

How Potential Investments may Change the Optimal Portfolio for the Exponential Utility

Walter Schachermayer*
Vienna University of Technology

Working Version, February 21, 2002
(First Version: December 28, 2000)

Abstract

We show that, for a utility function $U : \mathbb{R} \rightarrow \mathbb{R}$ having *reasonable asymptotic elasticity*, the optimal investment process $\widehat{H} \cdot S$ is a super-martingale under each equivalent martingale measure \mathbf{Q} , such that $\mathbf{E}[V(\frac{d\mathbf{Q}}{d\mathbf{P}})] < \infty$, where V is conjugate to U . Similar results for the special case of the exponential utility were recently obtained by Delbaen, Grandits, Rheinländer, Samperi, Schweizer, Stricker as well as Kabanov, Stricker.

This result gives rise to a rather delicate analysis of the “good definition” of “allowed” trading strategies H for the financial market S . One offspring of these considerations leads to the subsequent — at first glance paradoxical — example.

There is a financial market consisting of a deterministic bond and two risky financial assets $(S_t^1, S_t^2)_{0 \leq t \leq T}$ such that, for an agent whose preferences are modeled by expected exponential utility at time T , it is optimal to constantly hold one unit of asset S^1 . However, if we pass to the market consisting only of the bond and the first risky asset S^1 , and leaving the information structure unchanged, this trading strategy is not optimal any more: in this smaller market it is optimal to invest the initial endowment into the bond.

Key words: utility maximization, incomplete markets, duality.

JEL classification: G11, G12, C61

AMS 1991 subject classifications: Primary 90A09, 90A10; secondary 90C26.

*Support by the Austrian Science Foundation (FWF) under the Wittgenstein-Preis program Z36-MAT and grant SFB#010 and by the Austrian National Bank under grant 'Jubiläumfondprojekt Number 8699' is gratefully acknowledged. Thanks go to P. Grandits and Th. Rheinländer for discussions on the topic of this paper and specially to Ch. Stricker for pointing out an erroneous argument in a previous version of this paper. The present version has greatly benefited from numerous suggestions and comments by Y. Kabanov and Ch. Stricker as well as by two anonymous referees, for which I also thank. A final version of this paper has been submitted for publication under the title “A Super-Martingale Property of the Optimal Portfolio Process”.

1 Introduction

This paper continues the study of optimal investment in incomplete financial markets, where we consider a utility function $U : \mathbb{R} \rightarrow \mathbb{R}$ s.t.

$$U \text{ is smooth, strictly concave, } U'(-\infty) = \infty, \quad U'(\infty) = 0, \quad (1)$$

and satisfies the condition of *reasonable asymptotic elasticity*, i.e.,

$$\limsup_{x \rightarrow \infty} \frac{xU'(x)}{U(x)} < 1 \quad \text{and} \quad \liminf_{x \rightarrow -\infty} \frac{xU'(x)}{U(x)} > 1, \quad (2)$$

For the significance of the latter condition in the context of utility maximisation we refer to [S01]. The arch-example of a function U satisfying (1) and (2) is the exponential utility $U(x) = -e^{-\gamma x}$, for $\gamma > 0$.

We adopt the same setting as in [S01] to which we refer for unexplained notation: Fixing the time horizon $T \in]0, \infty[$, $(S_t)_{0 \leq t \leq T}$ will denote a locally bounded \mathbb{R}^d -valued semimartingale, modeling the price process of d financial assets. We work in discounted terms, i.e., the bond is assumed to be constant. As we consider only the case of a closed time index set $[0, T]$, all martingales will be uniformly integrable.

We denote by $\mathcal{M}^e(S)$ (resp. $\mathcal{M}^a(S)$) the set of probability measures \mathbf{Q} which are equivalent (resp. absolutely continuous with respect) to \mathbf{P} and such that S is a local \mathbf{Q} -martingale. We assume that

$$\mathcal{M}^e(S) \neq \emptyset \quad (3)$$

The basic problem is to find, for given initial endowment $x \in \mathbb{R}$, a trading strategy $(H_t)_{0 \leq t \leq T}$ such that the expected utility of the terminal wealth $x + (H \cdot S)_T = x + \int_0^T H_t dS_t$ becomes maximal:

$$\mathbf{E}[U(x + (H \cdot S)_T)] \longrightarrow \max! \quad (4)$$

We have been deliberately vague on the set of “allowed” trading strategies H over which we maximize in (4); in fact, the choice of the “good definition” of this class of trading strategies is rather subtle and constitutes the main topic of this paper.

The minimal requirement to impose on an “allowed” trading strategy H is that the stochastic integral $(H \cdot S)_t = \int_0^t H_u dS_u$ makes sense. Here the theory of stochastic integration (see, e.g., [P 90], [J 79], [RW 87]) tells us exactly what to impose on H : it has to be a predictable S -integrable process.

But, of course, this qualitative requirement is not enough as it does not rule out, e.g., doubling strategies, as was noticed by M. Harrison and S. Pliska ([HP 81]). In order to rule out such strategies, some additional condition is needed.

A strong condition is the subsequent concept of admissible strategies as introduced in [HP 81], modeling the situation of an agent with a finite credit line.

Definition 1.1 *A predictable S -integrable process H is an admissible trading strategy if the stochastic integral $(H \cdot S)_t = \int_0^t H_u dS_u$ is uniformly bounded from below.*

This notion turned out to be very useful for no-arbitrage arguments (compare [HP 81], [DS 94] and [DS 98b]). In the context of utility maximization for functions taking finite values only on \mathbb{R}_+ , while being $-\infty$ on \mathbb{R}_- (typical example: $U(x) = \ln(x)$), as analyzed, e.g., in [KLSX 91] and [KrS 99], this concept also proved to be the appropriate one.

In the present setting of utility functions taking finite values on all of \mathbb{R} , we also may and do use this class of trading strategies to give a precise meaning to the maximisation problem (4).

Definition 1.2 *We define the value function u associated to the optimization problem (4) by*

$$u(x) = \sup \{ \mathbf{E}[U(x + (H \cdot S)_T)], H \text{ admissible} \}, \quad x \in \mathbb{R}. \quad (5)$$

Note that the expectation is well-defined (taking possibly the value $+\infty$), and that $u(x)$ is an element of $[U(x), \infty]$.

In the present case of utility functions U taking finite values for all $x \in \mathbb{R}$, the class of admissible trading strategies is too narrow to find the optimizer in (5). In general, we cannot expect to find the optimal solution to (4) such that the random variable $(H \cdot S)_T$ is uniformly bounded from below. For example, in the classical Merton problem of optimal investment with respect to exponential utility in the Black-Scholes model, the optimal solution is not bounded from below.

Hence we have to look for a somewhat broader class of “allowed” trading strategies.

A possible approach is to impose some integrability condition on the process $H \cdot S$. But under which measure? Should we use the original measure \mathbf{P} , or some specific equivalent martingale measure \mathbf{Q} , or maybe all equivalent martingale measures? This issue was thoroughly addressed in [DGRSSS 00] and we shall elaborate further on this topic.

For a utility function $U : \mathbb{R} \rightarrow \mathbb{R}$ satisfying (1) we denote by $V : \mathbb{R}_+ \rightarrow \mathbb{R}$ its conjugate function

$$V(y) = \sup_x (U(x) - xy), \quad y > 0. \quad (6)$$

For example, for $U(x) = -e^{-\gamma x}$ we have $V(y) = \frac{y}{\gamma} (\ln(\frac{y}{\gamma}) - 1)$. The dual problem to (5) is given by

$$v(y) = \inf_{\mathbf{Q} \in \mathcal{M}^e(S)} \mathbf{E} [V(y \frac{d\mathbf{Q}}{d\mathbf{P}})], \quad y > 0. \quad (7)$$

Throughout the paper we shall make the following assumption:

Assumption 1.3 *For each $y > 0$, the dual value function $v(y)$ is finite and the minimizer $\widehat{\mathbf{Q}}(y) \in \mathcal{M}^e(S)$ for (7), called the minimax martingale measure, exists.*

As shown in [BF 00], Assumption 1.3 is satisfied under rather mild conditions. We remark that it is easy to verify that conditions (1), (2), (3) and Assumption 1.3 imply the assumptions of theorem 2.2 of [S 01]; hence under the present assumptions we may apply this theorem. This fact will repeatedly be used below.

Specializing to the case of exponential utility, it follows from the work of Csiszar [C 75] (see also [BF 00] and [S 01, remark 2.3]) that Assumption 1.3 is equivalent to the existence of $\mathbf{Q} \in \mathcal{M}^e(S)$ with finite relative entropy

$$H(\mathbf{Q}|\mathbf{P}) = \mathbf{E}_{\mathbf{P}} \left[\frac{d\mathbf{Q}}{d\mathbf{P}} \ln \left(\frac{d\mathbf{Q}}{d\mathbf{P}} \right) \right] = \mathbf{E}_{\mathbf{Q}} \left[\ln \left(\frac{d\mathbf{Q}}{d\mathbf{P}} \right) \right] < \infty. \quad (8)$$

In this case the measure $\widehat{\mathbf{Q}}(y)$ does not depend on $y > 0$, and minimizes the relative entropy among all absolutely continuous martingale measures, i.e.,

$$H(\widehat{\mathbf{Q}}|\mathbf{P}) = \min_{\mathbf{Q} \in \mathcal{M}^a(S)} H(\mathbf{Q}|\mathbf{P}) \quad (9)$$

Following [DGRSSS 00] we shall call $\widehat{\mathbf{Q}}$ the *entropy minimizing local martingale measure*.

Turning again to general utility functions U , we shall say that $\mathbf{Q} \in \mathcal{M}^a(S)$ has *finite V -expectation* if $\mathbf{E}[V(\frac{d\mathbf{Q}}{d\mathbf{P}})] < \infty$. It follows from the assumption of reasonable elasticity (2) that in this case $\mathbf{E}[V(y\frac{d\mathbf{Q}}{d\mathbf{P}})] < \infty$, for each $y > 0$ (see [S 01, Corollary 4.2]).

We now can introduce several possible definitions of “allowed” trading strategies.

Definition 1.4 *Under the above assumptions, fix the initial endowment $x \in \mathbb{R}$.*

- (i) *A predictable, S -integrable process H is in $\mathcal{H}_1(x)$, if $U(x + (H \cdot S)_T) \in L^1(\mathbf{P})$ and $H \cdot S$ is a super-martingale under the minimax martingale measure $\widehat{\mathbf{Q}}(y)$, where $y = u'(x)$.*
- (ii) *A predictable, S -integrable process H is in $\mathcal{H}_2(x)$, if $U(x + (H \cdot S)_T) \in L^1(\mathbf{P})$ and $H \cdot S$ is a super-martingale under each $\mathbf{Q} \in \mathcal{M}^a(S)$ with finite V -expectation $\mathbf{E}[V(\frac{d\mathbf{Q}}{d\mathbf{P}})]$.*
- (iii) *A predictable, S -integrable process H is in $\mathcal{H}_3(x)$, if $U(x + (H \cdot S)_T) \in L^1(\mathbf{P})$, $H \cdot S$ is a super-martingale under each $\mathbf{Q} \in \mathcal{M}^a(S)$ with finite V -expectation $\mathbf{E}[V(\frac{d\mathbf{Q}}{d\mathbf{P}})]$, and there exists a sequence $(H^n)_{n=1}^\infty$ of admissible trading strategies such that*

$$\lim_{n \rightarrow \infty} \mathbf{E}[U(x + (H \cdot S)_T \wedge (H^n \cdot S)_T)] = \mathbf{E}[U(x + (H \cdot S)_T)]. \quad (10)$$

The classes $\mathcal{H}'_1(x)$, $\mathcal{H}'_2(x)$ and $\mathcal{H}'_3(x)$ are defined by replacing in (i), (ii) and (iii) above the term “super-martingale” by the term “martingale”.

We remark that in the case of exponential utility the above concepts do not depend on the initial endowment x .

The concept \mathcal{H}'_1 was defined — for the case of exponential utility — under the name Θ_1 in [DGRSSS 00]; it also plays an important role in the results of [KrS 99] and [S 01] for the case of more general utility functions U . As was remarked in [DGRSSS 00], it is not very satisfactory to base the definition of the class of allowed strategies on the knowledge of the dual optimizer.

The concept \mathcal{H}'_2 corresponds to the class Θ_2 defined in [DGRSSS 00], where — for the exponential utility — it was required that $H \cdot S$ is a \mathbf{Q} -martingale, for each \mathbf{Q} with finite entropy. It follows from [DGRSSS 00] and the recent paper [KaS 00] that this latter concept works well for the exponential utility; but we shall see in Proposition 3.5 below that, for more general utility functions $U : \mathbb{R} \rightarrow \mathbb{R}$, the concept of a super-martingale is more appropriate than that of a martingale. This led us to define the class \mathcal{H}_2 .

Definition (iii) is in the spirit of [S 01, Definition 1.3]. In addition to the requirements of definition (ii) we also impose an approximability of $H \cdot S$ by a sequence $H^n \cdot S$, where each H^n is *admissible*. We remark that this concept is also related to the class Θ_3 defined in [DGRSSS 00].

Theorem 2.1 below asserts that it does not matter which of the classes \mathcal{H}_1 , \mathcal{H}'_1 , \mathcal{H}_2 or \mathcal{H}_3 (and, in the case of exponential utility, also \mathcal{H}'_2 and \mathcal{H}'_3) we choose for the utility maximization problem (4). We always end up with the same maximizer \widehat{H} , which in addition satisfies the proper duality relations with respect to the minimax martingale measure $\widehat{\mathbf{Q}}(y)$.

For the case of exponential utility the analogous result was proved in [DGRSSS 00] under a mild additional assumption ($\widehat{\mathbf{Q}}$ was supposed to satisfy a reverse Hölder condition $R_{L \log L}$). This assumption was shown to be superfluous in [KaS 00] (compare also [KaS 01b]).

The main result of the present paper is to establish the analogous result for general utility functions $U : \mathbb{R} \rightarrow \mathbb{R}$ satisfying (1) and (2). For expository reasons, we shall also indicate how the proof specializes to the case of exponential utility as the present arguments are somewhat different from those in [DGRSSS 00] and [KaS 00]; we then show how they may be extended to general utility functions. Roughly speaking, it turns out that some explicit calculations in the case of exponential utility are replaced by more conceptual arguments in the general setting (thus avoiding some calculations).

