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Numerical schemes and multivariate extrapolation of a two-dimensional anomalous sub-diffusion equation

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Abstract In this paper, a two-dimensional anomalous subdiffusion equation is considered. Two numerical schemes for solving this equation are presented. Their stability and convergence are discussed. A new multivariate extrapolation is introduced to improve the accuracy. Finally, a numerical example is given to demonstrate the effectiveness of the numerical schemes and confirm the theoretical analysis.

Keywords Anomalous dynamics; subdiffusion equation; multivariate extrapolation

1 Introduction

In recent years, fractional differential equations have featured prominently in many applications in physics ([1]-[3], [21], [24]-[27], [32], [34], [35]), chemistry ([18],[28]), and other fields of science and engineering ([4], [11],[16]). Techniques to handle these equations have been developed, for example, in ([5]-[10], [14], [15], [20], [22], [23], [29]-[31], [33]). However, up to now, numerical methods for higher-dimensional fractional partial differential equations are still limited ([12], [17], [36]). In this paper, we consider numerical methods for solving the following two-dimensional anomalous subdiffusion equation (2D-ASDE):

$$\frac{\partial u(x,y,t)}{\partial t} = {}_0D_t^{1-\gamma}\left(\kappa_1\frac{\partial^2 u(x,y,t)}{\partial x^2} + \kappa_2\frac{\partial^2 u(x,y,t)}{\partial y^2}\right) + f(x,y,t), \qquad (1)$$
$$0 < t \le T,\ 0 < x,\ y < L,$$

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with the initial-boundary conditions

$$u(x,y,0)=w(x,y),\ 0\le x,\ y\le L; \tag{2}$$

$$u(0,y,t)=\varphi_1(y,t),\ u(L,y,t)=\varphi_2(y,t),\ 0\le y\le L,\ 0\le t\le T; \tag{3}$$

$$u(x,0,t)=\psi_1(x,t),\ u(x,L,t)=\psi_2(x,t),\ 0\le x\le L,\ 0\le t\le T, \tag{4}$$

where $0<\gamma<1$. We assume that the diffusion coefficients satisfy $\kappa_1>0,\kappa_2>0$, and the operator ${}_0D_t^{1-\gamma}v(x,y,t)$ is the Riemann-Liouille fractional derivative of order $1-\gamma$ defined by

$${}_0D_t^{1-\gamma}v(x,y,t)=\frac{1}{\Gamma(\gamma)}\frac{\partial}{\partial t}\int_0^t\frac{v(x,y,\eta)}{(t-\eta)^{1-\gamma}}d\eta.$$

For the descretisation of the Riemann-Liouville fractional derivative, the Grünwald-Letnikov expansion has been exploited. This allows the use of the Riemann-Liouville definition during problem formulation, and then the use of the Grünwald-Letnikov expansion for obtaining the numerical solution [13].

In this paper, we always suppose that $u(x,y,t)\in U(R_3)$ is the exact solution for the problem (1)-(4), where

$$R_3=\{(x,y,t)|\ 0\le x,y\le L,\ 0\le t\le T\},$$

$$U(R_3)=\left\{u(x,y,t)|\ \frac{\partial^4u(x,y,t)}{\partial x^4},\frac{\partial^4u(x,y,t)}{\partial y^4},\frac{\partial^2u(x,y,t)}{\partial t^2}\in C(R_3)\right\}.$$

2 Two numerical schemes

We put $x_j=j\Delta_x, j=0,1,\ldots,M_1; y_l=l\Delta_y, l=0,1,\ldots,M_2$; and $t_k=k\Delta_t, k=0,1,\ldots,N$, respectively, where $\Delta_x=L/M_1$, $\Delta_y=L/M_2$ and $\Delta_t=T/N$. We introduce the following notations:

$$\delta_x^2w_{j,l}^n=w_{j-1,l}^n-2w_{j,l}^n+w_{j+1,l}^n;\ \delta_y^2w_{j,l}^n=w_{j,l-1}^n-2w_{j,l}^n+w_{j,l+1}^n.$$

Firstly, we present the following numerical schemes for solving the initial-boundary value problem of the 2D-ASDE (1)-(4):

2.1 An implicit numerical scheme

$$\frac{u_{j,l}^k-u_{j,l}^{k-1}}{\Delta_t}=\Delta_t^{\gamma-1}\sum_{m=0}^{k}\lambda_m\left(\kappa_1\frac{\delta_x^2u_{j,l}^{k-m}}{\Delta_x^2}+\kappa_2\frac{\delta_y^2u_{j,l}^{k-m}}{\Delta_y^2}\right)+f_{j,l}^k, \tag{5}$$

$$k=1,2,\ldots,N,\ j=1,2,\ldots,M_1-1,\ l=1,2,\ldots,M_2-1;$$

$$u_{j,l}^0=w(x_j,y_l),\ j=0,1,\ldots,M_1,\ l=0,1,\ldots,M_2; \tag{6}$$

$$u_{0,l}^k=\varphi_1(y_l,t_k),\ u_{M_1,l}^k=\varphi_2(y_l,t_k),\ l=0,1,\ldots,M_2,\ k=1,2,\ldots,N; \tag{7}$$

$$u_{j,0}^k=\psi_1(x_j,t_k),\ u_{j,M_2}^k=\psi_2(x_j,t_k),\ j=0,1,\ldots,M_1,\ k=1,2,\ldots,N, \tag{8}$$

where

$$\lambda_m=(-1)^m\begin{pmatrix}1-\gamma\\ m\end{pmatrix},\ m=0,1,\ldots,k;\ f_{j,l}^k\equiv f(x_j,y_l,t_k).$$

2.2 An explicit numerical scheme

$$\frac{u_{j,l}^{k+1}-u_{j,l}^{k}}{\Delta_t}=\Delta_t^{\gamma-1}\sum_{m=0}^{k}\lambda_m\left(\kappa_1\frac{\delta_x^2 u_{j,l}^{k-m}}{\Delta_x^2}+\kappa_2\frac{\delta_y^2 u_{j,l}^{k-m}}{\Delta_y^2}\right)+f_{j,l}^{k}, \tag{9}$$
$$k=0,1,\ldots,N-1,\ j=1,2,\ldots,M_1-1,\ l=1,2,\ldots,M_2-1;$$

$$u_{j,l}^{0}=w(x_j,y_l),\ j=0,1,\ldots,M_1,\ l=0,1,\ldots,M_2; \tag{10}$$

$$u_{0,l}^{k}=\varphi_1(y_l,t_k),\ u_{M_1,l}^{k}=\varphi_2(y_l,t_k),\ l=0,1,\ldots,M_2,\ k=1,2,\ldots,N; \tag{11}$$

$$u_{j,0}^{k}=\psi_1(x_j,t_k),\ u_{j,M_2}^{k}=\psi_2(x_j,t_k),\ j=0,1,\ldots,M_1,\ k=1,2,\ldots,N. \tag{12}$$

In the scheme (5)-(8) the density u takes the values at the time steps $k,(k=1,2,\cdots,N)$, while in the scheme (9)-(12) the density u takes the values at the time steps $k+1,(k=0,1,\cdots,N-1)$.

