Binary Time Series Generated by Chaotic Logistic Maps

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SUMMARY

This paper examines stochastic pairwise dependence structures in binary time series obtained from discretised versions of standard chaotic logistic maps. It is motivated by applications in communications modelling which make use of so-called chaotic binary sequences. The strength of non-linear stochastic dependence of the binary sequences is explored. In contrast to the original chaotic sequence, the binary version is non-chaotic with non-Markovian non-linear dependence, except in a special case. Marginal and joint probability distributions, and autocorrelation functions are elicited. Multivariate binary and more discretized time series from a single realisation of the logistic map are developed from the binary paradigm. Proposals for extension of the methodology to other cases of the general logistic map are developed. Finally, a brief illustration of the place of chaos-based binary processes in chaos communications is given.

Some key words: binary sequence; chaos; chaos communications; dependence; discretisation; invariant distribution; logistic map; randomness.

1. INTRODUCTION

Chaotic maps of the interval provide a rich array of dynamical behaviour and geometric features. Ruelle (1989) provides a succinct overview of the geometry of chaotic maps, while Mira (1987) discusses properties of maps of the interval as special and distinct cases of larger classes of dynamical systems. An aspect of dynamical systems which forms a central theme in the present paper is that of an (absolutely continuous) invariant measure: we refer the reader to the two books cited here, among others, for details.

The motivation of this paper is the use of binary chaotic sequences in the area of chaos-communications; see Kohda & Tsuneda (1994) for the first paper on binary chaotic sequences, Kohda (2002) for a current synthesis and Kennedy, Rovati & Setti (2000) for the first overview of the emerging field of chaos communications. A central theme in this area is that chaotic sequences are used in transmitting messages, replacing in a sense the periodic cyclic sequences of conventional systems. The focus in this paper is to produce one or more binary-valued sequences from a standard logistic map, according to the continuous values being in one of two sub-intervals of the map's domain defined by cut-points, one applying to each binary process. In communications modelling each discretised sequence is used to spread or carry the message sequence. It carries or encodes a message, also typically in binary form, and with the encoding done by linear transformation. An embryonic illustration is given in the final Section 8; this makes clear that calculation of bit error probability requires the joint binary distribution derived in the paper. There are security advantages when the spreading sequence is effectively random, making the transmitted sequence difficult to decode without full knowledge of the chaotic transmission mechanisms, but on the other hand dependence can increase system performance. Kohda & Tsuneda (1997) have shown that in the equally balanced binary case the discrete sequence, considered as a stochastic process, is fully independent, even though this can never be so for the underlying chaotic sequence. In the unbalanced cases, there is non-linear binary dependence. Information on the dependence is needed when studying the performance of the communications system, particularly with the presence of added noise in the transmission channel. The binary dependence studied in the paper is of the adjacent pairwise type . More general approaches to dependence in dynamical systems are demonstrated by Wolff (1994), for example. For treatments of autocorrelations arising from invariant measures of maps of the interval, the reader is referred to Preston (1983) and Young (1992).

An extension to a non-binary discretisation is also investigated, leading to a discrete uniform distribution as the counter part of the map's continuous invariant distribution. The methods herein can be extended to the study of dependency at higher lags and simultaneous discretisations, and we make some brief remarks at the end. We also propose a statistical method to handle the case when the logistic parameter is not 4 and, thus, tractable analytical calculations can not be performed.

To follow through from the motivation of the paper, we prefer to treat a chaotic system from the viewpoint of unpredictability; that is, to regard the resulting time series as a stochastic sequence when the initial value is assigned a probability distribution, whereupon sensitive dependence upon initial conditions of a chaotic map can be interpreted in terms of variance and prediction error. This relies on the map in question having specific ergodic properties, as in Collet (1996), and in particular for maps of the interval (Collet, 1994). Of course, the classical deterministic approach is concerned with the sensitive dependence upon initial conditions arising from periodic orbits being dense in the map's attractor: see, for example, Ruelle (1989). A fundamental theorem, due to Sharkovskii (1964), guarantees that the

existence of a period of some order necessarily implies the existence of periods of all orders within a strongly ordered lists of the natural numbers. In an important paper following on from that result, Li & Yorke (1975) considered the connections between existence of orbits of period three and chaotic behaviour. Zhang & Li (2000) examine local uniqueness of periodic orbits on a map's attractor and thus further elicit structure giving rise to sensitive dependence.
We employ invariant measures for chaotic systems, specifically that for the logistic map. In the more general sense, much is known about invariant measures for more general maps, such as in Bobok & Kuchta (1994), Misiurewicz (1981) and Mizera (1992), though the degree of technicality there is beyond the tractability of the present paper.

The standard logistic map is given by

$$\tau(x) = 4x(1-x) \qquad (0 < x < 1)\,, \tag{1.1}$$

and has two quadratic branches joining at $x = \frac{1}{2}$; denote by $\{X_t\}$ a stochastic sequence, or time series, arising from iteration of the difference equation $X_{t+1} = \tau(X_t)$ from an initial random variable X_0. It is well known that an absolutely continuous invariant distribution for this map exists and is given by the $beta\left(\frac{1}{2},\frac{1}{2}\right)$ distribution. By choosing X_0 to have the $beta\left(\frac{1}{2},\frac{1}{2}\right)$ distribution, the $\{X_t\}$ is a strictly stationary sequence of random variables with this distribution.