Having established the equivalence of the concepts \mathcal{H}_1 , \mathcal{H}'_1 , \mathcal{H}_2 and \mathcal{H}_3 with respect to the utility maximization problem (4) it will be natural to ask whether other (weaker) requirements of “allowed” trading strategies also yield the same conclusion. For example, consider the class of predictable S -integrable processes such that $H \cdot S$ is a martingale under *some* element $\mathbf{Q} \in \mathcal{M}^e(S)$. This class is closely related to the “workable contingent claims” as introduced in [DS 97]. One also might try variations of this requirement by imposing that $(H \cdot S)$ is a martingale — or a super-martingale — under *some* equivalent local martingale measure $\mathbf{Q} \in \mathcal{M}^e(S)$ with finite V -expectation.

Proposition 3.1, which presents the example described in the abstract, shows in a rather striking way that such hopes are in vain. These concepts do not allow for a good duality theory and lead to paradoxical results from an economic point of view. Two similar examples (Propositions 3.3 and 3.5) also show the sharpness of the assertion of Theorem 2.1.

2 The Main Result

Theorem 2.1 *Let $S = (S_t)_{0 \leq t \leq T}$ be a locally bounded \mathbb{R}^d -valued semimartingale, $U : \mathbb{R} \rightarrow \mathbb{R}$ a utility function satisfying (1), (2), (3) and assumption 1.3. For $x \in \mathbb{R}$, consider the optimization problem*

$$u_i(x) = \sup \mathbf{E} [U(x + (H \cdot S)_T)], \quad H \in \mathcal{H}_i(x). \quad (11)$$

For $i = 1, 2, 3$ the optimal solution $\widehat{H}_i \in \mathcal{H}_i(x)$ exists, is unique (in the sense that the process $((\widehat{H} \cdot S)_t)_{0 \leq t \leq T}$ is unique), coincides for all three cases and therefore may be denoted by \widehat{H} . In addition, \widehat{H} is also the unique optimizer in the class $\mathcal{H}'_1(x)$.

The value function $u(x)$ defined in (5) equals $u_i(x)$, for $i = 1, 2, 3$. Letting $y = u'(x)$, for $i = 1, 2, 3$, we have the following duality relation between $\widehat{H}(x)$ and the dual minimizer $\widehat{\mathbf{Q}}(y)$:

$$x + (\widehat{H} \cdot S)_T = -V' \left(y \frac{d\widehat{\mathbf{Q}}(y)}{d\mathbf{P}} \right) \quad \text{and} \quad y \frac{d\widehat{\mathbf{Q}}(y)}{d\mathbf{P}} = U' \left(x + (\widehat{H} \cdot S)_T \right). \quad (12)$$

In the case of the exponential utility function $U(x) = -e^{-\gamma x}$, \widehat{H} does not depend on x ; it is also the unique minimizer in the classes \mathcal{H}'_2 and \mathcal{H}'_3 ; relation (12) specializes to

$$(\widehat{H} \cdot S)_T = -\frac{1}{\gamma} \ln \left(\frac{y}{\gamma} \frac{d\widehat{\mathbf{Q}}}{d\mathbf{P}} \right) \quad \text{and} \quad \frac{d\widehat{\mathbf{Q}}}{d\mathbf{P}} = \frac{\gamma}{y} e^{-\gamma(\widehat{H} \cdot S)_T}. \quad (13)$$

where $y = u'(0)$.

The theorem essentially relies on Proposition 2.2 below which, for the case of exponential utility, was proved by Kabanov and Stricker [KaS00]. Admitting Proposition 2.2 for the moment the argument for Theorem 2.1 goes as follows.

Proof of Theorem 2.1 As remarked after Assumption 1.3 above, the present assumptions imply those of [S01, Theorem 2.2]. In particular, we know that, for $x \in \mathbb{R}$ and $y > 0$, satisfying $u'(x) = y$, the process

$$\widehat{X}_t(x) = \mathbf{E}_{\widehat{\mathbf{Q}}(y)} \left[-V' \left(y \frac{d\widehat{\mathbf{Q}}(y)}{d\mathbf{P}} \right) \middle| \mathcal{F}_t \right], \quad 0 \leq t \leq T, \quad (14)$$

well-defines a $\widehat{\mathbf{Q}}(y)$ -martingale, which is of the form $\widehat{X}(x) = x + \widehat{H}(x) \cdot S$, for some S -integrable predictable process $\widehat{H}(x)$ and

$$u(x) = \mathbf{E} \left[U \left(x + \left(\widehat{H}(x) \cdot S \right)_T \right) \right]. \quad (15)$$

Proposition 2.2 asserts that $\widehat{X}_t(x)$ is a super-martingale under each $Q \in \mathcal{M}^a(S)$ with finite V -expectation. Admitting this result, we shall show that $\widehat{H}(x)$ is the unique optimizer in the classes $\mathcal{H}_1(x)$, $\mathcal{H}'_1(x)$, $\mathcal{H}_2(x)$ and $\mathcal{H}_3(x)$.

The fact that $\widehat{H}(x)$ belongs to each of these classes is obvious for $\mathcal{H}_1(x)$, $\mathcal{H}'_1(x)$ and $\mathcal{H}_2(x)$; as regards $\mathcal{H}_3(x)$, it follows from [S01, Theorem 2.2] that $\widehat{H}(x)$ may be approximated by a sequence of admissible trading strategies in the sense of (10).

To show that $\widehat{H}(x)$ is the unique optimizer in each of these classes, we only have to show this for $\mathcal{H}_1(x)$, as this is the largest class.

We shall show that, in fact, the $\widehat{\mathbf{Q}}(y)$ -martingale $\widehat{X}(x) = x + \widehat{H}(x) \cdot S$ is optimal among the class of all $\widehat{\mathbf{Q}}(y)$ -super-martingales $(X_t)_{0 \leq t \leq T}$ starting at $X_0 = x$. Indeed, let X be a $\widehat{\mathbf{Q}}(y)$ -super-martingale and use the duality relation (see [S01, Theorem 2.2 (i)])

$$u(x) = \mathbf{E}[U(\widehat{X}_T(x))] = \mathbf{E} \left[V \left(y \frac{d\widehat{\mathbf{Q}}(y)}{d\mathbf{P}} \right) \right] + xy = v(y) + xy. \quad (16)$$

Applying the inequality

$$\begin{aligned} U(X_T(\omega)) &= \inf_{\eta > 0} \{V(\eta) + X_T(\omega)\eta\} \\ &\leq V \left(y \frac{d\widehat{\mathbf{Q}}(y)}{d\mathbf{P}}(\omega) \right) + X_T(\omega) y \frac{d\widehat{\mathbf{Q}}(y)}{d\mathbf{P}}(\omega), \end{aligned} \quad (17)$$

pointwise for $\omega \in \Omega$, we get from the $\widehat{\mathbf{Q}}(y)$ -super-martingale property of X that

$$\begin{aligned} \mathbf{E}[U(X_T)] &\leq \mathbf{E} \left[V \left(y \frac{d\widehat{\mathbf{Q}}(y)}{d\mathbf{P}} \right) \right] + y \mathbf{E}_{\widehat{\mathbf{Q}}(y)}[X_T] \\ &\leq \mathbf{E} \left[V \left(y \frac{d\widehat{\mathbf{Q}}(y)}{d\mathbf{P}} \right) \right] + xy \\ &= \mathbf{E}[U(\widehat{X}_T(x))], \end{aligned} \quad (18)$$

where the above estimate shows in particular that $\mathbf{E}[U(X_T)_+] < \infty$, so that $\mathbf{E}[U(X_T)]$ is well-defined. This readily shows that \widehat{H} is the unique optimizer in the class $\mathcal{H}_1(x)$. Hence, for $i = 1, 2, 3$, the value functions $u_i(x)$ coincide with the value function $u(x)$ as defined in (5).

As regards the fact that $\widehat{H}(x)$ also is in $\mathcal{H}'_2(x)$ and $\mathcal{H}'_3(x)$ for the case of exponential utility, we refer to [DGRSSS00] and [KaS00]. ■

Proposition 2.2 *Under the assumptions of Theorem 2.1, fix $\mathbf{Q} \in \mathcal{M}^a(S)$ with finite V -expectation $\mathbf{E}_{\mathbf{P}}[V(\frac{d\mathbf{Q}}{d\mathbf{P}})]$.*

The process $\widehat{X}(x) = x + \widehat{H}(x) \cdot S$, defined by (14), is a super-martingale under \mathbf{Q} .

We start with some auxiliary results. The subsequent lemma gives a general characterisation of local martingales X (or, more generally, stochastic integrals of local martingales), which are super-martingales. On the basis of a preliminary version of the present paper, Kabanov and Stricker have further elaborated on this topic [KaS 01a].

Lemma 2.3 *Let S be a local martingale on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbf{Q})$, H an S -integrable predictable process and $X = H \cdot S$.*

If, for every sequence $(\tau_n)_{n=1}^{\infty}$ of $[0, T] \cup \{\infty\}$ -valued stopping times increasing to $+\infty$, we have

$$\limsup_{n \rightarrow \infty} \mathbf{E}_{\mathbf{Q}} [X_{\tau_n} \mathbf{1}_{\{\tau_n < \infty, X_{\tau_n} \leq 0\}}] = 0, \quad (19)$$

then X is a local martingale and a super-martingale under \mathbf{Q} .

The proof of the lemma relies on the subsequent sublemma which is a straight forward consequence of a result of Ansel-Stricker [AS 94, Proposition 3.3] (compare also [KaS 01a]).

Sublemma 2.4 *Let $S = (S_t)_{0 \leq t \leq T}$ be a local martingale and suppose that, for $X = H \cdot S$, there is an integrable random variable $\vartheta \geq 0$ such that $X_t \geq -\vartheta$, a.s., for all $0 \leq t \leq T$.*

Then X is a local martingale and a super-martingale.

Proof of Lemma 2.3 Define the sequence of stopping times by $\sigma_0 = 0$, $n \geq 1$, and

$$\sigma_n = \inf\{t : X_t \leq -n\}, \quad (20)$$

which increases almost surely to infinity.

We assume that (19) holds true and define the random variables

$$\vartheta_n = \max((X_{\sigma_n})_- \mathbf{1}_{\{\sigma_n < \infty\}}, n). \quad (21)$$

It follows from (19) that, for n sufficiently large, we have $\mathbf{E}_{\mathbf{Q}}[\vartheta_n] < \infty$.

Note that the stopped process X^{σ_n} is bounded from below almost surely by the random variable ϑ_n ; indeed on $\{\sigma_n = \infty\}$ we have $\inf_{0 \leq t \leq T} X_t^{\sigma_n} \geq -n$, while on $\{\sigma_n < \infty\}$ we have $\inf_{0 \leq t \leq T} X_t^{\sigma_n} \geq -X_{\sigma_n} \mathbf{1}_{\{\sigma_n < \infty\}}$. Sublemma 2.4 therefore implies that each X^{σ_n} , is a super-martingale and that X is a local martingale under \mathbf{Q} .

We now show that (19) implies that, for every $[0, T]$ -valued stopping time σ , we have

$$E_{\mathbf{Q}}[|X_{\sigma}|] < \infty. \quad (22)$$

Indeed, otherwise

$$\begin{aligned} \infty &= \lim_{n \rightarrow \infty} \mathbf{E}_{\mathbf{Q}} [|X_{\sigma}| \mathbf{1}_{\{\sigma_n = \infty\}}] \\ &\leq \lim_{n \rightarrow \infty} \mathbf{E}_{\mathbf{Q}} [|X_{\sigma \wedge \sigma_n}|] \\ &\leq \lim_{n \rightarrow \infty} 2\mathbf{E}_{\mathbf{Q}} [(X_{\sigma \wedge \sigma_n})_-]. \end{aligned} \quad (23)$$

Defining inductively $\tau_0 = 0$ and $\tau_k = (\sigma_{n_k} \wedge \sigma) \cdot \mathbf{1}_{\{\sigma_{n_{k-1}} \leq \sigma\}} + \infty \cdot \mathbf{1}_{\{\sigma_{n_{k-1}} > \sigma\}}$, for $(n_k)_{k=1}^\infty$ increasing sufficiently fast to infinity, we find a sequence $(\tau_k)_{k=1}^\infty$ of stopping times increasing almost surely to infinity such that

$$\lim_{n \rightarrow \infty} \mathbf{E}_{\mathbf{Q}} [X_{\tau_k} \mathbf{1}_{\{\tau_k < \infty, X_{\tau_k} \leq 0\}}] = -\infty, \quad (24)$$

in contradiction to assumption (19).

Let us give the argument in some detail: suppose that $n_0 = 0, n_1, \dots, n_{k-1}$ and $\tau_0, \dots, \tau_{k-1}$ are defined. Note that by the super-martingale property of each X^{σ_n} we have that

$$\mathbf{E}_{\mathbf{Q}} \left[\left| X_{\sigma \wedge \sigma_{n_{k-1}}} \right| \right] < \infty \quad (25)$$

hence, in particular

$$\mathbf{E}_{\mathbf{Q}} \left[\left| X_{\sigma \wedge \sigma_{n_{k-1}}} \right| \cdot \mathbf{1}_{\{\sigma_{n_{k-1}} > \sigma\}} \right] = \mathbf{E}_{\mathbf{Q}} \left[|X_\sigma| \mathbf{1}_{\{\sigma_{n_{k-1}} > \sigma\}} \right] < \infty. \quad (26)$$

From (23) and

$$\mathbf{E}_{\mathbf{Q}} \left[(X_{\sigma \wedge \sigma_n})_- \mathbf{1}_{\{\sigma_{n_{k-1}} > \sigma\}} \right] = \mathbf{E}_{\mathbf{Q}} \left[(X_\sigma)_- \mathbf{1}_{\{\sigma_{n_{k-1}} > \sigma\}} \right] < \infty, \quad \text{for } n > n_{k-1}, \quad (27)$$

we deduce that

$$\lim_{n \rightarrow \infty} \mathbf{E}_{\mathbf{Q}} \left[(X_{\sigma \wedge \sigma_n})_- \cdot \mathbf{1}_{\{\sigma_{n_{k-1}} \leq \sigma\}} \right] = \infty \quad (28)$$

and therefore we may choose $n_k > n_{k-1}$ such that, for τ_k defined as above, we have

$$\mathbf{E}_{\mathbf{Q}} [X_{\tau_k} \mathbf{1}_{\{\tau_k < \infty, X_{\tau_k} \leq 0\}}] \leq -k, \quad (29)$$

which gives (24).

