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Stability and convergence of a new explicit finite-difference approximation for the variable-order nonlinear fractional diffusion equation

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Abstract

In this paper, we consider the variable-order nonlinear fractional diffusion equation

$$\frac{\partial u(x,t)}{\partial t} = B(x,t)_x R^{\alpha(x,t)} u(x,t) + f(u,x,t),$$

where ${}_x R^{\alpha(x,t)}$ is a generalized Riesz fractional derivative of variable order $\alpha(x,t)$ $(1 < \alpha(x,t) \le 2)$ and the nonlinear reaction term $f(u,x,t)$ satisfies the Lipschitz condition $|f(u_1,x,t) - f(u_2,x,t)| \le L|u_1 - u_2|$. A new explicit finite difference approximation is introduced. The convergence and stability of this approximation are proved. Finally, some numerical examples are provided to show that this method is computationally efficient. The proposed method and techniques are applicable to other variable-order nonlinear fractional differential equations.

Key words: variable order, fractional calculus, nonlinear fractional diffusion equation, convergence, stability, explicit difference approximation.

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1 Introduction

Fractional-order derivatives and integrals provide useful tools for the description of memory and hereditary properties of different substances [16,18–20,23]. Many physical processes appear to exhibit fractional-order behavior that may vary with time, or space, or space and time. The theory of pseudodifferential operators and equations has been used to deal with this situation [5,22]. The behavior of some diffusion processes in response to temperature changes may be better described using variable-order exponents in a pseudodifferential operator than time-varying coefficients [15,16]. Lorenzo and Hartley [15,16] presented the concept of variable-order fractional integration and differentiation. Multifractional pseudodifferential models have been considered in the representation of heterogeneous local behaviors. The solutions to such models are defined in fractional Besov spaces of variable order on $\mathbb{R}^n$ (Leopold [9]). Gaussian processes defined by elliptic pseudodifferential equations have been studied in Ruiz-Medina, Anh and Angulo [22]. The covariance function of these random processes defines the inner product of a fractional Sobolev space of variable order.

Fractional-order equations have recently been treated by a number of authors. Liu *et al.* [11,12] transformed the space fractional partial differential equation into a system of ordinary differential equations (method of lines) that was then solved using backward differentiation formulas. Roop [21] investigated the numerical approximation of the variational solution to the fractional advection-dispersion equation. Meerschaert *et al.* [17] examined finite difference approximations for fractional advection-dispersion flow equations. Shen *et al.* [25] proposed an explicit finite difference approximation for the space fractional diffusion equation and gave an error analysis. Liu *et al.* [13] discussed an approximation of the Lévy-Feller advection-dispersion process by a random walk and finite difference method. Liu *et al.* [14] derived an analysis of a discrete non-Markovian random walk approximation for the time fractional diffusion equation. Zhuang and Liu [28] analyzed an implicit difference approximation for the time fractional diffusion equation. Stability and convergence of the method were discussed. Lin and Liu [10] proposed the high order (2-6) approximations of the fractional ordinary differential equation and discussed the consistency, convergence and stability of these fractional high order methods. Yu *et al.* [27] developed a reliable algorithm of the Adomian decomposition method to solve the linear and nonlinear space-time fractional reaction-diffusion equations in the form of a rapidly convergent series with easily computable components. Recently, Chen *et al.* [2] presented a Fourier method for the fractional diffusion equation describing sub-diffusion, and gave the stability and global accuracy analyses of the difference approximation scheme. Zhuang *et al.* [29] also proposed a new solution and analytical techniques of implicit numerical methods for the anomalous sub-diffusion equation.

Fractional calculus allows the operators of integration and differentiation to have fractional order. The order may take on any real values. This fact enables us to consider the order of the fractional integrals and derivatives to be a function of time or space or space-time variables. Lorenzo and Hartley [15] suggested the concept of a variable-order operator. Glockle and Nonnenmacher [6] studied the relaxation processes and reaction kinetics of proteins that are described by fractional differential equations of order β. The order was found to have a temperature dependence. Electroviscous or electrorheological fluids [8] and polymer gels [26] are known to change their properties in response to changes in imposed electric field strength. The properties of magnetorheological elastomers respond to magnetic field strength [3]. From the field of damage modelling, it is noted that as the damage accumulates (with time) in a structure, the nonlinear stress/strain behavior changes. This behavior may be better described with variable-order calculus.

In order to present the model to be considered in this paper, we first define the concept of variable-order fractional derivatives. We assume that $1 < \underline{\alpha} \leq \alpha(x,t) \leq \overline{\alpha} \leq 2$, $X_a \leq x \leq X_b$, $0 < t \leq T$, and $u(x,t) = 0$, when $x \leq X_a$ or $x > X_b$. The Riesz fractional derivative ${}_x\overline{R}^{\alpha}$ is defined as [4]

$$\begin{aligned} {}_x\overline{R}^{\alpha} u(x,t) &= -(-\frac{d^2}{dx^2})^{\alpha/2} u(x,t) \\ &= \frac{-\sec(\alpha\pi/2)}{\Gamma(2-\alpha)} \left(\frac{1}{2}\frac{d^2}{d\theta^2} \int_{X_a}^{\theta} \frac{u(\xi,t)d\xi}{(\theta-\xi)^{\alpha-1}} + \frac{1}{2}\frac{d^2}{d\theta^2} \int_{\theta}^{X_b} \frac{u(\xi,t)d\xi}{(\xi-\theta)^{\alpha-1}} \right)_{\theta=x}. \end{aligned}$$

We can naturally expand this definition to a variable-order Riesz fractional derivative as

$$\begin{aligned} {}_x\overline{R}^{\alpha(x,t)} u(x,t) &= -(-\frac{d^2}{dx^2})^{\alpha(x,t)/2} u(x,t) \\ &= \frac{-\sec(\alpha(x,t)\pi/2)}{\Gamma(2-\alpha(x,t))} \left(\frac{1}{2}\frac{d^2}{d\theta^2} \int_{X_a}^{\theta} \frac{u(\xi,t)d\xi}{(\theta-\xi)^{\alpha(x,t)-1}} + \frac{1}{2}\frac{d^2}{d\theta^2} \int_{\theta}^{X_b} \frac{u(\xi,t)d\xi}{(\xi-\theta)^{\alpha(x,t)-1}} \right)_{\theta=x}. \end{aligned}$$