Since the map given by (1.1) is strictly deterministic, any (non-trivial) sequence arising from the map is completely dependent, in that each value is a function of its predecessor value. However, in the stochastic sense without knowledge of the generating mechanism, such sequences can appear to have little or no stochastic correlation; see, for example, Hall & Wolff (1995). This is because standard measures of autocorrelation are based on linear relationships, whereas the series is generated by the highly curved map given at (1.1); the lag-one scatter plot does, of course, have the curve given by (1.1) but with its horizontally linear approximation suggesting zero correlation. Hall & Wolff (1995) also found, for the standard logistic map, that there were no autocorrelations in $\{X_t\}$ at *any lag*; they did find a negative autocorrelation at lag-one in the sequence of squared values, although not at higher lags. However, Lawrance & Balakrishna (2002) have noted that after a mean adjustment to the chaotic sequence, so that any level effect is rightly eliminated, the squared sequence also has zero autocorrelations at all lags. As already stressed, these results do not imply the independence of the sequence or its binary version.

2. BINARY DISCRETISING OF THE STANDARD LOGISTIC MAP

The presence of dependence in binary sequences created from the binary discretisation of standard logistic map sequences is explored. A cut-point at a general value $c\,(0 < c < 1)$ is used to create a $(0,1)$ binary sequence $\{\tilde{X}_t\}$. The presence of dependence in the binary sequence $\{\tilde{X}_t\}$ for general c is investigated, with independence only when $c = \frac{1}{2}$.

Define the states of the binary sequence $\{\tilde{X}_t\}$ by the partition $C_1 = [0, c]$ for $\tilde{X}_t = 0$ and $C_2 = (c, 1]$ for $\tilde{X}_t = 1$. The marginal binary probabilities of $\tilde{X}_t$ are then

$$p_i(c) = pr(\tilde{X}_t = i) = pr(X_t \in C_i)\ ,\ i = 0,1. \tag{2.1}$$

From the beta invariant distribution of X_t, they become

$$p_0(c) = B(c)\ ,\quad p_1(c) = 1 - B(c) \tag{2.2}$$

where $B(c)$ is the beta $\left(\tfrac{1}{2}, \tfrac{1}{2}\right)$ cumulative distribution function given by

$$B(c) = \int_0^c \frac{dx}{\pi\sqrt{x(1-x)}} = \frac{2}{\pi}\sin^{-1}\sqrt{c}\ . \tag{2.3}$$

When $c = \tfrac{1}{2}$, $B(c) = \tfrac{1}{2}$ and when $c = \tfrac{3}{4}$, $B(c) = \tfrac{2}{3}$, special cases which are relevant later.

The lag-one joint probabilities of $\{\tilde{X}_t\}$ are given by

$$\begin{aligned} p_{i,j}(c) &= pr(\tilde{X}_t = i, \tilde{X}_{t+1} = j) = pr(X_t \in C_i,\, X_{t+1} \in C_j) \\ &= pr\left\{X_t \in C_i,\, \tau(X_t) \in C_j\right\},\ \ i = 0,1;\ j = 0,1 \end{aligned} \tag{2.4}$$

and just need calculations over the invariant distribution of $\left\{X_t\right\}$. By inspection of the standard logistic map (1.1), and noting the separation of cases at $c = \tfrac{3}{4}$, the non-zero fixed point of the map,

$$p_{0,0}(c) = pr(\tilde{X}_t = 0, \tilde{X}_{t+1} = 0) = \begin{cases} pr\left(0 \le X_t \le \tfrac{1}{2} - \tfrac{1}{2}\sqrt{1-c}\ \right) & 0 < c < \tfrac{3}{4} \\ pr\left(0 \le X_t \le \tfrac{1}{2} - \tfrac{1}{2}\sqrt{1-c}\ \right) + & \\ pr\left(\tfrac{1}{2} + \tfrac{1}{2}\sqrt{1-c} \le X_t \le c\right) & \tfrac{3}{4} \le c < 1. \end{cases} \tag{2.5}$$

Thus when $0 \le c < \tfrac{3}{4}$, (2.5) gives the joint $(0,0)$ probability, as

$$p_{0,0}(c) = pr\left(0 \le X_t \le \tfrac{1}{2} - \tfrac{1}{2}\sqrt{1-c}\right) = B\left(\tfrac{1}{2} - \tfrac{1}{2}\sqrt{1-c}\right) \tag{2.6}$$

This result simplifies by noting from the invariance of the marginal distribution of the standard logistic map, that

$$B\left(\tfrac{1}{2} - \tfrac{1}{2}\sqrt{1-c}\right) = \tfrac{1}{2}B(c) = 1 - B\left(\tfrac{1}{2} + \tfrac{1}{2}\sqrt{1-c}\right). \tag{2.7}$$

Thus, using the first result of (2.7) in (2.5),

$$p_{0,0}(c) = \tfrac{1}{2}B(c),\ \ 0 < c < \tfrac{3}{4}. \tag{2.8}$$

When $\tfrac{3}{4} \le c < 1$, (2.5) gives

$$\begin{aligned} p_{0,0}(c) &= pr\left(X_t \le \tfrac{1}{2} - \tfrac{1}{2}\sqrt{1-c}\right) + pr\left(X_t \le c, X_t \ge \tfrac{1}{2} + \tfrac{1}{2}\sqrt{1-c}\right) \\ &= B\left(\tfrac{1}{2} - \tfrac{1}{2}\sqrt{1-c}\right) + 1 - B\left(\tfrac{1}{2} + \tfrac{1}{2}\sqrt{1-c}\right). \end{aligned}$$

From again using (2.7),

$$p_{0,0}(c) = \left\{2B(c) - 1\right\},\ \tfrac{3}{4} \le c < 1. \tag{2.9}$$

Equations (2.8) and (2.9) thus specify the joint $(0,0)$ probabilities for state 0 followed by state 0.