Having thus established (22) we are ready to show the super-martingale property of X . It suffices to fix stopping times $0 \leq \rho \leq \sigma \leq T$ such that $\rho \leq \sigma_{n_0}$, for some $n_0 \in \mathbb{N}$, and to show that

$$\mathbf{E}_{\mathbf{Q}} [X_\sigma - X_\rho] \leq 0. \quad (30)$$

We deduce from the super-martingale property of X^{σ_n} that, for $n \geq n_0$,

$$\mathbf{E}_{\mathbf{Q}} [X_{\sigma \wedge \sigma_n} - X_\rho] \leq 0. \quad (31)$$

Using $\mathbf{E}_{\mathbf{Q}} [|X_\sigma|] < \infty$, we may deduce (30) from (31) and assumption (19) applied to the sequence of stopping times $\rho_n = (\sigma \wedge \sigma_n) \mathbf{1}_{\{\sigma_n \leq \sigma\}} + \infty \mathbf{1}_{\{\sigma_n > \sigma\}}$:

$$\begin{aligned} \mathbf{E}_{\mathbf{Q}} [X_\sigma] &= \lim_{n \rightarrow \infty} \mathbf{E}_{\mathbf{Q}} [X_\sigma \mathbf{1}_{\{\sigma_n > \sigma\}}] \\ &= \lim_{n \rightarrow \infty} \mathbf{E}_{\mathbf{Q}} [X_{\sigma \wedge \sigma_n} \mathbf{1}_{\{\sigma_n > \sigma\}}] \\ &= \lim_{n \rightarrow \infty} \left(\mathbf{E}_{\mathbf{Q}} [X_{\sigma \wedge \sigma_n}] - \mathbf{E}_{\mathbf{Q}} [X_{\sigma \wedge \sigma_n} \mathbf{1}_{\{\sigma_n \leq \sigma\}}] \right) \\ &\leq \lim_{n \rightarrow \infty} \left(\mathbf{E}_{\mathbf{Q}} [X_{\sigma \wedge \sigma_n}] - \mathbf{E}_{\mathbf{Q}} [X_{\rho_n} \mathbf{1}_{\{\rho_n < \infty, X_{\rho_n} \leq 0\}}] \right) \\ &\leq \mathbf{E}_{\mathbf{Q}} [X_\rho]. \quad \blacksquare \end{aligned} \quad (32)$$

Remark 2.5 We remark that the sufficient condition (19) also is necessary for a process $X = (X_t)_{0 \leq t \leq T}$ to be a super-martingale: indeed, if $X = (X_t)_{0 \leq t \leq T}$ is a super-martingale and the sequence of $[0, T] \cup \{\infty\}$ -valued stopping times $(\tau_n)_{0 \leq t \leq T}$ increases to ∞ , the sequence of random variables $(X_{\tau_n} \mathbf{1}_{\{\tau_n < \infty, X_{\tau_n} \leq 0\}})_{n=1}^{\infty}$ is uniformly integrable, which implies (19).

We have formulated Lemma 2.3 for processes indexed by the closed time interval $I = [0, T]$. This result also extends to the case of the open time interval $I = [0, \infty[$; we then have to require in condition (19) that the sequence $(\tau_n)_{n=1}^{\infty}$ increases *stationarily* to ∞ . Note that in the present case of $I = [0, T]$ a sequence $(\tau_n)_{n=1}^{\infty}$ of $[0, T] \cup \{\infty\}$ -valued stopping times, increasing to ∞ , automatically does so in a stationary way.

We now aboard the proof of Proposition 2.2 for the special case of the exponential utility $U(x) = -e^{-\gamma x}$ which will rely on the subsequent Lemma 2.6 pertaining to the well-known technique of “concatenation” (compare [KaS00, prop. 4.1]).

Lemma 2.6 *Under the assumptions of Theorem 2.1 let $\mathbf{Q} \in \mathcal{M}^a(S)$ with $H(\mathbf{Q}|\mathbf{P}) < \infty$ and τ a $[0, T] \cup \{\infty\}$ -valued stopping time. Denote by $(Z_t)_{0 \leq t \leq T}$ and $(\hat{Z}_t)_{0 \leq t \leq T}$ the density processes corresponding to \mathbf{Q} and $\hat{\mathbf{Q}}$ respectively, and define the probability measure \mathbf{Q}^τ by the following “concatenation operation”:*

$$\frac{d\mathbf{Q}^\tau}{d\mathbf{P}} = \begin{cases} Z_T & \text{if } \tau = \infty \\ Z_\tau \frac{\hat{Z}_T}{\hat{Z}_\tau} & \text{if } \tau < \infty \end{cases} \quad (33)$$

Then $\mathbf{Q}^\tau \in \mathcal{M}^a(S)$ and $H(\mathbf{Q}^\tau|\mathbf{P}) \leq H(\mathbf{Q}|\mathbf{P})$.

Proof Note that the random variable $Y = \frac{\hat{Z}_T}{\hat{Z}_\tau} \mathbf{1}_{\{\tau < \infty\}} + \mathbf{1}_{\{\tau = \infty\}}$ solves the conditional minimization problem

$$\mathbf{E}_{\mathbf{P}} [Y \ln(Y) | \mathcal{F}_\tau] \longrightarrow \min! \quad \text{a.s. on } \{\tau < \infty\}, \quad (34)$$

among all nonnegative random variables Y verifying $\mathbf{E}_{\mathbf{P}}[Y | \mathcal{F}_\tau] = 1$ a.s., and such that the process ${}^\tau S := S_t - S_{t \wedge \tau}$ “starting at τ ” is a local martingale with respect to the measure R defined by $\frac{dR}{d\mathbf{P}} = Y$.

Indeed, suppose there is such a function Y and an \mathcal{F}_τ -measurable subset $A \subseteq \{\tau < \infty\}$, $\mathbf{P}[A] > 0$, such that

$$\mathbf{E}_{\mathbf{P}} \left[\frac{\hat{Z}_T}{\hat{Z}_\tau} \ln \left(\frac{\hat{Z}_T}{\hat{Z}_\tau} \right) \middle| \mathcal{F}_\tau \right] > \mathbf{E}_{\mathbf{P}} [Y \ln(Y) | \mathcal{F}_\tau] \quad \text{a.s. on } A. \quad (35)$$

Then the probability measure $\tilde{\mathbf{Q}}$ defined by

$$\frac{d\tilde{\mathbf{Q}}}{d\mathbf{P}} = \tilde{Z}_T = \begin{cases} \hat{Z}_T & \text{on } \Omega \setminus A \\ \hat{Z}_\tau Y & \text{on } A \end{cases} \quad (36)$$

would be an element of $\mathcal{M}^a(S)$ with smaller entropy than $\widehat{\mathbf{Q}}$:

$$\begin{aligned}
& H(\widehat{\mathbf{Q}}|\mathbf{P}) - H(\widetilde{\mathbf{Q}}|\mathbf{P}) \\
&= \mathbf{E}_{\mathbf{P}} \left[\widehat{Z}_T \ln \left(\widehat{Z}_T \right) - \widetilde{Z}_T \ln \left(\widetilde{Z}_T \right) \right] \\
&= \mathbf{E}_{\mathbf{P}} \left[\left(\widehat{Z}_\tau \frac{\widehat{Z}_T}{\widehat{Z}_\tau} \ln \left(\widehat{Z}_\tau \frac{\widehat{Z}_T}{\widehat{Z}_\tau} \right) - \widehat{Z}_\tau Y \ln \left(\widehat{Z}_\tau Y \right) \right) \mathbf{1}_A \right] \\
&= \mathbf{E}_{\mathbf{P}} \left[\widehat{Z}_\tau \mathbf{1}_A \mathbf{E}_{\mathbf{P}} \left[\frac{\widehat{Z}_T}{\widehat{Z}_\tau} \left(\ln \left(\widehat{Z}_\tau \right) + \ln \left(\frac{\widehat{Z}_T}{\widehat{Z}_\tau} \right) \right) - Y \left(\ln \left(\widehat{Z}_\tau \right) + \ln(Y) \right) \middle| \mathcal{F}_\tau \right] \right] \\
&= \mathbf{E}_{\mathbf{P}} \left[\widehat{Z}_\tau \mathbf{1}_A \mathbf{E}_{\mathbf{P}} \left[\frac{\widehat{Z}_T}{\widehat{Z}_\tau} \ln \left(\frac{\widehat{Z}_T}{\widehat{Z}_\tau} \right) - Y \ln(Y) \middle| \mathcal{F}_\tau \right] \right] > 0.
\end{aligned} \tag{37}$$

This contradiction to the minimality of $\widehat{\mathbf{Q}}$ shows (34).

By the same argument we conclude that \mathbf{Q}^τ is the element of $\mathcal{M}^a(S)$ with minimal entropy such that $\mathbf{Q}^\tau|_{\mathcal{F}_\tau} = \mathbf{Q}|_{\mathcal{F}_\tau}$, which implies the assertion of the lemma. ■

Proof of Proposition 2.2 for the case of exponential utility

Assume that there is $\mathbf{Q} \in \mathcal{M}^a(S)$, $H(\mathbf{Q}|\mathbf{P}) < \infty$, such that $\widehat{X} = x + \widehat{H} \cdot S$ fails to be a super-martingale under \mathbf{Q} . Without loss of generality we may assume that $x = 0$ and that $\mathbf{Q} \in \mathcal{M}^e(S)$ (consider $\frac{\mathbf{Q} + \widehat{\mathbf{Q}}}{2}$).

From Lemma 2.3 we deduce that, if \widehat{X} fails to be a \mathbf{Q} -super-martingale, there exists a sequence $(\tau_n)_{n=1}^\infty$ of stopping times increasing to infinity such that

$$\limsup_{n \rightarrow \infty} \mathbf{E}_{\mathbf{Q}}[\widehat{X}_{\tau_n} \mathbf{1}_{\{\tau_n < \infty, \widehat{X}_{\tau_n} \leq 0\}}] < 0. \tag{39}$$

Of course, we may assume that $\widehat{X}_{\tau_n} \leq 0$ on $\{\tau_n < \infty\}$ so that we may replace $\{\tau_n < \infty, \widehat{X}_{\tau_n} \leq 0\}$ simply by $\{\tau_n < \infty\}$ in the above formula.

We know that \widehat{X} is a uniformly integrable martingale under $\widehat{\mathbf{Q}}$ and therefore

$$\lim_{n \rightarrow \infty} \mathbf{E}_{\widehat{\mathbf{Q}}}[\widehat{X}_{\tau_n} \mathbf{1}_{\{\tau_n < \infty\}}] = 0. \tag{40}$$

It follows that we have

$$\liminf_{n \rightarrow \infty} \mathbf{E}_{\mathbf{P}} \left[-\widehat{X}_{\tau_n} Z_{\tau_n} \mathbf{1}_{\{\tau_n < \infty, Z_{\tau_n} \geq \widehat{Z}_{\tau_n}\}} \right] > 0. \tag{41}$$

Now apply Lemma 2.6 to the probability measures $\mathbf{Q}^n \in \mathcal{M}^e(S)$

$$\frac{d\mathbf{Q}^n}{d\mathbf{P}} = \begin{cases} Z_{\tau_n} \frac{\widehat{Z}_T}{\widehat{Z}_{\tau_n}} & \text{for } \tau_n < \infty \\ Z_T & \text{for } \tau_n = \infty \end{cases} \tag{42}$$

We shall show that

$$\liminf_{n \rightarrow \infty} H(\mathbf{Q}^n|\mathbf{P}) > H(\mathbf{Q}|\mathbf{P}), \tag{43}$$

a contradiction to the assertion of Lemma 2.6 which will finish the proof.

$$\liminf_{n \rightarrow \infty} (H(\mathbf{Q}^n | \mathbf{P}) - H(\mathbf{Q} | \mathbf{P})) \quad (44)$$

$$= \liminf_{n \rightarrow \infty} \mathbf{E}_{\mathbf{P}} \left[\left(Z_{\tau_n} \frac{\widehat{Z}_T}{\widehat{Z}_{\tau_n}} \ln \left(Z_{\tau_n} \frac{\widehat{Z}_T}{\widehat{Z}_{\tau_n}} \right) - Z_T \ln(Z_T) \right) \mathbf{1}_{\{\tau_n < \infty\}} \right] \quad (45)$$

$$\geq \liminf_{n \rightarrow \infty} \left(\mathbf{E}_{\mathbf{P}} \left[Z_{\tau_n} \frac{\widehat{Z}_T}{\widehat{Z}_{\tau_n}} \ln \left(Z_{\tau_n} \frac{\widehat{Z}_T}{\widehat{Z}_{\tau_n}} \right) \mathbf{1}_{\{\tau_n < \infty, Z_{\tau_n} \geq \widehat{Z}_{\tau_n}\}} \right] + \right. \quad (46)$$

$$\left. \left(-\frac{1}{e} \right) \mathbf{P} \left[\tau_n < \infty, Z_{\tau_n} < \widehat{Z}_{\tau_n} \right] - \mathbf{E}_{\mathbf{P}} \left[Z_T \ln(Z_T) \mathbf{1}_{\{\tau_n < \infty\}} \right] \right) \quad (47)$$

$$\geq \liminf_{n \rightarrow \infty} \mathbf{E}_{\mathbf{P}} \left[Z_{\tau_n} \mathbf{1}_{\{\tau_n < \infty, Z_{\tau_n} \geq \widehat{Z}_{\tau_n}\}} \mathbf{E}_{\mathbf{P}} \left[\frac{\widehat{Z}_T}{\widehat{Z}_{\tau_n}} \ln \left(\widehat{Z}_T \right) \middle| \mathcal{F}_{\tau_n} \right] \right] \quad (48)$$

$$= \liminf_{n \rightarrow \infty} \mathbf{E}_{\mathbf{P}} \left[Z_{\tau_n} \mathbf{1}_{\{\tau_n < \infty, Z_{\tau_n} \geq \widehat{Z}_{\tau_n}\}} \mathbf{E}_{\mathbf{P}} \left[\frac{\widehat{Z}_T}{\widehat{Z}_{\tau_n}} \left(-\gamma \widehat{X}_T - \ln \left(\frac{y}{\gamma} \right) \right) \middle| \mathcal{F}_{\tau_n} \right] \right] \quad (49)$$

$$= \liminf_{n \rightarrow \infty} \mathbf{E}_{\mathbf{P}} \left[Z_{\tau_n} \mathbf{1}_{\{\tau_n < \infty, Z_{\tau_n} \geq \widehat{Z}_{\tau_n}\}} \left(-\gamma \widehat{X}_{\tau_n} - \ln \left(\frac{y}{\gamma} \right) \right) \right] > 0, \quad (50)$$

where we have used (13) in (49), and (41) as well as the fact that $(\widehat{Z}_t \widehat{X}_t)_{0 \leq t \leq T}$ is a uniformly integrable martingale under \mathbf{P} , in (50). ■

We now aboard the proof of Proposition 2.2 for general utility functions $U : \mathbb{R} \rightarrow \mathbb{R}$ having reasonable asymptotic elasticity, i.e., satisfying (1) and (2). We shall have to replace the explicit calculations for the “concatenation” above by some more conceptual arguments. As often encountered in mathematics things become somewhat easier by passing to a more general framework, as these conceptual arguments make the explicit — but cumbersome — calculations superfluous, which were used in the above arguments for the special case of the exponential utility.