3 Convergence of the numerical schemes

3.1 Convergence of the numerical scheme (5)-(8)

Lemma 1 *For arbitrary $\gamma, 0<\gamma<1$, it holds that [6]*
$\Delta_t^{\gamma-1}\sum\limits_{m=0}^{k}\lambda_m=\frac{1}{\Gamma(\gamma)}+O\left(\Delta_t\right).$

We define

$$R_{j,l}^{k}=\frac{\tilde{u}_{j,l}^{k}-\tilde{u}_{j,l}^{k-1}}{\Delta_t}-\Delta_t^{\gamma-1}\sum_{m=0}^{k}\lambda_m\left(\kappa_1\frac{\delta_x^2\tilde{u}_{j,l}^{k-m}}{\Delta_x^2}+\kappa_2\frac{\delta_y^2\tilde{u}_{j,l}^{k-m}}{\Delta_y^2}\right)$$
$$-f(x_j,y_l,t_k), \tag{13}$$

$$k=1,2,\ldots,N,\ j=1,2,\ldots,M_1-1,\ l=1,2,\ldots,M_2-1,$$

where $\tilde{u}_{j,l}^{n}\equiv u(x_j,y_l,t_n)$. Since $u(x,y,t)\in U(R_3)$, then

$$\frac{\delta_x^2\tilde{u}_{j,l}^{k-m}}{\Delta_x^2}=\frac{\partial^2 u(x_j,y_l,t_{k-m})}{\partial x^2}+O(\Delta_x^2),\ m=0,1,\ldots,k; \tag{14}$$

$$\frac{\delta_y^2\tilde{u}_{j,l}^{k-m}}{\Delta_y^2}=\frac{\partial^2 u(x_j,y_l,t_{k-m})}{\partial y^2}+O(\Delta_y^2),\ m=0,1,\ldots,k. \tag{15}$$

From [6], we know that

$${}_0D_t^{1-\gamma}g(t)|_{t=t_k}=\Delta_t^{\gamma-1}\sum_{m=0}^{k}\lambda_m g(t_k-m\Delta_t)+O(\Delta_t). \tag{16}$$

From Lemma 1, we obtain

$$
\begin{aligned}
&\Delta_t^{\gamma-1}\sum_{m=0}^{k}\lambda_m\left(\kappa_1\frac{\delta_x^2\tilde{u}_{j,l}^{k-m}}{\Delta_x^2}+\kappa_2\frac{\delta_y^2\tilde{u}_{j,l}^{k-m}}{\Delta_y^2}\right)\\
&=\Delta_t^{\gamma-1}\sum_{m=0}^{k}\lambda_m\left(\kappa_1\frac{\partial^2u(x_j,y_l,t_{k-m})}{\partial x^2}+\kappa_2\frac{\partial^2u(x_j,y_l,t_{k-m})}{\partial y^2}+O(\Delta_x^2+\Delta_y^2)\right)\\
&=\Delta_t^{\gamma-1}\sum_{m=0}^{k}\lambda_m\left(\kappa_1\frac{\partial^2u(x_j,y_l,t_{k-m})}{\partial x^2}+\kappa_2\frac{\partial^2u(x_j,y_l,t_{k-m})}{\partial y^2}\right)\\
&\quad+O(\Delta_x^2+\Delta_y^2)\Delta_t^{\gamma-1}\sum_{m=0}^{k}\lambda_m\\
&={}_0D_t^{1-\gamma}\left(\kappa_1\frac{\partial^2u(x_j,y_l,t_k)}{\partial x^2}+\kappa_2\frac{\partial^2u(x_j,y_l,t_k)}{\partial y^2}\right)+O(\Delta_t)\\
&\quad+O(\Delta_x^2+\Delta_y^2)\left(\frac{1}{\Gamma(\gamma)}+O(\Delta_t)\right)\\
&={}_0D_t^{1-\gamma}\left(\kappa_1\frac{\partial^2u(x_j,y_l,t_k)}{\partial x^2}+\kappa_2\frac{\partial^2u(x_j,y_l,t_k)}{\partial y^2}\right)+O(\Delta_t+\Delta_x^2+\Delta_y^2).
\end{aligned}
$$

In addition, since $u(x,y,t)\in U(R_3)$, then

$$\frac{\tilde{u}_{j,l}^{k}-\tilde{u}_{j,l}^{k-1}}{\Delta_t}=\frac{\partial u(x_j,y_l,t_k)}{\partial t}+O(\Delta_t).$$

According to the above analysis, we have

$$R_{j,l}^{k}=O(\Delta_t+\Delta_x^2+\Delta_y^2),$$
$$k=1,2,\ldots,N,\ j=1,2,\ldots,M_1-1,\ l=1,2,\ldots,M_2-1.$$

Since k,j,l are finite, there is a positive constant C_1 for all k,j,l such that

$$|R_{j,l}^{k}|\le C_1(\Delta_t+\Delta_x^2+\Delta_y^2). \tag{17}$$

Let

$$E_{j,l}^{k}=u(x_j,y_l,t_k)-u_{j,l}^{k},$$
$$k=1,2,\ldots,N,\ j=1,2,\ldots,M_1-1,\ l=1,2,\ldots,M_2-1$$

and

$$
\begin{aligned}
E^k&=\left[E_{1,1}^k,E_{1,2}^k,\ldots,E_{1,M_2-1}^k,\ldots,E_{M_1-1,1}^k,E_{M_1-1,2}^k,\ldots,E_{M_1-1,M_2-1}^k\right]^T,\\
R^k&=\left[R_{1,1}^k,R_{1,2}^k,\ldots,R_{1,M_2-1}^k,\ldots,R_{M_1-1,1}^k,R_{M_1-1,2}^k,\ldots,R_{M_1-1,M_2-1}^k\right]^T.
\end{aligned}
$$

From (13), we have

$$
\begin{aligned}
\tilde{u}_{j,l}^{k}&=\tilde{u}_{j,l}^{k-1}+\mu_1\sum_{m=0}^{k}\lambda_m\delta_x^2\tilde{u}_{j,l}^{k-m}+\mu_2\sum_{m=0}^{k}\lambda_m\delta_y^2\tilde{u}_{j,l}^{k-m}+\Delta_t f(x_j,y_l,t_k)\\
&\quad+\Delta_t R_{j,l}^{k}, k=1,2,\ldots,N, j=1,2,\ldots,M_1-1, l=1,2,\ldots,M_2-1,
\end{aligned}
$$

where $\mu_1=\kappa_1\frac{\Delta_t^\gamma}{\Delta_x^2}$ and $\mu_2=\kappa_2\frac{\Delta_t^\gamma}{\Delta_y^2}$. Subtracting (5) from the above equation gives

$$E_{j,l}^{k}=E_{j,l}^{k-1}+\mu_1\sum_{m=0}^{k}\lambda_m\delta_x^2E_{j,l}^{k-m}+\mu_2\sum_{m=0}^{k}\lambda_m\delta_y^2E_{j,l}^{k-m}+\Delta_t R_{j,l}^{k}, \tag{18}$$

$$k = 1, 2, \ldots, N,\ j = 1, 2, \ldots, M_1 - 1,\ l = 1, 2, \ldots, M_2 - 1.$$

For $k = 0, 1, \ldots, N$, we define the following grid functions:

$$E^k(x, y) = \begin{cases} E^k_{j,l}, \texttt{ when } (x, y) \in I_1; \\ 0, \texttt{ when } (x, y) \in I_2, \end{cases}$$

and

$$R^k(x, y) = \begin{cases} R^k_{j,l}, \texttt{ when } (x, y) \in I_1; \\ 0, \texttt{ when } (x, y) \in I_2, \end{cases}$$

where

$$I_1 = \left\{(x, y)|\ x_{j-\frac{1}{2}} < x \le x_{j+\frac{1}{2}},\ y_{l-\frac{1}{2}} < y \le y_{l+\frac{1}{2}}\right\},$$

