The analytical solution and numerical solution of the fractional diffusion-wave equation with damping

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Abstract

Fractional partial differential equations have been applied to many problems in physics, finance, and engineering. Numerical methods and error estimates of these equations are currently a very active area of research. In this paper we consider a fractional diffusion-wave equation with damping. We derive the analytical solution for the equation using the method of separation of variables. An implicit difference approximation is constructed. Stability and convergence are proved by the energy method. Finally, two numerical examples are presented to show the effectiveness of this approximation.

1 Introduction

Fractional differential equations have been widely used in recent years in various applications in science and engineering (see [4]; [5]; [6], [7]; [8]; [9]). The fractional diffusion equation and the fractional wave equation are two basic

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examples of these equations. The fractional diffusion equation was introduced in physics by Nigmatullin (see [19]; [20]) to describe diffusion in media with fractal geometry, which is a special type of porous media. He pointed out that many of the universal electromagnetic, acoustic, and mechanical responses can be more accurately modeled by the fractional diffusion-wave equation. Gorenflo et al. [21] presented the scale-invariant solutions for the time-fractional diffusion-wave equation in terms of the generalized Wright function. Agrawal [22, 23] extended this formulation to a diffusion-wave equation that contains a fourth-order space derivative term. Both semi-infinite and bounded space domains were considered. Mainardi et al. [24] presented the fundamental solution (Green function) for the space-time fractional diffusion equation. Agrawal [25] used the method of separation of variables to identify the eigenfunctions and to reduce the fractional diffusion-wave equation to a set of infinite equations each of which describes the dynamics of an eigenfunction. A Laplace transform technique was used to obtain the fractional Green function and a Duhamel integral type expression for the system's response. Anh and Leonenko [28] presented the Green functions and spectral representations of the mean-square solutions of the fractional diffusion-wave equations with random initial conditions. Orsingher and Zhao [26] discussed the space-fractional telegraph equation and obtained the Fourier transform of its fundamental solution. A symmetric process with discontinuous trajectories, whose transition function satisfies the space-fractional telegraph equation, was presented. Beghin and Orsingher [27] proved that the fundamental solution to the Cauchy problem for the fractional telegraph equation can be expressed as the distribution of the composition of two processes. Moreover they obtained explicit expressions for the probability distribution of a telegraph process. Orsingher and Beghin [29] studied the fundamental solutions to time-fractional telegraph equations and obtained the Fourier transform of the solutions. Chen et al. [30] discussed and derived the analytical solution of the time-fractional telegraph equation with three kinds of nonhomogeneous boundary conditions, namely, the Dirichlet, Neumann and Robin boundary conditions.

Compared with considerable work on the theoretical analysis, however, only a few authors researched numerical methods and numerical analysis of the fractional diffusion-wave equation. Povstenko [31] studied the solutions of time-fractional diffusion-wave equation in a half-space in the case of angular symmetry. William and Kassen [32] studied a generalized Crank-Nicolson scheme for the time discretization of a fractional wave equation, in combination with a space discretization by linear finite elements. Sun and Wu [33] gave a fully discrete difference scheme for the fractional diffusion-wave equation and proved that the difference scheme is uniquely solvable, unconditionally stable and convergent in the L_∞- norm.

In this paper, we will consider the fractional diffusion-wave equation with damping:

$$\begin{cases} {}_0D_t^\alpha u(x,t) + \lambda \frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + \mu\, s(x,t), & 0 < x < L,\ 0 \le t \le T, \\ u(x,0) = f(x),\ \frac{\partial u(x,0)}{\partial t} = g(x), & 0 \le x \le L, \\ u(0,t) = u(L,t) = 0, & 0 \le t \le T, \end{cases} \tag{1}$$

where $\lambda > 0$ and μ are constants, $f(0) = f(L) = 0$, $f(x)$ and $g(x)$ are both real-valued and sufficiently well-behaved functions. Here ${}_0D_t^\alpha u(x,t)$ is the Caputo derivative, which is defined as

$${}_0D_t^\alpha u(x,t) = \begin{cases} \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-s)^{1-\alpha} \frac{\partial^2 u(x,s)}{\partial s^2} ds, & 1 < \alpha < 2, \\ \frac{\partial^2 u(x,t)}{\partial t^2}, & \alpha = 2. \end{cases} \tag{2}$$

When $\alpha = 2$, Eq. (1) is the telegraph equation which governs electrical transmission in a telegraph cable:

$$\begin{cases} \frac{\partial^2 u(x,t)}{\partial t^2} + \lambda \frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + \mu\, s(x,t), & 0 < x < L,\ 0 \le t \le T, \\ u(x,0) = f(x),\ \frac{\partial u(x,0)}{\partial t} = g(x), & 0 \le x \le L, \\ u(0,t) = u(L,t) = 0, & 0 \le t \le T. \end{cases} \tag{3}$$

This equation can also be characterized as a wave equation, governing wave motion in a string, with a damping effect due to the term $\lambda \frac{\partial u(x,t)}{\partial t}$. That is, if $\lambda = 0$, Eq. (3) reduces to the wave equation, and if $\lambda \neq 0$ there is some initial directionality to the wave motion, but this effect rapidly disappears and the motion becomes completely random.

We will present analytical and numerical solutions for Eq. (1). The analytical solution is expressed through Mittag-Leffler type functions. This construction renders computation of the analytical solution difficult. This motivates us to give an implicit difference scheme for this problem. Their stability and convergence are proved by the energy method.

The structure of the paper is as follows. In Section 2, a method of separating variables is effectively implemented for solving Eq. (1). In Section 3, we present an an implicit difference approximation for this equation with initial and boundary conditions in a finite domain. In Sections 3 and 4, we discuss the stability and convergence of the difference approximation. Finally, numerical results are given to evaluate the method in Section 5.

2 Fundamental solution

For convenience, we introduce the following definitions and theorem, which are used later on in this paper.

Definition 1 (see [36]) A real or complex-valued function $f(x), x > 0$, is said to be in the space $C_\alpha, \alpha \in \mathbb{R}$, if there exists a real number $p > \alpha$ such that

$$f(x) = x^p f_1(x) \tag{4}$$

for a function $f_1(x)$ in $C([0, \infty])$.

Definition 2 (see [37]) A function $f(x)$, $x > 0$, is said to be in the space C_α^m, $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, if and only if $f^m \in C_\alpha$.