Similarly, for the joint $(1,1)$ probabilities from state 1 followed by state 1,

$$p_{1,1}(c) = pr(\tilde{X}_t = 1, \tilde{X}_{t+1} = 1) = pr\left(c < X_t < \tfrac{1}{2} + \tfrac{1}{2}\sqrt{1-c}\right)$$
$$= \begin{cases} 1 - \tfrac{3}{2}B(c) & 0 < c < \tfrac{3}{4} \\ 0 & \tfrac{3}{4} \le c < 1. \end{cases} \qquad (2.10)$$

The equations (2.8) - (2.10) allow the joint binary probability distribution of $(\tilde{X}_t, \tilde{X}_{t+1})$ to be exhibited as in Tables 2.1 and 2.2.

	0	1	$\tilde{X}_t$
0	$\frac{1}{2}B(c)$	$\frac{1}{2}B(c)$	$B(c)$
1	$\frac{1}{2}B(c)$	$1-\frac{3}{2}B(c)$	1- $B(c)$
$\tilde{X}_{t+1}$	$B(c)$	1- $B(c)$	1

Table 2.1 The joint distribution of $(\tilde{X}_t, \tilde{X}_{t+1})$ for $0 < c < \frac{3}{4}$.

	0	1	$\tilde{X}_t$
0	$2B(c)$ -1	1- $B(c)$	$B(c)$
1	1- $B(c)$	0	1- $B(c)$
$\tilde{X}_{t+1}$	$B(c)$	1- $B(c)$	1

Table 2.2. The joint distribution of $(\tilde{X}_t, \tilde{X}_{t+1})$ for $\frac{3}{4} \le c < 1$.

Note that in the case $c = \frac{1}{2}$ the joint distribution of Table 2.1 has $\frac{1}{4}$ for each entry, illustrating in this binary case the independence of $(\tilde{X}_t, \tilde{X}_{t+1})$. This is indicative that the balanced binary discretized version of the logistic map behaves exactly like an independent equi-balanced binary sequence. A more general comparison with a pair of independent Bernoulli trials can be made from the distribution of the number of +1 values, $T \equiv X_t + X_{t+1}$. It is then clear from Tables (2.1) and (2.2) that

$$P(T = i) = \left\{\tfrac{1}{2}B(c), B(c), 1 - \tfrac{3}{2}B(c)\right\}; \quad i = 0,1,2; \quad 0 < c < \tfrac{3}{4}$$
$$= \left\{2B(c) - 1, 2 - 2B(c), 0\right\}; \quad i = 0,1,2; \quad 0 < c < \tfrac{3}{4} \qquad (2.11)$$

which can be compared with the corresponding binomial expressions $B(c)^2$, $2B(c)\left(1 - B(c)\right)$, $\left(1 - B(c)\right)^2$. Unless $c = \frac{1}{2}$, when the two distributions are equal, there are appreciable differences. A brief illustration is given in Table 2.3.

c	Distribution	$P(0)$	$P(1)$	$P(2)$
0.2	Binary chaos	0.1476	0.2952	0.5572
0.2	Binomial	0.0871	0.4161	0.4968
0.7	Binary chaos	0.3155	0.6310	0.0535

0.7	Binomial	0.3981	0.4657	0.1362
0.85	Binary chaos	0.4936	0.5064	0.0000
0.85	Binomial	0.5577	0.3782	0.0641

Table 2.3. Comparison of distributions of T for discrete logistic chaos and binomial independence

The variable T of (2.11) will be employed in the communications illustration in Section 7.

3. AUTOCORRELATION OF THE BINARY DISCRETISED LOGISTIC MAP

The binary sequence $\{\tilde{X}_t\}$ derived from the standard logistic map with cut-point *c* is analyzed for its autocorrelation behaviour. The joint probabilities for $(\tilde{X}_t, \tilde{X}_{t+1})$ are given in Tables 2.1 and 2.2 but only (2.2) and (2.10) are needed for $\rho_1(c) = corr(\tilde{X}_t, \tilde{X}_{t+1})$. These give the result

$$\rho_1(c) = \begin{cases} \left[\frac{1}{2} - B(c)\right] / \left[1 - B(c)\right] & 0 < c < \frac{3}{4} \\ -\left[1 - B(c)\right] / B(c) & \frac{3}{4} \le c < 1. \end{cases} \tag{3.1}$$

This function is plotted in Figure 3.1. The curve passes through 0 at $c = \frac{1}{2}$, the case when $\{\tilde{X}_t\}$ is a fully random binary sequence. The range of values is $(-\frac{1}{2}, \frac{1}{2})$ and correlation takes its greatest negative value of $-\frac{1}{2}$ at $c = \frac{3}{4}$. This is when the cut point is at the fixed point of the map and the behaviour is initially that of slow oscillation about the fixed point and so alternating between C_1 and C_2. The original process has zero linear correlation at all lags, as does the mean corrected series $\{(X_t - \frac{1}{2})^2\}$, and thus neither form of autocorrelation indicates any non-linear dependency. The binary correlation here may be therefore regarded as a more realistic alternative non-linear measure of correlation for the logistic map model.