Replacing the two coefficients $\frac{1}{2}$ in the above definition by ρ and σ, we obtain the generalized Riesz fractional derivative of variable order

$$\begin{aligned} &{}_xR^{\alpha(x,t)} u(x,t) \\ &= \frac{-\sec(\alpha(x,t)\pi/2)}{\Gamma(2-\alpha(x,t))} \left(\rho\frac{d^2}{d\theta^2} \int_{X_a}^{\theta} \frac{u(\xi,t)d\xi}{(\theta-\xi)^{\alpha(x,t)-1}} + \sigma\frac{d^2}{d\theta^2} \int_{\theta}^{X_b} \frac{u(\xi,t)d\xi}{(\xi-\theta)^{\alpha(x,t)-1}} \right)_{\theta=x}, \end{aligned}$$

where $\rho \geq 0, \sigma \geq 0, \rho + \sigma = 1$. Clearly, when $\rho = 0.5, \sigma = 0.5$ we recover the variable-order Riesz fractional derivative and when $\rho = 1, \sigma = 0$ we have the variable-order Riemann-Liouville derivative multiplied by the coefficient

$\cos(\alpha(x,t)\pi/2)$, i.e.,

$$-\cos(\alpha(x,t)\pi/2)_x R^{\alpha(x,t)} u(x,t) = \frac{1}{\Gamma(2-\alpha(x,t))}\left(\frac{d^2}{d\theta^2}\int_{X_a}^{\theta}\frac{u(\xi,t)d\xi}{(\theta-\xi)^{\alpha(x,t)-1}}\right)_{\theta=x}.$$

Note that Lorenzo [15] presented a variable-structure differential form for this:

$$D_x^{\alpha(x,t)} u(x,t) = \lim_{h\to 0} h^{-\alpha(x,t)} \sum_{j=0}^{[(x-X_a)/h]} \frac{\Gamma(j-\alpha(x,t))}{\Gamma(-\alpha(x,t))\Gamma(j+1)} u(x-jh,t).$$

Using the gamma function relationships, we have that this is equivalent to the form

$$D_x^{\alpha(x,t)} u(x,t) = \lim_{h\to 0} h^{-\alpha(x,t)} \sum_{j=0}^{[(x-X_a)/h]} (-1)^j \binom{\alpha(x,t)}{j} u(x-jh,t).$$

Using the relationship between the Grünwald-Letnikov and Riemann-Liouville fractional derivatives [20], we obtain

$$\begin{aligned}&\frac{1}{\Gamma(2-\alpha)}\left(\frac{d^2}{d\theta^2}\int_{X_a}^{\theta}\frac{u(\xi,t)d\xi}{(\theta-\xi)^{\alpha-1}}\right)_{\theta=x}\\ &= \lim_{h\to 0} h^{-\alpha} \sum_{j=0}^{[(x-X_a)/h]} (-1)^j \binom{\alpha}{j} u(x-jh,t).\end{aligned}$$

Since h is independent of x and t for fixed x, t, we obtain

$$\begin{aligned}&\frac{1}{\Gamma(2-\alpha(x,t))}\left(\frac{d^2}{d\theta^2}\int_{X_a}^{\theta}\frac{u(\xi,t)d\xi}{(\theta-\xi)^{\alpha(x,t)-1}}\right)_{\theta=x}\\ &= \lim_{h\to 0} h^{-\alpha(x,t)} \sum_{j=0}^{[(x-X_a)/h]} (-1)^j \binom{\alpha(x,t)}{j} u(x-jh,t).\end{aligned}$$

Hence, this variable-order Riemann-Liouville definition conforms with the definition presented by Lorenzo [15].

In this paper, we consider the following variable-order nonlinear fractional diffusion equation (VO-NFDE):

$$\frac{\partial u(x,t)}{\partial t} = B(x,t)_x R^{\alpha(x,t)} u(x,t) + f(u,x,t), X_a < x < X_b, \quad 0 < t \le T, \quad (1)$$

with initial condition

$$u(x,0) = \varphi(x), X_a \le x \le X_b, \quad (2)$$

and boundary conditions

$$u(X_a,t) = u_a(t) = 0, \quad 0 \le t \le T, \tag{3}$$

$$u(X_b,t) = u_b(t) = 0, \quad 0 \le t \le T, \tag{4}$$

where $1 < \underline{\alpha} \le \alpha(x,t) \le \overline{\alpha} < 2$, $B(x,t) > 0$. Let $A(x,t) = -\sec(\alpha(x,t)\pi/2) \times B(x,t)$, and $0 < \underline{a} \le A(x,t) \le \bar{a}$. We assume that the function $f(u,x,t)$ is a source/sink term that satisfies the Lipschitz condition $|f(u_1,x,t)-f(u_2,x,t)| \le L|u_1-u_2|$. The fractional derivative in Eq. (1) is the generalized Riesz fractional derivative of variable order $\alpha(x,t)$. If $f(u,x,t) = g(u)$ and $\alpha(x,t) = \alpha$, the VO-NFDE can be rewritten as an alternative fractional reaction-diffusion equation:

$$\begin{aligned}\frac{\partial u(x,t)}{\partial t} = & \frac{A(x,t)}{\Gamma(2-\alpha)} \\ & \times \left(\rho \tfrac{d^2}{d\theta^2} \int_{X_a}^{\theta} \frac{u(\xi,t)d\xi}{(\theta-\xi)^{\alpha-1}} + \sigma \tfrac{d^2}{d\theta^2} \int_{\theta}^{X_b} \frac{u(\xi,t)d\xi}{(\xi-\theta)^{\alpha-1}} \right)_{\theta=x} + g(u).\end{aligned} \tag{5}$$

This model not only captures a faster spreading rate and the power-law invasion profiles observed in many applications [1], it is strongly motivated by a generalized central limit theorem for random movements with power-law probability densities.

The structure of the paper is as follows. In Section 2, an explicit finite difference approximation (EFDA) for the VO-NFDE is proposed. In Section 3, we introduce some lemmas that will be used during the analyses presented in later sections. The stability and convergence of the EFDA are discussed in Sections 4 and 5, respectively. Finally, some numerical examples are given in Section 6, which give evidence that the above approximation provides a computationally efficient method to solve a VO-NFDE.