Definition 3 (see [37]) A multivariate Mittag-Leffler function is defined as

$$E_{(a_1,\cdots,a_n),b}(z_1,\cdots,z_n) := \sum_{k=0}^{\infty} \sum_{\substack{l_1+\cdots+l_n=k,\\ l_1\geq 0,\cdots,l_n\geq 0}} \frac{k!}{l_1!\times\cdots\times l_n!} \frac{\prod_{i=1}^{n} z_i^{l_i}}{\Gamma(b+\sum_{i=1}^{n} a_i l_i)}, \tag{5}$$

in which $b > 0$, $a_i > 0$, $|z_i| < \infty$, $i = 1, \cdots, n$. In particular, if $n = 1$ a multivariate Mittag-Leffler function reduces to a Mittag-Leffler function

$$E_{a,b}(z_1) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(b+ka)}, a, b > 0, |z| < \infty. \tag{6}$$

Theorem 1 Let $\mu > \mu_1 > \cdots > \mu_n \geq 0$, $m_i - 1 < \mu_i \leq m_i$, $m_i \in \mathbb{N}_0, \lambda_i \in \mathbb{R}$, $i = 1, \cdots, n$. The initial value problem

$$\begin{cases} (D_*^{\mu} y)(x) - \sum_{i=1}^{n} \lambda_i (D_*^{\mu_i} y)(x) = g(x), \\ y^{(k)}(0) = c_k \in \mathbb{R}, k = 0, \cdots, m-1, \quad m-1 < \mu \leq m, \end{cases} \tag{7}$$

where D_*^{μ} is the Caputo derivative, the function $g(x)$ is assumed to lie in C_{-1} if $\mu \in \mathbb{N}$, in C_{-1}^1 if $\mu \notin \mathbb{N}$, and the unknown function $y(x)$, which is to be determined in the space C_{-1}^m, has the representation

$$y(x) = y_g(x) + \sum_{k=0}^{m-1} c_k u_k(x), x \geqslant 0, \tag{8}$$

where

$$y_g(x) = \int_0^x t^{\mu-1} E_{(.),\mu}(t) g(x-t) dt, \tag{9}$$

and

$$u_k(x) = \frac{x^k}{k!} + \sum_{i=l_k+1}^{n} \lambda_i x^{k+\mu-\mu_i} E_{(.),k+1+\mu-\mu_i}(x), k = 0, \cdots, m-1, \qquad (10)$$

fulfills the initial conditions $u_k^l(0) = \delta_{kl}, k, l = 0, \cdots, m-1$. Here,

$$E_{(.),\beta}(x) = E_{\mu-\mu_1,\cdots.\mu-\mu_n,\beta}(\lambda_1 x^{\mu-\mu_1}, \cdots, \lambda_n x^{\mu-\mu_n}). \qquad (11)$$

The natural numbers $l_k, k = 0, \cdots, m-1$, are determined from the condition

$$\begin{cases} m_{l_k} \geq k+1, \\ m_{l_k+1} \leq k+1. \end{cases} \qquad (12)$$

In the case $m_i \leq k, i = 1, \cdots, m-1$,we set $l_k := 0$, and if $m_i \geq k+1, i = 1, \cdots, m-1$,then $l_k := n$.

Proof. See [37].

In this section, we determine the solution of the following fractional diffusion-wave equation with damping:

$$\begin{array}{l} {}_0D_t^\alpha u(x,t) + a\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + \mu s(x,t), \\ 0 < x < L, t > 0, 1 < \alpha < 2, \end{array} \qquad (13)$$

with the initial conditions

$$u(x,0) = f(x), u_t(x,0) = g(x), 0 \leq x \leq L, \qquad (14)$$

and the nonhomogeneous boundary conditions

$$u(0,t) = \mu_1(t), u(L,t) = \mu_2(t), t > 0, \qquad (15)$$

using the method of separating variables, where $f(x), g(x)$ are continuous functions satisfying $f(0) = \mu_1(0)$, $f(L) = \mu_2(0)$, $\mu_1(t)$ and $\mu_2(t)$ are non-zero smooth functions with first order continuous derivative.

In order to solve the problem with nonhomogeneous boundary, we firstly transform the nonhomogeneous boundary condition into a homogeneous boundary condition. Let

$$u(x,t) = W(x,t) + V(x,t),$$

where $W(x,t)$ is a new unknown function and

$$V(x,t) = \mu_1(t) + \frac{(\mu_2(t) - \mu_1(t))x}{L} \qquad (16)$$

satisfies the boundary conditions

$$V(0,t)=\mu_1(t), V(L,t)=\mu_2(t). \tag{17}$$

The function $W(x,t)$ then satisfies the problem with homogeneous boundary conditions:

$$\begin{cases} {}_0D_t^\alpha W(x,t)+a\frac{\partial W(x,t)}{\partial t}=\frac{\partial^2 W(x,t)}{\partial x^2}+\mu\tilde{s}(x,t), & 0<x<L, t>0. \\ W(x,0)=\phi_1(x), \frac{\partial W(x,0)}{\partial t}=\psi_1(x), & 0\le x\le L, \\ W_1(0,t)=W_1(L,t)=0, & t\ge 0, \end{cases} \tag{18}$$

in which

$$\begin{array}{l} \tilde{s}(x,t)=-{}_0D_t^\alpha V(x,t)-a(\mu_1^{'}(t)+\frac{\mu_2^{'}(t)-\mu_1^{'}}{L}x)+\mu s(x,t), \\ \phi_1(x)=f(x)-\mu_1(0)-\frac{1}{L}[\mu_2(0)-\mu_1(0)]x, \\ \psi_1(x)=g(x)-\mu_1^{'}(0)-\frac{1}{L}[\mu_2^{'}(0)-\mu_1^{'}(0)]x. \end{array} \tag{19}$$

We solve the corresponding homogeneous equation (18) ($\tilde{s}(x,t)$ being replaced by 0) with the boundary conditions by the method of separation of variables.