The correlation according to (3.1) at $c = 0$ is near $\frac{1}{2}$, but there is no real cut-point and the discretised values are all 1's. At $c = 1$, the correlation according to (3.1) is zero, again an artificial situation, since the process will be all 0's in this case. From the derivative of (3.1) $\rho(c)$ it is not continuous as $c \to 0^+$ or 1^-; the limits are practically artificial and the process is degenerate at these values.

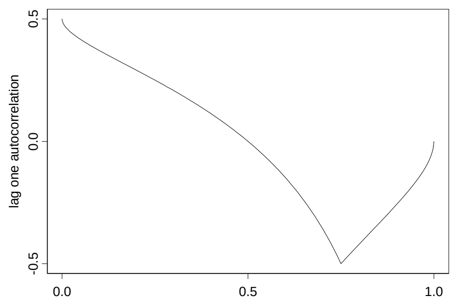

Figure 3.1. The lag one correlation, $\rho_1(c)$, of the binary discretised chaotic standard logistic map using a cut-point *c*, as given by equation (3.1).

Figure 3.1 can also be interpreted as a calibration in c of the departure from independence when the cut-point is not $\frac{1}{2}$ as in the balanced binary case.

4. MULTIPLE BINARY DISCRETISATION OF THE LOGISTIC MAP

Using a single continuous chaotic sequence may seem wasteful in some potential applications, such as arise in multiple channel communications, and use of different cut-points allows multiple binary sequences to be created. Let $\left\{\tilde{X}_t^{c_1}\right\}$ and $\left\{\tilde{X}_t^{c_2}\right\}$ be two such binary sequences of the same realization of the standard logistic map using cut-points c_1 and c_2, respectively.

Two different bivariate derived sequences can be suggested, the lag-zero sequence $\left\{\tilde{X}_t^{c_1}, \tilde{X}_t^{c_2}\right\}$ and the lag-one sequence $\left\{\tilde{X}_t^{c_1}, \tilde{X}_{t+1}^{c_2}\right\}$. The first is a strictly vector version of the binary sequence constructed in Section 2 but cannot have $c_1 = c_2$ because then $\tilde{X}_t^{c_1} \equiv \tilde{X}_t^{c_2}$ for all t. The second is not different from a single sequence when $c_1 = c_2$ because of the results in Sections 2 and 3. Interest is in the cross-dependency and independence of $\tilde{X}_t^{c_1}$ and $\tilde{X}_t^{c_2}$.

In the case of $\left\{\tilde{X}_t^{c_1}, \tilde{X}_t^{c_2}\right\}$, the following joint probability expressions can be seen,

$$
\begin{aligned}
&pr\left(\tilde{X}_t^{c_1} = 0, \tilde{X}_t^{c_2} = 0\right) = pr\left(0 < X_t < \min(c_1, c_2)\right), \\
&pr\left(\tilde{X}_t^{c_1} = 0, \tilde{X}_t^{c_2} = 1\right) = \begin{cases} 0 & c_1 < c_2 \\ pr\left(c_2 < X_t < c_1\right) & c_1 > c_2 \end{cases}, \\
&pr\left(\tilde{X}_t^{c_1} = 1, \tilde{X}_t^{c_2} = 0\right) = \begin{cases} pr\left(c_1 < X_t < c_2\right) & c_1 < c_2 \\ 0 & c_1 > c_2 \end{cases}, \\
&pr\left(\tilde{X}_t^{c_1} = 1, \tilde{X}_t^{c_2} = 1\right) = pr\left(\max(c_1, c_2) < X_t < 1\right).
\end{aligned} \tag{4.1}
$$

These probabilities can all be expressed in terms of the beta $\left(\frac{1}{2}, \frac{1}{2}\right)$ distribution functions $B(c_1)$ and $B(c_2)$; hence from (4.1) and (2.2) the correlation of $\tilde{X}_t^{c_1}$ and $\tilde{X}_t^{c_2}$ is given by

$$
\rho_0(c_1, c_2) = \begin{cases} \left[\dfrac{B(c_1)\{1 - B(c_2)\}}{\{1 - B(c_1)\}B(c_2)}\right]^{1/2} & c_1 \le c_2 \\ \left[\dfrac{\{1 - B(c_1)\}B(c_2)}{B(c_1)\{1 - B(c_2)\}}\right]^{1/2} & c_1 > c_2. \end{cases} \tag{4.2}
$$

For instance, when $c_1 = \frac{1}{2}$, $c_2 = \frac{3}{4}$, $\rho_0 = 1/\sqrt{2}$. A graph of $\rho_0(c_1, c_2)$ versus c_1 and c_2 is given in Figure 4.1. The ridge along $c_1 = c_2$ indicates the perfect correlation of identical processes.

