 6.2.7, that maps are not generally shadowing on their ω-limit sets. However, it would seem to be implied by the various results throughout this (and other) work that shadowing of some form is likely to be required in any general description of ω-limit sets in terms of ICT.

7.3 The Criticality of Critical Points

In Chapter 4, to extract a characterization of ω-limit sets based on the property of internal chain transitivity (ICT), we rely heavily upon the fact that the itinerary map is continuous at points whose orbits do not contain critical points, and focus on ω-limit sets containing only such points. We note that far from being vacuous, this case contains an uncountable number of examples, and also provides an example of when the set of itineraries of a subset of the interval is a shift space. Then in Chapter 6 we note that for certain expanding maps with shadowing, a closed, invariant, ICT set is always an ω-limit set, and the existence of shadowing for interval maps depends upon the behaviour of the critical points of the map. Moreover we prove that locally pre-critical maps are h-shadowing on a closed, invariant, non-empty set provided the set contains no image of a critical point. Thus we see that in each approach we have taken, the existence (or otherwise) of critical points in the object set, together with the behaviour of the map with regards to its critical points, is crucial to obtaining the result we require.

Geometrically speaking, for a set to be the ω-limit set of a point x we must have that the orbit of x meets arbitrarily small neighbourhoods of each point in the set infinitely often. If we consider the tent map as an example, we see that for the image of the critical point (call this $f(c)$), this can only occur from below, since no points get mapped above $f(c)$. Eliminating critical points from the set avoids this problem, but to include them (and thus all iterates thereof) we must give special consideration to how the map behaves on a set containing their images to ensure that it is an ω-limit set.

Several authors have addressed this problem (see for example [1], [4], [9], [10]), and we note that the solutions they provide are far from trivial, but the full analysis of their results is outside the scope of this work. In forthcoming research we will attempt to tackle the issue of ICT, ω-limit sets and critical points by refining our analytic methods further, particularly that of symbolic dynamics as is described below.

7.4 Symbolic Dynamics

A large proportion of the new material in this work uses symbolic dynamics constructively in proving results about ω-limit sets (Chapter 4). We have refined existing material in this area, proving results which make this form of analysis possible, and in so doing we obtain certain results which we are yet to fully explain using more conventional methods (Chapters 5 and 6).

A key difficulty in using symbolic analysis is that we have relied upon the continuity of the itinerary map away from pre-critical points, which immediately limits our scope to sets which don't contain pre-critical points. We believe, however, that there is scope for using symbolic dynamics to prove more general results, and to avoid such restrictions in the future we will define a new topology on the shift space, with regards to which the itinerary map is continuous at every point. Thus topological properties of subsets of the interval, such as weak incompressibility (see Chapter 2), will be evident in their images under the itinerary map, and we can use the structure of the shift space to our advantage (as in Theorem 4.4.8). The situation is still far from simple however, as the new topology (unlike the Tychonoff product of the discreet topology used in Chapter 4) does not admit a metric. This is work in progress which we hope will result in a greater understanding of ω-limit sets of interval maps.

Appendices

A: Redundancy of Sensitive Dependence in Devaney's Definition of Chaos

In [5], Banks et al prove that for maps on metric spaces SDIC is redundant in the definition of chaos, implied by transitivity and density of periodic points. Furthermore in [51] it is shown that for maps of the interval, transitivity implies both a dense set of periodic points and SDIC. Here we prove the first of the two results.

Theorem. *For a metric space (X, d), if a continuous map $f : X \to X$ is topologically transitive, with the set of periodic points for f dense in X, then f has sensitive dependence on initial conditions (and is thus chaotic).*

Proof. Suppose that $f : X \to X$ is topologically transitive and its periodic points are dense in X. Notice that for any periodic points q and r with disjoint orbits $\theta(q)$ and $\theta(r)$, there is a minimum distance δ_0 between the two orbits, since each has a finite number of points. Thus for every $x \in X$, x is a distance at least $\delta_0/2$ away from either $\theta(q)$ or $\theta(r)$. We will show that f has sensitive dependence to initial conditions with sensitivity constant $\delta := \delta_0/8$.