For a fixed $[0, T]$ -valued stopping time τ we shall develop the notion of the conditional value functions $U^\tau(x)$ and $V^\tau(y)$ at time τ , and the corresponding dynamic programming principles.

We first isolate a slight variation of Theorem 2.2 of [S01]: in the setting of this theorem, replace T by an almost surely finite stopping time τ , and the utility function U by a family $(U_\omega(\cdot))_{\omega \in \Omega}$ of utility functions satisfying (1), and depending on $\omega \in \Omega$ in an \mathcal{F} -measurable way.

We suppose that there is a constant $C > 0$, such that

$$y |V'_\omega(y)| \leq C V_\omega(y), \quad y > 0, \quad (51)$$

almost surely, where $(V_\omega(\cdot))_{\omega \in \Omega}$ denotes the family of conjugate functions. We refer to [KrS99, Corollary 6.1] and [S01, Corollary 4.2] for the relation of (51) to (2).

Under these assumptions we study the optimisation problem

$$u(x) = \sup \{ \mathbf{E} [U_\omega(x + (H \cdot S)_\tau)] : H \text{ admissible} \}. \quad (52)$$

The statement as well as the proof of [S01, Theorem 2.2] carry over — mutatis mutandis — in a straight-forward way: if $u(x) < \infty$, for some $x \in \mathbb{R}$, the optimizer $\widehat{F}_\tau(x)$ to (52) exists, and the duality relation between the primal and dual optimizer now is given, for $y = u'(x)$, by

$$\widehat{F}_\tau(x)(\omega) = -V'_\omega \left(y \frac{d\widehat{\mathbf{Q}}(y)}{d\mathbf{P}}(\omega) \right), \quad \text{for a.e. } \omega \in \Omega. \quad (53)$$

Next we consider the financial market modeled by the process ${}^\tau S = (S_t - S_{t \wedge \tau})_{0 \leq t \leq T}$, starting at τ , and the optimisation problem (5), with S replaced by ${}^\tau S$:

$$u^\tau(x) = \sup \{ \mathbf{E} [U(x + (H \cdot {}^\tau S)_T)] : H \text{ admissible} \}. \quad (54)$$

Clearly (1), (2), (3), and Assumption (1.3) imply that the assumptions of [S01, Theorem 2.2] are still satisfied for the optimisation problem (54). Denote by $\widehat{F}_T^\tau(x)$ the optimizer to (54).

We may define the family of *conditional value functions* $(U_\omega^\tau(x))_{\omega \in \Omega}$ by

$$U^\tau(x) = \text{ess sup} \{ \mathbf{E} [U(x + (H \cdot {}^\tau S)_T) | \mathcal{F}_\tau] : H \text{ adm.} \} = \mathbf{E} \left[U \left(\widehat{F}_T^\tau(x) \right) \middle| \mathcal{F}_\tau \right]. \quad (55)$$

Let us be slightly pedantic on this issue: for each rational number $x \in \mathbb{Q}$ choose an \mathcal{F}_τ -measurable representant, still denoted by $\omega \mapsto U_\omega^\tau(x)$, of the equivalence class defined in (55). Clearly $\mathbb{Q} \ni x \mapsto U_\omega^\tau(x)$ defines a finite concave function on \mathbf{Q} , for almost each $\omega \in \Omega$, which therefore may uniquely be extended to a concave function, defined for all $x \in \mathbb{R}$. Therefore (55) defines an (equivalence class of) \mathcal{F}_τ -measurable function(s) $\omega \mapsto U_\omega^\tau(\cdot)$ taking their values in the set of concave functions defined on \mathbb{R} .

For any \mathcal{F}_τ -measurable real function X_τ , taking finitely many values $\{x_1, \dots, x_N\}$ we may define

$$\widehat{F}_T^\tau(X_\tau) = \sum_{i=1}^n \widehat{F}_T^\tau(x_i) \mathbf{1}_{\{X_\tau = x_i\}}, \quad (56)$$

and this definition extends in an obvious way to general \mathcal{F}_τ -measurable real functions X_τ as $\widehat{F}_T^\tau(x_n) \rightarrow \widehat{F}_T^\tau(x)$ almost surely, for $x_n \rightarrow x$, (compare [S01, Proof of Theorem 2.2]). We then have

$$\mathbf{E} [U_\omega^\tau(X_\tau)] = \mathbf{E} \left[U \left(\widehat{F}_T^\tau(X_\tau) \right) \right], \quad (57)$$

whenever one of these expectations makes sense, and

$$\mathbf{E} [U_\omega^\tau(X_\tau)] \geq \mathbf{E} [U(X_\tau + (H \cdot {}^\tau S)_T)], \quad (58)$$

for each predictable process H such that, conditionally w.r. to \mathcal{F}_τ , the process $(X_\tau + (H \cdot {}^\tau S)_t)_{\tau \leq t \leq T}$ is bounded from below.

Hence the value function $u(x)$ defined in (5) satisfies the dynamic programming equation

$$u(x) = \sup \{ \mathbf{E} [U_\omega^\tau(x + (H \cdot S)_\tau)] : H \text{ admissible} \}. \quad (59)$$

We also formulate the dual problem

$$v(y) = \inf \{ \mathbf{E} [V_\omega^\tau(y \frac{d\mathbf{Q}}{d\mathbf{P}})] : \mathbf{Q} \in \mathcal{M}^a(S^\tau) \}, \quad (60)$$

where V_ω^τ is conjugate to U_ω^τ , and $\mathcal{M}^a(S^\tau)$ refers to the stopped process $(S_t^\tau)_{0 \leq t \leq T} = (S_{t \wedge \tau})_{0 \leq t \leq T}$.

Denote by $\mathcal{M}({}^\tau S)$ the set of all probability measures \mathbf{Q} on \mathcal{F} , absolutely continuous with respect to \mathbf{P} such that ${}^\tau S$ is a local \mathbf{Q} -martingale, and $\mathbf{Q}|_{\mathcal{F}_\tau} = \mathbf{P}|_{\mathcal{F}_\tau}$. We then have

$$V_\omega^\tau(y) = \text{ess inf}_{\mathbf{Q} \in \mathcal{M}({}^\tau S)} \mathbf{E} [V(y \frac{d\mathbf{Q}}{d\mathbf{P}}) | \mathcal{F}_\tau], \quad y > 0. \quad (61)$$

Indeed, (61) is just the conditional version of the conjugacy of the value functions u and v in [S01, Theorem 2.2] (the precise interpretation of (61) is similar as after (55) above). The optimizer $\widehat{\mathbf{Q}}^\tau(y) \in \mathcal{M}(\tau S)$ to (61) uniquely exists, for $y > 0$, and we again may define $\widehat{\mathbf{Q}}^\tau(Z_\tau)$, for an \mathcal{F}_τ -measurable \mathbb{R}_+ -valued function Z_τ , similarly as in (56), to obtain the dynamic programming equation

$$V^\tau(Z_\tau) = \mathbf{E} \left[V \left(Z_\tau \frac{d\widehat{\mathbf{Q}}^\tau(Z_\tau)}{d\mathbf{P}} \right) \middle| \mathcal{F}_\tau \right]. \quad (62)$$

In particular, if $(Z_t)_{0 \leq t \leq T}$ is the density process of some $\mathbf{Q} \in \mathcal{M}^a(S)$, we have

$$V^\tau(Z_\tau) \leq \mathbf{E} [V(Z_T) | \mathcal{F}_\tau], \quad (63)$$

as $\frac{Z_\tau}{Z_T}$ is the density of some $\mathbf{Q} \in \mathcal{M}(\tau S)$. This inequality may be viewed as the abstract version of Lemma 2.6.

We now show that, under the assumption $U(0) > 0$, the family $(V_\omega^\tau(\cdot))_{\omega \in \Omega}$ verifies (51). Indeed, if U satisfies (1) and (2) and $U(0) > 0$, we have by [S01, Corollary 4.2] the existence of $C > 0$ such that

$$y |V'(y)| \leq CV(y), \quad y > 0. \quad (64)$$

This inequality carries over to V_ω^τ , for $\omega \in \Omega$:

$$\begin{aligned} y |(V_\omega^\tau)'(y)| &= \mathbf{E} \left[y \frac{d\widehat{\mathbf{Q}}^\tau(y)}{d\mathbf{P}} V' \left(y \frac{d\widehat{\mathbf{Q}}^\tau(y)}{d\mathbf{P}} \right) \middle| \mathcal{F}_\tau \right] \\ &\leq C \mathbf{E} \left[V \left(y \frac{d\widehat{\mathbf{Q}}^\tau(y)}{d\mathbf{P}} \right) \middle| \mathcal{F}_\tau \right] = CV_\omega^\tau(y), \end{aligned} \quad (65)$$

where in the first equality we again have applied a conditional version of [S01, Theorem 2.2(v)].

Denoting by $(\widehat{X}_t(x))_{0 \leq t \leq T}$ and $(\widehat{Z}_t(y))_{0 \leq t \leq T}$ the optimal processes associated to the original problem (5) and (7), the stopped processes $(\widehat{X}_t^\tau(x))_{0 \leq t \leq T}$ and $(\widehat{Z}_t^\tau(y))_{0 \leq t \leq T}$ are the optimal processes associated to (59) and (60). Indeed, this dynamic programming principle now follows from (58) and (63). Whence (53) implies that, for $y = u'(x)$,

$$\widehat{X}_\tau(x) = - (V_\omega^\tau)' \left(\widehat{Z}_\tau(y) \right), \quad \text{a.s.} \quad (66)$$

Let us resume the essentials of the above discussion, which we shall use in the proof of Proposition 2.2 below:

Proposition 2.7 *Under the assumptions of Theorem 2.1, let τ be a $[0, T]$ -valued stopping time, and assume that $U(0) > 0$.*

Define the \mathcal{F}_τ -measurable family $(U_\omega^\tau(\cdot))_{\omega \in \Omega}$ of conditional value functions by (55), and let $(V_\omega^\tau(\cdot))_{\omega \in \Omega}$ denote the family of conjugate functions.

Denoting by $(\widehat{X}_t(x))_{0 \leq t \leq T}$ and $(\widehat{Z}_t(y))_{0 \leq t \leq T}$ the optimal processes associated to (5) and (7) we have, for $y = u'(x)$, by (66)

$$\widehat{X}_\tau(x) = - (V_\omega^\tau)' \left(\widehat{Z}_\tau(y) \right), \quad \text{a.s.} \quad (67)$$

For the density process $(Z_t)_{0 \leq t \leq T}$ of any $\mathbf{Q} \in \mathcal{M}^a(S)$ we have, by (63)

$$V^\tau(Z_\tau) \leq \mathbf{E}[V(Z_T) | \mathcal{F}_\tau]. \quad (68)$$

There is a constant $c > 0$ such that, for almost all $\omega \in \Omega$, by (65)

$$cy |(V_\omega^\tau)'(y)| \leq V_\omega^\tau(y), \quad \text{for } y > 0. \quad (69)$$

Proof of Proposition 2.2 for the general case

First note that there is no loss of generality in assuming that $U(0) > 0$, as the assertion is clearly invariant under adding a constant to U . Suppose that there is $\mathbf{Q} \in \mathcal{M}^a(S)$, $\mathbf{E}[V(\frac{d\mathbf{Q}}{d\mathbf{P}})] < \infty$, and $x \in \mathbb{R}$, such that $\widehat{X}(x)$ fails to be a \mathbf{Q} -super-martingale.

Similarly as in (41) above we conclude from Lemma 2.3 that there is a sequence $(\tau_n)_{n=1}^\infty$ of stopping times, increasing to ∞ , such that

$$\liminf_{n \rightarrow \infty} \mathbf{E}_{\mathbf{P}} \left[-\widehat{X}_{\tau_n}(x) Z_{\tau_n} \mathbf{1}_{\{\tau_n < \infty, Z_{\tau_n} \geq y \widehat{Z}_{\tau_n}(y)\}} \right] > 0, \quad (70)$$

where Z_t is the density process of \mathbf{Q} , $\widehat{Z}_t(y)$ the density process of $\widehat{\mathbf{Q}}(y)$ and $u'(x) = y$.