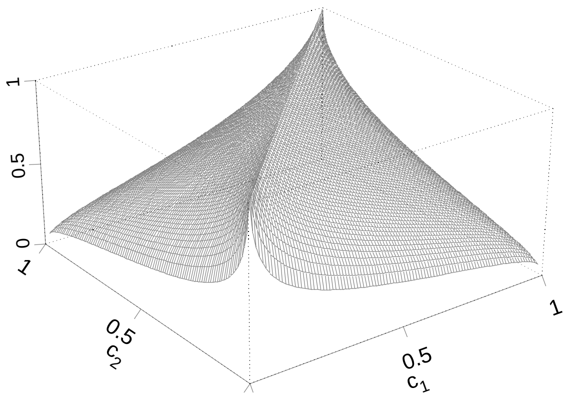

Figure 4.1. The cross-correlation at lag zero, $\rho_0(c_1, c_2)$, of two discretised chaotic processes induced from a common process using cut-points c_1 and c_2, as given by equation (4.2).

When considering $\left\{X_t^{c_1}, X_{t+1}^{c_2}\right\}$ the required formulae are more numerous. Attention will be confined to obtaining $pr\left(\tilde{X}_t^{c_1}=1, \tilde{X}_{t+1}^{c_2}=1\right)$ which is enough for the correlation of $X_t^{c_1}$ and $X_{t+1}^{c_2}$ to be given. The required probability can first be written as

$$pr\left(\tilde{X}_t^{c_1}=1, \tilde{X}_{t+1}^{c_2}=1\right)=pr\left(X_t>c_1, \tau(X_t)>c_2\right)=pr\left(X_t>c_1, \tfrac{1}{2}-\tfrac{1}{2}\sqrt{1-c_2}<X_t<\tfrac{1}{2}+\tfrac{1}{2}\sqrt{1-c_2}\right) \tag{4.3}$$

From considering three cases, (4.3) becomes

$$pr\left(\tilde{X}_t^{c_1}=1, \tilde{X}_{t+1}^{c_2}=1\right)=\begin{cases}1-B(c_2) & c_1<\tfrac{1}{2}-\tfrac{1}{2}\sqrt{1-c_2}\\ 1-\tfrac{1}{2}B(c_2)-B(c_1) & \left|c_1-\tfrac{1}{2}\right|<\tfrac{1}{2}\sqrt{1-c_2}\\ 0 & c_1>\tfrac{1}{2}+\tfrac{1}{2}\sqrt{1-c_2}\end{cases} \tag{4.4}$$

The explicit correlation $\rho_1(c_1, c_2)$ of $(\tilde{X}_t^{c_1}, \tilde{X}_{t+1}^{c_2})$ can now be calculated as

$$\rho_1(c_1,c_2)=\begin{cases}\left(\dfrac{B(c_1)}{B(c_2)}\dfrac{[1-B(c_2)]}{[1-B(c_1)]}\right)^{\frac{1}{2}} & c_1<\tfrac{1}{2}-\tfrac{1}{2}\sqrt{1-c_2}\\ \dfrac{[\tfrac{1}{2}-B(c_1)]}{\left([1-B(c_1)][1-B(c_2)]\right)^{\frac{1}{2}}}\left(\dfrac{B(c_2)}{B(c_1)}\right)^{\frac{1}{2}} & \left|c_1-\tfrac{1}{2}\right|<\tfrac{1}{2}\sqrt{1-c_2}.\\ -\left(\dfrac{[1-B(c_1)][1-B(c_2)]}{B(c_1)B(c_2)}\right)^{\frac{1}{2}} & c_1>\tfrac{1}{2}+\tfrac{1}{2}\sqrt{1-c_2}\end{cases} \tag{4.5}$$

These results are illustrated in Figures 4.2. This expression reduces to (3.1) when $c_1=c_2=c$ and the processes are identical: in numerical experiments a graph of the diagonal elements of $\rho_1(c_1,c_2)$ *versus* $c_1=c_2=c$ was identical to Figure 3.1. When $c_1=\tfrac{1}{2}$, by (4.2), the processes are independent.

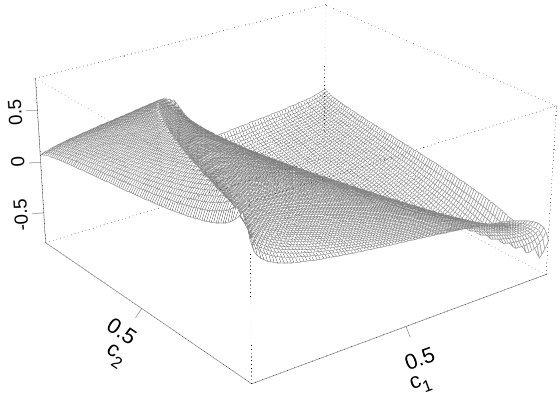

Figure 4.2. The cross-correlation at lag one, $\rho_1(c_1, c_2)$, of two discretised chaotic processes induced from a common process using cut-points c_1 and c_2, as given by equation (4.5).