$$I_2 = \left\{(x, y)|\ (x, y) \in I_2^{(x)},\ \texttt{or } (x, y) \in I_2^{(y)}\right\},$$

$$I_2^{(x)} = \left\{(x, y)|\ 0 \le x \le \frac{\Delta_x}{2},\ \texttt{or } L - \frac{\Delta_x}{2} < x \le L\right\},$$

$$I_2^{(y)} = \left\{(x, y)|\ 0 \le y \le \frac{\Delta_y}{2},\ \texttt{or } L - \frac{\Delta_y}{2} < y \le L\right\},$$

in which, $j = 1, 2, \ldots, M_1 - 1,\ l = 1, 2 \ldots, M_2 - 1$. Then $E^k(x, y)$ and $R^k(x, y)$ can be expanded in Fourier series as

$$E^k(x, y) = \sum_{l_1, l_2 = -\infty}^{\infty} \alpha_k(l_1, l_2) e^{i2\pi(l_1 x/L + l_2 y/L)},\ k = 0, 1, \ldots, N,$$

and

$$R^k(x, y) = \sum_{l_1, l_2 = -\infty}^{\infty} \beta_k(l_1, l_2) e^{i2\pi(l_1 x/L + l_2 y/L)},\ k = 0, 1, \ldots, N,$$

respectively, where

$$\alpha_k(l_1, l_2) = \frac{1}{L} \int\limits_{0 \le x, y \le L} E^k(x, y) e^{-i2\pi(l_1 x/L + l_2 y/L)} dxdy,$$

$$\beta_k(l_1, l_2) = \frac{1}{L} \int\limits_{0 \le x, y \le L} R^k(x, y) e^{-i2\pi(l_1 x/L + l_2 y/L)} dxdy.$$

Applying the Parseval identity, we get

$$\|E^k\|_2 \equiv \left(\sum_{j=1}^{M_1-1} \sum_{l=1}^{M_2-1} \Delta_x \Delta_y |E^k_{j,l}|^2 \right)^{\frac{1}{2}} = \left(\sum_{l_1, l_2 = -\infty}^{\infty} |\alpha_k(l_1, l_2)|^2 \right)^{\frac{1}{2}}, \quad (19)$$

$$k = 0, 1, \ldots, N$$

and

$$\|R^k\|_2 \equiv \left(\sum_{j=1}^{M_1-1} \sum_{l=1}^{M_2-1} \Delta_x \Delta_y |R^k_{j,l}|^2 \right)^{\frac{1}{2}} = \left(\sum_{l_1, l_2 = -\infty}^{\infty} |\beta_k(l_1, l_2)|^2 \right)^{\frac{1}{2}}, \quad (20)$$

$$k = 0, 1, \ldots, N.$$

We now suppose that

$$E_{j,l}^k = \alpha_k e^{i(\sigma_1 j \Delta_x + \sigma_2 l \Delta_y)}, \tag{21}$$

and

$$R_{j,l}^k = \beta_k e^{i(\sigma_1 j \Delta_x + \sigma_2 l \Delta_y)}, \tag{22}$$

respectively, where $\sigma_1 = 2\pi l_1/L, \sigma_2 = 2\pi l_2/L$. Substituting the above expressions into (18) gives

$$\alpha_k = \alpha_{k-1} - \mu \sum_{m=0}^{k} \lambda_m \alpha_{k-m} + \Delta_t \beta_k,\ k = 1, 2, \ldots, N, \tag{23}$$

where $\mu = 4\left(\mu_1 \sin^2 \frac{\sigma_1 \Delta_x}{2} + \mu_2 \sin^2 \frac{\sigma_2 \Delta_y}{2}\right) \geq 0$.

Lemma 2 *The coefficients $\lambda_m\,(m = 0, 1, \ldots)$ possess the following properties [15]:*
(1) $\lambda_0 = 1;\ \lambda_1 = \gamma - 1;\ \lambda_m < 0,\ m = 1, 2, \ldots;$
(2) $\sum\limits_{m=0}^{\infty} \lambda_m = 0$ and $\forall n \in N,\ -\sum\limits_{m=1}^{n} \lambda_m < 1.$

Using Lemma 2, equation (23) can be written as

$$\alpha_k = \frac{1 + (1-\gamma)\mu}{1+\mu} \alpha_{k-1} - \frac{\mu}{1+\mu} \sum_{m=2}^{k} \lambda_m \alpha_{k-m} + \frac{\Delta_t \beta_k}{1+\mu}, k = 1, 2, \ldots, N. \tag{24}$$

Proposition 1 *Let $\alpha_k\,(k = 1, 2, \ldots, N)$ be the solution of equation (24). Then there exists a positive constant C_2 such that*

$$|\alpha_k| \leq C_2 k \Delta_t |\beta_1|,\ k = 1, 2, \ldots, N.$$

Proof From $E^0 = 0$ and (19), we have

$$\alpha_0 \equiv \alpha_0(l_1, l_2) = 0. \tag{25}$$

By the convergence of the series on the right-hand side of (20), there is a positive constant C_2 such that

$$|\beta_k| \equiv |\beta_k(l_1, l_2)| \leq C_2 |\beta_1| \equiv C_2 |\beta_1(l_1, l_2)|,\ k = 1, 2, \ldots, N. \tag{26}$$

When $k = 1$, (24) and (25) give

$$\alpha_1 = \frac{\Delta_t \beta_1}{1+\mu}.$$

Since $\mu \geq 0$, (26) leads to

$$|\alpha_1| \leq \Delta_t |\beta_1| \leq C_2 \Delta_t |\beta_1|.$$

We assume that

$$|\alpha_n| \leq C_2 n \Delta_t |\beta_1|,\ n = 1, 2, \ldots, k-1.$$

Noticing that $0 < \gamma < 1$ and $\mu \geq 0$, from Lemma 2, (24) and (26), we obtain

$$
\begin{aligned}
|\alpha_k| &\leq \tfrac{1+(1-\gamma)\mu}{1+\mu}|\alpha_{k-1}| + \tfrac{\mu}{1+\mu}\sum_{m=2}^{k}|\lambda_m||\alpha_{k-m}| + \tfrac{\Delta_t|\beta_k|}{1+\mu} \\
&\leq \left[\tfrac{1+(1-\gamma)\mu}{1+\mu}(k-1) + \tfrac{\mu}{1+\mu}\sum_{m=2}^{k}|\lambda_m|(k-m) + \tfrac{1}{1+\mu}\right] C_2\Delta_t|\beta_1| \\
&\leq \left[\tfrac{1+(1-\gamma)\mu}{1+\mu}(k-1) + \tfrac{\mu}{1+\mu}(k-1)\sum_{m=2}^{k}|\lambda_m| + \tfrac{1}{1+\mu}\right] C_2\Delta_t|\beta_1| \\
&= \left\{\tfrac{1+(1-\gamma)\mu}{1+\mu}(k-1) + \tfrac{\mu}{1+\mu}(k-1)\left[-\sum_{m=1}^{k}\lambda_m - (1-\gamma)\right] + \tfrac{1}{1+\mu}\right\} C_2\Delta_t|\beta_1| \\
&\leq \left\{\tfrac{1+(1-\gamma)\mu}{1+\mu}(k-1) + \tfrac{\mu}{1+\mu}(k-1)[1-(1-\gamma)] + \tfrac{1}{1+\mu}\right\} C_2\Delta_t|\eta_1| \\
&= \left[(k-1) + \tfrac{1}{1+\mu}\right] C_2\Delta_t|\beta_1| \\
&\leq C_2 k\Delta_t|\beta_1|.
\end{aligned}
$$