2 An explicit finite difference approximation for the VO-NFDE

For numerical approximations, we define $t_j = j\tau$ to be the integration time $0 \le t_j \le n\tau = T$, and $h = X/m$ the grid size in the spatial dimension, with $x_i = ih$ for $i = 0, 1, \cdots, m$. We use the shifted Grünwald formula for the $\alpha(x,t)-$fractional derivative approximation:

$$
\begin{aligned}
&\frac{1}{\Gamma(2-\alpha(x_i,t_j))}\left(\frac{d^2}{d\theta^2}\int_{X_a}^{\theta}\frac{u(\xi,t)d\xi}{(\theta-\xi)^{\alpha(x,t)-1}}\right)_{\theta=x_i}\\
\approx{}& h^{-\alpha_{i+1}^j}\sum_{k=0}^{i+1}\omega_{i+1-k}^{(\alpha_{i+1}^j)}u(x_k,t_j), \qquad (6)\\
&\frac{1}{\Gamma(2-\alpha(x_{i+1},t_j))}\left(\frac{d^2}{d\theta^2}\int_{X_a}^{\theta}\frac{u(\xi,t)d\xi}{(\theta-\xi)^{\alpha(x,t)-1}}\right)_{\theta=x_{i+1}}\\
\approx{}& h^{-\alpha_{i-1}^j}\sum_{k=i-1}^{m}\omega_{k-(i-1)}^{(\alpha_{i-1}^j)}u(x_k,t_j), \qquad (7)
\end{aligned}
$$

where $\omega_k^{\alpha_i^j}=(-1)^k\binom{\alpha_i^j}{k}$, $\alpha_i^j=\alpha(x_i,t_j)$. Substituting (7) into (1) yields

$$
\begin{aligned}
\frac{u_i^{j+1}-u_i^j}{\tau}={}&\rho a_i^j h^{-\alpha_{i+1}^j}\sum_{k=0}^{i+1}\omega_{i+1-k}^{(\alpha_{i+1}^j)}u_k^j+\sigma a_i^j h^{-\alpha_{i-1}^j}\sum_{k=i-1}^{m}\omega_{k-(i-1)}^{(\alpha_{i-1}^j)}u_k^j+f_i^j,\\
&i=1,\cdots m-1,\ j=0,\cdots n-1, \qquad (8)
\end{aligned}
$$

where $A(x_i,t_j)=a_i^j$, $f_i^j=f(u_i^j,x_i,t_j)$. Put $r_i^j=\tau/h^{\alpha_i^j}$. We then obtain the following explicit difference approximation:

$$
\begin{aligned}
u_i^{j+1}={}&u_i^j+\rho[r_{i+1}^j a_i^j\omega_{i+1}^{(\alpha_{i+1}^j)}u_0^j+\cdots+\\
&r_{i+1}^j a_i^j\omega_2^{(\alpha_{i+1}^j)}u_{i-1}^j+r_{i+1}^j a_i^j\omega_1^{(\alpha_{i+1}^j)}u_i^j+r_{i+1}^j a_i^j\omega_0^{(\alpha_{i+1}^j)}u_{i+1}^j\ ]\\
&+\sigma[r_{i-1}^j a_i^j\omega_0^{(\alpha_{i-1}^j)}u_{i-1}^j+r_{i-1}^j a_i^j\omega_1^{(\alpha_{i-1}^j)}u_i^j+r_{i-1}^j a_i^j\omega_2^{(\alpha_{i-1}^j)}u_{i+1}^j\\
&+\cdots+r_{i-1}^j a_i^j\omega_{m-(i-1)}^{(\alpha_{i-1}^j)}u_m^j\ ]+\tau f_i^j, \qquad (9)\\
&i=1,\cdots m-1,\ j=0,\cdots n-1.
\end{aligned}
$$

Let $U^{j+1}=(u_{m-1}^{j+1},u_{m-2}^{j+1},\cdots,u_1^{j+1})^T$,

$B^j=$

$$
\begin{pmatrix}
\rho r_m^j a_{m-1}^j\omega_0^{(\alpha_m^j)}u_m^j+\rho r_m^j a_{m-1}^j\omega_m^{(\alpha_m^j)}u_0^j\\
\rho r_{m-1}^j a_{m-2}^j\omega_{m-1}^{(\alpha_{m-1}^j)}u_0^j\\
\vdots\\
\rho r_3^j a_2^j\omega_3^{(\alpha_3^j)}u_0^j\\
\rho r_2^j a_1^j\omega_2^{(\alpha_2^j)}u_0^j
\end{pmatrix}
+
\begin{pmatrix}
\sigma r_{m-2}^j a_{m-1}^j\omega_2^{(\alpha_{m-2}^j)}u_m^j\\
\sigma r_{m-2}^j a_{m-1}^j\omega_2^{(\alpha_{m-2}^j)}u_m^j\\
\vdots\\
\sigma r_1^j a_2^j\omega_{m-1}^{(\alpha_1^j)}u_m^j\\
\sigma r_0^j a_1^j\omega_m^{(\alpha_0^j)}u_m^j+\sigma r_0^j a_1^j\omega_m^{(\alpha_0^j)}u_0^j
\end{pmatrix},
$$

$F^j=(\tau f(u_{m-1}^j,x_{m-1},t_j),\cdots,\tau f(u_1^j,x_1,t_j))^T$,

$$P^j = \rho \begin{pmatrix} r_m^j a_{m-1}^j \omega_1^{(\alpha_m^j)} & r_m^j a_{m-1}^j \omega_2^{(\alpha_m^j)} & r_m^j a_{m-1}^j \omega_3^{(\alpha_m^j)} & \cdots \\ r_{m-1}^j a_{m-2}^j \omega_0^{(\alpha_{m-1}^j)} & r_{m-1}^j a_{m-2}^j \omega_1^{(\alpha_{m-1}^j)} & r_{m-1}^j a_{m-2}^j \omega_2^{(\alpha_{m-1}^j)} & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ \cdots & 0 & r_2^j a_1^j \omega_0^{(\alpha_2^j)} & r_2^j a_1^j \omega_1^{(\alpha_2^j)} \end{pmatrix}$$

$$+\sigma \begin{pmatrix} r_{m-2}^j a_{m-1}^j \omega_1^{(\alpha_{m-2}^j)} & r_{m-2}^j a_{m-1}^j \omega_0^{(\alpha_{m-2}^j)} & 0 & \cdots \\ r_{m-3}^j a_{m-2}^j \omega_2^{(\alpha_{m-3}^j)} & r_{m-3}^j a_{m-2}^j \omega_1^{(\alpha_{m-3}^j)} & r_{m-3}^j a_{m-2}^j \omega_0^{(\alpha_{m-3}^j)} & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ \cdots & r_0^j a_1^j \omega_3^{(\alpha_0^j)} & r_0^j a_1^j \omega_2^{(\alpha_0^j)} & r_0^j a_1^j \omega_1^{(\alpha_0^j)} \end{pmatrix}$$

$+I_{m-1}$.

Then the EFDA (9) can be written in matrix form

$$U^{j+1} = P^j U^j + B^j + F^j. \tag{10}$$

3 Preliminaries

In this section, some lemmas are given that will be used in the approximations of variable-order derivatives.