5. FINER DISCRETISING OF THE LOGISTIC MAP

It has been seen in Section 2 that discretising the logistic map at $c = \frac{1}{2}$, where its branches join, produces an independent balanced binary sequence. In this section finer discretising is considered, but keeping the distribution balanced in the sense of being discretely uniform. The consequent lack of independence is investigated in terms of autocorrelation. A discrete sequence $\{\tilde{X}_t\}$ with the finite alphabet ($j = 0,1,...,k-1$) of size $k \geq 2$ is obtained from the continuous output of a standard logistic map. Using the cut points $(c_0, c_1, \cdots, c_{k-1}, c_k)$, where $0 = c_0 < c_1 < \cdots < c_{k-1} < c_k = 1$, $\tilde{X}_t$ is defined by

$$\tilde{X}_t = i \text{ for } c_i \leq X_t < c_{i+1}, i = 0,1,\ldots,k-1. \tag{5.1}$$

Choosing these cut points such that

$$B(c_j) = \frac{j}{k} \; \left(j = 0,1,\cdots,k-1 \right), \tag{5.2}$$

where $B(\cdot)$ is the cumulative distribution function, as at (2.3), of the beta $\left(\frac{1}{2},\frac{1}{2}\right)$ density function, confers a discrete uniform distribution on $\tilde{X}_t$; it is a transformation of the logistic map akin to the inverse sign transformation giving a tent map with a continuous invariant uniform distribution. From (2.3) it can be shown that

$$c_j = \left(\sin(\frac{\pi j}{2k}) \right)^2, \; j = 0,1,\cdots,k. \tag{5.3}$$

For convenience, k is chosen to be even, although with some messy modifications, the ensuing arguments will also still hold for odd k. A version of $\tilde{X}_t$ which is uniformly discretely distributed and equi-spread over $[0,1]$, is then defined as

$$\tilde{U}_t = (2\tilde{X}_t + 1)/2k. \tag{5.4}$$

However, it is simpler to work with the alphabet ($j = 0,1,...,k-1$) directly and use $\tilde{X}_t$. The study of $\{\tilde{X}_t\}$, which has k states, then requires the entries of the associated $k \times k$ joint (i, j) probability matrix of $\tilde{X}_t$. Explicit expressions are required for

$$\begin{aligned} pr\left(\tilde{X}_t = i, \tilde{X}_{t+1} = j\right) &= pr\left(X_t \subset [c_i, c_{i+1}], X_{t+1} \subset [c_j, c_{j+1}]\right) \\ &= pr\left(X_t \subset [c_i, c_{i+1}], \tau(X_t) \subset [c_j, c_{j+1}]\right). \end{aligned} \tag{5.5}$$

This entails computing pre-images of the cut-points under the logistic map $\tau(x)$ given by (1.1). These are given by

$$\tau^{-1}\left(c_j\right) = \frac{1}{2}\left(1 \pm \sqrt{1-c_j}\right) \equiv (c_j^-, c_j^+) = \frac{1}{2}\left(1 \pm \cos\left(\frac{\pi j}{2k}\right)\right). \tag{5.6}$$

From using the explicit form of the cut-points (5.3), along with some elementary trigonometry concerning double angles, it can be seen that

$$c_j = c_{2j}^- \quad \left(j = 0, \cdots, \tfrac{k}{2}\right) \qquad c_{k-j} = c_{2j}^+ \quad \left(j = 0, \cdots, \tfrac{k}{2}\right). \tag{5.7}$$

Further, by monotonicity of the map τ about $\frac{1}{2}$, we see

$$c_j < c_{2j+1}^- < c_{j+1} \text{ and } c_{k-j-1} < c_{2j+1}^+ < c_{k-j} \tag{5.8}$$

for $j = 0, \cdots, \frac{k}{2}$, where the inequalities are strict. These results indicate that conditional on $\tilde{X}_t = i$, $\tilde{X}_{t+1}$ can only take one of two values, $2i$ or $2i+1$ when i is less than $k/2$ and $2(k-i)-2$ or $2(k-i)-1$ when i is greater or equal $k/2$. The joint probabilities (5.5) reduce to probabilities that a single variate from the original sequence lies between particular pairs of cut-points, as in (5.8), whence

$$pr\left(\tilde{X}_t = i, \tilde{X}_{t+1} = j\right) = \begin{cases} \frac{1}{2k} & j = 2i,\ 2i+1; \quad i = 0, \cdots, \frac{k}{2} - 1 \\ \frac{1}{2k} & j = 2(k-i)-1;\ j = 2(k-i)-2\ ;\ i = \frac{k}{2}, \cdots, k-1 \\ 0 & \text{otherwise.} \end{cases} \tag{5.9}$$

This result shows that the finer uniformly discretised sequence $\{\tilde{X}_t\}$ is not independent, as in the balanced binary case. It is somewhat enlightening to see the joint probabilities (5.9) in matrix form, as

$$\begin{bmatrix} 0 & 0 & \cdots & \frac{1}{2k} & \frac{1}{2k} & \cdots & 0 & 0 \\ 0 & 0 & \cdots & \frac{1}{2k} & \frac{1}{2k} & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & \frac{1}{2k} & \cdots & 0 & 0 & \cdots & \frac{1}{2k} & 0 \\ 0 & \frac{1}{2k} & \cdots & 0 & 0 & \cdots & \frac{1}{2k} & 0 \\ \frac{1}{2k} & 0 & \cdots & 0 & 0 & \cdots & 0 & \frac{1}{2k} \\ \frac{1}{2k} & 0 & \cdots & 0 & 0 & \cdots & 0 & \frac{1}{2k} \end{bmatrix}. \tag{5.10}$$