If we let $W(x,t)=X(x)T(t)$ and substitute $W(x,t)$ by it in (18), we obtain an ordinary linear differential equation for $X(x)$:

$$X^{''}(x)+\lambda X(x)=0, X(0)=X(L)=0, \tag{20}$$

and a fractional ordinary linear differential equation with the Caputo derivative for $T(t)$:

$${}_0D_t^\alpha T(t)+aT^{'}(t)+\lambda T(t)=0, \tag{21}$$

where the parameter λ is a positive constant. The Sturm-Liouville problem given by (20) has eigenvalues

$$\lambda_n=\frac{n^2\pi^2}{L^2}, \quad n=1,2,\cdots$$

and corresponding eigenfunctions

$$X_n(x)=\sin\frac{n\pi x}{L}, \quad n=1,2,\cdots.$$

Now we seek a solution of the nonhomogeneous problem (18) of the form

$$W(x,t)=\sum_{n=1}^{\infty}B_n(t)\sin\frac{n\pi x}{L}. \tag{22}$$

We assume that the series can be differentiated term by term. In order to determine $B_n(t)$, we expand $\tilde{s}(x,t)$ as a Fourier series by the eigenfunctions $\{\sin\frac{n\pi x}{L}\}$:

$$\tilde{s}(x,t) = \sum_{n=1}^{\infty} \tilde{s}_n(t) \sin\frac{n\pi x}{L}, \tag{23}$$

where

$$\tilde{s}_n(t) = \frac{2}{L}\int_0^L \tilde{s}(x,t)\sin\frac{n\pi x}{L}dx. \tag{24}$$

Substituting (22), (23) into (18) yields

$$\begin{aligned}\sum_{n=1}^{\infty} \sin\tfrac{n\pi x}{L}\, {}_0D_t^{\alpha} B_n(t) + a\sum_{n=1}^{\infty}\sin\tfrac{n\pi x}{L} B_n^{'}(t) &= \tfrac{-n^2\pi^2 k}{L^2}\sum_{n=1}^{\infty}\sin\tfrac{n\pi x}{L}B_n(t)\\ &+ \sum_{n=1}^{\infty}\tilde{s}_n(t)\sin\tfrac{n\pi x}{L}.\end{aligned} \tag{25}$$

By equating the coefficients of both sides we get

$${}_0D_t^{\alpha} B_n(t) + aB_n^{'}(t) + \frac{n^2\pi^2 k}{L^2}B_n(t) = \tilde{s}_n(t). \tag{26}$$

Since $W(x,t)$ satisfies the initial conditions in (18), we must have

$$\begin{cases} \sum\limits_{n=0}^{\infty} B_n(0)\sin\frac{n\pi x}{L} = \phi_1(x), & 0 < x < L,\\ \sum\limits_{n=0}^{\infty} B_n^{'}(0)\sin\frac{n\pi x}{L} = \psi_1(x), & 0 < x < L, \end{cases} \tag{27}$$

which yields

$$\begin{cases} B_n(0) = \frac{2}{L}\int\limits_0^L \phi_1(x)\sin\frac{n\pi x}{L}dx, & n = 1, 2, \cdots,\\ B_n^{'}(0) = \frac{2}{L}\int\limits_0^L \psi_1(x)\sin\frac{n\pi x}{L}dx, & n = 1, 2, \cdots. \end{cases} \tag{28}$$

For each value of n, (26) and (28) make up a fractional initial value problem. According to Theorem 1, the fractional initial value problem has the solution

$$\begin{aligned}B_n(t) &= \int\limits_0^t \tau^{2\alpha-1}E_{(2\alpha-1,2\alpha),2\alpha}(-a\tau^{2\alpha-1}, \tfrac{-\pi^2 n^2}{L^2}\tau^{2\alpha})\tilde{s}_n(t-\tau)d\tau\\ &+B_n(0)u_0(t) + B_n^{'}(0)u_1(t),\end{aligned} \tag{29}$$

in which

$$u_0(t) = 1 - \frac{\pi^2 n^2}{L^2} t^{2\alpha} E_{(2\alpha-1,2\alpha),1+2\alpha}(-at^{2\alpha-1}, -\frac{\pi^2 n^2}{L^2} t^{2\alpha}), \tag{30}$$

$$\begin{aligned} u_1(t) \quad = \quad & t - at^{2\alpha} E_{(2\alpha-1,2\alpha),1+2\alpha}(-at^{2\alpha-1}, -\tfrac{\pi^2 n^2}{L^2} t^{2\alpha}) \\ & -\tfrac{\pi^2 n^2}{L^2} t^{1+2\alpha} E_{(2\alpha-1,2\alpha),2+2\alpha}(-at^{2\alpha-1}, -\tfrac{\pi^2 n^2}{L^2} t^{2\alpha}), \end{aligned} \tag{31}$$

where the multivariate Mittag-Leffler function is given in Definition 3. Hence we get the solution of the initial-boundary value problem (18) in the form

$$\begin{aligned} W(x,t) \quad &= \quad \sum_{n=1}^{\infty} B_n(t) \sin \tfrac{n\pi x}{L} \\ &= \quad \sum_{n=1}^{\infty} [\int_0^t \tau^{2\alpha-1} E_{(\alpha,2\alpha),2\alpha}(-a\tau^{\alpha}, -\tfrac{\pi^2 n^2}{L^2} \tau^{2\alpha}) \tilde{s}_n(t-\tau) d\tau \\ & \quad + B_n(0) u_0(t) + B_n^{'}(0) u_1(t)] \sin \tfrac{n\pi x}{L}, \end{aligned} \tag{32}$$

where the functions $u_0(t)$ and $u_1(t)$ are given in (30) and (31), respectively. Therefore, we obtain the solution of problem (13)-(15) as

$$\begin{aligned} & u(x,t) = \sum_{n=1}^{\infty} [\int_0^t \tau^{2\alpha-1} E_{(2\alpha-1,2\alpha),2\alpha}(-a\tau^{2\alpha-1}, -\tfrac{\pi^2 n^2}{L^2} \tau^{2\alpha}) \tilde{s}_n(t-\tau) d\tau \\ & + B_n(0) u_0(t) + B_n^{'}(0) u_1(t)] \sin \tfrac{n\pi x}{L} + \mu_1(t) + \tfrac{(\mu_2(t)-\mu_1(t))x}{L}. \end{aligned} \tag{33}$$

The convergence of the series (32) and (33) and the finiteness of the integrals in these equations were discussed in [38].

3 An implicit finite difference approximation scheme

Let us consider an interval $[0, L]$, and define $h = \frac{L}{M}$ to be the grid size in the x-direction. For a positive integer M, we denote $x_i = ih, 0 \le i \le M$. We define $t_n = n\tau (n > 0)$ to be the integration time in $0 \le t \le T$ and τ to be the grid step in time. Define $u_j^n = u(jh, n\tau)$. Suppose $u^n = (u_1^n, \cdots, u_M^n)$ is an M- dimensional vector. For convenience, let us introduce the notations

(1) $\nabla_t u_i^n = u_i^n - u_i^{n-1}$; (2) $\nabla_x u_i^n = u_i^n - u_{i-1}^n$;

(3) $\delta_x^2 u_i^n = u_{i+1}^n - 2u_i^n + u_{i-1}^n$; (4) $\| u^n \|_\infty = \max_{0 \le i \le M} |u_i^n|$;

(5) $\| u^n \|_2 = (h \sum_{j=1}^{M-1} (u_j^n)^2)^{\frac{1}{2}}$; (6) $\| u^n \|_1 = [h \sum_{i=1}^{M-1} (\frac{u_i^n - u_i^{n-1}}{h})^2]^{1/2}$.