Lemma 1 *From the discrete scheme (7), we have*

$$\begin{aligned} &\frac{1}{\Gamma(2-\alpha(x_i,t_j))}\left(\frac{d^2}{d\theta^2}\int_{X_a}^{\theta}\frac{u(\xi,t)d\xi}{(\theta-\xi)^{\alpha(x_i,t_j)-1}}\right)_{\theta=x_i} \\ &= h^{-\alpha_{i+1}^j}\sum_{k=0}^{i+1}\omega_{i+1-k}^{(\alpha_{i+1}^j)}u(x_k,t_j)+O(h) \end{aligned}$$

and

$$\begin{aligned} &\frac{1}{\Gamma(2-\alpha(x_i,t_j))}\left(\frac{d^2}{d\theta^2}\int_{\theta}^{X_b}\frac{u(\xi,t)d\xi}{(\xi-\theta)^{\alpha(x_i,t_j)-1}}\right)_{\theta=x_i} \\ &= h^{-\alpha_{i-1}^j}\sum_{k=i-1}^{m}\omega_{k-(i-1)}^{(\alpha_{i-1}^j)}u(x_k,t_j)+O(h). \end{aligned}$$

Proof. From [20], we have

$$\frac{1}{\Gamma(2-\alpha)}\left(\frac{d^2}{d\theta^2}\int_{X_a}^{\theta}\frac{u(\xi,t)d\xi}{(\theta-\xi)^{\alpha-1}}\right)_{\theta=x} = h^{-\alpha}\sum_{k=0}^{[(x-X_a)/h]}(-1)^k\binom{\alpha}{k}u(x-kh,t)+O(h).$$

Let $\frac{1}{\Gamma(2-\alpha(x,t))}\left(\frac{d^2}{d\theta^2}\int_{X_a}^{\theta}\frac{u(\xi,t)d\xi}{(\theta-\xi)^{\alpha(x,t)-1}}\right)_{\theta=x} = g(x,t)$. Then,

$$g(x,t) = h^{-\alpha(x,t)}\sum_{k=0}^{[(x-X_a)/h]}(-1)^k\binom{\alpha(x,t)}{k}u(x-kh,t)+O(h)$$

holds for any $x,t \in [X_a, X_b]\times[0,T]$. Thus

$$g(x_i,t_j) = h^{-\alpha(x_i,t_j)}\sum_{k=0}^{i}(-1)^k\binom{\alpha(x_i,t_j)}{k}u(x_{i-k},t_j)+O(h)$$

and

$$g(x_{i+1},t_j) = h^{-\alpha(x_{i+1},t_j)}\sum_{k=0}^{i+1}(-1)^k\binom{\alpha(x_{i+1},t_j)}{k}u(x_{i+1-k},t_j)+O(h),$$

i.e.,

$$\begin{aligned}&\frac{1}{\Gamma(2-\alpha(x_i,t_j))}\left(\frac{d^2}{d\theta^2}\int_{X_a}^{\theta}\frac{u(\xi,t)d\xi}{(\theta-\xi)^{\alpha(x_i,t_j)-1}}\right)_{\theta=x_i}\\ =\,& h^{-\alpha_i^j}\sum_{k=0}^{i}\omega_{i-k}^{(\alpha_i^j)}u(x_k,t_j)+O(h),\end{aligned} \tag{11}$$

and

$$\begin{aligned}&\frac{1}{\Gamma(2-\alpha(x_{i+1},t_j))}\left(\frac{d^2}{d\theta^2}\int_{X_a}^{\theta}\frac{u(\xi,t)d\xi}{(\theta-\xi)^{\alpha(x_{i+1},t_j)-1}}\right)_{\theta=x_{i+1}}\\ =\,& h^{-\alpha_{i+1}^j}\sum_{k=0}^{i+1}\omega_{i+1-k}^{(\alpha_{i+1}^j)}u(x_k,t_j)+O(h).\end{aligned} \tag{12}$$

Hence

$$\begin{aligned}
&\frac{1}{\Gamma(2-\alpha(x_i,t_j))}\left(\frac{d^2}{d\theta^2}\int_{X_a}^{\theta}\frac{u(\xi,t)d\xi}{(\theta-\xi)^{\alpha(x_i,t_j)-1}}\right)_{\theta=x_i}\\
&-h^{-\alpha_{i+1}^j}\sum_{k=0}^{i+1}\omega_{i+1-k}^{(\alpha_{i+1}^j)}u(x_k,t_j)\\
=&\frac{1}{\Gamma(2-\alpha(x_i,t_j))}\left(\frac{d^2}{d\theta^2}\int_{X_a}^{\theta}\frac{u(\xi,t)d\xi}{(\theta-\xi)^{\alpha(x_i,t_j)-1}}\right)_{\theta=x_i}\\
&-\frac{1}{\Gamma(2-\alpha(x_{i+1},t_j))}\left(\frac{d^2}{d\theta^2}\int_{X_a}^{\theta}\frac{u(\xi,t)d\xi}{(\theta-\xi)^{\alpha(x_{i+1},t_j)-1}}\right)_{\theta=x_{i+1}}+O(h).
\end{aligned}$$

Assuming that $g(x,t)$ is continuous in $[X_a,X_b]\times[0,T]$, we have

$$g(x_i,t_j)-g(x_{i+1},t_j)+O(h)\leq C\left|x_i-x_{i+1}\right|+O(h)=O(h).$$

Hence,

$$\begin{aligned}
&\frac{1}{\Gamma(2-\alpha(x_i,t_j))}\left(\frac{d^2}{d\theta^2}\int_{X_a}^{\theta}\frac{u(\xi,t)d\xi}{(\theta-\xi)^{\alpha(x_i,t_j)-1}}\right)_{\theta=x_i}\\
=&h^{-\alpha_{i+1}^j}\sum_{k=0}^{i+1}\omega_{i+1-k}^{(\alpha_{i+1}^j)}u(x_k,t_j)+O(h).
\end{aligned}\tag{13}$$

Similarly we have

$$\begin{aligned}
&\frac{1}{\Gamma(2-\alpha(x_i,t_j))}\left(\frac{d^2}{d\theta^2}\int_{\theta}^{X_b}\frac{u(\xi,t)d\xi}{(\xi-\theta)^{\alpha(x_i,t_j)-1}}\right)_{\theta=x_i}\\
=&h^{-\alpha_{i-1}^j}\sum_{k=i-1}^{m}\omega_{k-(i-1)}^{(\alpha_{i-1}^j)}u(x_k,t_j)+O(h).
\end{aligned}$$