Conditionally on X_t this can be seen as a discrete version of the tent map with marginal distribution discretely uniform over $(0,1,....,k-1)$. One may argue heuristically to obtain the matrix in (5.12), in that the selection of cutpoints was made to ensure equibalance, the non-zero joint probabilities in each cell must be equal. In addition, by virtue of the continuity and non-linearity of the underlying map, there must be at least some connected states in the joint distribution of the discretised process. Hence, there are not k states each with probability mass $1/k$, but rather $2k$ states each with probability mass $1/(2k)$. Furthermore, the marginal probability that the process will be in a given accessible state is $1/k$, by construction in (5.2), thus the matrix in (5.12) becomes the one-step transition probability matrix when all non-zero entries are replaced by $\frac{1}{2k}$. That is to say the effect of discretising a continuous-valued map into k equi-probable discrete states still permits one to predict that, for example, state i will map either to state $2i$ or to state $2i+1$ but not otherwise, by (5.10), $(i=0,1,...,\frac{k}{2}-1)$. However, discretisation destroys any further ability to predict the precise image of the state (because of expanding properties of chaotic maps). Thus, macroscopic prediction is possible, but microscopic prediction is not. At the finer scale, the best one can say is that state i will map either to state $2i$ or to state $2i+1$ with equal probability. One might call this notion *local independence*. Of course, in the $k=2$ case with equal balance, we have complete independence, as evaluated in Section 2. When the discretisation does not result in a discrete uniform marginal distribution, the transition distributions will be more complicated, transitions to varying numbers of states being possible.

6. DEPENDENCY AFTER FINER DISCRETISING OF THE LOGISTIC MAP

The variables $\{\tilde{U}_t\}$, as defined at (5.4), will be employed when making comparisons of dependency to the standard logistic map since both are defined over the unit interval. The joint distribution of $(\tilde{U}_t,\ \tilde{U}_{t+1})$ is given by (5.10). It can then be shown that the first-order autocorrelation of $\{\tilde{U}_t\}$ is exactly zero for all k. This is in agreement with the corresponding result for the standard logistic map.

A simulation using k=8 on series of length n=1000 rendered 95% confidence intervals for small lag autocorrelations which all contained zero. The intervals were comparable with those suggested by classical asymptotic Normal theory; e.g., for the first-order autocorrelation, [-0.074, 0.057].

For higher order correlations of $\{\tilde{U}_t\}$, it can be found explicitly, helped by MapleV, that

$$corr(\tilde{U}_t^2,\tilde{U}_{t+1}^2)=-\frac{1}{8}\frac{(k^2-4)(7k^2+8)}{(4k^2-1)(k^2-1)}\,. \qquad (6.1)$$

For large k, this approaches $-\frac{7}{32}=-\frac{3\frac{1}{2}}{16}$, which is the corresponding result for the tent map, and close to the logistic map value of $-\frac{4}{17}$, as found in Hall & Wolff (1995). As mentioned in Lawrance & Balakrishna (2001), variables in quadratic correlations need to be adjusted by their means in order for the correlation to properly measure nonlinear dependency. Then (6.1) is replaced by

$$corr\left\{\left(\tilde{U}_t-E(\tilde{U}_t)\right)^2,\left(\tilde{U}_{t+1}-E(\tilde{U}_t)\right)^2\right\}=\frac{1}{4}\frac{(k^2-16)}{(k^2-1)}\,. \qquad (6.2)$$

For large k, this result also approaches the corresponding value for the tent map, that is $\frac{1}{4}$, which is quite different to the value of zero for the logistic map. The discrepancy is thus associated with the different invariant distributions, beta for the logistic map and uniform for the tent map. It is interesting to note that the adjusted quadratic correlation (6.2) is degenerate when $k = 2$ and zero for $k = 4$, raising the possibility of independence with this latter discretisation and worth further investigation.

7. A COMMUNICATIONS SETTING OF LOGISTIC BINARY CHAOS

A current engineering research area in which chaotic processes are essential elements is chaos-based communications systems; see Kennedy, Rovati & Setti (2000) for an overview and Lawrance & Balakrishna (2001) for a particular study. Such systems use chaotic sequences as message carriers, both directly and after discretisation, instead of regular wave forms. To explain simply, one such system called *antipodal chaos shift keying* (APCSK), is concerned with using chaotic binary carrier sequences to transmit binary message bits. In the case to be considered chaotic binary carrier variables $\{\tilde{X}_i\}$ take values $\pm E$ with probabilities $(c, 1-c)$, thus having mean $\mu = (2c-1)E$ and the message bit takes values $b = \pm 1$; when $c = 0, 1$ the system is referred to as *binary phase shift keying* (BPSK). To achieve good system performance, each bit is transmitted N times, that is, it is said to have been spread over the chaotic segment $\left(\tilde{X}_1, \tilde{X}_2, ..., \tilde{X}_N\right)$. A binary bit is transmitted as $\left(b\tilde{X}_1, b\tilde{X}_2, ..., b\tilde{X}_N\right)$, so the carrier segment stays the same when the bit value $b = 1$ is transmitted and is reversed in sign when the bit value $b = -1$ is transmitted. Noise in the transmission channel is assumed to be white Gaussian, with variance σ^2. Thus the received signal is $\left(R_1, R_2, ..., R_N\right)$ where $R_i = b\tilde{X}_i + \varepsilon_i$, $i = 1, 2, ..., N$ where the ε-term denotes the Gaussian noise. This is not enough information to estimate b, but in the simplest system it is possible to generate $\left(\tilde{X}_1, \tilde{X}_2, ..., \tilde{X}_N\right)$ at the receiver as well. When this is the case, an estimate of b is given by $\hat{b}$, as in Lawrance & Balakrishna (2001), where