In order to construct the implicit finite difference approximation scheme, we firstly discretize the Caputo derivative ${}_0D_t^{\alpha} u(x,t)$.

Lemma 1 Suppose the third order partial derivative of $u(x,t)$ with respect to x exists in the interval $[0,t_n]$. Let

$$ {}_0\widetilde{D}_t^{\alpha}u(x,t)|_{x_i}^{t_n} = \frac{1}{\tau\Gamma(2-\alpha)}[a_0 u_t^{'}(x_i,t_n) - \sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})u_t^{'}(x_i,t_k) $$
$$ -a_{n-1}u_t^{'}(x_i,t_0)] $$

Then (see [33])

$$ {}_0D_t^{\alpha}u(x,t)|_{x_i}^{t_n} = {}_0\widetilde{D}_t^{\alpha}u(x,t)|_{x_i}^{t_n} + O(\tau^{3-\alpha}), $$

with $a_k = \frac{\tau^{2-\alpha}}{2-\alpha}[(k+1)^{2-\alpha} - k^{2-\alpha}]$.

In order to discretize Eq. (1) at the points $(x_i, \frac{t_n+t_{n-1}}{2})$ we firstly introduce the following lemma.

Lemma 2 If the second-order partial derivatives of the function $u(x,t)$ with respect to the variables t and x are continuous, and (x_i,t_n) are mesh points, then

(1) $u(x_i, \frac{t_n+t_{n-1}}{2}) = \frac{u(x_i,t_n)+u(x_i,t_{n-1})}{2} + O(\tau^2)$,

(2) $u_t^{'}(x_i, \frac{t_n+t_{n-1}}{2}) = \frac{u(x_i,t_n)-u(x_i,t_{n-1})}{\tau} + O(\tau^2)$,

Proof. (1) Using Taylor's theorem

$$ \begin{aligned} u(x_i,t_n) = \;& u(x_i,\tfrac{t_n+t_{n-1}}{2}) + u_t^{'}(x_i,\tfrac{t_n+t_{n-1}}{2})\tfrac{\tau}{2} + \tfrac{1}{2!}u_{tt}^{''}(x_i,\tfrac{t_n+t_{n-1}}{2})(\tfrac{\tau}{2})^2 \\ & + \tfrac{1}{3!}u_{ttt}^{'''}(x_i,\tfrac{t_n+t_{n-1}}{2})(\tfrac{\tau}{2})^3 + O(\tau^4), \end{aligned} \tag{34} $$

$$ \begin{aligned} u(x_i,t_{n-1}) = \;& u(x_i,\tfrac{t_n+t_{n-1}}{2}) + u_t^{'}(x_i,\tfrac{t_n+t_{n-1}}{2})(-\tfrac{\tau}{2}) + \tfrac{1}{2!}u_{tt}^{''}(x_i,\tfrac{t_n+t_{n-1}}{2})(-\tfrac{\tau}{2})^2 \\ & + \tfrac{1}{3!}u_{ttt}^{'''}(x_i,\tfrac{t_n+t_{n-1}}{2})(-\tfrac{\tau}{2})^3 + O(\tau^4), \end{aligned} \tag{35} $$

Adding (34) and (35) yields

$$ u(x_i,t_n) + u(x_i,t_{n-1}) = 2u(x_i,\frac{t_n+t_{n-1}}{2}) + O(\tau^2), $$

i.e.

$$ u(x_i,\frac{t_n+t_{n-1}}{2}) = \frac{u(x_i,t_n)+u(x_i,t_{n-1})}{2} + O(\tau^2). $$

(2) Subtracting (34) from (35) we derive

$$ u(x_i,t_n) - u(x_i,t_{n-1}) = u_t^{'}(x_i,\frac{t_n+t_{n-1}}{2})\tau + O(\tau^3). $$

Therefore

$$ u_t^{'}(x_i,\frac{t_n+t_{n-1}}{2}) = \frac{u(x_i,t_n)-u(x_i,t_{n-1})}{\tau} + O(\tau^2). $$

Thus we prove Lemma 2.

Applying Lemma 2 we have

$$\begin{aligned}\frac{u_t'(x_i,t_n)+u_t'(x_i,t_{n-1})}{2} &= u_t'(x_i,\tfrac{t_n+t_{n-1}}{2})+O(\tau^2)\\ &= \frac{u(x_i,t_n)-u(x_i,t_{n-1})}{\tau}+O(\tau^2)\\ &= \frac{\nabla_t u_i^n}{\tau}+O(\tau^2).\end{aligned} \tag{36}$$

From Lemma 1 and Lemma 2 and [33], we easily get

$$\begin{aligned}&{}_0D_t^{\alpha}u(x,t)|_{x_i}^{\frac{t_n+t_{n-1}}{2}}\\ &={}_0\,\tilde{D}_t^{\alpha}u(x,t)|_{x_i}^{\frac{t_n+t_{n-1}}{2}}+O(\tau^{3-\alpha})\\ &=\frac{1}{\tau\Gamma(2-\alpha)}[a_0\frac{\nabla_t u_i^n}{\tau}-\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})\frac{\nabla_t u_i^k}{\tau}-a_{n-1}u_t'(x_i,t_0)]+O(\tau^{3-\alpha}).\end{aligned} \tag{37}$$

According to Taylor's theorem,

$$\frac{\partial^2 u(x_i,t_n)}{\partial x^2}=\frac{\delta_x^2 u_i^n}{h^2}+O(h^2),$$

$$\frac{\partial^2 u(x_i,t_{n-1})}{\partial x^2}=\frac{\delta_x^2 u_i^{n-1}}{h^2}+O(h^2),$$

and using Lemma 2 we obtain

$$\begin{aligned}\frac{\partial^2 u(x_i,\frac{t_n+t_{n-1}}{2})}{\partial x^2} &= [\frac{\partial^2 u(x_i,t_n)}{\partial x^2}+\frac{\partial^2 u(x_i,t_{n-1})}{\partial x^2}]/2+O(\tau^2)\\ &= \frac{\delta_x^2 u_i^n+\delta_x^2 u_i^{n-1}}{2h^2}+O(\tau^2)+O(h^2).\end{aligned} \tag{38}$$

