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FUNDAMENTAL SOLUTION AND DISCRETE RANDOM WALK MODEL FOR THE TIME-SPACE RIESZ FRACTIONAL ADVECTION-DISPERSION EQUATION

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ABSTRACT. In this paper, we consider a time-space Riesz fractional advection-dispersion equation (TSRFADE). The TSRFADE is obtained from the standard advection-dispersion equation by replacing the first-order time derivative by the Caputo fractional derivative of order $\alpha \in (0, 1]$, the first-order and second-order space derivatives by the Riesz fractional derivatives of order $\beta_1 \in (0, 1)$ and of order $\beta_2 \in (1, 2]$, respectively. We derive the fundamental solution for the TSRFADE with an initial condition (TSRFADE-IC). The fundamental solution can be interpreted as a spatial probability density function evolving in time. We also investigate a discrete random walk model based on an explicit finite difference approximation for the TSRFADE-IC.

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1. Introduction

In recent years fractional differential equations have received much interest due to their usefulness in physical applications [1,9,23,28]. Oldham and Spanier [21], Miller and Ross [16], and Podlubny [22] provide the history and a comprehensive treatment of this subject. Diffusion with an additional velocity field and diffusion under the influence of a constant external force field are both modelled by the advection-dispersion equation. The space-fractional advection-dispersion equation, in which the dispersive flux is described by a fractional space derivative, has been applied to modeling the anomalous or super diffusion of solutes observed in heterogeneous porous media [2,4,18]. A straightforward extension of

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the continuous time random walk (CTRW) model leads to a fractional advection-dispersion equation (FADE) [3].

The FADE is an approach to the kinetics of chaotic dynamics, based on the results of Zaslavsky [24,25]. The derived equation is also called fractional kinetic equation (FKE). The investigation of the type and properties of the FKE is at its beginning stage [26]. In [19], the authors provide an important physical interpretation of the FKE.

Nowadays the connection between random walk and fractional order dynamics is well known. Gorenflo and Mainardi [6,7] interpreted the two-level difference scheme resulting from the Grünwald-Letnikov discretization of fractional derivatives as a random walk model discrete in space and time. Liu et al. [12] proposed a random walk model for approximating a Lévy-Feller advection-dispersion process. The fundamental solutions of these equations exhibit useful scaling properties that make them attractive for applications. Liu et al. [11] derived the fundamental solution of the time fractional advection dispersion equation. Mainardi et al. [13] and Huang et al. [8] considered the space-time fractional diffusion equation and the space-time fractional advection-dispersion equation, respectively. They presented explicit representations of these equations, and proved a general representation of the Green functions for which the fundamental solution can be interpreted as a spatial probability density. Mainardi et al. [14] also showed how the fundamental solutions of the time-fractional diffusion equation and the symmetric, space-fractional diffusion equation for the Cauchy and signalling problems provide probability density functions related to certain stable distributions.

In this paper, we consider the following time-space Riesz fractional advection-dispersion equation (TSRFADE):

$$ {}_tD_*^{\alpha}u(x,t) = B_1\frac{\partial^{\beta_1}}{\partial|x|^{\beta_1}}u(x,t) + B_2\frac{\partial^{\beta_2}}{\partial|x|^{\beta_2}}u(x,t) \qquad x\in\mathbf{R},\mathbf{t}\in\mathbf{R}^+, \tag{1} $$

where α $(0<\alpha\le 1)$, β_1 $(0<\beta<1)$ and β_2 $(1<\beta\le 2)$ are real parameters. The coefficients B_1 and B_2 are both positive constants. The time fractional derivative ${}_tD_*^{\alpha}u(x,t)$ is the Caputo fractional derivative of order α $(0<\alpha\le 1)$ defined by [22]

$$ {}_tD_*^{\alpha}u(x,t) = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)}\displaystyle\int_0^t \frac{\partial u(x,\eta)}{\partial\eta}\frac{d\eta}{(t-\eta)^{\alpha}}, & 0<\alpha<1, \\ \dfrac{\partial u(x,t)}{\partial t}, & \alpha=1, \end{cases} \tag{2} $$

while the space fractional derivatives $\frac{\partial^{\beta_1}}{\partial|x|^{\beta_1}}$ and $\frac{\partial^{\beta_2}}{\partial|x|^{\beta_2}}$ are Riesz space-fractional derivatives of order β_1 and β_2 respectively, defined by[6]

$$\frac{\partial^{\beta}}{\partial|x|^{\beta}} \equiv {}_xD^{\beta} = -c({}_{-\infty}D_x^{\beta} + {}_xD_{+\infty}^{\beta}), \tag{3}$$

where the coefficient $c = \frac{1}{2\cos(\beta\pi/2)}$, and

$${}_{-\infty}D_x^{\beta}u(x,t) = \left(\frac{d}{dx}\right)^m [{}_{-\infty}I_x^{m-\beta}u(x,t)], \tag{4}$$

$${}_xD_{+\infty}^{\beta}u(x,t) = (-1)^m \left(\frac{d}{dx}\right)^m [{}_xI_{+\infty}^{m-\beta}u(x,t)] \tag{5}$$

with $m \in \mathbf{N}$, $m-1 < \beta \leq m$, as the left-side and right-side Riemann-Liouville fractional derivatives. In the above expressions the fractional operators ${}_{-\infty}I_x^{\beta}u(x,t)$ and ${}_xI_{+\infty}^{\beta}u(x,t)$ are defined as the left- and right-side Weyl fractional integrals [20]:

$$\begin{cases} {}_{-\infty}I_x^{\beta}u(x,t) = \frac{1}{\Gamma(\beta)}\int\limits_{-\infty}^{x}(x-\xi)^{\beta-1}u(\xi,t)d\xi, \quad \beta > 0, \\ {}_xI_{+\infty}^{\beta}u(x,t) = \frac{1}{\Gamma(\beta)}\int\limits_{x}^{+\infty}(\xi-x)^{\beta-1}u(\xi,t)d\xi, \quad \beta > 0. \end{cases} \tag{6}$$

The TSRFADE is obtained from the standard advection-dispersion equation by replacing the first-order time derivative by the Caputo fractional derivative of order $\alpha \in (0,1]$, and the first-order and second-order space derivatives by the Riesz space fractional derivatives of order $\beta_1 \in (0,1)$ and of order $\beta_2 \in (1,2]$, respectively. When $\alpha = 1$, $\beta_1 = 1$ and $\beta_2 = 2$, the above equation reduces to the classical advection-dispersion equation.