Applying Proposition 2.7 to the stopping time $\tau_n \wedge T$, and using the monotonicity of $y \mapsto (V_\omega^{\tau_n})'(y)$, we obtain

$$\liminf_{n \rightarrow \infty} \mathbf{E}_{\mathbf{P}} \left[V \left(\frac{d\mathbf{Q}}{d\mathbf{P}} \right) \mathbf{1}_{\{\tau_n < \infty, Z_{\tau_n} \geq y \widehat{Z}_{\tau_n}(y)\}} \right] \quad (71)$$

$$\stackrel{(68)}{\geq} \liminf_{n \rightarrow \infty} \mathbf{E}_{\mathbf{P}} \left[V_\omega^{\tau_n}(Z_{\tau_n}) \mathbf{1}_{\{\tau_n < \infty, Z_{\tau_n} \geq y \widehat{Z}_{\tau_n}(y)\}} \right] \quad (72)$$

$$\stackrel{(69)}{\geq} \liminf_{n \rightarrow \infty} c \mathbf{E}_{\mathbf{P}} \left[Z_{\tau_n} (V_\omega^{\tau_n})'(Z_{\tau_n}) \mathbf{1}_{\{\tau_n < \infty, Z_{\tau_n} \geq y \widehat{Z}_{\tau_n}(y)\}} \right] \quad (73)$$

$$\geq \liminf_{n \rightarrow \infty} c \mathbf{E}_{\mathbf{P}} \left[Z_{\tau_n} (V_\omega^{\tau_n})'(y \widehat{Z}_{\tau_n}(y)) \mathbf{1}_{\{\tau_n < \infty, Z_{\tau_n} \geq y \widehat{Z}_{\tau_n}(y)\}} \right] \quad (74)$$

$$\stackrel{(67)}{=} \liminf_{n \rightarrow \infty} c \mathbf{E}_{\mathbf{P}} \left[-\widehat{X}_{\tau_n}(x) Z_{\tau_n} \mathbf{1}_{\{\tau_n < \infty, Z_{\tau_n} \geq y \widehat{Z}_{\tau_n}(y)\}} \right] \stackrel{(70)}{>} 0, \quad (75)$$

a contradiction to the \mathbf{P} -integrability of $V(\frac{d\mathbf{Q}}{d\mathbf{P}})$ and the fact that $(\tau_n)_{n=1}^\infty$ increases to ∞ . ■

3 The Role of Potential Investments: a Puzzling Example

We start by describing the building block of the example.

Let $(\Omega, \mathcal{F}, (\mathcal{G}_t)_{t \geq 0}, \mathbf{Q})$ be a filtered probability space on which a process $W = (W_t)_{t \geq 0}$ and a sequence $(\xi_n)_{n=1}^\infty$ of $\{0, 1\}$ -valued random variables is defined such that

- (i) W is a standard Brownian motion under \mathbf{Q} starting at $W_0 = a_0$ for a constant a_0 to be specified below, and $(\mathcal{G}_t)_{t \geq 0}$ is the (saturated) filtration generated by W .
- (ii) $(\xi_n)_{n=1}^\infty$ is an independent sequence of $\{0, 1\}$ -valued random variables, independent of W , such that $\mathbf{Q}[\xi_n = 1] = q_n$, for some sequence $q_n \in]0, 1[$ to be specified below.

We also fix sequences of real numbers $(a_n)_{n=0}^\infty$ and $(b_n)_{n=1}^\infty$ such that $(a_n)_{n=0}^\infty$ is strictly decreasing to $-\infty$, and $(b_n)_{n=1}^\infty$ satisfies $b_n > a_{n-1}$. In the application below we shall have $a_n \approx -2^n$ and $b_n \approx \ln(n)$. By the notation $f(n) \approx g(n)$ we mean that $\left(\frac{f(n)}{g(n)}\right)_{n=1}^\infty \in [c^{-1}, c]$, for some $c > 1$.

Define the stopping times $(\sigma_n)_{n=0}^\infty$ by $\sigma_0 = 0$ and

$$\sigma_n = \inf \{t \geq \sigma \ni: W_{\sigma_{n-1}} = a_{n-1} \text{ and } (W_t = a_n \text{ or } b_n)\}, \quad \text{for } n \geq 1. \quad (76)$$

Letting $d_n = \frac{a_{n-1} - b_n}{a_n - b_n} \in]0, 1[$ and $e_n = \prod_{j=1}^n d_j$, one verifies inductively that

$$\mathbf{Q} [W_{\sigma_n} = a_n, \sigma_n < \infty] = e_n, \quad \text{for } n \geq 0. \quad (77)$$

Note that for $a_n \approx -2^n$, $b_n \approx \ln(n)$ we have $d_n \approx \frac{1}{2}$ and $e_n \approx 2^{-n}$. If e_n tends to zero, the stopping times $(\sigma_n)_{n=1}^\infty$ increase a.s. to ∞ , which we assume from now on.

The filtration $(\mathcal{F}_t)_{t \geq 0}$ will be the smallest (right continuous and saturated) filtration containing $(\mathcal{G}_t)_{t \geq 0}$ and such that $\xi_n \mathbf{1}_{\{W_{\sigma_n} = a_n, \sigma_n < \infty\}}$ is \mathcal{F}_{σ_n} -measurable. Let

$$\tau = \inf_n \{\sigma_n : W_{\sigma_n} = b_n \text{ or } \xi_n = 1\}, \quad (78)$$

and

$$\tau_n = \sigma_n \wedge \tau, \quad \text{for } n \geq 0, \quad (79)$$

so that $(\tau_n)_{n=1}^\infty$ defines a sequence of stopping times with respect to the filtration $(\mathcal{F}_t)_{t \geq 0}$ stationarily increasing to the finite stopping time τ .

We now define the process S which is simply Brownian motion W starting at a_0 and stopped at time τ , i.e.,

$$S_t := (W^\tau)_t = W_{t \wedge \tau}, \quad t \geq 0. \quad (80)$$

Note that $(\mathcal{F}_{t \wedge \tau})_{t \geq 0}$ is the (right continuous, saturated) filtration generated by S .

Using the notation

$$A_n = \{\tau_n = \tau \text{ and } S_{\tau_n} = a_n\}, \quad B_n = \{\tau_n = \tau \text{ and } S_{\tau_n} = b_n\}, \quad (81)$$

we find an almost sure partition $((A_n)_{n=1}^\infty, (B_n)_{n=1}^\infty)$ of Ω into disjoint sets.

Here is the interpretation for this construction: we obtain the process S by letting a Brownian motion W start at $W_0 = a_0$ and run until time σ_1 , when it first hits $a_1 < a_0$ or $b_1 > a_0$. If $S_{\sigma_1} = W_{\sigma_1} = b_1$, i.e., on the set B_1 , we stop and everything is over; if $S_{\sigma_1} = W_{\sigma_1} = a_1$ we flip a coin modeled by the random variable ξ_1 . If $\xi_1 = 1$, i.e., on the set A_1 , we stop again, but if $\xi_1 = 0$ we continue by letting S run like the Brownian motion W until it first hits a_2 or b_2 ; again we only continue if we hit a_2 and, in addition, $\xi_2 = 0$. Proceeding in an obvious way we stop with probability one at a finite time, namely at the stopping time τ .

Note that S is a martingale with respect to the measure \mathbf{Q} and the filtration $(\mathcal{F}_t)_{t \geq 0}$.

From now on we suppose that $\lim_{n \rightarrow \infty} e_n a_n \neq 0$, and that there is $n_0 \geq 0$ such that, for $n \geq n_0$, we have $a_n < 0$, $b_n > 0$ and $\sum_{n \geq n_0} b_n / |a_{n-1}| < \infty$, which is verified for the choices of a_n and b_n indicated above. Under these assumptions we have that the sequence of random variables $(S_{\tau_n})_{n \geq 0}$ remains bounded in $L^1(\mathbf{Q})$. Indeed $\{S_{\tau_{n+1}} \neq S_{\tau_n}\} \subseteq \{S_{\tau_n} = a_n\}$ and an elementary calculation reveals that, for $n \geq n_0$,

$$\|S_{\tau_{n+1}} \mathbf{1}_{\{S_{\tau_n} = a_n\}}\|_{L^1(\mathbf{Q})} \leq \left(1 + 2 \frac{b_{n+1}(1 - d_{n+1})}{|a_n|}\right) \|S_{\tau_n} \mathbf{1}_{\{S_{\tau_n} = a_n\}}\|_{L^1(\mathbf{Q})} \quad (82)$$

so that

$$\|S_{\tau_{n+1}}\|_{L^1(\mathbf{Q})} \leq \left(1 + 2 \frac{b_{n+1}(1 - d_{n+1})}{|a_n|}\right) \|S_{\tau_n}\|_{L^1(\mathbf{Q})}, \quad (83)$$

which readily implies the boundedness of $(\|S_{\tau_n}\|_{L^1(\mathbf{Q})})_{n=1}^\infty$, and in particular the \mathbf{Q} -integrability of S_τ .

We now pass to the choice of $(q_n)_{n=1}^\infty$. Under the above assumptions we have that the martingale S is uniformly integrable with respect to \mathbf{Q} iff

$$\sum_{n=1}^\infty q_n = \sum_{n=1}^\infty \mathbf{Q}[\xi_n = 1] = \infty. \quad (84)$$

Indeed, this condition is equivalent to the fact that $(S_{\tau_n})_{n=1}^\infty$ converges to S_τ in the norm of $L^1(\mathbf{Q})$, which in turn is equivalent to

$$\lim_{n \rightarrow \infty} \mathbf{E}[S_{\tau_n} \mathbf{1}_{\{S_{\tau_n} \neq S_\tau\}}] = \quad (85)$$

$$= \lim_{n \rightarrow \infty} \left[a_n e_n \prod_{j=1}^{n+1} (1 - q_j) \right] = 0. \quad (86)$$

The construction of the building block for the subsequent Proposition 3.1 now is finished, up to some cosmetics (which may be skipped at a first reading): the reader might dislike the feature that the time index set for the process S is \mathbb{R}_+ , and not the compact interval $[0, T]$. The remedy is very easy: fix $T > 0$, and make the deterministic time change

$$\tilde{S}_{T(1-e^{-t})} := S_t, \quad t \geq 0, \quad (87)$$

so that \tilde{S} is defined over the time index set $[0, T[$. By the a.s. finiteness of the stopping time τ we have that the trajectories of \tilde{S} become eventually constant on $[0, T[$ a.s. and may therefore be continuously extended to the time index set $[0, T]$ thus obtaining a process $(\tilde{S}_t)_{0 \leq t \leq T}$ with all the features of the above process S and such that \tilde{S} is a local \mathbf{Q} -martingale. We now have that \tilde{S} is a \mathbf{Q} -martingale iff S is uniformly \mathbf{Q} -integrable which was characterized by (84) above. By abuse of notation we shall still write S for the time-transformed process \tilde{S} .

Proposition 3.1 *There are semimartingales $S^1 = (S_t^1)_{0 \leq t \leq T}$ and $S^2 = (S_t^2)_{0 \leq t \leq T}$, defined on and adapted to $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbf{P})$, and a probability measure $\mathbf{Q} \sim \mathbf{P}$ with the following properties:*

- (i) $H(\mathbf{Q}|\mathbf{P}) < \infty$.
- (ii) S^1 is a martingale under \mathbf{Q} and a local martingale, but not a supermartingale, under \mathbf{P} .
- (iii) S^2 is a martingale under \mathbf{Q} , but not a local martingale under \mathbf{P} .
- (iv) Denoting by S^{large} the \mathbb{R}^2 -valued process (S^1, S^2) , we have that \mathbf{Q} is the unique equivalent local martingale measure for S^{large} . The equality

$$S_T^1 = -\ln\left(\frac{d\mathbf{Q}}{d\mathbf{P}}\right) \quad (88)$$

holds true, and therefore the process $(S_t^1)_{0 \leq t \leq T}$ equals the investment process $(\widehat{X}_t)_{0 \leq t \leq T}$, starting at $\widehat{X}_0 = S_0^1$, which is optimal with respect to the utility function $U(x) = -e^{-x}$.

(v) Denoting by S^{small} the \mathbb{R} -valued process S^1 , we have that \mathbf{P} is a local martingale measure for S^{small} . Therefore the optimal investment process with initial endowment $\widehat{X}_0 = S_0^1$ equals $\widehat{X}_t \equiv S_0^1$. The optimality pertains to the utility function $U(x) = -e^{-x}$ (and, in fact, to any increasing, strictly concave utility function U).

Before aboarding the proof, let us comment on the result: In the case of the “large” financial market S^{large} , the optimal investment consists in constantly holding one unit of the first asset S^1 and not touching the second asset S^2 . Nevertheless by passing to the “small” financial market S^{small} , which consists of this first asset only, this strategy is not optimal any more! In this case the optimal strategy is not to invest at all into a risky asset and to keep the money in the bond $B_t \equiv \text{const}$.

Hence the role of the asset S^2 may be compared to a catalyst in chemistry: its sheer presence changes the situation without entering into the chemical reaction.

As we shall see below, one reason for this seemingly paradoxical result is that in the former case the optimal strategy $H = (1, 0)$ of holding one unit of the first asset can be approximated by a sequence $H^n = (H^{n,1}, H^{n,2})$ of admissible strategies in the sense of Definition 1.4 (iii) and (iii') above. The second asset is needed for these approximating strategies and such an approximation is not possible by only dealing in the first asset.

Proof of Proposition 3.1 To define the process S^1 we shall do the construction preceding the statement of Proposition 3.1, where we shall specify the numbers a_n, b_n appropriately: let $a_0 = -1$ and

$$a_n := -\ln \left(\frac{e^{2^n}}{n(n+1) \prod_{j=1}^{n-1} (1 - e^{-2^j})} \right) \approx -2^n, \quad n \geq 1, \quad (89)$$

$$b_n := -\ln \left(\frac{1}{n \prod_{j=1}^{n-1} (1 - e^{-2^j})} \right) \approx \ln(n), \quad n \geq 1, \quad (90)$$

where $\prod_{j=1}^0 := 1$. Note that $(a_n)_{n=0}^\infty$ decreases to $-\infty$ and $(b_n)_{n=1}^\infty$ increases to $+\infty$ and $-1 = a_0 < b_1 = 0$.