It follows from Lemma 2 that

$$u_t'(x_i,\frac{t_n+t_{n-1}}{2})=\frac{\nabla_t u_i^n}{\tau}+O(\tau^2). \tag{39}$$

Using subsequently (37), (38) and (39), we derive the expression of Eq (1) at mesh points $(x_i,\frac{t_n+t_{n-1}}{2})$:

$$\begin{aligned}&\frac{1}{\tau\Gamma(2-\alpha)}[a_0\frac{\nabla_t u_i^n}{\tau}-\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})\frac{\nabla_t u_i^k}{\tau}-a_{n-1}g_i]+\lambda\frac{\nabla_t u_i^n}{\tau}\\ &=\frac{\delta_x^2 u_i^n+\delta_x^2 u_i^{n-1}}{2h^2}+\mu\, s(x_i,\frac{t_n+t_{n-1}}{2})+R_i^n,\end{aligned} \tag{40}$$

with initial and boundary conditions

$$u_i^0=f(x_i), 0\le i\le M,$$

$$u_0^n = u_M^n = 0, n \geq 1.$$

Here $R_i^n = C(\tau^{3-\alpha} + h^2)$ is the local truncation error, C is a constant. In this way, we get the implicit finite difference scheme for Eq. (1) at the points $(x_i, \frac{t_n + t_{n-1}}{2})$ as

$$\begin{aligned} &\tfrac{1}{\tau\Gamma(2-\alpha)}[a_0 \tfrac{\nabla_t u_i^n}{\tau} - \sum_{k=1}^{n-1}(a_{n-k-1} - a_{n-k})\tfrac{\nabla_t u_i^k}{\tau} - a_{n-1} g_i] + \lambda \tfrac{\nabla_t u_i^n}{\tau} \\ &= \tfrac{\delta_x^2 u_i^n + \delta_x^2 u_i^{n-1}}{2h^2} + \mu\, s(x_i, \tfrac{t_n + t_{n-1}}{2}) \end{aligned} \tag{41}$$

with initial and boundary conditions

$$u_i^0 = f(x_i), 0 \leq i \leq M,$$

$$u_0^n = u_M^n = 0, n \geq 1.$$

Lemma 3 Let $u_i^n (0 \leq i \leq M, n \geq 1)$ be the numerical solution of the difference scheme (41), then the difference scheme (41) is uniquely solvable.

Proof. Eq. (41) can be written as

$$\begin{aligned} &-\tfrac{1}{2h^2} u_{i+1}^n + (\tfrac{\tau^{-\alpha}}{\Gamma(3-\alpha)} + \tfrac{\lambda}{\tau} + \tfrac{1}{h^2}) u_i^n - \tfrac{1}{2h^2} u_{i-1}^n \\ &= \tfrac{1}{2h^2} u_{i+1}^{n-1} + (\tfrac{\tau^{-\alpha}}{\Gamma(3-\alpha)} + \tfrac{\lambda}{\tau} - \tfrac{1}{h^2} - \tfrac{\tau^{-\alpha}}{\Gamma(3-\alpha)} a_{n-1}) u_i^{n-1} + \tfrac{1}{2h^2} u_{i-1}^{n-1} \\ &\quad - \sum_{k=1}^{n-2} \tfrac{\tau^{-\alpha}}{\Gamma(3-\alpha)} (a_k - a_{k+1}) u_i^k + \tfrac{a_{n-1}}{\tau\Gamma(2-\alpha)} g_i + \mu s(x_i, \tfrac{t_n + t_{n-1}}{2}), \\ &\quad i = 1, \cdots, M-1. \end{aligned} \tag{42}$$

These equations, together with the boundary conditions $u_0^n = u_M^n = 0$, result in a linear system of equations whose coefficient matrix is strictly diagonally dominant and irreducible. Hence the difference scheme (41) is uniquely solvable.

4 Stability analysis

Theorem 2 Let $u_i^n (0 \leq i \leq M, n \geq 1)$ denote the exact solution for the implicit finite difference scheme (41), then the implicit finite difference scheme (41) is unconditionally stable.

Proof. Multiplying Eq. (41) by $h\tau \bigtriangledown_t u_i^n$ and summing for i from 1 to $M-1$ and for n from 1 to N we obtain

$$
\begin{aligned}
&\tfrac{1}{\tau\Gamma(2-\alpha)} \sum_{i=1}^{M-1} \sum_{n=1}^{N} [a_0 \tfrac{\bigtriangledown_t u_i^n}{\tau} - \sum_{k=1}^{n-1} (a_{n-k-1} - a_{n-k}) \tfrac{\bigtriangledown_t u_i^k}{\tau} - a_{n-1} g_i] h\tau \bigtriangledown_t u_i^n \\
&+\lambda \sum_{i=1}^{M-1} \sum_{n=1}^{N} \tfrac{\bigtriangledown_t u_i^n}{\tau} \cdot h\tau \bigtriangledown_t u_i^n \\
&= \sum_{n=1}^{N} \sum_{i=1}^{M-1} \tfrac{\delta_x^2 u_i^n + \delta_x^2 u_i^{n-1}}{2h^2} h\tau \bigtriangledown_t u_i^n + \sum_{n=1}^{N} \sum_{i=1}^{M-1} \mu s(x_i, \tfrac{t_n + t_{n-1}}{2}) h\tau \bigtriangledown_t u_i^n.
\end{aligned}
\tag{43}
$$

From [33] we have

$$
\begin{aligned}
&\tau \sum_{n=1}^{N} \sum_{i=1}^{M-1} \tfrac{\delta_x^2 u_i^n + \delta_x^2 u_i^{n-1}}{2h^2} h \bigtriangledown_t u_i^n \\
&= -\tfrac{1}{2} \sum_{n=1}^{N} [h \sum_{i=1}^{M} (\tfrac{u_i^n - u_i^{n-1}}{h})^2 - h \sum_{i=1}^{M} (\tfrac{u_i^{n-1} - u_i^{n-1}}{h})^2] \\
&= -\tfrac{1}{2} (\| u^N \|_1 - \| u^0 \|_1).
\end{aligned}
\tag{44}
$$