The structure of the paper is as follows. In Section 2, using the method of Laplace and Fourier transforms, the fundamental solution of the TSRFADE with initial condition (TSRFADE-IC) is derived. In Section 3, we derive an explicit representation of the Green functions which the fundamental solution can be interpreted as a spatial probability density. In Section 4, an explicit finite difference approximation for TSRFADE-IC is proposed. A discrete random walk model for the TSRFADE-IC is presented in Section 5.

2. The fundamental solution of the TSRFADE-IC

In this section, we consider the following TSRFADE-IC:

$$\begin{cases} {}_tD_*^{\alpha}u(x,t) = B_1\dfrac{\partial^{\beta_1}}{\partial|x|^{\beta_1}}u(x,t) + B_2\dfrac{\partial^{\beta_2}}{\partial|x|^{\beta_2}}u(x,t), & x \in \mathbf{R}, \mathbf{t} \in \mathbf{R}^+, \\ u(x,0) = g(x), & x \in \mathbf{R}. \end{cases} \tag{7}$$

We assume $u(x,t)$ and $g(x)$ are both real-valued and sufficiently well-behaved functions. We will derive the fundamental solution of the TSRFADE-IC (7) by applying the Laplace and Fourier transforms to (7) with respect to variables t and x. Let us recall the following fundamental formulas proved in [13,22]:

$$L\{{}_tD_*^{\alpha}f(t);s\} = s^{\alpha}\tilde{f}(s) - \sum_{k=0}^{m-1} s^{\alpha-1-k}f^{(k)}(0^+), \quad m-1 < \alpha \le m,$$

$$F\{{}_xD_0^{\beta}f(x);k\} = -|k|^{\beta}\widehat{f}(k),$$

where $L \equiv$ Laplace transform, $F \equiv$ Fourier transform, with $Lf = \tilde{f}$, $Ff = \widehat{f}$. Applying the Laplace transform to (7), we obtain the following nonhomogeneous differential equation :

$$s^{\alpha}\widetilde{u}(x,s) - s^{\alpha-1}g(x) = B_{1x}D^{\beta_1}\widetilde{u}(x,s) + B_{2x}D^{\beta_2}\widetilde{u}(x,s). \tag{8}$$

Applying the Fourier transform to (8) with respect to variable x by taking into account the Fourier transform for the Riesz fractional derivative yields

$$s^{\alpha}\widehat{\widetilde{u}}(k,s) - s^{\alpha-1}\widehat{g}(k) = B_1(-|k|^{\beta_1})\widehat{\widetilde{u}}(k,s) + B_2(-|k|^{\beta_2})\widehat{\widetilde{u}}(k,s). \tag{9}$$

From (9) we obtain

$$\widehat{\widetilde{u}}(k,s) = \frac{s^{\alpha-1}\widehat{g}(k)}{s^{\alpha} + B_1|k|^{\beta_1} + B_2|k|^{\beta_2}}. \tag{10}$$

By using the known Laplace transform

$$E_{\alpha}(-\lambda t^{\alpha}) \overset{\mathcal{L}}{\longleftrightarrow} \frac{s^{\alpha-1}}{s^{\alpha}+\lambda}, \quad Re(s) >| \lambda |^{\frac{1}{\alpha}}, \tag{11}$$

where $E_{\alpha}(z)$ is Mittag-Leffler function, defined by [20]

$$E_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)}, \quad \alpha > 0, \quad z \in \mathbf{C},$$

We easily recognize from (11) that

$$E_{\alpha}((-(B_1|k|^{\beta_1} + B_2|k|^{\beta_2})t^{\alpha})\widehat{g}(k) \overset{\mathcal{L}}{\longleftrightarrow} \frac{s^{\alpha-1}\widehat{g}(k)}{s^{\alpha} + B_1|k|^{\beta_1} + B_2|k|^{\beta_2}}, \tag{12}$$

i.e.,

$$E_{\alpha}((-(B_1|k|^{\beta_1} + B_2|k|^{\beta_2})t^{\alpha})\widehat{g}(k) \overset{\mathcal{L}}{\longleftrightarrow} \widehat{\widetilde{u}}(k,s).$$

Inverting the Laplace transform in (12), we get

$$\widehat{u}(k,t) = E_{\alpha}((-(B_1|k|^{\beta_1} + B_2|k|^{\beta_2})t^{\alpha})\widehat{g}(k). \tag{13}$$

Invert the Fourier transform in (13) yields

$$\begin{aligned} u(x,t) &= \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} E_\alpha((-(B_1|k|^{\beta_1}+B_2|k|^{\beta_2})t^\alpha)e^{-ikx}\widehat{g}(k)dk \\ &= \frac{1}{2\pi}\left[\int\limits_{-\infty}^{+\infty} E_\alpha((-(B_1|k|^{\beta_1}+B_2|k|^{\beta_2})t^\alpha)e^{-ikx}\int\limits_{-\infty}^{+\infty} e^{iky}g(y)dy\right]dk \\ &= \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\left\{\int\limits_{-\infty}^{+\infty} e^{-ikx}E_\alpha((-(B_1|k|^{\beta_1}+B_2|k|^{\beta_2})t^\alpha)e^{iky}g(y)dy\right\}dk \\ &= \int\limits_{-\infty}^{+\infty}\left\{\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} E_\alpha((-(B_1|k|^{\beta_1}+B_2|k|^{\beta_2})t^\alpha)e^{-ik(x-y)}dk\right\}g(y)dy \\ &= \int\limits_{-\infty}^{+\infty} G_{\alpha,\beta_1,\beta_2}(x-y,t)g(y)dy, \end{aligned} \tag{14}$$

where

$$G_{\alpha,\beta_1,\beta_2}(x-y,t) = \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} E_\alpha((-(B_1|k|^{\beta_1}+B_2|k|^{\beta_2})t^\alpha)e^{-ik(x-y)}dk. \tag{15}$$

Let $x^{'} = x - y$, then (15) is simplified to

$$G_{\alpha,\beta_1,\beta_2}(x^{'},t) = \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} E_\alpha((-(B_1|k|^{\beta_1}+B_2|k|^{\beta_2})t^\alpha)e^{-ikx^{'}}dk, \tag{16}$$

which is the Green function of (16).