The measure \mathbf{Q} will be defined by letting

$$q_n = \mathbf{Q}[\xi_n = 1] = \frac{1}{n+1}, \quad n \geq 1, \quad (91)$$

while the measure \mathbf{P} will be defined in an analogous way by letting

$$p_n = \mathbf{P}[\xi_n = 1] = e^{-2^n}, \quad n \geq 1. \quad (92)$$

Let $d_n = \frac{a_{n-1} - b_n}{a_n - b_n} \in]0, 1[$ and $e_n = \prod_{j=1}^n d_n$ be as above (so that we have $d_n \approx \frac{1}{2}$ and $e_n \approx 2^{-n}$). We then obtain the subsequent quantities for the measures \mathbf{Q} and \mathbf{P} :

$$\mathbf{P}[A_n] = e_n e^{-2^n} \prod_{j=1}^{n-1} (1 - e^{-2^j}) \approx 2^{-n} e^{-2^n}, \quad (93)$$

$$\mathbf{P}[B_n] = e_{n-1} (1 - d_n) \prod_{j=1}^{n-1} (1 - e^{-2^j}) \approx 2^{-n}, \quad (94)$$

and

$$\mathbf{Q}[A_n] = e_n \frac{1}{n+1} \prod_{j=1}^{n-1} \left(1 - \frac{1}{j+1}\right) = \frac{e_n}{n(n+1)} \approx \frac{2^{-n}}{n^2} \quad (95)$$

$$\mathbf{Q}[B_n] = e_{n-1}(1-d_n) \prod_{j=1}^{n-1} \left(1 - \frac{1}{j+1}\right) = \frac{e_{n-1}(1-d_n)}{n} \approx \frac{2^{-n}}{n}. \quad (96)$$

Observe that we have arranged things in such a way that we have

$$a_n = -\ln \left(\frac{\mathbf{Q}[A_n]}{\mathbf{P}[A_n]} \right), \quad b_n = -\ln \left(\frac{\mathbf{Q}[B_n]}{\mathbf{P}[B_n]} \right), \quad \text{for } n \geq 1. \quad (97)$$

Next we define the random variable S_T^1 such that (88) holds true. Noting that $\frac{d\mathbf{Q}}{d\mathbf{P}}$ is constant on each A_n and B_n this amounts to

$$S_T^1 = \begin{cases} a_n & \text{on } A_n \\ b_n & \text{on } B_n. \end{cases} \quad (98)$$

The process $(S_t^1)_{0 \leq t \leq T}$ is defined by

$$S_t^1 = \mathbf{E}_{\mathbf{Q}}[S_T^1 | \mathcal{F}_t], \quad 0 \leq t \leq T. \quad (99)$$

It now is obvious that this process starts at $S_0^1 = a_0 = -1$ and equals the continuous process S described in the construction preceding the proposition. Indeed, this process is a \mathbf{Q} -martingale with terminal value S_T^1 . We infer from (84), (91) and (92) that S^1 is a (uniformly integrable) \mathbf{Q} -martingale, but only a local martingale under \mathbf{P} .

The process $(S_t^2)_{0 \leq t \leq T}$ is defined by

$$S_t^2 = \sum_{n=1}^{\infty} 2^{-n} (\xi_n - q_n) \mathbf{1}_{[\tau_n, T]}(t) \mathbf{1}_{\{W_{\tau_n} = a_n\}}, \quad 0 \leq t \leq T. \quad (100)$$

Clearly S^2 is a bounded martingale under \mathbf{Q} as $\mathbf{E}_{\mathbf{Q}}[\xi_n] = q_n$. It is also obvious that \mathbf{Q} is the unique equivalent martingale measure for the process $S^{\text{large}} = (S^1, S^2)$ on the sigma-algebra \mathcal{F}_τ generated by S^{large} . Indeed, a local martingale measure \mathbf{Q} for the process S in the construction preceding the proposition is unique up to the choice of q_n (which may, in general, be any \mathcal{F}_{τ_n} -measurable $]0, 1[$ -valued function, defined on $\{\tau_n < \infty, W_{\tau_n} = a_n\}$). The requirement that S^2 is a \mathbf{Q} -martingale and the predictability of the stopping time $\rho_n := \inf\{t : S_t^1 = a_n\}$ forces these function to equal $q_n = \frac{1}{n+1}$ a.s. on $\{\tau_n < \infty, W_{\tau_n} = a_n\}$.

We now turn to the verification of assertions (i)–(v):

(i) We have

$$H(\mathbf{Q}|\mathbf{P}) = \sum_{n=1}^{\infty} \left(\mathbf{Q}[A_n] \ln \left(\frac{\mathbf{Q}[A_n]}{\mathbf{P}[A_n]} \right) + \mathbf{Q}[B_n] \ln \left(\frac{\mathbf{Q}[B_n]}{\mathbf{P}[B_n]} \right) \right) \quad (101)$$

$$\approx \sum_{n=1}^{\infty} \left(\frac{2^{-n}}{n^2} 2^n + \frac{2^{-n}}{n} \ln \left(\frac{1}{n} \right) \right) < \infty. \quad (102)$$

(ii) S^1 is a uniformly integrable martingale under \mathbf{Q} by (99); on the other hand, the fact that $\sum_{j=1}^{\infty} p_n = \sum_{n=1}^{\infty} e^{-2^n} < \infty$ implies that S^1 fails to be a uniformly integrable martingale under \mathbf{P} by (84) above.

(iii) obvious.

(iv) We have already observed that \mathbf{Q} is the unique equivalent martingale measure for S^{large} and that (88) holds true. Hence it follows from the general theory (see [S01, theorem 2.1]) that $\widehat{X}_t := S_t^1$ is the optimal investment process for the exponential utility function $U(x) = -e^{-x}$ and initial endowment $\widehat{X}_0 = a_0$.

(v) For S^{small} we have that $\mathbf{P} \in \mathcal{M}^e(S^{\text{small}})$ and therefore it is the entropy-minimizing local martingale measure for S^{small} . Again it follows from the general theory that $\widehat{X}_t \equiv S_0$ is the optimal investment process for the initial endowment $\widehat{X}_0 = S_0$.

Indeed, in this case the assertion reduces to a triviality, using only the monotonicity and concavity of U and Jensen's inequality. If H is any admissible trading strategy, we have $\mathbf{E}_{\mathbf{P}}[S_0 + (H \cdot S)_T] \leq S_0$, and therefore $\mathbf{E}_{\mathbf{P}}[U(S_0 + (H \cdot S)_T)] \leq U(S_0)$. ■

Remark 3.2 It is interesting to explicitly identify an approximating sequence of admissible strategies $(H^n)_{n=1}^{\infty}$ for the optimal investment $\widehat{H} = (1, 0)$ in the case of $S^{\text{large}} = (S^1, S^2)$ (which we know to exist by Theorem 2.1) and to give some economic interpretation for it.

Let us start by considering $\widetilde{H}^n = (\mathbf{1}_{[0, \tau_n]}, 0)$, i.e., the strategy of holding one unit of asset S^1 up to time τ_n and then selling it. Clearly this is an admissible strategy and we have

$$\left(\widetilde{H}^n \cdot S^{\text{large}}\right)_T = \begin{cases} a_i & \text{on } A_i, & \text{for } i \leq n \\ b_i & \text{on } B_i, & \text{for } i \leq n \\ a_n & \text{on } A_j \cup B_j, & \text{for } j > n. \end{cases} \quad (103)$$

But this is a very poor investment from the point of view of exponential utility $U(x) = -e^{-x}$. Indeed, $\mathbf{P}[\bigcup_{j>n} (A_j \cup B_j)] \approx 2^{-n}$ and therefore

$$\mathbf{E} \left[U \left(\widehat{H}_n \cdot S^{\text{large}} \right)_T \right] \leq \mathbf{P} \left(\bigcup_{j>n} (A_j \cup B_j) \right) (-e^{-a_n}) \approx 2^{-n} (-e^{2^n}) \approx -e^{2^n}. \quad (104)$$

How to remedy this situation? By only considering investments in S^1 , i.e., in the “small” financial market S^{small} , there is little one can do about it. This should be intuitively clear and is formally implied by Proposition 3.1.

But in the case of the “large” financial market S^{large} it is possible to find a remedy by trading on S^2 .

The stopping time $\rho_n = \inf\{t : S_t^1 = a_n\}$ is predictable and therefore it does make sense to invest into the asset S^2 at time $(\rho_n)_-$ an appropriate quantity and to sell it again at time ρ_n . In other words, in the “large” financial market we are allowed to make bets on the outcome of the random variable ξ_n . These bets are priced by the measure \mathbf{Q} for which we have

$$\mathbf{E}_{\mathbf{Q}}[\xi_n = 1] = q_n = \frac{1}{n+1}. \quad (105)$$

In other words, it is possible to make a bet which pays $-a_n$ in the case $\{\xi_n = 0\}$, i.e., on the sets $\bigcup_{j>n}(A_j \cup B_j)$, and na_n in the case $\{\xi_n = 1\}$, i.e., on the set A_n .

In fact, the good remedy for the above trading strategy \tilde{H}_n is slightly more tricky, as it does not suffice to make a bet on ξ_n , but rather on ξ_N for some $N \gg n$.

To be more precise: fix $n \in \mathbb{N}$ and, for $N > n$, consider the contingent claim

$$f_N = -a_N \mathbf{1}_{\{\bigcup_{j>N} A_j \cup B_j\}} - \epsilon_N \mathbf{1}_{A_n}, \quad (106)$$

where we choose $\epsilon_N > 0$ such that $\mathbf{E}_{\mathbf{Q}}[f_N] = 0$, which yields

$$\epsilon_N \mathbf{Q}[A_n] = -a_N \mathbf{Q} \left[\bigcup_{j>N} A_j \cup B_j \right], \quad (107)$$

so that

$$\epsilon_N \approx n^2 2^n / N, \quad (108)$$

which tends to zero for $N \rightarrow \infty$. Denote by $L^N = (L^{N,1}, L^{N,2})$ the trading strategy which replicates the contingent claim f_N . This is possible as the market S^{large} is complete.

Defining the trading strategy K^N as $\tilde{H}^n + L^N$, where $\tilde{H}^n = (\mathbf{1}_{[0, \tau_n]}, 0)$, note that K^N is admissible and satisfies

$$(K^N \cdot S^{\text{large}})_T = \begin{cases} a_i & \text{on } A_i, & \text{for } i \leq N, i \neq n \\ a_n - \epsilon_N & \text{on } A_n \\ b_i & \text{on } B_i, & \text{for } i \leq N \\ 0 & \text{on } A_j \cup B_j, & \text{for } j > N. \end{cases} \quad (109)$$

Hence $\mathbf{E}_{\mathbf{P}}[U((K^N \cdot S^{\text{large}})_T)]$ tends to the optimal $\mathbf{E}_{\mathbf{P}}[U(S_T^1)]$, as N tends to infinity. Speaking economically: after having followed the optimal trading strategy S^1 up to time τ_n the investor operating in the large financial market may continue to invest *with a finite credit line in S^1 and S^2* , in such a way, to finally obtaining almost the optimal expected utility.

Here is another observation on this puzzling example: the optimal investment process $\hat{X} = S^1$ for S^{large} fails to be a \mathbf{P} -martingale on the time index set $[0, T]$; but it is easily verified that it is a \mathbf{P} -martingale on the open time index set $[0, T[$. In particular we have

$$\mathbf{E}_{\mathbf{P}}[\hat{X}_t] = a_0 = -1, \quad \text{for } 0 \leq t < T, \quad \text{and } \mathbf{E}_{\mathbf{P}}[\hat{X}_T] > -1. \quad (110)$$

By passing to the utility process $(U(\hat{X}_t))_{0 \leq t \leq T}$ we deduce from Jensen's inequality that $\mathbf{E}_{\mathbf{P}}[U(\hat{X}_t)]$ is a decreasing function of $t \in [0, 1[$. It is not hard to verify that $\lim_{t \nearrow T} \mathbf{E}_{\mathbf{P}}[U(\hat{X}_t)] = -\infty$; on the other hand we have $\mathbf{E}_{\mathbf{P}}[U(\hat{X}_T)] > U(\hat{X}_0) = -e$.

Speaking economically, the optimal investor is not bothered by the fact that her expected utility is continuously decreasing in time $t \in [0, T[$ and in very bad shape at time $T - \epsilon$, as she knows that it will jump up in a discontinuous way at time T . An explanation for this apparently dangerous attitude is that — potentially — she always has the possibility of the above described bets on ξ_n which can save her. In the absence of these bets, she cannot dare to follow the investment strategy \hat{X} . Nevertheless, in the large market S^{large} it is optimal for her never to bet on ξ_n .

Another feature of this example is the following: the reader might have wondered whether one may weaken the requirements of “allowability” or “permittedness” of a trading strategy H as given in Definition 1.4 by only requiring that the process $H \cdot S$ is a uniformly integrable martingale for *some* equivalent martingale measure $\mathbf{Q} \in \mathcal{M}^e(S)$. This class of trading strategies is closely related to the class of trading strategies studied in [DS97] under the name of “workable” trading strategies.

The present example shows in a rather striking way that such hopes are in vain (even when we are requiring in addition that the above \mathbf{Q} has finite entropy): by considering S^{small} and the process $X_t = S_t^1$, we find a process which is a uniformly integrable martingale under the measure $\mathbf{Q} \in \mathcal{M}^e(S)$, and such that, for $U(x) = -e^{-x}$,

$$\mathbf{E}[U(X_1)] > U(-1) = -e. \quad (111)$$

However, item (v) asserts that the optimal investment equals $\widehat{X}_t \equiv -1$.

This shows in particular that the duality theory for the optimization problem breaks down if we would weaken the requirement on the “permitted” trading strategies to allow for all trading strategies such that the stochastic integral is a uniformly integrable martingale with respect to some equivalent martingale measure.

A closely related topic is the following: A basic feature in portfolio optimization, which we implicitly encountered above, and which sometimes is referred to as the “Fundamental Theorem of Utility Maximization” [K00, corr. 2.7], states — roughly speaking — that a trading strategy H is optimal iff $U'((H \cdot S)_T)$ is — up to a normalizing factor $y > 0$ — the density of a local martingale measure for S . The present example shows that one has to be rather careful when trying to make a precise theorem of this general principle: for $S^{\text{small}} = S^1$ we have the situation that $H \equiv 1$ is a trading strategy such that, for $H \cdot S^1 = S^1$ and $\frac{d\mathbf{Q}}{d\mathbf{P}} = U'((H \cdot S)_T) = \exp(-S_T^1)$, we have that \mathbf{Q} is a martingale measure for S^1 and that $H \cdot S^1$ is a uniformly integrable \mathbf{Q} -martingale. Nevertheless the trading strategy $H \equiv 1$ is not optimal, as $H \equiv 1$ is not permitted.

In [GR00, lemma 3.3 and proposition 3.4] additional conditions were isolated, which are sufficiently strong to rule out the above example, and under which the above “Fundamental Theorem of Utility Maximization” becomes a precise mathematical theorem (for the case of exponential utility and a locally bounded semimartingale S).