Since

$$
\begin{aligned}
&\frac{1}{\tau\Gamma(2-\alpha)}\sum_{i=1}^{M-1}\sum_{n=1}^{N}[a_0\frac{\bigtriangledown_t u_i^n}{\tau}-\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})\frac{\bigtriangledown_t u_i^k}{\tau}-a_{n-1}g_i]\cdot h\tau\bigtriangledown_t u_i^n\\
&\geq\frac{h}{\tau\Gamma(2-\alpha)}\sum_{i=1}^{M-1}[\sum_{n=1}^{N}a_0(\bigtriangledown_t u_i^n)^2-\frac{1}{2}\sum_{n=2}^{N}\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})((\bigtriangledown_t u_i^k)^2\\
&\quad+(\bigtriangledown_t u_i^n)^2)-\frac{1}{2}\sum_{n=1}^{N}a_{n-1}(g_i\tau)^2-\frac{1}{2}\sum_{n=1}^{N}a_{n-1}(\bigtriangledown_t u_i^n)^2]\\
&=\frac{h}{\tau\Gamma(2-\alpha)}\sum_{i=1}^{M-1}[\sum_{n=1}^{N}a_0(\bigtriangledown_t u_i^n)^2-\frac{1}{2}\sum_{k=1}^{N-1}\sum_{n=k+1}^{N}(a_{n-k-1}-a_{n-k})(\bigtriangledown_t u_i^k)^2\\
&\quad-\frac{1}{2}\sum_{n=2}^{N}(a_0-a_{n-1})(\bigtriangledown_t u_i^n)^2-\frac{1}{2}\sum_{n=1}^{N}a_{n-1}(g_i\tau)^2-\frac{1}{2}\sum_{n=1}^{N}a_{n-1}(\bigtriangledown_t u_i^n)^2]\\
&=\frac{h}{\tau\Gamma(2-\alpha)}\sum_{i=1}^{M-1}[\sum_{n=1}^{N}a_0(\bigtriangledown_t u_i^n)^2-\frac{1}{2}\sum_{k=1}^{N-1}(a_0-a_{N-k})(\bigtriangledown_t u_i^k)^2\\
&\quad-\frac{1}{2}\sum_{n=2}^{N}(a_0-a_{n-1})(\bigtriangledown_t u_i^n)^2-\frac{1}{2}\sum_{n=1}^{N}a_{n-1}(g_i\tau)^2-\frac{1}{2}\sum_{n=1}^{N}a_{n-1}(\bigtriangledown_t u_i^n)^2]\\
&=\frac{h}{\tau\Gamma(2-\alpha)}\sum_{i=1}^{M-1}[\sum_{n=1}^{N}a_0(\bigtriangledown_t u_i^n)^2-\frac{1}{2}\sum_{k=1}^{N}(a_0-a_{N-k})(\bigtriangledown_t u_i^k)^2\\
&\quad-\frac{1}{2}\sum_{n=1}^{N}(a_0-a_{n-1})(\bigtriangledown_t u_i^n)^2-\frac{1}{2}\sum_{n=1}^{N}a_{n-1}(g_i\tau)^2-\frac{1}{2}\sum_{n=1}^{N}a_{n-1}(\bigtriangledown_t u_i^n)^2]\\
&=\frac{h}{2\tau\Gamma(2-\alpha)}\sum_{i=1}^{M-1}[\sum_{n=1}^{N}a_{N-n}(\bigtriangledown_t u_i^n)^2-\sum_{n=1}^{N}a_{n-1}(g_i\tau)^2)]\\
&\geq\frac{h}{2\tau\Gamma(2-\alpha)}\sum_{i=1}^{M-1}[a_{N-1}\sum_{n=1}^{N}(\bigtriangledown_t u_i^n)^2-\frac{(N\tau)^{2-\alpha}}{2-\alpha}(g_i\tau)^2]\\
&=\frac{h}{2\tau\Gamma(2-\alpha)}\sum_{i=1}^{M-1}[t_N^{1-\alpha}\tau\sum_{n=1}^{N}(\bigtriangledown_t u_i^n)^2-\frac{(N\tau)^{2-\alpha}}{2-\alpha}(g_i\tau)^2],
\end{aligned}
\tag{45}
$$

the left-hand side of (43) is bounded below by

$$
\begin{aligned}
&\frac{1}{\tau\Gamma(2-\alpha)}\sum_{i=1}^{M-1}\sum_{n=1}^{N}[a_0\frac{\bigtriangledown_t u_i^n}{\tau}-\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})\frac{\bigtriangledown_t u_i^k}{\tau}-a_{n-1}g_i]h\bigtriangledown_t u_i^n\\
&+\lambda\sum_{i=1}^{M-1}\sum_{n=1}^{N}\frac{\bigtriangledown_t u_i^n}{\tau}\cdot h\tau\bigtriangledown_t u_i^n\\
&\geq\frac{h}{2\tau\Gamma(2-\alpha)}\sum_{i=1}^{M-1}[t_{N-1}^{1-\alpha}\tau\sum_{n=1}^{N}(\bigtriangledown_t u_i^n)^2-\frac{(N\tau)^{2-\alpha}}{2-\alpha}(g_i\tau)^2]+\lambda h\sum_{i=1}^{M-1}\sum_{n=1}^{N}(\bigtriangledown_t u_i^n)^2\\
&=\frac{t_N^{1-\alpha}h+2\lambda\Gamma(2-\alpha)h}{2\Gamma(2-\alpha)}\sum_{i=1}^{M-1}\sum_{n=1}^{N}(\bigtriangledown_t u_i^n)^2-\frac{t_N^{2-\alpha}h}{2\tau\Gamma(3-\alpha)}\sum_{i=1}^{M-1}(g_i\tau)^2.
\end{aligned}
\tag{46}
$$

The second term on the right-hand side of (43) is bounded above by

$$\begin{aligned}
&\sum_{n=1}^{N}\sum_{i=1}^{M-1}\mu\cdot h\cdot\tau\cdot s(x_i,\tfrac{t_n+t_{n-1}}{2})\bigtriangledown_t u_i^n\\
&=h\tau\sum_{n=1}^{N}\sum_{i=1}^{M-1}2\mu\sqrt{\tfrac{\tau\Gamma(2-\alpha)}{2(t_N^{1-\alpha}+2\lambda\Gamma(2-\alpha))}}s(x_i,\tfrac{t_n+t_{n-1}}{2})\cdot\sqrt{\tfrac{t_N^{1-\alpha}+2\lambda\Gamma(2-\alpha)}{2\tau\Gamma(2-\alpha)}}\bigtriangledown_t u_i^n\\
&\leq h\tau\sum_{n=1}^{N}\sum_{i=1}^{M-1}\tfrac{\mu^2\tau\Gamma(2-\alpha)}{2(t_N^{1-\alpha}+2\lambda\Gamma(2-\alpha))}(s(x_i,\tfrac{t_n+t_{n-1}}{2}))^2\\
&\quad+\sum_{n=1}^{N}\sum_{i=1}^{M-1}\tfrac{t_N^{1-\alpha}h+2\lambda\Gamma(2-\alpha)h}{2\Gamma(2-\alpha)}\cdot(\bigtriangledown_t u_i^n)^2.
\end{aligned}\tag{47}$$