3. Green function's probability interpretation

Eq. (16) is equivalent to the form

$$G_{\alpha,\beta_1,\beta_2}(x,t) = \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} E_\alpha((-(B_1|k|^{\beta_1}+B_2|k|^{\beta_2})t^\alpha)e^{-ikx}dk. \tag{17}$$

It is the fundamental solution of (7) corresponding to $g(x) = \delta(x)$ (the Dirac delta function). The Green function can be characterized by

$$\widehat{G}_{\alpha,\beta_1,\beta_2}(k,t) = E_\alpha[-(B_1|k|^{\beta_1}+B_2|k|^{\beta_2})t^\alpha], \tag{18}$$

and

$$\widetilde{\widehat{G}}_{\alpha,\beta_1,\beta_2}(k,s) = \frac{s^{\alpha-1}}{s^\alpha + B_1|k|^{\beta_1}+B_2|k|^{\beta_2}}. \tag{19}$$

In view of the conjugate property of the Mittag-Leffler function, we obtain

$$\begin{aligned} \overline{\widehat{G}_{\alpha,\beta_1,\beta_2}(k,t)} &= \overline{E_\alpha[-(B_1|k|^{\beta_1}+B_2|k|^{\beta_2})t^\alpha]} \\ &= E_\alpha[\overline{-(B_1|k|^{\beta_1}+B_2|k|^{\beta_2})t^\alpha}] \\ &= E_\alpha[-(B_1|k|^{\beta_1}+B_2|k|^{\beta_2})t^\alpha] \\ &= \widehat{G}_{\alpha,\beta_1,\beta_2}(k,t). \end{aligned} \tag{20}$$

Furthermore, we easily recognize

$$\widehat{G}_{\alpha,\beta_1,\beta_2}(0,t) = E_\alpha(0) = 1, \qquad t \geq 0. \tag{21}$$

Provided that $G_{\alpha,\beta_1,\beta_2}(x,t)$ does exist as inverse Fourier transform of (18), equations (20)-(21) mean that $G_{\alpha,\beta_1,\beta_2}(x,t)$ is real and normalized, i.e.,

$$G_{\alpha,\beta_1,\beta_2}(x,t) \in \mathbf{R}, \qquad \int_{-\infty}^{+\infty} G_{\alpha,\beta_1,\beta_2}(x,t)dx = 1.$$

It remains to prove that $G_{\alpha,\beta_1,\beta_2}(x,t)$ is nonnegative, which ensures that the Green function is the spatial probability density for different values of the relevant parameters α, β_1, β_2. It can be obtained by deriving its explicit representation as indicated below.

Let us first consider the special case $0 < \beta_1 < 1$, $1 < \beta_2 \leq 2$ and $\alpha = 1$ (space Riesz fractional advection-dispersion equation). In this case, reducing the Mittag-Leffer function in (18) is reduced to the exponential function:

$$\begin{aligned} \widehat{G}_{1,\beta_1,\beta_2}(k,t) &= E_1[-(B_1|k|^{\beta_1}+B_2|k|^{\beta_2})t] \\ &= e^{-(B_1|k|^{\beta_1}+B_2|k|^{\beta_2})t} \\ &= \widehat{G}_{1,\beta_1}(k,t)\cdot \widehat{G}_{1,\beta_2}(k,t), \end{aligned} \tag{22}$$

where

$$\widehat{G}_{1,\beta_1}(k,t) = e^{-B_1|k|^{\beta_1}t}, \qquad \widehat{G}_{1,\beta_2}(k,t) = e^{-B_2|k|^{\beta_2}t}.$$

To derive the Green function in the space and time domain, we recover the characteristic function of the class of symmetric stable probability distribution (see [13,14] the reader about the properties and more details about the stable probability distributions), and using the similar notation as that in [13], we have

$$\widehat{L_\beta}(k) = e^{-|k|^\beta}, \tag{23}$$

where β can be $\beta_1 (0 < \beta_1 < 1)$ or $\beta_2 (1 < \beta_1 \leq 2)$.

From [13,14], and using the symmetry relation $L_\beta(x) = L_\beta(-x)$, we see that it is a symmetric stable probability density function (pdf), with solution

$$L_{\beta_1}(x) = \begin{cases} \dfrac{1}{\pi x}\displaystyle\sum_{n=1}^{\infty}(-x^{-\beta_1})^n \frac{\Gamma(1+n\beta_1)}{n!}\sin\left(-\frac{n\pi\beta_1}{2}\right), & x \neq 0, \\ \dfrac{1}{\pi\beta_1}\Gamma\left(\dfrac{1}{\beta_1}\right), & x = 0. \end{cases} \tag{24}$$

$$L_{\beta_2}(x) = \begin{cases} \dfrac{1}{\pi\beta_2}\displaystyle\sum_{n=0}^{\infty}(-1)^n \dfrac{\Gamma\left(\frac{2n+1}{\beta_2}\right)}{(2n)!}x^{2n}, & x \neq 0, \\ \dfrac{1}{\pi\beta_2}\Gamma\left(\dfrac{1}{\beta_2}\right), & x = 0. \end{cases} \tag{25}$$

By the Fourier transform of a convolution of two functions

$$F\{h(x) * f(x); k\} = F\left\{\int_{-\infty}^{+\infty} h(x-y)f(y)dy; k\right\} = \widehat{h}(k)\widehat{f}(k),$$

and the scaling property of Fourier transform

$$F\{f(ax); k\} = |a|^{-1}\widehat{f}(k/a), \quad a \in \mathbf{R},$$

we have

$$\begin{aligned} & G_{1,\beta_1,\beta_2}(x,t) \\ = \; & G_{1,\beta_1}(x,t) * G_{1,\beta_2}(x,t) \\ = \; & \left[(B_1 t)^{-\frac{1}{\beta_1}} L_{\beta_1}\left(\frac{x}{(B_1 t)^{\frac{1}{\beta_1}}}\right)\right] * \left[(B_2 t)^{-\frac{1}{\beta_2}} L_{\beta_2}\left(\frac{x}{(B_2 t)^{\frac{1}{\beta_2}}}\right)\right], \end{aligned} \tag{26}$$

which is nonnegative.