One more observation: in the context of Definition 1.4 (iii) and (iii') one might ask whether the approximating trading strategies $(H^n)_{n=1}^\infty$ for the optimal trading strategy \widehat{H} can be chosen by stopping \widehat{H} , i.e., by letting $H^n = \widehat{H} \cdot \mathbf{1}_{[0, \tau_n]}$ for a suitable chosen sequence of stopping times increasing to infinity. The example of S^{large} shows that — in general — this is not possible; in fact, we deduce from (v) that, even allowing to multiply $\widehat{H} = (1, 0)$ by more general predictable processes than $\mathbf{1}_{[0, \tau_n]}$, this does not help to find an approximating sequence $(H^n)_{n=1}^\infty$ of admissible integrands.

Let us now discuss the role of the process S^2 , in more detail, which mainly serves to “make the martingale measure \mathbf{Q} unique”. For convenience we have modeled it as a process with jumps at predictable times: the idea of the above construction is most intuitively explained by “flipping a coin”. This modelization leads to the above trading strategies of “buying at $(\rho_n)_-$ and selling at ρ_n ”.

For readers who don't like discontinuous processes and trading strategies as above we point out that the phenomenon of the above example has nothing to do with these

features: instead of flipping a coin one can just as well run a Brownian motion until it first hits plus or minus one.

This is the modification of the above construction we have in mind: define S^1 up to time σ_1 just as above. Now define the process S^2 to equal zero up to time σ_1 and then to run like the Brownian motion W on the event $\{S_{\sigma_1}^1 = a_1\}$ (otherwise S^2 remains equal to zero for ever). Let S^2 run until the stopping time $\nu_1 > \sigma_1$ when S^2 first hits $+1$ or -1 . On the stochastic interval $[\sigma_1, \nu_1]$ the process S^1 is kept constant; after ν_1 the process S^1 runs again like the Brownian motion W on the set $\{S_{\nu_1}^2 = -1\}$ (which now corresponds to $\{\xi_1 = 0\}$) while S^2 remains constant; on the set $\{S_{\nu_1}^2\}$ we stop everything.

Continuing in an obvious way we obtain a continuous process S^2 which plays exactly the same role as the discontinuous process S^2 constructed above (we can also make S^2 bounded by replacing $+1$ and -1 by $+2^{-n}$ and -2^{-n} at the n 'th step to define the stopping time ν_n). The different weight on the outcomes of ξ_n under \mathbf{P} and \mathbf{Q} now is achieved by Girsanov's theorem which allows to give different probabilities to the events $\{S_{\nu_1}^2 = 1\}$ and $\{S_{\nu_1}^2 = -1\}$. We leave the details to the energetic reader.

Summing up, we have sketched a modification of the above construction such that S^2 is a continuous bounded process; also note that we thus may base the above construction on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbf{P})$ generated by the one-dimensional Brownian motion W .

In the next example we show that, under the assumptions of Theorem 2.1 and for the case of exponential utility, it may happen that there is some $\mathbf{Q} \in \mathcal{M}^e(S)$ under which the optimal investment process $\widehat{X} = \widehat{H} \cdot S$ fails to be a supermartingale. Hence the condition on the finiteness of $H(\mathbf{Q}|\mathbf{P})$, or, more generally, of $E[V(\frac{d\mathbf{Q}}{d\mathbf{P}})] < \infty$, cannot be dropped in Definition 1.4 (ii) and (ii').

Example 3.3 *There is a continuous \mathbb{R} -valued financial market $(S_t)_{0 \leq t \leq T}$ such that the entropy-minimal element $\widehat{\mathbf{Q}} \in \mathcal{M}^e(S)$ exists, and there is some equivalent martingale measure $\mathbf{Q} \in \mathcal{M}^e(S)$, with $H(\mathbf{Q}|\mathbf{P}) = \infty$, for which the optimal investment process \widehat{X} with initial endowment $\widehat{X}_0 = 0$ is only a local \mathbf{Q} -martingale, but not a \mathbf{Q} -supermartingale.*

The construction is similar in spirit to the above one, but somewhat simpler. Again we have a stochastic base $(\Omega, \mathcal{G}, \widehat{\mathbf{Q}})$ on which there is defined a standard $\widehat{\mathbf{Q}}$ -Brownian motion $W = (W_t)_{t \geq 0}$, this time starting at $W_0 = 0$, and a sequence $(\xi_n)_{n=1}^\infty$ of independent $\{0, 1\}$ -valued random variables which are independent of W .

The stopping times $(\sigma_n)_{n=1}^\infty$ are defined inductively by $\sigma_0 = 0$ and

$$\begin{aligned} \sigma_n &= \inf \{t \geq \sigma_{n-1} : W_{\sigma_{n-1}} = -(2^{n-1} - 1) \\ &\quad \text{and } (W_t = -(2^n - 1) \text{ or } 1)\}. \end{aligned} \quad (112)$$

Let $\tau = \inf_n \{\sigma_n : W_{\sigma_n} = 1 \text{ or } \xi_n = 1\}$ and

$$\tau_n = \sigma_n \wedge \tau, \quad \text{for } n \geq 0. \quad (113)$$

The process $S = (S_t)_{t \geq 0}$ again is defined as the stopped Brownian motion W^τ , $(\mathcal{F}_{t \wedge \tau})_{t \geq 0}$ as the natural filtration generated by S and $\mathcal{F} = \mathcal{F}_\tau$. We also make a deterministic time change, so that S , is defined over the closed finite time interval $[0, T]$.

We have not yet specified the probabilities $\widehat{q}_n = \widehat{\mathbf{Q}}[\xi_n = 1]$; note that we again have that S is a uniformly integrable martingale under $\widehat{\mathbf{Q}}$ iff

$$\sum_{n=1}^{\infty} \widehat{q}_n = \sum_{n=1}^{\infty} \widehat{\mathbf{Q}}[\xi_n = 1] = \infty. \quad (114)$$

We now define $\widehat{q}_n = 1 - e^{-2^n}$, for each $n \in \mathbb{N}$, so that S is a uniformly integrable $\widehat{\mathbf{Q}}$ -martingale. We also define a measure \mathbf{Q} on (Ω, \mathcal{F}) in a similar way, but letting $q_n = \mathbf{Q}[\xi_n = 1] = 2^{-n}$, so that S is only a local \mathbf{Q} -martingale but not uniformly \mathbf{Q} -integrable, and therefore not a \mathbf{Q} -supermartingale.

We still have to define the measure \mathbf{P} on (Ω, \mathcal{F}) . We do this via

$$\frac{d\mathbf{P}}{d\widehat{\mathbf{Q}}} = c^{-1} e^{S_T}, \quad (115)$$

where the normalizing constant c is chosen such that

$$c = \mathbf{E}_{\widehat{\mathbf{Q}}}[e^{S_T}]. \quad (116)$$

The above expectation is well-defined as S_T is bounded from above; hence \mathbf{P} is a probability measure.

Clearly the measures $\widehat{\mathbf{Q}}$ and \mathbf{P} are equivalent. It is slightly less obvious that \mathbf{Q} also is equivalent to $\widehat{\mathbf{Q}}$. To verify this, let $A_n = \{S_T = -(2^n - 1)\}$, $B_n = \{S_T = 1 = W_{\sigma_n}\}$ and $B = \cup_{n=1}^{\infty} B_n = \{S_T = 1\}$. We then have

$$\begin{aligned} \mathbf{Q}[A_n] &= 2^{-n} \prod_{i=1}^{n-1} (1 - q_i) q_n \approx 2^{-2n}, & \mathbf{Q}[B_n] &= 2^{-n} \prod_{i=1}^{n-1} (1 - q_i) \approx 2^{-n} \\ \widehat{\mathbf{Q}}[A_n] &= 2^{-n} \prod_{i=1}^{n-1} (1 - \widehat{q}_i) \widehat{q}_n \approx 2^{-n} e^{-2^n}, & \widehat{\mathbf{Q}}[B_n] &= 2^{-n} \prod_{i=1}^{n-1} (1 - \widehat{q}_i) \approx 2^{-n} e^{-2^n} \\ \mathbf{P}[A_n] &= c^{-1} \cdot e^{-(2^n - 1)} \widehat{\mathbf{Q}}[A_n] \approx 2^{-n} e^{-2^{n+1}}, & \mathbf{P}[B_n] &= c^{-1} e \widehat{\mathbf{Q}}[B_n] \approx 2^{-n} e^{-2^n} \end{aligned}$$

Under all three measures, the sequence $((A_n)_{n=1}^{\infty}, (B_n)_{n=1}^{\infty})$ forms an almost sure partition of (Ω, \mathcal{F}) into sets of strictly positive measure.

For each fixed A_n (resp. B_n), the measures \mathbf{Q} and $\widehat{\mathbf{Q}}$, restricted to $\mathcal{F}_{|A_n}$ (resp. $\mathcal{F}_{|B_n}$) are equivalent. Indeed, \mathbf{Q} and $\widehat{\mathbf{Q}}$ are clearly equivalent on each \mathcal{F}_{τ_n} . Note that $\mathcal{F}_{\tau_n|A_n}$ (resp. $\mathcal{F}_{\tau_n|B_n}$) coincides with $\mathcal{F}_{|A_n}$ (resp. $\mathcal{F}_{|B_n}$). Hence \mathbf{Q} and $\widehat{\mathbf{Q}}$ are equivalent measures on (Ω, \mathcal{F}) .

To check that $H(\widehat{\mathbf{Q}}|\mathbf{P})$ is finite, we may explicitly calculate it:

$$\begin{aligned} H(\widehat{\mathbf{Q}}|\mathbf{P}) &= \sum_{n=1}^{\infty} \widehat{\mathbf{Q}}[A_n] \ln \left(\frac{\widehat{\mathbf{Q}}[A_n]}{\mathbf{P}[A_n]} \right) + \widehat{\mathbf{Q}}[B] \ln \left(\frac{\widehat{\mathbf{Q}}[B]}{\mathbf{P}[B]} \right) \\ &\approx \sum_{n=1}^{\infty} 2^{-n} e^{-2^n} \ln(e^{2^n}) + \widehat{\mathbf{Q}}[B] (\ln(c) - 1) < \infty \end{aligned} \quad (117)$$

A similar calculation reveals that $H(\mathbf{Q}|\mathbf{P}) = \infty$. More generally, for any local martingale measure $\widetilde{\mathbf{Q}} \in \mathcal{M}^a(S)$, such that S fails to be a $\widetilde{\mathbf{Q}}$ -martingale, we have $H(\widetilde{\mathbf{Q}}|\mathbf{P}) = \infty$. Indeed, letting $E_n = \cup_{i=n+1}^{\infty} (A_i \cup B_i) = \{S_{\tau_n} = -2^n - 1 \text{ and } \xi_n = 0\}$, the local $\widetilde{\mathbf{Q}}$ -martingale S is a martingale iff $\lim_{n \rightarrow \infty} (2^n - 1) \widetilde{\mathbf{Q}}[E_n] = 0$. As $S^{\tau_{n+1}}$ is bounded, it is

a $\tilde{\mathbf{Q}}$ -martingale, which implies that $\tilde{\mathbf{Q}}[B_{n+1}] = \tilde{\mathbf{Q}}[E_n]/2$. Hence, if S fails to be a $\tilde{\mathbf{Q}}$ -martingale, we may find $\alpha > 0$ and an infinite subset $I \subseteq \mathbb{N}$ such that $\tilde{\mathbf{Q}}[B_n] \geq \alpha 2^{-n}$, for $n \in I$. This implies

$$\begin{aligned} H(\tilde{\mathbf{Q}}|\mathbf{P}) &\geq \sum_{n \in I} \tilde{\mathbf{Q}}[B_n] \ln \frac{\tilde{\mathbf{Q}}[B_n]}{\mathbf{P}[B_n]} - e^{-1} \\ &\approx \alpha \sum_{n \in I} 2^{-n} \ln \left(\frac{\alpha \cdot 2^{-n}}{2^{-n} e^{-2^n}} \right) = \infty. \end{aligned} \quad (118)$$

Next we show that $\hat{\mathbf{Q}}$ is indeed the entropy-minimal element in $\mathcal{M}^a(S)$, as indicated by the notation, and that $\hat{X}(0) = S$ is the optimal process starting at $S_0 = 0$.

Fixing again $\tilde{\mathbf{Q}} \in \mathcal{M}^a(S)$ we shall show that $H(\tilde{\mathbf{Q}}|\mathbf{P}) \geq H(\hat{\mathbf{Q}}|\mathbf{P})$. To do so, we may obviously assume that $H(\tilde{\mathbf{Q}}|\mathbf{P}) < \infty$ so that S is a $\tilde{\mathbf{Q}}$ -martingale. Therefore, letting $V(y) = y(\ln(y) - 1)$ and $U(x) = -e^{-x}$ we find

$$\begin{aligned} H(\tilde{\mathbf{Q}}|\mathbf{P}) - 1 &= \mathbf{E} \left[V \left(\frac{d\tilde{\mathbf{Q}}}{d\mathbf{P}} \right) \right] \\ &= \mathbf{E} \left[\sup_{\xi \in \mathbb{R}} \left(U(\xi) - \xi \frac{d\tilde{\mathbf{Q}}}{d\mathbf{P}} \right) \right] \\ &\geq \mathbf{E} \left[U(\ln(c^{-1}) + S_T) - (\ln(c^{-1}) + S_T) \frac{d\tilde{\mathbf{Q}}}{d\mathbf{P}} \right] \\ &= c \mathbf{E}[\exp(-S_T)] - \mathbf{E}_{\tilde{\mathbf{Q}}}[S_T] + \ln(c) \\ &= c \mathbf{E}[\exp(-S_T)] + \ln(c), \end{aligned} \quad (120)$$

with equality holding true iff $\tilde{\mathbf{Q}} = \hat{\mathbf{Q}}$, i. e., iff $-V'(\frac{d\tilde{\mathbf{Q}}}{d\mathbf{P}}) = -V'(\frac{d\hat{\mathbf{Q}}}{d\mathbf{P}}) = -\ln(\frac{d\hat{\mathbf{Q}}}{d\mathbf{P}}) = \ln(c^{-1}) + S_T$ holds true almost surely.