Subtracting (47) from (43) yields

$$\begin{aligned}
&-\tfrac{1}{2}(\parallel u^N\parallel_1^2-\parallel u^0\parallel_1^2)+h\tau\sum_{n=1}^{N}\sum_{i=1}^{M-1}\tfrac{\mu^2\tau\Gamma(2-\alpha)}{2(t_N^{1-\alpha}+2\lambda\Gamma(2-\alpha))}(s(x_i,\tfrac{t_n+t_{n-1}}{2}))^2\\
&\geq-\tfrac{t_N^{2-\alpha}h}{2\tau\Gamma(3-\alpha)}\sum_{i=1}^{M-1}(g_i\tau)^2,
\end{aligned}\tag{48}$$

that is,

$$\begin{aligned}
\parallel u^N\parallel_1^2\;\leq\;&\parallel u^0\parallel_1^2+h\sum_{n=1}^{N}\sum_{i=1}^{M-1}\tfrac{\mu^2\tau^2\Gamma(2-\alpha)}{t_N^{1-\alpha}+2\lambda\Gamma(2-\alpha)}(s(x_i,\tfrac{t_n+t_{n-1}}{2}))^2\\
+\;&\tfrac{t_N^{2-\alpha}h}{2\tau\Gamma(3-\alpha)}\sum_{i=1}^{M-1}(g_i\tau)^2.
\end{aligned}\tag{49}$$

So we can define the energy norm as $\parallel u^n\parallel_E=\parallel u^n\parallel_1=\sqrt{h\sum_{i=1}^{M}(\frac{u_i^n-u_i^{n-1}}{h})^2}$. Considering the formula (see [34])

$$\parallel u^n\parallel_\infty\leq\frac{\sqrt{L}}{2}\parallel u^n\parallel_E,$$

we can directly obtain that the implicit finite difference (41) is unconditionally stable.

5 Convergence analysis

We denote the exact solution of the partial differential equation (1) by $u(x_i,t_n)$, the exact solution of the finite difference equation (41) by u_i^n, and the error by $e_i^n=u(x_i,t_n)-u_i^n, e^n=(e_1^n,\cdots,e_{M-1}^n)$. Subtracting (41) from

(40) we obtain

$$\begin{cases} \frac{1}{\tau\Gamma(2-\alpha)}[a_0\frac{\nabla_t e_i^n}{\tau} - \sum\limits_{k=1}^{n-1}(a_{n-k-1} - a_{n-k})\frac{\nabla_t e_i^k}{\tau}] + \frac{\lambda}{\tau} \nabla_t e_i^n & \\ = \frac{\delta_x^2 e_i^n + \delta_x^2 e_i^{n-1}}{2h^2} + R_i^n, & 1 \leq i \leq M-1, n \geq 1, \\ e_i^0 = 0, & 0 \leq i \leq M, \\ e_0^n = e_M^n = 0, & n \geq 1. \end{cases} \tag{50}$$

From (49) we get

$$\| e^n \|_1^2 \leq h \sum_{k=1}^{n} \sum_{i=1}^{M-1} \frac{\tau^2\Gamma(2-\alpha)}{t_n^{1-\alpha} + 2\lambda\Gamma(2-\alpha)}(R_i^k)^2.$$

Because $R_i^k = C(h^2 + \tau^{3-\alpha})$, we get

$$\begin{aligned} \| e^n \|_1^2 &\leq \frac{h\tau^2\Gamma(2-\alpha)}{t_n^{1-\alpha}+2\lambda\Gamma(2-\alpha)} \sum_{k=1}^{n} \sum_{i=1}^{M-1} (R_i^k)^2 \\ &\leq \frac{h\tau\Gamma(2-\alpha)}{t_n^{1-\alpha}+2\lambda\Gamma(2-\alpha)} n(M-1) \mid C \mid^2 (h^2 + \tau^{3-\alpha})^2 \\ &= \frac{|C|^2 n\tau\Gamma(2-\alpha)h(M-1)}{t_n^{1-\alpha}+2\lambda\Gamma(2-\alpha)}(h^2 + \tau^{3-\alpha})^2 \\ &\leq \mid C \mid^2 \Gamma(2-\alpha)t_n^\alpha L(h^2 + \tau^{3-\alpha})^2 \\ &\leq \mid C \mid^2 \Gamma(2-\alpha)LT^\alpha(h^2 + \tau^{3-\alpha})^2. \end{aligned} \tag{51}$$

Furthermore

$$\| e^n \|_\infty \leq \frac{\sqrt{L}}{2} \| e^n \|_1;$$

thus

$$\| e^n \|_\infty \leq \frac{\mid C \mid L\sqrt{\Gamma(2-\alpha)T^\alpha}}{2}(h^2 + \tau^{3-\alpha}).$$

So as $h \to 0, \tau \to 0$ we have $\| e^n \|_\infty \to 0$. This proves that the finite difference scheme is convergent.

6 Numerical results

Example 1. In order to show the approximation order of Eq. (1), we construct an example with an analytic solution. Consider the following fractional wave equation with damping ($\alpha = 1.7, \lambda = 1.0$)

$$\begin{cases} {}_0D_t^{1.7}u(x,t) + \frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + s(x,t), & 0 < x < 2,\ t \geq 0, \\ u(x,0) = 0, \frac{\partial u(x,0)}{\partial t} = 0, & 0 \leq x \leq 2, \\ u(0,t) = u(2,t) = 0, & t \geq 0, \end{cases} \tag{52}$$

Table 1: Comparison of maximum errors (MERR) and error rate (ER) at time $t = 1.0, \alpha = 1.7$

h	τ	MERR	ER
0.1	0.05	4.43333328E-003	-
0.05	0.025	1.86513124E-003	2.3770
0.025	0.0125	7.73686093E-004	2.4107
0.0125	0.00625	3.18271105E-004	2.4309
0.00625	0.003125	1.30274618E-004	2.4438
0.003125	0.0015625	5.31628461E-005	2.4505

where

$$s(x,t) = \frac{2x(2-x)}{\Gamma(1.3)} t^{0.3} + 2t\, x(2-x) + 2t^2.$$

The exact solution of the equation is $u(x,t) = t^2 x(2-x)$.