We now present a composition rule which allows us to express the general Green function of the time-space Riesz fractional advection-dispersion equation (with the restriction $0 < \alpha \leq 1$) as an integral involving the two Green functions corresponding to space Riesz fractional advection-dispersion equation and time fractional diffusion equation [13]. For this purpose, we note that the Fourier-Laplace transform of the Green function (19) can be re-written in integral form as [13,15,19]

$$\begin{aligned} \widetilde{\widehat{G}}_{\alpha,\beta_1,\beta_2}(k,s) &= \frac{s^{\alpha-1}}{s^\alpha + B_1|k|^{\beta_1} + B_2|k|^{\beta_2}} \\ &= s^{\alpha-1}\int_0^{+\infty} e^{-u\left[s^\alpha + B_1|k|^{\beta_1} + B_2|k|^{\beta_2}\right]}du \\ &= \int_0^{+\infty} e^{-\left[B_1|k|^{\beta_1} + B_2|k|^{\beta_2}\right]u} \cdot \left(s^{\alpha-1}e^{-s^\alpha u}\right) du \\ &= \int_0^{+\infty} \widehat{G}_{1,\beta_1,\beta_2}(k,u) \cdot \widetilde{G}_{2\alpha}(u,s)du, \end{aligned} \tag{27}$$

where

$$\widetilde{G}_\lambda(x,s) = s^{\lambda/2-1}e^{-|x|s^{\lambda/2}}, \qquad x \in \mathbf{R}, \qquad Re(s) > 0,$$

with solution

$$G_\lambda(x,t) = t^{-\lambda/2}M_{\lambda/2}\left(|x|/t^{\lambda/2}\right), \qquad x \in \mathbf{R}, \qquad t \geq 0, \tag{28}$$

where $M_{\lambda/2}$ denotes the M function (of the Wright type) of order $\lambda/2$, whose general properties can be found in [5,13]. The function M_λ is defined for any order $\lambda \in (0,1)$ and $\forall z \in \mathbf{C}$ by

$$M_\lambda(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma[-\lambda n + (1-\lambda)]} = \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{(-z)^{n-1}}{(n-1)!}\Gamma(\lambda n)\sin(\pi\lambda n).$$

Going back to the time-space domain (by inversing to (27)), we obtain the relation

$$G_{\alpha,\beta_1,\beta_2}(x,t) = \int_0^{+\infty} G_{1,\beta_1,\beta_2}(x,u)G_{2\alpha}(u,t)du. \tag{29}$$

Hence, Eq.(29) is a formula separating variables. It states that the Green function for the space-time Riesz fractional advection-dispersion equation of order α, β_1, β_2, with $0 < \alpha \le 1$, $0 < \beta_1 < 1$ and $1 < \beta_2 \le 2$, can be expressed in terms of the Green function for the space Riesz fractional advection-dispersion equation of order β_1, β_2 and the Green function for the time fractional diffusion equation of order 2α. We can ensure the Green function $G_{\alpha,\beta_1,\beta_2}(x,t)$ is nonnegative by the nonnegative properties of G_{1,β_1,β_2} and $G_{2\alpha}$.

Then we can conclude that the Green function can be interpreted as a spatial pdf, evolving in time.

4. An explicit finite difference approximation for the TSRFADE-IC

We now consider a discrete form of (7) both in time and space. We introduce a spatial grid $-\infty < ... < x_{i-2} < x_{i-1} < x_i < x_{i+1} < x_{j+2} < ... < \infty$ with the step $h = x_k - x_{k-1}$. We denote the value of the function $u(x)$ at the point x_k as $u_k = u(x_k)$, for $k \in \mathbf{Z}$. Using the relationship between the Riemann-Liouville derivative and the Grünwald-Letnikov scheme [22], we discretize the Riesz fractional derivative $\frac{\partial^{\beta_1} u}{\partial |x|^{\beta_1}}$ by the Grünwald-Letnikov scheme in the case $0 < \beta_1 < 1$, and the Riesz fractional derivative $\frac{\partial^{\beta_2} u}{\partial |x|^{\beta_2}}$ by the shifted Grünwald-Letnikov scheme in the case $1 < \beta \le 2$:

$$\begin{aligned} \frac{\partial^{\beta_1}}{\partial |x|^{\beta_1}} u_i &\equiv {}_xD^{\beta_1} u_i \\ &= -c_1[{}_{-\infty}D_x^{\beta_1} u_i + {}_xD_{+\infty}^{\beta_1} u_i] \\ &\approx -c_1[{}_hD_-^{\beta_1} u_i + {}_hD_+^{\beta_1} u_i], \end{aligned} \tag{30}$$

where

$${}_hD_\pm^{\beta_1} u_i = \frac{1}{h^{\beta_1}} \sum_{k=0}^{+\infty} (-1)^k \begin{pmatrix} \beta_1 \\ k \end{pmatrix} u_{i\mp k}, \quad 0 < \beta_1 < 1, \tag{31}$$

$$\begin{aligned}\frac{\partial^{\beta_2}}{\partial|x|^{\beta_2}}u_i &\equiv {}_xD^{\beta_2}u_i \\ &= -c_2[{}_{-\infty}D_x^{\beta_2}u_i + {}_xD_{+\infty}^{\beta_2}u_i] \\ &\approx -c_2[{}_hD_-^{\beta_2}u_i + {}_hD_+^{\beta_2}u_i],\end{aligned} \tag{32}$$

where

$${}_hD_{\pm}^{\beta_2}u_i = \frac{1}{h^{\beta_2}}\sum_{k=0}^{+\infty}(-1)^k\begin{pmatrix}\beta_2\\k\end{pmatrix}u_{i\mp k\pm 1}, \quad 1<\beta_2\le 2, \tag{33}$$

and the coefficients $c_1 = \dfrac{1}{2\cos(\beta_1\pi/2)}$, $c_2 = \dfrac{1}{2\cos(\beta_2\pi/2)}$.

Notice that the shifted scheme (33), for the case $\beta_2 = 2$, leads to the standard symmetric three-point difference scheme.