This completes the proof of Proposition 1 via mathematical induction. □

Theorem 1 *Let $u(x, y, t) \in U(R_3)$ be the exact solution for the problem (1)-(4). Then the numerical scheme (5)-(8) is L_2-convergent with convergence order $O(\Delta_t + \Delta_x^2 + \Delta_y^2)$.*

Proof From (17) and the first equality in (20), we have

$$\|R^k\|_2 \leq \sqrt{M_1\Delta_x}\sqrt{M_2\Delta_y}C_1(\Delta_t + \Delta_x^2 + \Delta_y^2) = C_1L(\Delta_t + \Delta_x^2 + \Delta_y^2), \tag{27}$$

$$k = 1, 2, \ldots, N.$$

In view of Proposition 1, (19), (20) and (27), we get

$$\|E^k\|_2 \leq C_2k\Delta_t\|R^1\|_2 \leq C_1C_2k\Delta_tL(\Delta_t + \Delta_x^2 + \Delta_y^2)$$

because $k\Delta_t \leq T$. Hence

$$\|E^k\|_2 \leq C(\Delta_t + \Delta_x^2 + \Delta_y^2),$$

where $C = C_1C_2TL$. □

3.2 Convergence of the numerical scheme (9)-(12)

We define

$$
\begin{aligned}
R_{j,l}^{k+1} = &\frac{\tilde{u}_{j,l}^{k+1} - \tilde{u}_{j,l}^{k}}{\Delta_t} - \Delta_t^{\gamma-1}\sum_{m=0}^{k}\lambda_m\left(\kappa_1\frac{\delta_x^2\tilde{u}_{j,l}^{k-m}}{\Delta_x^2} + \kappa_2\frac{\delta_y^2\tilde{u}_{j,l}^{k-m}}{\Delta_y^2}\right) \\
&-f(x_j, y_l, t_k),
\end{aligned} \tag{28}
$$

$$k = 0, 1, \ldots, N-1,\ j = 1, 2, \ldots, M_1 - 1,\ l = 1, 2, \ldots, M_2 - 1,$$

where $\tilde{u}_{j,l}^n = u(x_j, y_l, t_n)$. Since $u(x, y, t) \in U(R_3)$, then

$$\frac{\tilde{u}_{j,l}^{k+1} - \tilde{u}_{j,l}^{k}}{\Delta_t} = \frac{\partial u(x_j, y_l, t_k)}{\partial t} + O(\Delta_t).$$

Again from (14)-(16), it can be derived that

$$|R_{j,l}^{k+1}| \le C_3(\Delta_t + \Delta_x^2 + \Delta_y^2), \tag{29}$$

$$k = 0, 1, \dots, N-1,\ j = 1, 2, \dots, M_1 - 1,\ l = 1, 2, \dots, M_2 - 1,$$

where C_3 is a positive constant. From (28), we have

$$\tilde{u}_{j,l}^{k+1} = \tilde{u}_{j,l}^{k} + \mu_1 \sum_{m=0}^{k} \lambda_m \delta_x^2 \tilde{u}_{j,l}^{k-m} + \mu_2 \sum_{m=0}^{k} \lambda_m \delta_y^2 \tilde{u}_{j,l}^{k-m} + \Delta_t f(x_j, y_l, t_k)$$
$$+\Delta_t R_{j,l}^{k+1},$$

$$k = 0, 1, \dots, N-1,\ j = 1, 2, \dots, M_1 - 1,\ l = 1, 2, \dots, M_2 - 1,$$

where μ_1 and μ_2 are as defined in Section 3.1.

Subtracting (9) from the above equation, we obtain

$$E_{j,l}^{k+1} = E_{j,l}^{k} + \mu_1 \sum_{m=0}^{k} \lambda_m \delta_x^2 E_{j,l}^{k-m} + \mu_2 \sum_{m=0}^{k} \lambda_m \delta_y^2 E_{j,l}^{k-m} + \Delta_t R_{j,l}^{k+1}, \tag{30}$$

$$k = 0, 1, \dots, N-1,\ j = 1, 2, \dots, M_1 - 1,\ l = 1, 2, \dots, M_2 - 1.$$

It can be seen that (19), (20), (25) and (26) are still valid for the numerical scheme (9)-(12).

Similarly, we also assume that $E_{j,l}^{k}$ and $R_{j,l}^{k}$ are of the form (21) and (22), respectively. Substituting (21) and (22) into (30) gives

$$\alpha_{k+1} = \alpha_k - \mu \sum_{m=0}^{k} \lambda_m \alpha_{k-m} + \Delta_t \beta_{k+1},\ k = 0, 1, \dots, N-1, \tag{31}$$

where μ is as defined in Section 3.1.

Using Lemma 2, equation (31) can be rewritten as

$$\alpha_{k+1} = (1-\mu)\alpha_k - \mu \sum_{m=1}^{k} \lambda_m \alpha_{k-m} + \Delta_t \beta_{k+1},\ k = 0, 1, \dots, N-1. \tag{32}$$

Proposition 2 *Let $\alpha_{k+1}(k = 0, 1, \dots, N-1)$ be the solutions of equations (32). If $\mu_1 + \mu_2 \le \frac{1}{4}$, then there is a positive constant C_2 such that*

$$|\alpha_{k+1}| \le C_2 k \Delta_t |\beta_1|,\ k = 0, 1, \dots, N-1.$$

Proof When $k = 0$, from (32) and (25), we have

$$\alpha_1 = \Delta_t \beta_1.$$

By (26) we have

$$|\alpha_1| = \Delta_t |\beta_1| \le C_2 \Delta_t |\beta_1|.$$

Suppose that

$$|\alpha_n| \le C_2 n \Delta_t |\beta_1|,\ \ n = 1, 2, \dots, k.$$

Noting that $\mu_1 + \mu_2 \leq \frac{1}{4}$ yields $0 \leq \mu \leq 1$, from Lemma 2 and (32), we have

$$\begin{aligned}
|\alpha_{k+1}| &\leq (1-\mu)|\alpha_k| + \mu \sum_{m=1}^{k} |\lambda_m||\alpha_{k-m}| + \Delta_t|\beta_{k+1}| \\
&\leq \left[(1-\mu)k + \mu \sum_{m=1}^{k} |\lambda_m|(k-m) + 1\right] C_2\Delta_t|\beta_1| \\
&\leq \left[(1-\mu)k + \mu k \sum_{m=1}^{k} |\lambda_m| + 1\right] C_2\Delta_t|\beta_1| \\
&= \left[(1-\mu)k + \mu k \left(-\sum_{m=1}^{k} \lambda_m\right) + 1\right] C_2\Delta_t|\beta_1| \\
&\leq [(1-\mu)k + \mu k + 1] C_2\Delta_t|\beta_1| \\
&= C_2(k+1)\Delta_t|\beta_1|.
\end{aligned}$$

This proves Proposition 2 via mathematical induction. □

From (29) and the first equality in (20) we have

$$\|R^{k+1}\|_2 \leq \sqrt{M_1\Delta_x}\sqrt{M_2\Delta_y}C_3(\Delta_t + \Delta_x^2 + \Delta_y^2) = C_3L(\Delta_t + \Delta_x^2 + \Delta_y^2), \qquad (33)$$

$$k = 0, 1, \ldots, N-1.$$

According to Proposition 2, (19), (20) and (33), we have

$$\|E^{k+1}\|_2 \leq C_2(k+1)\Delta_t\|R^1\|_2 \leq C_2C_3(k+1)\Delta_tL(\Delta_t + \Delta_x^2 + \Delta_y^2),$$

$$k = 0, 1, \ldots, N-1.$$

As $(k+1)\Delta_t \leq T$, then

$$\|E^{k+1}\|_2 \leq \tilde{C}(\Delta_t + \Delta_x^2 + \Delta_y^2), \; k = 0, 1, \ldots, N-1,$$

where $\tilde{C} = C_2C_3TL$. This leads to

Theorem 2 *Let $u(x, y, t) \in U(R_3)$ be the exact solution for the problem (1)-(4). If $\mu_1 + \mu_2 \leq \frac{1}{4}$, then the numerical scheme (9)-(12) is L_2-convergent with convergence order $O(\Delta_t + \Delta_x^2 + \Delta_y^2)$.*