So pick any $x \in X$, and let N be any open neighbourhood of x. Define

$$U := N \cap B_\delta(x)$$

Since the periodic points are dense, there is a periodic point $p \in U$; of period n say. i.e. $f^n(p) = p$ but $f^k(p) \neq p$ for any $k < n$. By the initial observation (and without loss of generality), let $\theta(q)$ be the orbit which is a distance at least $\delta_0/2 = 4\delta$ from x.

Define

$$V := \bigcap_{i=0}^{n} f^{-i}\left[B_\delta\big(f^i(q)\big)\right]$$

So V is the set of all points mapped under f^i to within δ of $f^i(q)$ for *every* $0 \leq i \leq n$. V is a finite intersection of open sets, so is open, and since f is continuous V is an open neighbourhood of q. f is topologically transitive, so there is a $y \in U$ such that for some $k \geq 0$, $f^k(y) \in V$.

Let j equal the integer part of $k/n+1$. Thus $k/n+1 \geq j$, so $nj - k \leq n$. Also, $j > k/n$ so $nj - k > 0$ and we can conclude that $nj - k \geq 1$. Putting this together we get that $1 \leq nj - k \leq n$.

Now

$$f^{nj}(y) = f^{nj-k}\big(f^k(y)\big) \in f^{nj-k}(V) \subseteq B_\delta\big(f^{nj-k}(q)\big)$$

by definition of V. We know that $f^{nj}(p) = p$, and by the triangle inequality we also have

$$\begin{aligned} d\big(x, f^{nj-k}(q)\big) &\leq d(x,p) + d\big(p, f^{nj-k}(q)\big) \\ &\leq d(x,p) + d\big(p, f^{nj}(y)\big) + d\big(f^{nj}(y), f^{nj-k}(q)\big) \end{aligned}$$

So

$$
\begin{aligned}
d\big(p, f^{nj}(y)\big) &\geq d\big(x, f^{nj-k}(q)\big) - d\big(f^{nj}(y), f^{nj-k}(q)\big) - d(x,p) \\
&> 4\delta - \delta - \delta
\end{aligned}
$$

since $f^{nj-k}(q) \in \theta(q)$, $f^{nj}(y) \in B_\delta\big(f^{nj-k}(q)\big)$ and $p \in B_\delta(x)$ as noted above. So

$$
d\big(p, f^{nj}(y)\big) = d\big(f^{nj}(p), f^{nj}(y)\big) > 2\delta
$$

But

$$
d\big(f^{nj}(p), f^{nj}(x)\big) + d\big(f^{nj}(x), f^{nj}(y)\big) \geq d\big(f^{nj}(p), f^{nj}(y)\big) > 2\delta
$$

so either $d\big(f^{nj}(p), f^{nj}(x)\big) > \delta$ or $d\big(f^{nj}(x), f^{nj}(y)\big) > \delta$. And since both p and y are in N, we have found an element of N which is eventually mapped a distance at least δ away from x, so we are done. $\square$

B: Partial Orders

A set P is a *partially ordered set* (with partial order $\leq$) if for all $x, y, z \in P$ we have the following properties:

1. $x \leq x$;
2. If $x \leq y$ and $y \leq z$ then $x \leq z$;
3. If $x \leq y$ and $y \leq x$ then $x = y$.

A set P is a *totally ordered set*, or *chain* (with total order $\leq$) if it is a partially ordered set and furthermore for every $x, y \in P$ we have that either $x \leq y$

or $y \leq x$. For example, if we define a partial order $\preceq$ on a set X by saying that for subsets $U, V \subset X$, $U \preceq V$ if $U \subseteq V$, the collection of subsets of X is partially ordered under this rule. Moreover, a collection of nested sets forms a chain.

We also state a theorem by Hausdorff about partially ordered sets, which can be found in the Appendices of [43] for example.

Theorem (Hausdorff's Maximal Principle)**.** *Every partially ordered set contains a maximal chain.*

We note here that Hausdorff's Maximality Principle is equivalent to the Axiom of Choice, which states that for any collection X of non-empty sets, we can define a function f on X such that for each $C \in X$, $f(C)$ is a well-defined element of C.

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Index