The Riesz fractional advection term $\dfrac{\partial^{\beta_1}}{\partial|x|^{\beta_1}}u_i$ $(0<\beta_1<1)$ in (30) can be approximated in the following form:

$$\begin{aligned}&\frac{\partial^{\beta_1}}{\partial|x|^{\beta_1}}u_i \\ \approx\ & -c_1\left[\frac{1}{h^{\beta_1}}\sum_{k=0}^{+\infty}(-1)^k\begin{pmatrix}\beta_1\\k\end{pmatrix}u_{i-k} + \frac{1}{h^{\beta_1}}\sum_{k=0}^{+\infty}(-1)^k\begin{pmatrix}\beta_1\\k\end{pmatrix}u_{i+k}\right] \\ =\ & -\frac{c_1}{h^{\beta_1}}\left[\sum_{k=0}^{+\infty}(-1)^k\begin{pmatrix}\beta_1\\k\end{pmatrix}u_{i-k} + \sum_{k=0}^{+\infty}(-1)^k\begin{pmatrix}\beta_1\\k\end{pmatrix}u_{i+k}\right] \\ =\ & -2\frac{c_1}{h^{\beta_1}}u_i - \frac{c_1}{h^{\beta_1}}\left[\sum_{k=1}^{+\infty}(-1)^k\begin{pmatrix}\beta_1\\k\end{pmatrix}u_{i-k} + \sum_{k=1}^{+\infty}(-1)^k\begin{pmatrix}\beta_1\\k\end{pmatrix}u_{i+k}\right] \\ =\ & -2\frac{c_1}{h^{\beta_1}}u_i - \frac{c_1}{h^{\beta_1}}\sum_{k=-\infty,k\neq 0}^{+\infty}(-1)^k\begin{pmatrix}\beta_1\\k\end{pmatrix}u_{i+k} \\ =\ & \frac{1}{h^{\beta_1}}\sum_{k=-\infty}^{+\infty}\omega_k^{(\beta_1)}u_{i+k},\end{aligned} \tag{34}$$

where

$$\begin{cases}\omega_0^{(\beta_1)} = -2c_1, \\ \omega_{\pm k}^{(\beta_1)} = (-1)^{k+1}\begin{pmatrix}\beta_1\\k\end{pmatrix}c_1, \quad k=1,2,\ldots\end{cases} \tag{35}$$

The Riesz fractional diffusion term $\frac{\partial^{\beta_2}}{\partial|x|^{\beta_2}}u_i$ $(1 < \beta_2 \le 2)$ in (7) can be approximated in the following form:

$$\begin{aligned}
&\frac{\partial^{\beta_2}}{\partial|x|^{\beta_2}}u_i\\
\approx\ & -c_2\left[\frac{1}{h^{\beta_2}}\sum_{k=0}^{+\infty}(-1)^k\binom{\beta_2}{k}u_{i-k+1}+\frac{1}{h^{\beta_2}}\sum_{k=0}^{+\infty}(-1)^k\binom{\beta_2}{k}u_{i+k-1}\right]\\
=\ & \left[2\frac{c_2}{h^{\beta_2}}\binom{\beta_2}{1}u_i\right]+\left[-\frac{c_2}{h^{\beta_2}}\left(\binom{\beta_2}{2}+1\right)u_{i+1}\right]\\
& +\left[-\frac{c_2}{h^{\beta_2}}\left(\binom{\beta_2}{2}+1\right)u_{i-1}\right]\\
& +\left[\frac{c_2}{h^{\beta_2}}\sum_{k=3}^{+\infty}(-1)^{k-1}\binom{\beta_2}{k}u_{i-k+1}+\frac{c_2}{h^{\beta_2}}\sum_{k=3}^{+\infty}(-1)^{k-1}\binom{\beta_2}{k}u_{i+k-1}\right],
\end{aligned} \tag{36}$$

while

$$\begin{aligned}
&\left[\frac{c_2}{h^{\beta_2}}\sum_{k=3}^{+\infty}(-1)^{k-1}\binom{\beta_2}{k}u_{i-k+1}+\frac{c_2}{h^{\beta_2}}\sum_{k=3}^{+\infty}(-1)^{k-1}\binom{\beta_2}{k}u_{i+k-1}\right]\\
&=\frac{c_2}{h^{\beta_2}}\left[\sum_{k=2}^{+\infty}(-1)^k\binom{\beta_2}{k+1}u_{i-k}+\sum_{k=2}^{+\infty}(-1)^k\binom{\beta_2}{k+1}u_{i+k}\right].
\end{aligned} \tag{37}$$

Thus

$$\frac{\partial^{\beta_2}}{\partial|x|^{\beta_2}}u_i \approx \frac{1}{h^{\beta_2}}\sum_{k=-\infty}^{+\infty}\omega_k^{(\beta_2)}u_{i+k}, \tag{38}$$

where

$$\begin{cases}
\omega_0^{(\beta_2)} = 2\binom{\beta_2}{1}c_2,\\
\omega_{\pm1}^{(\beta_2)} = -\left[\binom{\beta_2}{2}+1\right]c_2,\\
\omega_{\pm k}^{(\beta_2)} = (-1)^k\binom{\beta_2}{k+1}c_2, \quad k = 2,3,\ldots.
\end{cases} \tag{39}$$

We introduce a temporal grid $0 = t_0 < t_1 < ... < t_n < t_{n+1} < ... < \infty$ with the grid step $\tau = t_{n+1} - t_n$. At a point x_k at the moment of time t_n we denote the function $u(x,t)$ as $u_k^n = u(x_k, t_n)$, for $k \in \mathbf{Z}$ and $n \in \mathbf{N}$. Adopting the discrete scheme in [10], we discretize the Caputo time fractional derivative as

$$\begin{aligned}
{}_tD_*^{\alpha}u_i^{n+1} &= \frac{1}{\Gamma(1-\alpha)}\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}}(t_{n+1}-\eta)^{-\alpha}\frac{\partial u(x_i,\eta)}{\partial\eta}d\eta \\
&= \frac{1}{\Gamma(1-\alpha)}\sum_{j=0}^{n}\frac{u_i^{n+1}-u_i^{n}}{\tau}\int_{t_j}^{t_{j+1}}(t_{n+1}-\eta)^{-\alpha}d\eta+O(\tau^{2-\alpha}) \\
&= \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{j=0}^{n}b_{n-j}[u_i^{n+1}-u_i^{n}]+O(\tau^{2-\alpha}) \qquad (40)\\
&= \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{j=0}^{n}b_{j}[u_i^{n+1-j}-u_i^{n-j}]+O(\tau^{2-\alpha}),
\end{aligned}$$

where $b_j=(j+1)^{1-\alpha}-j^{1-\alpha}, j=0,1,2,\cdots,n$.