Lemma 2 *The coefficients* $\omega_k^{(\alpha(x,t))}=(-1)^k\binom{\alpha(x,t)}{k}$, $1<\alpha(x,t)<2$, $k=0,1,\cdots$ *satisfy the conditions*
(1) $\omega_0^{(\alpha(x,t))}=1$*;*
(2) $\omega_1^{(\alpha(x,t))}<0$ *and* $\omega_k^{(\alpha(x,t))}>0$ *for any* $k>1$*.*
(3) $\omega_1^{(\alpha(x,t))}+\omega_2^{(\alpha(x,t))}+\omega_3^{(\alpha(x,t))}+\cdots=-\omega_0^{(\alpha(x,t))}$*.*

Proof. From $\omega_k^{(\alpha(x,t))}=(-1)^k\binom{\alpha(x,t)}{k}$, we have

$$\omega_0^{(\alpha(x,t))}=\binom{\alpha(x,t)}{0}=1,$$

$$\omega_1^{(\alpha(x,t))}=(-1)\binom{\alpha(x,t)}{1}=\frac{-\alpha(x,t)}{1}<0.$$

For $k\geq 2$, we note that

$$\omega_k^{(\alpha(x,t))} = (-1)^k \binom{\alpha(x,t)}{k}$$
$$= \frac{-\alpha(x,t)}{1} \cdot \frac{-(\alpha(x,t)-1)}{2} \cdot \frac{-(\alpha(x,t)-2)}{3} \cdots \frac{-(\alpha(x,t)-k+1)}{k} > 0.$$

Noting that $\omega_0^{(\alpha(x,t))} + \omega_1^{(\alpha(x,t))} z + \omega_2^{(\alpha(x,t))} z^2 + \cdots = \sum_{k=0}^{+\infty} \omega_k^{(\alpha(x,t))} z^k = (1-z)^{\alpha(x,t)}$, we have

$$\sum_{k=0}^{+\infty} \omega_k^{(\alpha(x,t))} = (1-1)^{\alpha(x,t)} = 0.$$

Hence,

$$\sum_{k=1}^{+\infty} \omega_k^{(\alpha(x,t))} = -\omega_0^{(\alpha(x,t))}.$$

4 Convergence of the EFDA

Let $\overrightarrow{U^{J+1}} = (u(x_{m-1}, t_{j+1}), u(x_{m-2}, t_{j+1}), \cdots, u(x_1, t_{j+1}))^T$ be the exact solution of the VO-TFDE at time level $t = t_{j+1}$. Define the error vector at time level $t = t_{j+1}$ as

$$\overrightarrow{e^{j+1}} = \overrightarrow{U^{J+1}} - \overrightarrow{U^{j+1}}, \tag{14}$$
$$\overrightarrow{e^0} = \overrightarrow{0}. \tag{15}$$

Theorem 3 *The EFDA* (10) *for the VO-TFDE* (1) − (4) *is convergent, and* $|u_i^j - u(x_i, t_j)| = O(\tau + h)$ *for any* i, j, *when* $\tau \le h^{\overline{\alpha}} / \left(\overline{a} \cdot \overline{\alpha}(\rho + \sigma)\right)$.

Proof. From (10), we have

$$\overrightarrow{e^{j+1}} = P^j \overrightarrow{e^j} + \overrightarrow{F_e^j} + \tau O(\tau + h),$$

where

$$\begin{aligned}\overrightarrow{F_e^j} =& (\tau f(u(x_{m-1}, t_j), x_{m-1}, t_j) - \tau f(u_{m-1}^j, x_{m-1}, t_j),\\ & \cdots, \tau f(u(x_1, t_j), x_1, t_j) - \tau f(u_1^j, x_1, t_j))^T\\ \le& (\tau L_{m-1}^j e_{m-1}^j, \cdots, \tau L_1^j e_1^j)^T\\ \le& \triangle F^j \overrightarrow{e^j},\end{aligned}$$

where $\triangle F^j = diag(\tau L_{m-1}^j, \cdot, \tau L_1^j)$. Noting that $\left|L_i^j\right| \le L$ for any i, j, we have

$$
\begin{aligned}
\left\|P^j\right\|_\infty = & \max_{1\le i\le m-1}\Big(\left|\sigma a_i^j r_{i-1}^j \omega_{m-(i-1)-1}^{(\alpha_{i-1}^j)}\right| + \cdots + \left|\sigma a_i^j r_{i-1}^j \omega_3^{(\alpha_{i-1}^j)}\right| \\
& + \left|\rho r_{i+1}^j a_i^j \omega_0^{(\alpha_{i+1}^j)} + \sigma a_i^j r_{i-1}^j \omega_2^{(\alpha_{i-1}^j)}\right| \\
& + \left|\rho r_{i+1}^j a_i^j \omega_1^{(\alpha_{i+1}^j)} + \sigma a_i^j r_{i-1}^j \omega_1^{(\alpha_{i-1}^j)} + 1\right| \\
& + \left|\rho r_{i+1}^j a_i^j \omega_2^{(\alpha_{i+1}^j)} + \sigma a_i^j r_{i-1}^j \omega_0^{(\alpha_{i-1}^j)}\right| \\
& + \left|\rho r_{i+1}^j a_i^j \omega_3^{(\alpha_{i+1}^j)}\right| + \cdots + \left|\rho r_{i+1}^j a_i^j \omega_i^{(\alpha_{i+1}^j)}\right|\Big) \\
\le & \Big(\left|\sigma a_k^j r_{k-1}^j \omega_{m-(k-1)-1}^{(\alpha_{k-1}^j)}\right| + \cdots + \left|\rho r_{k+1}^j a_k^j \omega_k^{(\alpha_{k+1}^j)}\right|\Big).
\end{aligned}
$$