4 Stability of the numerical schemes

4.1 Stability of the numerical scheme (5)-(8)

We rewrite (5) as

$$u_{j,l}^k = u_{j,l}^{k-1} + \mu_1 \sum_{m=0}^{k} \lambda_m \delta_x^2 u_{j,l}^{k-m} + \mu_2 \sum_{m=0}^{k} \lambda_m \delta_y^2 u_{j,l}^{k-m} + \tau f_{j,l}^k, \qquad (34)$$

$$k = 1, 2, \ldots, N, \; j = 1, 2, \ldots, M_1 - 1, \; l = 1, 2, \ldots, M_2 - 1,$$

where μ_1 and μ_2 are as defined in Section 3.1.

Let $U_{j,l}^k$ be an approximation of the solution for the numerical scheme (5)-(8), and let

$$\rho_{j,l}^k = u_{j,l}^k - U_{j,l}^k, \; k = 0, 1, \ldots, N, \; j = 1, 2, \ldots, M_1 - 1, \; l = 1, 2, \ldots, M_2 - 1,$$

and

$$\rho^k = \left[\rho^k_{1,1}, \rho^k_{1,2}, \dots, \rho^k_{1,M_2-1}, \dots, \rho^k_{M_1-1,1}, \rho^k_{M_1-1,2}, \dots, \rho^k_{M_1-1,M_2-1}\right]^T,$$

respectively. We obtain the following roundoff error equation:

$$\rho^k_{j,l} = \rho^{k-1}_{j,l} + \mu_1 \sum_{m=0}^{k} \lambda_m \delta_x^2 \rho^{k-m}_{j,l} + \mu_2 \sum_{m=0}^{k} \lambda_m \delta_y^2 \rho^{k-m}_{j,l}, \tag{35}$$

$$k = 1, 2, \dots, N,\ j = 1, 2, \dots, M_1 - 1,\ l = 1, 2, \dots, M_2 - 1.$$

For $k = 0, 1, \dots, N$, we define the following grid function:

$$\rho^k(x, y) = \begin{cases} \rho^k_{j,l}, \texttt{ when } (x, y) \in I_1; \\ 0, \texttt{ when } (x, y) \in I_2, \end{cases}$$

where $j = 1, 2, \dots, M_1 - 1,\ l = 1, 2 \dots, M_2 - 1$, whereas I_1 and I_2 are as defined in Section 3.1. Then $\rho^k(x, y)$ can be expressed as a Fourier series:

$$\rho^k(x, y) = \sum_{l_1, l_2 = -\infty}^{\infty} \xi_k(l_1, l_2) e^{i2\pi(l_1 x/L + l_2 y/L)},\ k = 0, 1, \dots, N,$$

where $\xi_k(l_1, l_2) = \frac{1}{L} \int\limits_{0 \le x, y \le L} \rho^k(x, y) e^{-i2\pi(l_1 x/L + l_2 y/L)} dxdy$. The Parseval identity yields

$$\|\rho^k\|_2 \equiv \left(\sum_{j=1}^{M_1-1} \sum_{l=1}^{M_2-1} \Delta_x \Delta_y |\rho^k_{j,l}|^2 \right)^{\frac{1}{2}} = \left(\sum_{l_1, l_2 = -\infty}^{\infty} |\xi_k(l_1, l_2)|^2 \right)^{\frac{1}{2}}, \tag{36}$$

$$k = 0, 1, \dots, N.$$

We now assume that the solution of equation (35) has the following form:

$$\rho^k_{j,l} = \xi_k e^{i(\sigma_1 j \Delta_x + \sigma_2 l \Delta_y)},$$

where σ_1 and σ_2 are as defined in Section 3.1. Substituting the above expression into (35) gives

$$\xi_k = \xi_{k-1} - \mu \sum_{m=0}^{k} \lambda_m \xi_{k-m},\ k = 1, 2, \dots, N, \tag{37}$$

where $\mu = 4(\mu_1 \sin^2 \frac{\sigma_1 \Delta_x}{2} + \mu_2 \sin^2 \frac{\sigma_2 \Delta_y}{2}) \ge 0$. Using Lemma 2, equation (37) can be written as

$$\xi_k = \frac{1 + (1 - \gamma)\mu}{1 + \mu} \xi_{k-1} - \frac{\mu}{1 + \mu} \sum_{m=2}^{k} \lambda_m \xi_{k-m},\ k = 1, 2, \dots, N. \tag{38}$$

Proposition 3 *Let $\xi_k (k = 1, 2, \dots, N)$ be the solution of equation (38), then*

$$|\xi_k| \le |\xi_0|,\ k = 1, 2, \dots, N.$$

Proof For $k = 1$, from (38), we have

$$\xi_1 = \frac{1 + (1-\gamma)\mu}{1+\mu}\xi_0.$$

Noting that $0 < \gamma < 1$ and $\mu \geq 0$ we get

$$|\xi_1| \leq \frac{1 + (1-\gamma)\mu}{1+\mu}|\xi_0| \leq |\xi_0|.$$

Assume that

$$|\xi_n| \leq |\xi_0|, \ n = 1, 2, \ldots, k-1.$$

Then from (38) and Lemma 2 we obtain

$$\begin{aligned}
|\xi_k| &\leq \tfrac{1+(1-\gamma)\mu}{1+\mu}|\xi_{k-1}| + \tfrac{\mu}{1+\mu}\sum_{m=2}^{k}|\lambda_m||\xi_{k-m}| \\
&\leq \left[\tfrac{1+(1-\gamma)\mu}{1+\mu} + \tfrac{\mu}{1+\mu}\sum_{m=2}^{k}|\lambda_m|\right]|\xi_0| \\
&= \left\{\tfrac{1+(1-\gamma)\mu}{1+\mu} + \tfrac{\mu}{1+\mu}\left[-\sum_{m=1}^{k}\lambda_m - (1-\gamma)\right]\right\}|\xi_0| \\
&\leq \left\{\tfrac{1+(1-\gamma)\mu}{1+\mu} + \tfrac{\mu}{1+\mu}\left[1 - (1-\gamma)\right]\right\}|\xi_0| \\
&= |\xi_0|.
\end{aligned}$$

This completes the proof of Proposition 3 by mathematical induction. □

Using Proposition 3 and (36), it can be concluded that the solution of the roundoff error equation (35) satisfies

$$\|\rho^k\|_2 \leq \|\rho^0\|_2, \ k = 1, 2, \ldots, N.$$

Then, we have the following result:

Theorem 3 *The numerical scheme (5)-(8) is unconditionally stable.*

4.2 Stability of the numerical scheme (9)-(12)

For simplification, we rewrite (9) as

$$u_{j,l}^{k+1} = u_{j,l}^{k} + \mu_1 \sum_{m=0}^{k} \lambda_m \delta_x^2 u_{j,l}^{k-m} + \mu_2 \sum_{m=0}^{k} \lambda_m \delta_y^2 u_{j,l}^{k-m} + \Delta_t f_{j,l}^{k}, \tag{39}$$

$$k = 0, 1, \ldots, N-1, \ j = 1, 2, \ldots, M_1 - 1, \ l = 1, 2, \ldots, M_2 - 1.$$