Now we replace (7) with the following explicit finite difference approximation (EFDA):

$$\begin{aligned}
&\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{j=0}^{n}b_j[u_i^{n+1-j}-u_i^{n-j}] \\
&= \frac{B_1}{h^{\beta_1}}\sum_{k=-\infty}^{+\infty}\omega_k^{(\beta_1)}u_{i+k}^{n}+\frac{B_2}{h^{\beta_2}}\sum_{k=-\infty}^{+\infty}\omega_k^{(\beta_2)}u_{i+k}^{n}. \qquad (41)
\end{aligned}$$

After simplifications we obtained the final form

$$\begin{aligned}
u_i^{n+1} &= b_nu_i^0+\sum_{j=0}^{n-1}(b_j-b_{j+1})u_i^{n-j}+B_1\mu_1\sum_{k=-\infty}^{+\infty}\omega_k^{(\beta_1)}u_{i+k}^{n} \\
&\quad +B_2\mu_2\sum_{k=-\infty}^{+\infty}\omega_k^{(\beta_2)}u_{i+k}^{n}, \qquad (42)
\end{aligned}$$

where $\mu_1=\dfrac{\tau^{\alpha}\Gamma(2-\alpha)}{h^{\beta_1}}$, $\mu_2=\dfrac{\tau^{\alpha}\Gamma(2-\alpha)}{h^{\beta_2}}$.

Note that the coefficients possess the following properties:

Property 1: The coefficients $b_j, j=1,2,\cdots$, satisfy [17,27]

(1)$b_j>0, j=1,2,\cdots;$

(2)$b_j>b_{j+1}, j=0,1,\cdots.$

Property 2: The coefficients $\omega_k^{(\beta_1)}$ and $\omega_k^{(\beta_2)}$ $(k\in\mathbf{Z})$ satisfy

(1)$\omega_0^{(\beta_1)}<0$, $\omega_0^{(\beta_2)}<0$;

(2)$\omega_{\pm k}^{(\beta_1)}\geq 0$, $\omega_{\pm k}^{(\beta_1)}\geq 0$ for $k=1,2,\ldots;$

(3) $\sum_{k=-\infty}^{+\infty}\omega_k^{(\beta_1)}=\sum_{k=-\infty}^{+\infty}\omega_k^{(\beta_2)}=0.$

Proof. The property (1) is easily obtained from the expressions (35) and (39).

Note that in the case $0<\beta_1<1$, we have

$$(-1)^{k+1}\begin{pmatrix}\beta_1\\k\end{pmatrix}=\left|\begin{pmatrix}\beta_1\\k\end{pmatrix}\right|\geq 0,\quad c_1=\frac{1}{2\cos(\beta_1\pi/2)}>0,$$

i.e.,

$$\omega_{\pm k}^{(\beta_1)} \geq 0, \quad k = 1, 2, \ldots$$

In the case $1 < \beta_2 \leq 2$, we have

$$(-1)^k \begin{pmatrix} \beta_2 \\ k+1 \end{pmatrix} = \left| \begin{pmatrix} \beta_2 \\ k+1 \end{pmatrix} \right| \leq 0, \quad c_2 = \frac{1}{2\cos(\beta_2 \pi/2)} < 0,$$

i.e.,

$$\omega_{\pm k}^{(\beta_2)} \geq 0, \quad k = 1, 2, \ldots$$

And

$$\begin{aligned} \sum_{k=-\infty}^{+\infty} \omega_k^{(\beta_1)} &= \omega_0^{(\beta_1)} + \sum_{k=1}^{+\infty} \omega_{\pm k}^{(\beta_1)} \\ &= -2c_1 + \sum_{k=1}^{+\infty} 2 \left[(-1)^{k+1} \begin{pmatrix} \beta_1 \\ k \end{pmatrix} c_1 \right] \\ &= 2 \left[\sum_{k=0}^{+\infty} (-1)^{k+1} \begin{pmatrix} \beta_1 \\ k \end{pmatrix} c_1 \right] \\ &= 2 \left[(-1) \sum_{k=0}^{+\infty} (-1)^k \begin{pmatrix} \beta_1 \\ k \end{pmatrix} c_1 \right] \\ &= 0; \end{aligned}$$

$$\begin{aligned} \sum_{k=-\infty}^{+\infty} \omega_k^{(\beta_2)} &= \omega_0^{(\beta_2)} + \sum_{k=1}^{+\infty} \omega_{\pm k}^{(\beta_2)} \\ &= 2 \begin{pmatrix} \beta_2 \\ 1 \end{pmatrix} c_2 - 2 \left[\begin{pmatrix} \beta_2 \\ 2 \end{pmatrix} + 1 \right] c_2 + 2 \sum_{k=2}^{+\infty} (-1)^k \begin{pmatrix} \beta_2 \\ k+1 \end{pmatrix} c_2 \\ &= -2 \sum_{k=0}^{+\infty} (-1)^k \begin{pmatrix} \beta_2 \\ k \end{pmatrix} c_2 \\ &= 0. \end{aligned}$$

5. A discrete random walk model for the TSRFADE-IC

In this section, we present a discrete random walk model for the TSRFADE-IC

$$u(x, 0) = \delta(x) \quad (x \in \mathbf{R}), \tag{43}$$

where $\delta(x)$ is the Dirac delta function. The dependent variable u is then discretized by introducing $y_j(t_n)$ as

$$y_j(t_n) \approx \int_{x_j - h/2}^{x_j + h/2} u(x, t_n) dx \approx h u(x_j, t_n). \tag{44}$$

Inserting the above expression in the EFDA (42) yields the transition law

$$\begin{aligned} y_i(t_{n+1}) &= b_n y_i(t_0) + \sum_{j=0}^{n-1}(b_j - b_{j+1})y_i(t_{n-j}) + B_1\mu_1 \sum_{k=-\infty}^{+\infty} \omega_k^{(\beta_1)} y_{i+k}(t_n) \\ &\quad + B_2\mu_2 \sum_{k=-\infty}^{+\infty} \omega_k^{(\beta_2)} y_{i+k}(t_n) \\ &= [1 - b_1 + B_1\mu_1\omega_0^{(\beta_1)} + B_2\mu_2\omega_0^{(\beta_2)}]y_i(t_n) \\ &\quad + \sum_{j=1}^{n-1}(b_j - b_{j+1})y_i(t_{n-j}) + b_n y_i(t_0) \qquad (45) \\ &\quad + B_1\mu_1 \sum_{k=-\infty, k\neq 0}^{+\infty} \omega_k^{(\beta_1)} y_{i+k}(t_n) + B_2\mu_2 \sum_{k=-\infty, k\neq 0}^{+\infty} \omega_k^{(\beta_2)} y_{i+k}(t_n) \\ &\quad (k \in \mathbf{Z}, \mathbf{n} \in \mathbf{N}). \end{aligned}$$

In [7], the authors have showed that a discrete random walk model for the strictly space fractional diffusion is a discrete Markovian random walk model for the Lévy stable motion, while the strictly time fractional diffusion is a discrete non-Markovian case. Combining the approach of the strictly space fractional diffusion and the strictly time fractional diffusion, the authors also constructed a discrete random walk model for the strictly space-time fractional diffusion equation, which is also a non-Markovian case.