For h given,we choose τ to satisfy $\rho r_{i+1}^j a_i^j \omega_1^{(\alpha_{i+1}^j)} + \sigma a_i^j r_{i-1}^j \omega_1^{(\alpha_{i-1}^j)} + 1 \ge 0$, i.e., $\tau \le \frac{-1}{\rho a_i^j \omega_1^{(\alpha_{i+1}^j)}/h^{\alpha_{i+1}^j} + \sigma a_i^j \omega_1^{(\alpha_{i-1}^j)}/h^{\alpha_{i-1}^j}}$ for any i, j. From

$$
\frac{h^{\overline{\alpha}}}{\overline{a\alpha}(\rho+\sigma)} \le \frac{-1}{\rho a_i^j \omega_1^{(\alpha_{i+1}^j)}/h^{\alpha_{i+1}^j} + \sigma a_i^j \omega_1^{(\alpha_{i-1}^j)}/h^{\alpha_{i-1}^j}},
$$

we have that $\tau \le -1/(\rho a_i^j \omega_1^{(\alpha_{i+1}^j)}/h^{\alpha_{i+1}^j} + \sigma a_i^j \omega_1^{(\alpha_{i-1}^j)}/h^{\alpha_{i-1}^j})$ for any i, j when $\tau \le h^{\overline{\alpha}}/\left(\overline{a\alpha}(\rho+\sigma)\right)$. Under this condition, $\rho r_{i+1}^j a_i^j \omega_1^{(\alpha_{i+1}^j)} + \sigma a_i^j r_{i-1}^j \omega_1^{(\alpha_{i-1}^j)} + 1 \ge 0$ for any i, j. This implies that

$$
\begin{aligned}
\left\|P^j\right\|_\infty = & 1 + \sum_{i=0}^{k+1} \rho r_{k+1}^j a_k^j \omega_i^{(\alpha_{k+1}^j)} + \sum_{i=0}^{m-(k-1)} \sigma r_{k-1}^j a_k^j \omega_i^{(\alpha_{k-1}^j)} \\
= & 1 + \rho r_{k+1}^j a_k^j \sum_{i=0}^{k+1} \omega_i^{(\alpha_{k+1}^j)} + \sigma r_{k-1}^j a_k^j \sum_{i=0}^{m-(k-1)} \omega_i^{(\alpha_{k-1}^j)} \\
= & 1 + \rho r_{k+1}^j a_k^j \sum_{i=k+2}^{\infty} \omega_i^{(\alpha_{k+1}^j)} + \sigma r_{k-1}^j a_k^j \sum_{i=m-k+2}^{\infty} \omega_i^{(\alpha_{k-1}^j)} \\
\le & 1.
\end{aligned}
$$

Hence, when $\tau \le h^{\overline{\alpha}}/\left(\overline{a}\cdot\overline{\alpha}(\rho+\sigma)\right)$,

$$
\begin{aligned}
\left\|\overrightarrow{e^{j+1}}\right\|_\infty &\le \left\|P^j + \triangle F^j\right\|_\infty \left\|\overrightarrow{e^{j}}\right\|_\infty + C\tau(\tau+h) \\
&\le (1+\tau L)\left\|\overrightarrow{e^{j}}\right\|_\infty + C\tau(\tau+h) \\
&\le (1+\tau L)\Big((1+\tau L)\left\|\overrightarrow{e^{j-1}}\right\|_\infty + C\tau(\tau+h)\Big) + C\tau(\tau+h) \\
&= (1+\tau L)^2\left\|\overrightarrow{e^{j-1}}\right\|_\infty + C\tau(\tau+h)((1+\tau L)+1) \\
&\le (1+\tau L)^3\left\|\overrightarrow{e^{j-2}}\right\|_\infty + C\tau(\tau+h)((1+\tau L)^2+(1+\tau L)+1) \\
&\le \cdots \\
&\le (1+\tau L)^{j+1}\left\|\overrightarrow{e^{0}}\right\|_\infty + C\tau(\tau+h)((1+\tau L)^j+\cdots+1) \\
&\le (1+\tau L)^{n}\left\|\overrightarrow{e^{0}}\right\|_\infty + C\tau(\tau+h)((1+\tau L)^{n-1}+\cdots+1) \\
&\le e^{LT}\left\|\overrightarrow{e^{0}}\right\|_\infty + C\tau(\tau+h)(ne^{LT}) = O(\tau+h).
\end{aligned}
$$

This completes the proof.

5 Stability of the EFDA

Let $\overrightarrow{W^{j+1}}$ and $\overrightarrow{U^{j+1}}$ be the numerical solutions with initial values given by $\overrightarrow{W^0}$ and $\overrightarrow{U^0}$, respectively.

Theorem 4 $|\overrightarrow{W^{j+1}}-\overrightarrow{U^{j+1}}| \le C\|\overrightarrow{W^0}-\overrightarrow{U^0}\|_\infty$ *for any* j*, when* $\tau \le h^{\overline{\alpha}}/\left(\overline{a}\cdot\overline{\alpha}(\rho+\sigma)\right)$.

Proof. Let $\overrightarrow{W^{j+1}} - \overrightarrow{U^{j+1}} = \overrightarrow{\varepsilon^{j+1}}$. From (10), we have $\overrightarrow{\varepsilon^{j+1}} = P^j\overrightarrow{\varepsilon^j} + \overrightarrow{F^j_\varepsilon}$, where

$$
\begin{aligned}
\overrightarrow{F^j_\varepsilon} &= (\tau f({}_*u^j_{m-1}, x_{m-1}, t_j) - \tau f(u^j_{m-1}, x_{m-1}, t_j), \\
&\quad \cdots, \tau f({}_*u^j_1, x_1, t_j) - \tau f(u^j_1, x_1, t_j))^T \\
&\triangleq (\tau L^j_{m-1}\varepsilon^j_{m-1}, \cdots, \tau L^j_1\varepsilon^j_1)^T \\
&\triangleq \triangle F^j\overrightarrow{\varepsilon^j}.
\end{aligned}
$$

From the proof of convergence in Section 4, we have $\|\triangle F^j + P^j\|_\infty \le 1+\tau L$, when $\tau \le h^{\overline{\alpha}}/\left(\overline{a}\ \overline{\alpha}(a+b)\right)$. Thus,

$$
\begin{aligned}
\left\|\overrightarrow{\varepsilon^{j+1}}\right\|_\infty &\le \left\|\triangle F^j + P^j\right\|_\infty \left\|\overrightarrow{\varepsilon^{j}}\right\|_\infty \le (1+\tau L)\left\|\overrightarrow{\varepsilon^{j}}\right\|_\infty \\
&\le \cdots \le (1+\tau L)^n\left\|\overrightarrow{\varepsilon^{0}}\right\|_\infty \le e^{LT}\left\|\overrightarrow{\varepsilon^{0}}\right\|_\infty.
\end{aligned}
$$

Therefore, for the EFDA (10), if there is a perturbation in U^0, the small change would not cause large error in the numerical solution.

We then obtain the following result:

Theorem 5 *If* $\tau \leq h^{\overline{\alpha}} / (\overline{a} \cdot \overline{\alpha}(\rho + \sigma))$*, then the EFDA (10) is stable.*

6 Numerical examples

To demonstrate the effectiveness of the EFDA for VO-NFDE, we present some numerical examples.