Similarly, it can be derived that the roundoff error equation of the numerical scheme (9)-(12) is given by

$$\rho_{j,l}^{k+1} = \rho_{j,l}^{k} + \mu_1 \sum_{m=0}^{k} \lambda_m \delta_x^2 \rho_{j,l}^{k-m} + \mu_2 \sum_{m=0}^{k} \lambda_m \delta_y^2 \rho_{j,l}^{k-m}, \tag{40}$$

$$k = 0, 1, \ldots, N-1, \ j = 1, 2, \ldots, M_1 - 1, \ l = 1, 2, \ldots, M_2 - 1.$$

Similar to the derivation in Section 4.1, we now also suppose that the solution of the roundoff error equation (40) has the form

$$\rho_{j,l}^k = \xi_k e^{i(\sigma_1 j \Delta_x + \sigma_2 l \Delta_y)},$$

where σ_1 and σ_2 are as defined in Section 3.1. Substituting the above expression into (40), we obtain

$$\begin{aligned} \xi_{k+1} &= \xi_k - 4\mu_1 \sin^2 \frac{\sigma_1 \Delta_x}{2} \sum_{m=0}^{k} \lambda_m \xi_{k-m} - 4\mu_2 \sin^2 \frac{\sigma_2 \Delta_y}{2} \sum_{m=0}^{k} \lambda_m \xi_{k-m} \\ &= \xi_k - \mu \sum_{m=0}^{k} \lambda_m \xi_{k-m}, \ k = 0, 1, \dots, N-1, \end{aligned} \tag{41}$$

where μ is as defined in Section 4.1. Applying Lemma 2, equation (41) can be rewritten as

$$\xi_{k+1} = (1-\mu)\xi_k - \mu \sum_{m=1}^{k} \lambda_m \xi_{k-m}, \ k = 0, 1, \dots, N-1. \tag{42}$$

Proposition 4 *Let $\xi_{k+1} (k = 0, 1, \dots, N-1)$ be the solutions of equations (42). If $\mu_1 + \mu_2 \le \frac{1}{4}$, then*

$$|\xi_{k+1}| \le |\xi_0|, \ k = 0, 1, \dots, N-1.$$

Proof When $k = 0$, from (42), we have

$$\xi_1 = (1-\mu)\xi_0.$$

Since $\mu_1 + \mu_2 \le \frac{1}{4}$ yields $0 \le \mu \le 1$, then

$$|\xi_1| \le (1-\mu)|\xi_0| \le |\xi_0|.$$

Suppose that

$$|\xi_n| \le |\xi_0|, \ n = 1, 2, \dots, k.$$

From (42) and Lemma 2, we obtain

$$\begin{aligned} |\xi_{k+1}| &\le (1-\mu)|\xi_k| + \mu \sum_{m=1}^{k} |\lambda_m||\xi_{k-m}| \\ &\le \left(1 - \mu + \mu \sum_{m=1}^{k} |\lambda_m|\right) |\xi_0| \\ &= \left(1 - \mu - \mu \sum_{m=1}^{k} \lambda_m\right) |\xi_0| \\ &\le (1 - \mu + \mu)|\xi_0| \\ &= |\xi_0|. \end{aligned}$$

The proof of Proposition 4 is completed via mathematical induction. □

Similar to the proof of Theorem 3, we also have the following result:

Theorem 4 *If $\mu_1 + \mu_2 \le \frac{1}{4}$, then the numerical scheme (9)-(12) is stable.*

5 Solvability of the numerical scheme (5)-(8)

Let

$$u^k = [u^k_{1,1}, u^k_{1,2}, \ldots, u^k_{1,M_2-1}, \ldots, u^k_{M_1-1,1}, u^k_{M_1-1,2}, \ldots, u^k_{M_1-1,M_2-1}]^T,$$
$$k = 0, 1, \ldots, N.$$

Theorem 5 *The numerical scheme (5)-(8) is uniquely solvable.*

Proof It can be seen that the corresponding homogeneous linear algebraic equations for the numerical scheme (5)-(8) are given as follows

$$u^k_{j,l} = u^{k-1}_{j,l} + \mu_1 \sum_{m=0}^{k} \lambda_m \delta_x^2 u^{k-m}_{j,l} + \mu_2 \sum_{m=0}^{k} \lambda_m \delta_y^2 u^{k-m}_{j,l},$$
$$k = 1, 2, \ldots, N,\ j = 1, 2, \ldots, M_1 - 1,\ l = 1, 2, \ldots, M_2 - 1,$$
$$u^0_{j,l} = 0,\ j = 0, 1, \ldots, M_1,\ l = 0, 1, \ldots, M_2;$$
$$u^k_{0,l} = u^0_{M_1,l} = 0,\ l = 1, 2, \ldots, M_2 - 1,\ k = 1, 2, \ldots, N;$$
$$u^k_{j,0} = u^k_{j,M_2} = 0,\ j = 1, 2, \ldots, M_1 - 1,\ k = 1, 2, \ldots, N.$$

Similar to the proof of Theorem 3, we also can derive that

$$\|u^k\|_2 \le \|u^0\|_2,\ \ k = 1, 2, \ldots, N.$$

Using $u^0 = 0$, we obtain

$$u^k = 0,\ k = 1, 2, \ldots, N.$$

□

6 The multivariate extrapolation and its application

Theorem 6 *Suppose that the error*

$$\begin{aligned} & g^* - g(\Delta_x, \Delta_y, \Delta_t) \\ = \ & \alpha_0 \Delta_x^{\lambda_0} + \alpha_1 \Delta_x^{\lambda_1} + \cdots + \alpha_{n_1} \Delta_x^{\lambda_{n_1}} + \beta_0 \Delta_y^{\mu_0} + \beta_1 \Delta_y^{\mu_1} + \cdots + \beta_{n_2} \Delta_y^{\mu_{n_2}} \\ & + \gamma_0 \Delta_t^{\nu_0} + \gamma_1 \Delta_t^{\nu_1} + \cdots + \gamma_{n_3} \Delta_t^{\nu_{n_3}}, \end{aligned} \tag{43}$$

where α_i, $\lambda_i (i = 0, 1, \cdots n_1)$; β_i, $\mu_i (i = 0, 1, \cdots n_2)$; γ_i, $\nu_i (i = 0, 1, \cdots n_3)$ are constants which are independent of $\Delta_x, \Delta_y, \Delta_t$, whereas

$$\alpha_i \ne 0,\ i = 0, 1, \cdots n_1;\ \beta_i \ne 0,\ i = 0, 1, \cdots n_2;\ \gamma_i \ne 0,\ i = 0, 1, \cdots n_3,$$

$$0 < \lambda_0 < \lambda_1 < \cdots < \lambda_{n_1};\ 0 < \mu_0 < \mu_1 < \cdots < \mu_{n_2};\ 0 < \nu_0 < \nu_1 < \cdots < \nu_{n_3}.$$