We will show that the random walk admits an interpretation as (45). Now we introduce the concept of a discrete random walk model. A discrete random walk on the grids $(jh|j \in \mathbf{Z})$ is obtained by defining the random variables:

$$S_n = hY_1 + hY_2 + \cdots + hY_n, \quad (n \in \mathbf{N}), \qquad (46)$$

where $S_0 = 0$, $Y_1, Y_2, \cdots Y_n$ are independent identically distributed random variables. The recursion $S_{n+1} = S_n + hY_{n+1}$ then results.

By a suitable normalization, when time proceeds from $t = t_n$ to t_{n+1}, the sojourn probabilities $y_i(t_n)$ are redistributed according to the general rule (45), the coefficients before $y_i(t_n)$ represent the probabilities of transition from x_{i+k} to x_i (likewise from x_i to x_{i+k}). Using the definition and the property of the Dirac delta function $\delta(x)$, we have

$$y_j(0) = \int_{x_j-h/2}^{x_j+h/2} u(x,0)dx = \int_{x_j-h/2}^{x_j+h/2} \delta(x)dx = \begin{cases} 1, & j = 0, \\ 0, & j \neq 0. \end{cases} \qquad (47)$$

This means that the random walker starts at point $x_0 = 0$. Actually, the formula (45) can be interpreted as a discrete random model only if all coefficients (which represent the probabilities of transition) are nonnegative and the sum of these coefficients are equal to 1.

In the following section, we will consider whether these coefficients satisfy the two above conditions.

Under the condition

$$\tau^{\alpha} \leq \frac{2 - 2^{1-\alpha}}{2\left[\dfrac{B_1 c_1}{h^{\beta_1}} - \dfrac{B_2 c_2 \beta_2}{h^{\beta_2}}\right]}, \tag{48}$$

the transfer coefficient $[1 - b_1 + B_1\mu_1\omega_0^{(\beta_1)} + B_2\mu_2\omega_0^{(\beta_2)}] \geq 0$ in (45). Using the property 1 and property 2, we can easily get that all the coefficients in (45) are nonnegative.

Proposition 1. The solution $y_i(t_{n+1})$ preserves nonnegativity if all coefficients are nonnegative; hence if the coefficient of the term $y_i(t_n)$ is non-negative, i.e., the condition (48) is satisfied.

Applying the property 2, we can obtain that the sum of these coefficients is equal to 1.

In fact,

$$\begin{aligned} & b_n + \sum_{j=0}^{n-1}(b_j - b_{j+1}) + B_1\mu_1 \sum_{k=-\infty}^{+\infty} \omega_k^{(\beta_1)} + B_2\mu_2 \sum_{k=-\infty}^{+\infty} \omega_k^{(\beta_2)} \\ = & \ b_n + \sum_{j=0}^{n-1}(b_j - b_{j+1}) \\ = & \ b_0 \\ = & \ 1 \end{aligned}$$

Thus, we can get the following result by using mathematical induction.

Proposition 2. The solution $y_i(t_n)$ is conservative, i.e.,

$$\sum_{i=-\infty}^{+\infty} |y_i(t_0)| < \infty \Rightarrow \sum_{i=-\infty}^{+\infty} y_i(t_n) = \sum_{i=-\infty}^{+\infty} y_i(t_0)\ , \qquad n \in \mathbf{N}. \tag{49}$$

Proof. Let $S_n = \sum_{i=-\infty}^{+\infty} y_i(t_n)$ for $n \geq 0$, then from (45) we obtain

$$\begin{aligned} S_1 &= \sum_{i=-\infty}^{+\infty} y_i(t_1) \\ &= b_0 \sum_{i=-\infty}^{+\infty} y_i(t_0) + B_1\mu_1 \sum_{i=-\infty}^{+\infty}\sum_{k=-\infty}^{+\infty} \omega_k^{(\beta_1)} y_{i+k}(t_0) \\ &\quad + B_2\mu_2 \sum_{i=-\infty}^{+\infty}\sum_{k=-\infty}^{+\infty} \omega_k^{(\beta_2)} y_{i+k}(t_0) \\ &= b_0 \sum_{i=-\infty}^{+\infty} y_i(t_0) \\ &= S_0. \end{aligned}$$

Now assume that when $k \le n$, the equation (49) is true, i.e., $S_0 = S_1 = \cdots = S_n$. When $k = n+1$, we have

$$\begin{aligned} S_{n+1} &= \sum_{i=-\infty}^{+\infty} y_i(t_{n+1}) \\ &= b_n \sum_{i=-\infty}^{+\infty} y_i(t_0) + \sum_{j=0}^{n-1}(b_j - b_{j+1}) \sum_{i=-\infty}^{+\infty} y_i(t_{n-j}) \\ &\quad + B_1\mu_1 \sum_{i=-\infty}^{+\infty}\sum_{k=-\infty}^{+\infty} \omega_k^{(\beta_1)} y_{i+k}(t_n) + B_2\mu_2 \sum_{i=-\infty}^{+\infty}\sum_{k=-\infty}^{+\infty} \omega_k^{(\beta_2)} y_{i+k}(t_n) \\ &= b_n S_0 + \sum_{j=0}^{n-1}(b_j - b_{j+1}) S_0 \\ &= [b_n + \sum_{j=0}^{n-1}(b_j - b_{j+1})] S_0 \\ &= S_0, \end{aligned}$$

i.e., we have proved that when $n = k+1$, the equation (49) is also true and the induction is completed.

Remark 1. Nonnegativity preservation and conservativity implies that our scheme can be interpreted as a redistribution scheme of clumps $y_i(t_n)$.