Example 1: Variable-order nonlinear equation with $\alpha(x,t) = 1.5 + 0.5e^{-(xt)^2-1}$. Then

$$\frac{\partial u(x,t)}{\partial t} = -0.5\cos(\alpha(x,t)\pi/2)_x R^{\alpha(x,t)} u(x,t) + f(u,x,t), \tag{16}$$

where $f(u,x,t) = \frac{1}{80}(t+1)^{-1}u - \left(\frac{\Gamma(3)x^{2-\alpha(x,t)}(t+1)}{20\Gamma(3-\alpha(x,t))} - \frac{\Gamma(4)x^{3-\alpha(x,t)}(t+1)}{160\Gamma(4-\alpha(x,t))}\right)$, $\varphi(x) = \frac{x^2(8-x)}{80}$, $X_a = 0$ and $X_b = 8$, $T = 1.0$. We take $\rho = 1$, $\sigma = 0$. The exact solution of the above equation is

$$u(x,t) = \frac{x^2(8-x)(t+1)}{80}. \tag{17}$$

A comparison of the numerical solution of the EFDA and the exact solution is listed in Table 6.1 and is shown in Figure 6.1. It can be seen that the EFDA is in excellent agreement with the exact solution and the maximum error $C(h + \tau)$.

The evolution results for the VO-NFDE using the EFDA is shown in Figure 6.2. One observes that the system exhibits diffusion behaviors that the solution continuously depends on the space and time variables.

When we take τ, h so that $\tau > h^{\overline{\alpha}} / (\overline{a}\overline{\alpha}(\rho + \sigma))$, that is, the stability condition is not satisfied, the results listed in Table 6.2 show that the numerical solution of EFDA is unstable.

Example 2: We consider a variable-order fractional reaction-diffusion equation using Fisher's growth equation and a Riesz fractional derivative of variable order $\alpha(x,t) = 1.5 + 0.5e^{-(xt)^2-1}$. In this case,

$$\frac{\partial u(x,t)}{\partial t} = D_x R^{\alpha(x,t)} u(x,t) + ru(x,t)(1 - u(x,t)/K), \tag{18}$$

Table 1
Numerical solution and error $(h = 0.08, \tau = 0.0052)$

$x(t = 0.9948)$	EFDA	Exact solution	Error
0.80	0.11490000	0.11824342	0.00334342
1.60	0.40853333	0.40945134	0.00091801
2.40	0.80430001	0.80339167	0.00090833
3.20	1.22560000	1.22390141	0.00169859
4.00	1.59583330	1.59361140	0.00222194
4.80	1.83840001	1.83575041	0.00264959
5.60	1.87670004	1.87367728	0.00302272
6.40	1.63413334	1.63079523	0.00333810
7.20	1.03410006	1.03101940	0.00308060
The maximum error is 0.00341230 in $x = 6.27, t = 1.0$			

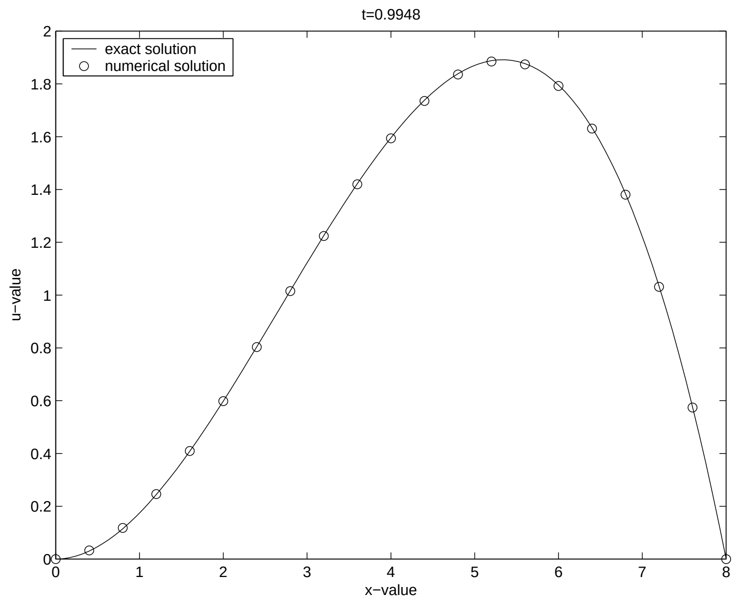

Fig. 1. A comparison of the EFDA and the exact solution in Example 1.

where $x \in [-50, 50]$, and $t \in [0, 25], \rho = \sigma = 0.5$.

Numerical simulations were performed with $K = 1$, $r = 0.25$ and $D = 0.1$ with a smooth step-like initial function $u(x, 0) = \varphi(x)$, which takes the constant value $u = 0.8$ around the origin and rapidly decays to 0 away from the origin.

Figure 6.3 shows that the use of the conventional diffusion term $\alpha = 2$ in (18) produces a rapidly decaying solution away from the origin. However, if one replaces the diffusion term by a fractional one, even with an order α close to

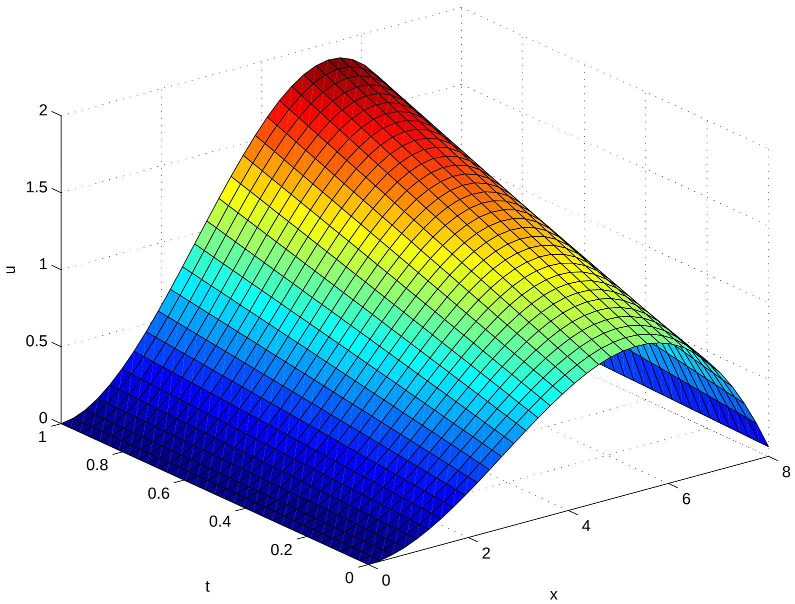

Fig. 2. The numerical solution of the EFDA in Example 1.