Then

$$\begin{aligned} & g^* - g_m(\Delta_x, \Delta_y, \Delta_t) \\ = \ & O(\Delta_x^{\lambda_m} + \Delta_y^{\mu_m} + \Delta_t^{\nu_m}),\ m = 1, 2, \cdots, \min\{n_1, n_2, n_3\}, \end{aligned} \tag{44}$$

where

$$g_0(\Delta_x, \Delta_y, \Delta_t) = g(\Delta_x, \Delta_y, \Delta_t), \tag{45}$$

$$\begin{aligned} & g_m(\Delta_x, \Delta_y, \Delta_t) \\ = & \Big[g_{m-1}(\xi\Delta_x, \eta\Delta_y, \zeta\Delta_t) - \xi^{\lambda_{m-1}} g_{m-1}(\Delta_x, \eta\Delta_y, \zeta\Delta_t) \\ & + \xi^{\lambda_{m-1}} \eta^{\mu_{m-1}} g_{m-1}(\Delta_x, \Delta_y, \zeta\Delta_t) - \eta^{\mu_{m-1}} g_{m-1}(\xi\Delta_x, \Delta_y, \zeta\Delta_t) \\ & + \xi^{\lambda_{m-1}} \zeta^{\nu_{m-1}} g_{m-1}(\Delta_x, \eta\Delta_y, \Delta_t) - \zeta^{\nu_{m-1}} g_{m-1}(\xi\Delta_x, \eta\Delta_y, \Delta_t) \\ & + \eta^{\mu_{m-1}} \zeta^{\nu_{m-1}} g_{m-1}(\xi\Delta_x, \Delta_y, \Delta_t) - \xi^{\lambda_{m-1}} \eta^{\mu_{m-1}} \zeta^{\nu_{m-1}} \\ & g_{m-1}(\Delta_x, \Delta_y, \Delta_t)\Big] / [(1-\xi^{\lambda_{m-1}})(1-\eta^{\mu_{m-1}})(1-\zeta^{\nu_{m-1}})], \end{aligned} \tag{46}$$

$$m = 1, 2, \cdots, \min\{n_1, n_2, n_3\}.$$

The parameters ξ, η *and* ζ *are selected so that* $1-\xi^{\lambda_{m-1}}, 1-\eta^{\mu_{m-1}}, 1-\zeta^{\nu_{m-1}} \neq 0$.

Proof When $m = 1$, from (43) we have

$$\begin{aligned} & g^* - g(\xi\Delta_x, \Delta_y, \Delta_t) \\ = & \alpha_0 \xi^{\lambda_0} \Delta_x^{\lambda_0} + \alpha_1 \xi^{\lambda_1} \Delta_x^{\lambda_1} + \cdots + \alpha_j \xi^{\lambda_j} \Delta_x^{\lambda_j} + \beta_0 \Delta_y^{\mu_0} + \beta_1 \Delta_y^{\mu_1} + \cdots + \beta_k \Delta_y^{\mu_k} \\ & + \gamma_0 \Delta_t^{\nu_0} + \gamma_1 \Delta_t^{\nu_1} + \cdots + \gamma_l \Delta_t^{\nu_l}. \end{aligned} \tag{47}$$

By (47) − (43) × ξ^{λ_0} we obtain

$$\begin{aligned} & g^* - \frac{g(\xi\Delta_x, \Delta_y, \Delta_t) - \xi^{\lambda_0} g(\Delta_x, \Delta_y, \Delta_t)}{1-\xi^{\lambda_0}} \\ = & O(\Delta_x^{\lambda_1}) + \beta_0 \Delta_y^{\mu_0} + \beta_1 \Delta_y^{\mu_1} + \cdots + \beta_k \Delta_y^{\mu_k} + \gamma_0 \Delta_t^{\nu_0} + \gamma_1 \Delta_t^{\nu_1} + \cdots + \gamma_l \Delta_t^{\nu_l}. \end{aligned} \tag{48}$$

Again from (48), we have

$$\begin{aligned} & g^* - \frac{g(\xi\Delta_x, \eta\Delta_y, \Delta_t) - \xi^{\lambda_0} g(\Delta_x, \eta\Delta_y, \Delta_t)}{1-\xi^{\lambda_0}} \\ = & O(\Delta_x^{\lambda_1}) + \beta_0 \eta^{\mu_0} \Delta_y^{\mu_0} + \beta_1 \eta^{\mu_1} \Delta_y^{\mu_1} + \cdots + \beta_k \eta^{\mu_k} \Delta_y^{\mu_k} \\ & + \gamma_0 \Delta_t^{\nu_0} + \gamma_1 \Delta_t^{\nu_1} + \cdots + \gamma_l \Delta_t^{\nu_l}, \end{aligned} \tag{49}$$

By (49) − (48) × η^{μ_0} we obtain

$$\begin{aligned} & g^* - \Bigg[\frac{g(\xi\Delta_x, \eta\Delta_y, \Delta_t) - \xi^{\lambda_0} g(\Delta_x, \eta\Delta_y, \Delta_t)}{(1-\xi^{\lambda_0})(1-\eta^{\mu_0})} \\ & -\eta^{\mu_0} \frac{g(\xi\Delta_x, \Delta_y, \Delta_t) - \xi^{\lambda_0} g(\Delta_x, \Delta_y, \Delta_t)}{(1-\xi^{\lambda_0})(1-\eta^{\mu_0})}\Bigg] \\ = & O(\Delta_x^{\lambda_1} + \Delta_y^{\mu_1}) + \gamma_0 \Delta_t^{\nu_0} + \gamma_1 \Delta_t^{\nu_1} + \cdots + \gamma_l \Delta_t^{\nu_l}. \end{aligned} \tag{50}$$

Again from (50), we obtain

$$\begin{aligned} & g^* - \Bigg[\frac{g(\xi\Delta_x, \eta\Delta_y, \zeta\Delta_t) - \xi^{\lambda_0} g(\Delta_x, \eta\Delta_y, \zeta\Delta_t)}{(1-\xi^{\lambda_0})(1-\eta^{\mu_0})} \\ & -\eta^{\mu_0} \frac{g(\xi\Delta_x, \Delta_y, \zeta\Delta_t) - \xi^{\lambda_0} g(\Delta_x, \Delta_y, \zeta\Delta_t)}{(1-\xi^{\lambda_0})(1-\eta^{\mu_0})}\Bigg] \\ = & O(\Delta_x^{\lambda_1} + \Delta_y^{\mu_1}) + \gamma_0 \zeta^{\nu_0} \Delta_t^{\nu_0} + \gamma_1 \zeta^{\nu_1} \Delta_t^{\nu_1} + \cdots + \gamma_l \zeta^{\nu_l} \Delta_t^{\nu_l}. \end{aligned} \tag{51}$$

Finally, by (51) − (50) × η^{μ_0} we obtain

$$
\begin{aligned}
g^* \;&- \Big[g(\xi\Delta_x, \eta\Delta_y, \zeta\Delta_t) - \xi^{\lambda_0} g(\Delta_x, \eta\Delta_y, \zeta\Delta_t) + \xi^{\lambda_0}\eta^{\mu_0} g(\Delta_x, \Delta_y, \zeta\Delta_t)\\
&-\eta^{\mu_0} g(\xi\Delta_x, \Delta_y, \zeta\Delta_t) + \xi^{\lambda_0}\zeta^{\nu_0} g(\Delta_x, \eta\Delta_y, \Delta_t) - \zeta^{\nu_0} g(\xi\Delta_x, \eta\Delta_y, \Delta_t)\\
&+\eta^{\mu_0}\zeta^{\nu_0} g(\xi\Delta_x, \Delta_y, \Delta_t) - \xi^{\lambda_0}\eta^{\mu_0}\zeta^{\nu_0} g(\Delta_x, \Delta_y, \Delta_t)\Big]\\
&/[(1-\xi^{\lambda_0})(1-\eta^{\mu_0})(1-\zeta^{\nu_0})] = O(\Delta_x^{\lambda_1} + \Delta_y^{\mu_1} + \Delta_t^{\nu_1}).
\end{aligned}
$$

So, when $m = 1$, (44) holds. □

The multivariate extrapolation (45)-(46), which is suitable for multi-variables, is an extension of the traditional Richardson extrapolation, which is only suitable for one variable.