Similarly to Gerenflo's [7] discussion, the interpretation of our redistribution scheme is as follows: the clump $y_i(t_{n+1})$ arises as a weighted-memory average of the (previous) n+1 values $y_i(t_m)$ (with $m = n, n-1, \ldots, 1, 0$) and the(left and right sides) infinite values $y_j(t_n)$ (with $j \in (-\infty, +\infty), j \in \mathbb{Z}$), and positive weights of the two parts

$$\begin{array}{ll} b_n, b_{n-1} - b_n, \ldots, b_1 - b_2, & 1 - b_1 + B_1\mu_1\omega_0^{(\beta_1)} + B_2\mu_2\omega_0^{(\beta_2)}, \\ B_1\mu_1\omega_k^{(\beta_1)} + B_2\mu_2\omega_k^{(\beta_2)}, & k \in (-\infty, +\infty), k \in \mathbb{Z}, \end{array} \tag{50}$$

subject to the condition (48). For a random walk interpretation we consider the $y_i(t_n)$ as probabilities of sojourn at point x_i in instant t_n requiring the normalization condition $\sum_{i=-\infty}^{+\infty} y_i(t_0) = 1$.

For $n = 0$ equation (45) gives that a particle sitting at x_i in instant t_0 jumps, when t proceeds from t_0 to t_1, with probability $B_1\mu_1\omega_k^{(\beta_1)} + B_2\mu_2\omega_k^{(\beta_2)}$ to either the left or right side point x_{i+k} or x_{i-k} ($k \in (-\infty, +\infty), k \in \mathbb{Z}$), and with probability $b_0 + B_1\mu_1\omega_0^{(\beta_1)} + B_2\mu_2\omega_0^{(\beta_2)}$ it remains at x_i.

For $n \ge 1$, we have showed that all coefficients (probabilities) are non-negative, with unit sum. Having a particle, sitting in x_i at instant t_n, there is a probability $1 - b_1 + B_1\mu_1\omega_0^{(\beta_1)} + B_2\mu_2\omega_0^{(\beta_2)}$ to be again at x_j at instant t_{n+1}, and a probability $B_1\mu_1\omega_k^{(\beta_1)} + B_2\mu_2\omega_k^{(\beta_2)}$ to go to either the left or right side point x_{i+k} or x_{i-k} ($k \in (-\infty, +\infty), k \in \mathbb{Z}$). The striking difference here is the appearance of arbitrarily large jumps for such discrete model referred to as Lévy flight. However the sum of these contributions is $1 - b_1 = 2 - 2^{1-\alpha} \le 1$ and excluding the case $\alpha = 1$ in which we recover standard diffusion(Markovian process), for $\alpha < 1$ we have to consider the previous time levels(non-Markovian process). Then, from levels t_{n-1} we obtain the contribution $b_1 - b_2$ for the probability of staying at x_i also at time t_{n+1}, from level t_{n-2} we obtain the contribution $b_2 - b_3$ for the probability of staying at x_i at $t_{n+1}, \ldots$, from level t_1 we get the contribution $b_{n-1} - b_n$ for the probability of staying at x_i at t_{n+1}, and finally, from level $t_0 = 0$ we get the contribution b_n for the probability of staying at x_i at t_{n+1}. Thus, the whole history up to t_n decides probabilistically where the particle will be at instant t_{n+1}.

Let us consider the problem of simulating the transition from time level t_n to t_{n+1}: assume the particle is sitting at x_i at time t_n. Generate a random number equidistributed in $0 \leq \rho < 1$, and subdivide the interval $[0, 1)$ as follows: from left to right beginning at zero, the adjacent intervals of length $1 - b_1, b_1 - b_2, \ldots, b_{n-1} - b_n, b_n$, are considered left-closed, right-open. The sum of these intervals is 1. We divide further the first interval(of length $1-b_1$)into infinite subintervals of length $\cdots, B_1\mu_1\omega_{-k}^{(\beta_1)} + B_2\mu_2\omega_{-k}^{(\beta_2)}, \cdots, B_1\mu_1\omega_{-1}^{(\beta_1)} + B_2\mu_2\omega_{-1}^{(\beta_2)}, 1 - b_1 + B_1\mu_1\omega_0^{(\beta_1)} + B_2\mu_2\omega_0^{(\beta_2)}, B_1\mu_1\omega_1^{(\beta_1)} + B_2\mu_2\omega_1^{(\beta_2)}, \cdots, B_1\mu_1\omega_k^{(\beta_1)} + B_2\mu_2\omega_k^{(\beta_2)}, \cdots$. Then we look into which of the above intervals the random number falls. If it is in first interval with length $1 - b_1$, then we look in which subinterval, and correspondingly move the particle to x_{i-k}, or leave it at x_i or move to x_{i+k}. If the random number falls into one of the intervals with length $b_1 - b_2, \ldots, b_{n-1} - b_n, b_n$, then we move the particle back to its previous position $x(t_{n-j})$, which by chance could be identical with $x_i = x(t_n)$. If the random number falls into the rightmost interval with length b_n then we move the particle back to its initial position $x(t_0)$, for which we recommend $x(t_0) = 0$, meaning $y_i(t_0) = \delta_{i0}$, in accordance with the initial condition $u(x, 0) = \delta(x)$ for STRFADE. Besides the diffusive part $\left\{\cdots, B_1\mu_1\omega_{-k}^{(\beta_1)} + B_2\mu_2\omega_{-k}^{(\beta_2)}, \cdots, B_1\mu_1\omega_{-1}^{(\beta_1)} + B_2\mu_2\omega_{-1}^{(\beta_2)}, 1 - b_1 + B_1\mu_1\omega_0^{(\beta_1)} + B_2\mu_2\omega_0^{(\beta_2)}, B_1\mu_1\omega_1^{(\beta_1)} + B_2\mu_2\omega_1^{(\beta_2)}, \cdots, B_1\mu_1\omega_k^{(\beta_1)} + B_2\mu_2\omega_k^{(\beta_2)}, \cdots\right\}$, which lets the particle appear arbitrarily large jumps, we have for $0 < \alpha < 1$ the memory part which gives a tendency to return to former positions even if they are far away. Due to property 1, of course, the probability to return to a far away point gets smaller and smaller the larger the time lapse is from the instant when the particle was there.

From the above analysis, we notice that in analogy to the case of the strictly time fractional diffusion, the "discrete" diffusion in space occurs only between the time-level t_n and t_{n+1} and the memory part of the process only straight-backwards in time. However, in contrast to the strictly time fractional diffusion, the discrete diffusion (or the random walker) can now go to any grid point in space, not only to immediate neighboring grid points.

6. Conclusions

In this paper we derived the fundamental solution for the space-time Riesz fractional advection-dispersion equation with initial condition, and this solution can be interpreted as a spatial probability density function evolving in time. We also discretized the space-time Riesz fractional advection-dispersion equation and generated a discrete random walk model for this STRFADE-IC.

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