At the mesh points, we denote $u(x_i,t_n)$ and u_i^n as exact solution and numerical solution of Eq. (52) respectively. Let the maximum error $E^n = \max\limits_{0\le i\le M} \mid u(x_i,t_n) - u_i^n \mid$, and the error rate $R \approx \log_2(\frac{E^n}{E^{2n}})$.Table 1 shows the numerical errors at $\alpha = 1.7, t = 1$ between the exact solution and the numerical solutions obtained. It can be seen that

$$R = \frac{error_1}{error_2} \approx (\frac{\tau_1}{\tau_2})^{1.3} = 2^{1.3}.$$

Thus we obtain that the order of convergence of the numerical method is $O(h^2 + \tau^{3-\alpha}) = O(h^2 + \tau^{1.3})$. These results are in good agreement with our theoretical analysis.

Example 2. We consider the following fractional wave equation with damping:

$$\begin{cases} {}_0D_t^{1.7}u(x,t) + \frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + \sin(x), & 0 < x < 2,\ t \ge 0, \\ u(x,0) = 0, \frac{\partial u(x,0)}{\partial t} = 0, & 0 \le x \le 2, \\ u(0,t) = u(2,t) = 0, & t \ge 0, \end{cases} \tag{53}$$

The evolution results for $\alpha = 1.7, 0 \le t \le 1.0, 0 \le x \le 2; 1.1 \le \alpha \le 2, 0 \le t \le 1.0, x = 1.8$, and $1.1 \le \alpha \le 2, t = 1.0, 0 \le x \le 2$ are shown in Figures 1, 2, and 3, respectively. Figures 1-3 show that the system exhibits diffusion-wave behaviors. From Figure 3, it can be seen that the solution continuously depends on the time fractional derivative.

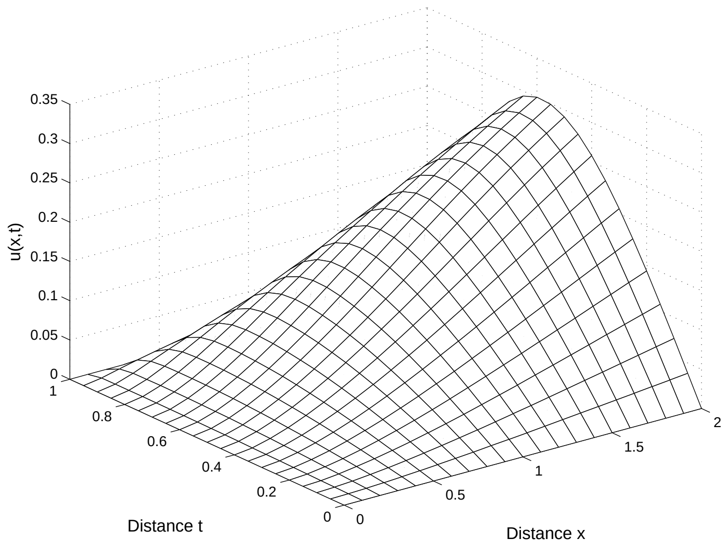

Figure 1: The numerical approximation when $\alpha = 1.7$.

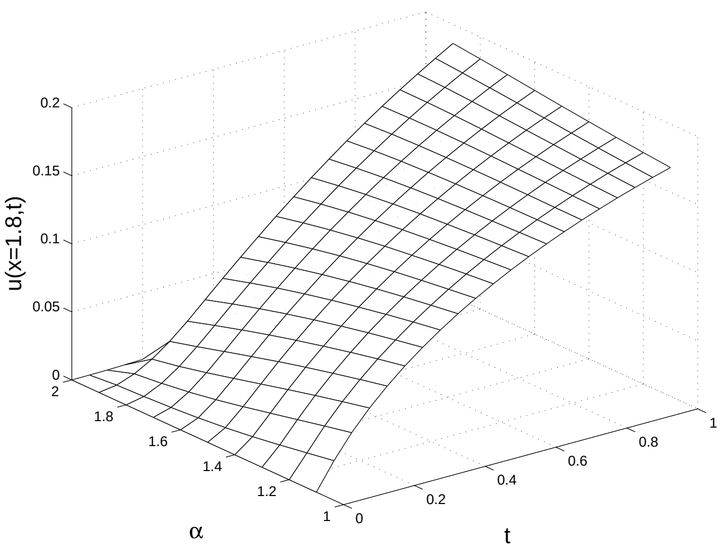

Figure 2: The numerical approximation $u(x, t)$ for various α when $x = 1.8$.

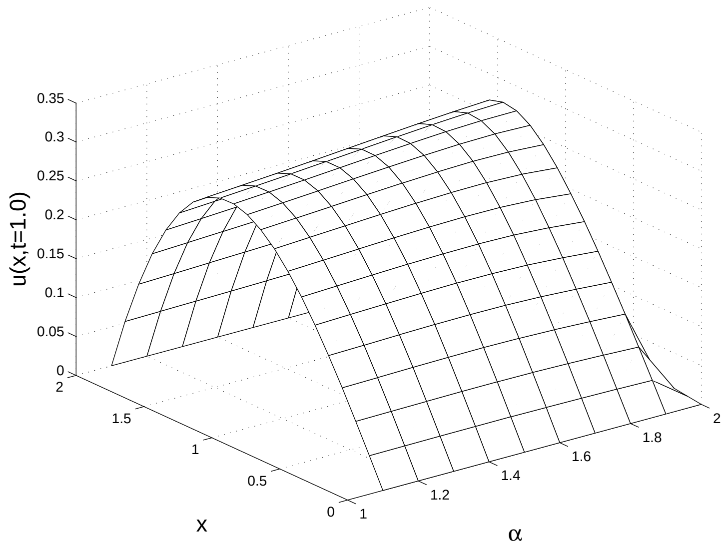

Figure 3: The numerical approximation $u(x,t)$ for various α when $t = 1.0$.

7 Conclusions

In this paper, a fractional diffusion-wave equation with damping has been described and demonstrated. We derive the analytical solution of the equation using the method of separation of variables. The analytical solution is expressed through Mittag-Leffler type functions. An implicit difference approximation is constructed. Stability and convergence are proved by the energy method. Two numerical examples are presented to show the effectiveness of the difference method. The energy method and analytical techniques can also be extended to other fractional partial differential equations.

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