We notice that the spatial error of the numerical scheme (5)-(8) is of second order, whereas the temporal error is of first order. Similar to (45)-(46), we can derive a temporal extrapolation as follows:

$$
g_m(\Delta_x, \Delta_y, \Delta_t) = \frac{g_{m-1}(\Delta_x, \Delta_y, \zeta\Delta_t) - \zeta^{\nu_{m-1}} g_{m-1}(\Delta_x, \Delta_y, \Delta_t)}{1-\zeta^{\nu_{m-1}}}, \qquad (52)
$$

$$
m = 1, 2, \cdots, n_3,
$$

where $g_0(\Delta_x, \Delta_y, \Delta_t) = g(\Delta_x, \Delta_y, \Delta_t),\ \zeta \neq 1$.

Applying the extrapolation (52) ($\zeta = 1/2$) to the numerical scheme (5)-(8), we obtain the following algorithm:

$$
u_1^{j,l,k}(\Delta_x, \Delta_y, \Delta_t) = 2u_{j,l}^{2k}\left(\Delta_x, \Delta_y, \frac{\Delta_t}{2}\right) - u_{j,l}^{k}(\Delta_x, \Delta_y, \Delta_t), \qquad (53)
$$

where $u_{j,l}^{k}(\Delta_x, \Delta_y, \Delta_t)$ represents the numerical solution at the point (x_j, y_l, t_k) by the spatial steps $\Delta_x,\ \Delta_y$ and temporal step Δ_t in the numerical scheme (5)-(8), whereas $u_1^{j,l,k}(\Delta_x, \Delta_y, \Delta_t)$ is the numerical solution at the point (x_j, y_l, t_k) in the algorithm (53).

Again applying the extrapolation (52) ($\zeta = 1/3$) to the numerical scheme (5)-(8), we obtain also the following algorithm:

$$
\widetilde{u}_1^{j,l,k}(\Delta_x, \Delta_y, \Delta_t) = \frac{3u_{j,l}^{3k}\left(\Delta_x, \Delta_y, \frac{\Delta_t}{3}\right) - u_{j,l}^{k}(\Delta_x, \Delta_y, \Delta_t)}{2}, \qquad (54)
$$

where $u_{j,l}^{k}(\Delta_x, \Delta_y, \Delta_t)$ is as defined in (53), whereas $\widetilde{u}_1^{j,l,k}(\Delta_x, \Delta_y, \Delta_t)$ represents the numerical solution at the point (x_j, y_l, t_k) in the algorithm (54).

7 Numerical results

In order to demonstrate our theoretical analysis, we solve the following initial-boundary value problem for the 2D-ASDE:

$$
\begin{aligned}
\frac{\partial u(x,y,t)}{\partial t} &= {}_0D_t^{1-\gamma}\left(\frac{\partial^2 u(x,y,t)}{\partial x^2} + \frac{\partial^2 u(x,y,t)}{\partial y^2}\right) + e^{x+y}\Big[(1+\gamma)t^{\gamma}\\
&\quad -2\frac{\Gamma(2+\gamma)}{\Gamma(1+2\gamma)}\Big],\ 0 < t \le 1,\ 0 < x, y < 1, \qquad (55)
\end{aligned}
$$

Table 1 The maximum error E_∞ of the numerical scheme (9)-(12)

Δ_t	$\Delta_x = \Delta_y$	$\gamma = 0.7$	$\gamma = 0.8$
1/400	1/4	1.0115×10^{-2}	7.3628×10^{-4}
1/4000	1/8	2.4627×10^{-4}	2.4657×10^{-4}

Table 2 The maximum error E_∞ of the numerical scheme (5)-(8)

γ	$\Delta_t = \Delta_x^2 = \Delta_y^2 = 1/16$	$\Delta_t = \Delta_x^2 = \Delta_y^2 = 1/64$
0.5	1.9352×10^{-3}	6.4532×10^{-4}
0.6	2.8525×10^{-3}	8.6152×10^{-4}
0.7	3.7660×10^{-3}	1.0722×10^{-3}
0.8	4.7275×10^{-3}	1.2942×10^{-3}
0.9	6.0864×10^{-3}	1.6072×10^{-3}

Table 3 The maximum error E_∞ of the algorithm (53)

γ	$\Delta_t = \Delta_x^2 = \Delta_y^2 = 1/16$	$\Delta_t = \Delta_x^2 = \Delta_y^2 = 1/64$
0.5	1.1470×10^{-3}	2.9328×10^{-4}
0.6	1.1259×10^{-3}	2.8887×10^{-4}
0.7	1.1000×10^{-3}	2.8467×10^{-4}
0.8	1.0762×10^{-3}	2.8019×10^{-4}
0.9	1.0569×10^{-3}	2.7870×10^{-4}

$$u(x, y, 0) = 0, \ \ 0 \le x, y \le 1; \tag{56}$$

$$u(0, y, t) = e^y t^{1+\gamma}, \ \ u(1, y, t) = e^{1+y} t^{1+\gamma}, \ \ 0 \le y \le 1, \ \ 0 < t \le 1; \tag{57}$$

$$u(x, 0, t) = e^x t^{1+\gamma}, \ \ u(x, 1, t) = e^{1+x} t^{1+\gamma}, \ \ 0 \le x \le 1, \ \ 0 < t \le 1. \tag{58}$$

The exact solution of the problem (55)-(58) is

$$u(x, y, t) = e^{x+y} t^{1+\gamma}.$$

Define

$$E_\infty = \max_{0 \le k \le N} \left\{ \|E^k\|_2 \right\}.$$

Table 1 shows the maximum error of the numerical solutions for the problem (55)-(58) using the numerical scheme (9)-(12) for different Δ_t, $\Delta_x = \Delta_y$ and γ.

Table 2 shows the maximum error of the numerical solutions for the problem (55)-(58) using the numerical scheme (5)-(8) for different $\Delta_t = \Delta_x^2 = \Delta_y^2$ and γ.

Table 3 shows the maximum error of the numerical solutions of the problem (55)-(58) using the algorithm (53) for different $\Delta_t = \Delta_x^2 = \Delta_y^2$ and γ.

Table 4 shows the maximum error of the numerical solutions of the problem (55)-(58) using the algorithm (54) for different $\Delta_t = \Delta_x^2 = \Delta_y^2$ and γ.

Comparing Table 3 and Table 4 with Table 2, it can be seen that the accuracy of the numerical results for the algorithms (53) and (54) are improved.

Table 4 The maximum error E_∞ of the algorithm (54)

γ	$\Delta_t = \Delta_x^2 = \Delta_y^2 = 1/16$	$\Delta_t = \Delta_x^2 = \Delta_y^2 = 1/64$
0.5	1.1299×10^{-3}	2.8964×10^{-4}
0.6	1.1112×10^{-3}	2.8650×10^{-4}
0.7	1.0905×10^{-3}	2.8413×10^{-4}
0.8	1.0716×10^{-3}	2.8172×10^{-4}
0.9	1.0552×10^{-3}	2.8079×10^{-4}

8 Conclusions

In this paper, two numerical schemes for solving the 2D-ASDE have been presented. Their stability, convergence and solvability have been analyzed. A new multivariate extrapolation and its application have also been introduced. The theoretical analyses have been verified by some numerical results.

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