Having established that $\hat{\mathbf{Q}}$ is the entropy-minimal element of $\mathcal{M}^a(S)$, we now show the optimality of $\hat{X}(0) = S$. By Theorem 2.1 it suffices to show that $\mathbf{E}[U(S_T)] \geq \mathbf{E}[U(X_T)]$ for every random variable X_T satisfying $\mathbf{E}_{\hat{\mathbf{Q}}}[X_T] \leq 0$. This is done via the dual version of the above argument.

$$\begin{aligned} \mathbf{E}[U(X_T)] &= \mathbf{E} \left[\inf_{\eta > 0} (V(\eta) + \eta X_T) \right] \\ &\leq \mathbf{E} \left[V \left(c^{-1} \left(\frac{d\hat{\mathbf{Q}}}{d\mathbf{P}} \right) \right) + c^{-1} \frac{d\hat{\mathbf{Q}}}{d\mathbf{P}} X_T \right] \\ &\leq \mathbf{E} \left[V \left(c^{-1} \left(\frac{d\hat{\mathbf{Q}}}{d\mathbf{P}} \right) \right) \right] \\ &= c^{-1} \left(H(\hat{\mathbf{Q}}|\mathbf{P}) - \ln(c) - 1 \right), \end{aligned} \quad (121)$$

with equality holding true iff $X_T = S_T$ almost surely.

We now have proved all the assertions in the statement of Example 3.3. ■

Remark 3.4 The process S above fails to be a \mathbf{Q} -supermartingale but it is a \mathbf{Q} -submartingale as it is a local martingale bounded from above. This raises the question, whether one can produce a similar example as above such that S fails to be a \mathbf{Q} -submartingale. The answer is yes: it suffices to define the measures $\mathbf{Q}, \widehat{\mathbf{Q}}$ and the process S precisely as above. The only difference is that we now change the sign in the Definition (115) of \mathbf{P} , i.e.

$$\frac{d\mathbf{P}}{d\widehat{\mathbf{Q}}} = c^{-1}e^{-S_T}, \quad (122)$$

where the normalizing constant c now is given by $c = \mathbf{E}_{\widehat{\mathbf{Q}}}[e^{-S_T}]$. As we have $\widehat{\mathbf{Q}}[A_n] \approx 2^{-n}e^{-2^n}$, the expectation is indeed finite. We again have $H(\widehat{\mathbf{Q}}|\mathbf{P}) < \infty$ while $H(\mathbf{Q}|\mathbf{P}) = \infty$.

In this setting $\widehat{\mathbf{Q}}$ again is the entropy-minimal element of $\mathcal{M}^a(S)$ and the optimal process $\widehat{X}(0)$ equals $\widehat{X}(0) = -S$, which therefore is a martingale under $\widehat{\mathbf{Q}}$, but only a local martingale, and not a sub-martingale under \mathbf{Q} .

The verification is analogous to the above arguments and left to the reader.

The subsequent final example is somewhat related to the preceding remark and shows that, for utility functions U different from the exponential one, the term “ \mathbf{Q} -supermartingale” in Definition 1.4(ii) and (iii) cannot be replaced by the term “ \mathbf{Q} -martingale”. Hence the result shown in [DGRSSS 00] for the case of exponential utility, that the optimal investment process $\widehat{H} \cdot S$ also is a \mathbf{Q} -submartingale, for each $\mathbf{Q} \in \mathcal{M}^e(S)$ with finite relative entropy $H(\mathbf{Q}|\mathbf{P})$, is a special feature of the exponential utility.

However, the fact that this result does not extend to other utility functions should not be viewed as a drawback: indeed, the question whether a real-valued local martingale X is a submartingale is related to the behaviour of the process X when X_t assumes values close to $+\infty$, while the supermartingale property is related to the behaviour of X when X_t assumes values close to $-\infty$. It is the latter aspect which — in the present context — is the delicate issue from an economic point of view, as this aspect is related to the situation when the agent’s wealth is deep into the red. On the other hand, there seems to be no obstacle from an economic point of view, when the investment process $(X_t)_{0 \leq t \leq T}$ assumes values close to $+\infty$, i.e., when the agent becomes very rich. In other words, it is the \mathbf{Q} -supermartingale property asserted in Theorem 2.1 which is economically relevant, and the failure of the \mathbf{Q} -submartingale property displayed by the subsequent example does not lead to any major economic paradoxes.

Example 3.5 *There is a utility function $U : \mathbb{R} \rightarrow \mathbb{R}$ satisfying (1) and (2), and a continuous \mathbb{R} -valued financial market $(S_t)_{0 \leq t \leq T}$, such that there is $\mathbf{Q} \in \mathcal{M}^e(S)$ with finite V -expectation $\mathbf{E}[V(\frac{d\mathbf{Q}}{d\mathbf{P}})] < \infty$, and there is $x \in \mathbb{R}$, such that the optimal investment process $\widehat{X}_t(x) = x + (\widehat{H} \cdot S)_t$ fails to be a \mathbf{Q} -submartingale.*

The example will mainly depend on the values of the utility function $U(x)$ for $x \geq 1$ as we shall do the construction such that $\widehat{X}_t \geq 1$. We define U by $U(x) = \ln(x)$, for $x \geq 1$, and extend the definition of $U(x)$ in an arbitrary manner to the entire real line such that (1) and (2) hold true. This implies that the conjugate function V satisfies $V(y) = -\ln(y) - 1$, for $0 < y \leq 1$. We may and do also assume that $\sum_{n=1}^{\infty} 2^{-n} \cdot V(n) < \infty$; indeed, it suffices, e.g., to choose the extension of U such that $U(x) = -e^{-x}$, for x sufficiently close to $-\infty$, so that $V(y) = y(\ln(y) - 1)$, for y sufficiently large.

Now we proceed similarly as in Example 3.3 and remark 3.4: let $W = (W_t)_{t \geq 0}$ be a standard Brownian motion, starting this time at $W_0 = 2$, define $\sigma_0 = 0$,

$$\sigma_n = \inf\{t \geq \sigma_{n-1} : W_{\sigma_{n-1}} = 2^{n-1} + 1 \text{ and } (W_t = 2^n + 1 \text{ or } 1)\}, \quad n \geq 1, \quad (123)$$

and

$$\tau = \inf_n \{\sigma_n : W_{\sigma_n} = 1 \text{ or } \xi_n = 1\}. \quad (124)$$

Again we let $S = W^\tau$ and note that $S_t \geq 1$, for all t , so that $U(S_t) = \ln(S_t)$. Define the probability measure \mathbf{Q} and $\widehat{\mathbf{Q}}$ similarly as above by

$$\widehat{q}_n = \widehat{\mathbf{Q}}[\xi_n = 1] = \frac{1}{2} \quad \text{and} \quad q_n = \mathbf{Q}[\xi_n = 1] = 2^{-n}, \quad (125)$$

so that S is a local martingale under \mathbf{Q} and $\widehat{\mathbf{Q}}$ which, by the same token as in (86) above, is uniformly integrable under $\widehat{\mathbf{Q}}$ but not under \mathbf{Q} . Hence S fails to be a \mathbf{Q} -submartingale. Define \mathbf{P} by

$$\frac{d\mathbf{P}}{d\widehat{\mathbf{Q}}} = c^{-1} S_T \quad (126)$$

which is the formula corresponding to (115) for the present utility function U as in this case $U'(x) = x^{-1}$, for $x \geq 1$. The normalizing constant $c > 0$ is chosen such that \mathbf{P} is a probability measure i.e., $c = \mathbf{E}_{\widehat{\mathbf{Q}}}[S_T]$, which is easily seen to be finite.

Letting again $A_n = \{S_T = 2^n + 1\}$, for $n \geq 1$, $B_n = \{S_T = 1 = W_{\sigma_n}\}$ and $B = \{S_T = 1\}$ we find

$$\begin{array}{ll} \mathbf{Q}[A_n] \approx 2^{-2n} & \mathbf{Q}[B_n] \approx 2^{-n} \\ \widehat{\mathbf{Q}}[A_n] = 2^{-2n} & \widehat{\mathbf{Q}}[B_n] = 2^{-2n+1} \\ \mathbf{P}[A_n] \approx 2^{-n} & \mathbf{P}[B_n] \approx 2^{-2n} \end{array}$$

Hence we have

$$\mathbf{E} \left[V \left(\frac{d\mathbf{Q}}{d\mathbf{P}} \right) \right] = \sum_{n=1}^{\infty} \mathbf{P}[A_n] V \left(\frac{\mathbf{Q}[A_n]}{\mathbf{P}[A_n]} \right) + \sum_{n=1}^{\infty} \mathbf{P}[B_n] V \left(\frac{\mathbf{Q}[B_n]}{\mathbf{P}[B_n]} \right) < \infty \quad (127)$$

and

$$\mathbf{E} \left[V \left(\frac{d\widehat{\mathbf{Q}}}{d\mathbf{P}} \right) \right] = \sum_{n=1}^{\infty} \mathbf{P}[A_n] V \left(\frac{\widehat{\mathbf{Q}}[A_n]}{\mathbf{P}[A_n]} \right) + \sum_{n=1}^{\infty} \mathbf{P}[B_n] V \left(\frac{\widehat{\mathbf{Q}}[B_n]}{\mathbf{P}[B_n]} \right) < \infty. \quad (128)$$

A similar reasoning as in Example (3.3) above reveals that $\widehat{\mathbf{Q}}$ minimizes $\mathbf{E}[V(\frac{d\mathbf{Q}}{d\mathbf{P}})]$ over $\mathbf{Q} \in \mathcal{M}^a(S)$ and that $\widehat{X}(2) = S$ is the optimal process starting at $S_0 = 2$.

This time, however, we have exhibited a measure $\mathbf{Q} \in \mathcal{M}^e(S)$ with *finite* V -expectation such that S fails to be a \mathbf{Q} -submartingale.

This finishes the proof of the assertions on Example 3.5. ■

References

- [AS 94] J.P. Ansel, C. Stricker, (1994), *Couverture des actifs contingents et prix maximum*. Ann. Inst. Henri Poincaré, Vol. 30, pp. 303–315.
- [BF 00] F. Bellini, M. Frittelli, (2000), *On the existence of minimax martingale measures*. preprint.
- [C 75] I. Csiszar, (1975), *I-Divergence, Geometry of Probability Distributions, and Minimization Problems*. Annals of Probability, Vol. 3, No. 1, pp. 146–158.
- [DGRSSS 00] F. Delbaen, P. Grandits, T. Rheinländer, D. Samperi, M. Schweizer, C. Stricker, (2000), *Exponential hedging and entropic penalties*. Preprint, to appear in Mathematical Finance.
- [DS 94] F. Delbaen, W. Schachermayer, (1994), *A General Version of the Fundamental Theorem of Asset Pricing*. Math. Annalen, Vol. 300, pp. 463–520.
- [DS 97] F. Delbaen, W. Schachermayer, (1997), *The Banach Space of Workable Contingent Claims in Arbitrage Theory*. Annales de l' I.H.P., Vol. 33, No. 1, pp. 113–144.
- [DS 98a] F. Delbaen, W. Schachermayer, (1998), *A Simple Counter-Example to Several Problems in the Theory of Asset Pricing, which arises in many incomplete markets*. Mathematical Finance, Vol. 8, pp. 1–12.
- [DS 98b] F. Delbaen, W. Schachermayer, (1998), *The Fundamental Theorem of Asset Pricing for Unbounded Stochastic Processes*. Mathematische Annalen, Vol. 312, pp. 215–250.
- [E 80] M. Emery, (1980), *Compensation de processus à variation finie non localement intégrables*. Séminaire de Probabilités XIV, Springer Lecture Notes in Mathematics, Vol. 784, pp. 152–160.
- [F 00] M. Frittelli, (2000), *The minimal entropy martingale measure and the valuation problem in incomplete markets*. Mathematical Finance, Vol. 10, pp. 39–52.
- [GR 00] P. Grandits, T. Rheinländer, (2000), *On the minimal Entropy Martingale Measure*. Preprint, to appear in Annals of Probability.
- [HP 81] J.M. Harrison, S.R. Pliska, (1981), *Martingales and Stochastic integrals in the theory of continuous trading*. Stoch. Proc. & Appl., Vol. 11, pp. 215–260.
- [J 79] J. Jacod, (1979), *Calcul stochastique et problèmes de martingales*. Lecture Notes in Mathematics, Vol. 714.
- [K 00] J. Kallsen, (2000), *Utility-Based Derivative Pricing in Incomplete Markets*. Preprint.
- [KaS 00] Yu.M. Kabanov, C. Stricker, (2000), *On the optimal portfolio for the exponential utility maximization: remarks to the six-author paper*. Preprint of the Université de Franche-Comté.

- [KaS01a] Yu.M. Kabanov, Ch. Stricker, (2001), *On the true submartingale property d'après Schachermayer*. Preprint of the Université de Franche-Comté.
- [KaS01b] Yu.M. Kabanov, Ch. Stricker, (2001), *Simple strategies in exponential utility maximization*. Preprint of the Université de Franche-Comté.
- [KLSX91] I. Karatzas, J.P. Lehoczky, S.E. Shreve, G.L. Xu, (1991), *Martingale and duality methods for utility maximisation in an incomplete market*. SIAM Journal of Control and Optimisation, Vol. 29, pp. 702–730.
- [KrS99] D. Kramkov, W. Schachermayer, (1999), *The Asymptotic Elasticity of Utility Functions and Optimal Investment in Incomplete Markets*. Annals of Applied Probability, Vol. 9, No. 3, pp. 904–950.
- [P90] P. Protter, (1990), *Stochastic Integration and Differential Equations*. Applications of Mathematics, Vol. 21.
- [RW87] L.C.G. Rogers, D. Williams, (1987), *Diffusions, Markov Processes and Martingales, Volume 2: Itô Calculus*. Wiley, Chichester New York.
- [S93] W. Schachermayer, (1993), *A Counter-Example to several Problems in the Theory of Asset Pricing*. Math. Finance, Vol. 3, pp. 217–229.
- [S01] W. Schachermayer, (2001), *Optimal Investment in Incomplete Markets when Wealth may Become Negative*. Annals of Applied Probability, Vol. 11, No. 3, pp. 694–734.

Walter Schachermayer

Department of Financial and Actuarial Mathematics
 Vienna University of Technology
 Wiedner Hauptstrasse 8-10 / 105
 A-1040 Vienna
 Austria

wschach@fam.tuwien.ac.at
<http://www.fam.tuwien.ac.at/>