Table 2
Numerical solution and error ($h = 0.08, \tau = 0.03$)

j	t	$\|U^j\|_\infty$
0	0.0000E+000	0.948076800000000
3	9.9091E-002	1.03424601076074
6	0.1818	1.12028829531262
9	0.2727	1.20623855225105
12	0.3636	1.29218159285461
15	0.4545	242.619401668829
18	0.5455	100646.384788015
21	0.6363	48248992.0260945
24	0.7273	18972231041.3812
27	0.8182	8115682797939.26
30	0.9091	3.097677346662629E+015
33	1.0000	1.081812227575542E+018

2, the solution displays heavy tails and fast spreading, similar to the results reported in del Castillo-Negrete *et al.* [4] for the one-sided ordinary fractional reaction-diffusion equation. Figure 6.4 shows the approximate solution of (18) for different times. From Figure 6.4, it is seen that the system exhibits diffusion behaviors that the solution continuously depends on the space and time variables.

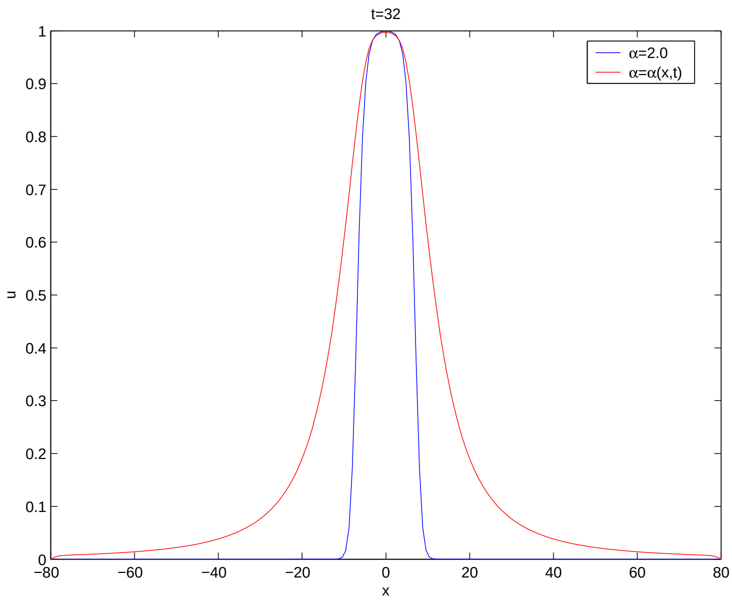

Fig. 3. Solution of the fractional Fisher equation with $\alpha = 2.0$ which shows heavier tails in the fractional case $\alpha = \alpha(x,t) < 2$.

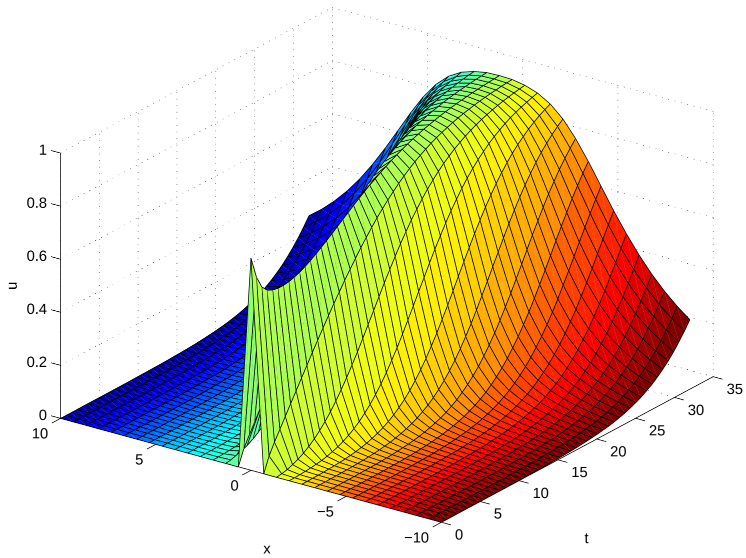

Fig. 4. The approximate solution of (18) for different times.

Example 3: We consider a variable-order nonlinear Riesz fractional diffusion equation with $\alpha(x,t) = 1.5 + 0.25\cos(x)\sin(2t)$:

$$\frac{\partial u(x,t)}{\partial t} = 0.5{}_xR^{\alpha(x,t)}u(x,t) + f(u,x,t), \tag{19}$$

where $f(u,x,t) = 0.5(u - u^2), \varphi(x) = 0.5\left(e^{-x^2} - e^{-100}\right)$, and $x \in [-10, 10]$, $T = 5$, $\rho = \sigma = 0.5$. Figure 6.5 shows the approximate solution of (19). From

Figure 6.5, it is seen that the system exhibits diffusion behaviors such that the solution continuously depends on the space and time variables.

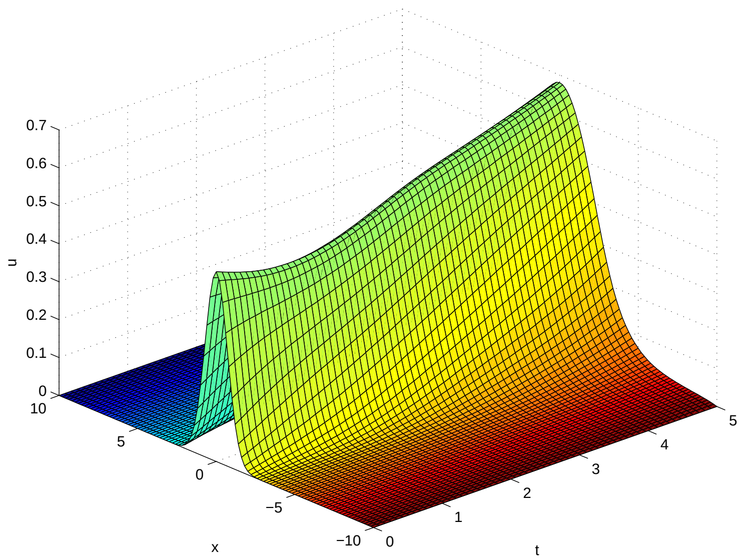

Fig. 5. The approximation solution of (19).

7 Conclusions

In this paper, a variable-order nonlinear fractional diffusion equation is investigated. A new explicit finite difference approximation for this equation has been described and demonstrated. The convergence and stability of this scheme are established. Numerical examples have been presented to demonstrate the effectiveness of the method. We believe that the proposed method and techniques can be applied to other kinds of variable-order nonlinear fractional differential equations.

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