$$\hat{b} = 1 \text{ if } C(\tilde{X}, R) \geq 0, \ \hat{b} = -1 \text{ if } C(\tilde{X}, R) < 0 \tag{7.1}$$

and

$$C(\tilde{X}, R) \equiv \sum_{i=1}^{N} \left(\tilde{X}_i - \mu\right)\left(R_i - \mu\right).$$

The probabilistic properties of $\{\tilde{X}_i\}$ are required to obtain an assessment of performance of the system. The standard measure of *bit error rate* is formed from the conditional probabilities given by $P\left(\hat{b} = -1 \middle| b = 1\right)$ and $P\left(\hat{b} = 1 \middle| b = -1\right)$. For the first of these

$$BER(N) \equiv P(\hat{b} = -1 \,|\, b = 1) = P\left\{C(\tilde{X}, R) < 0 \,|\, b = 1\right\} \tag{7.2}$$

and equally for the second, so

$$BER(N) = P\left\{C(\tilde{X}, R) < 0 \,|\, b = 1\right\} = P\left\{\sum_{i=1}^{N}\left(\tilde{X}_i - \mu\right)^2 + \sum_{i=1}^{N}\varepsilon_i\left(\tilde{X}_i - \mu\right) < 0\right\}. \tag{7.3}$$

The Gaussian noise assumption allows this to be expressed as

$$BER(N) = E\left\{\Phi\left(-\sqrt{\sum_{i=1}^{N}\left(\tilde{X}_i - \mu\right)^2} \Big/ \sigma\right)\right\}. \tag{7.4}$$

With the binary logistic chaos distribution of the $\{X_i\}$ which take values $\pm E$, and in the simplest of illustration with $N = 2$, (7.4) can be written

$$BER = E\left\{\Phi\left(-\left(E\sqrt{2}/\sigma\right)\sqrt{\left[2+(2c-1)^2 - 2(2c-1)T_2\right]}\right)\right\}. \tag{7.5}$$

Here $T_2 = \left(\tilde{X}_1 + \tilde{X}_2\right)/E$ is based on the sum of the adjacent discretized chaotic variables and takes the values $(-2, 0, +2)$ with the probabilities given at (2.11), and the expectation is over T_2. The quantity $E\sqrt{2}/\sigma$ is the appropriate chaotic signal to noise ratio which is customarily expressed in decibels as $SNR = 20\log\left(E\sqrt{2}/\sigma\right)$. The bit error probability (7.5) can then be considered as $BER(c, SNR)$ and evaluated as a sum of three cross-product terms. Computations from (7.5) then give the so-called waterfall curves in Figure 7.1.

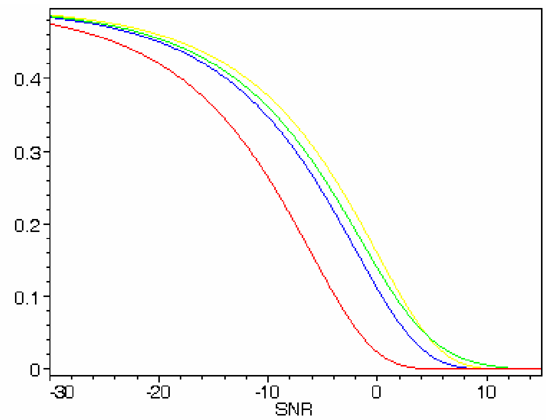

Figure 7.1 Plots of the bit error probability as a function of SNR *in the order* $c = 0, 0.75, 0.3, 0.5$ *from lowest to highest at* $SNR = 0$.

Notice that the lowest bit error curve is always when $c = 0$ and the spreading sequence is just a constant value, $-E$; this thus refers to a BPSK type of communication system. The highest bit errors for SNR less than approximately 5 are seen to be when $c = 0.5$ for which the binary spreading sequence is independent; for higher SNR values the order in reducing bit error is $c = 0.3, 0.5, 0.75, 0$. While the BPSK system gives the best performance, for realistic systems there are performance advantages from dependence, most strongly at the practically important upper end. For c in the range $(0.75,1)$ the curves lie between those shown and offer no further advantages. Of course, the minimal spreading used for illustration here, $N = 2$, is not realistic; in practice bit errors need to less than 10^{-4} and this requires much larger values of N.

8. FINAL COMMENTS

A number of further issues remains in the area of chaotic binary dependent time series. The most obvious issue is the extension of the dependency study to higher lags and simultaneous discretisations: the methods presented here would appear to be applicable in most cases, subject to more tedious and complicated calculations, but nevertheless using the same techniques. There are possible, although less explicit, generalisations when the logistic multiplier is less than 4 and yet the process is still fully chaotic; the work by Hall & Wolff (1995) could possibly be extended in the discrete direction. Specifically, whereas the (absolutely continuous) invariant density in the case of parameter 4 has two asymptotes (at 0 and 1), both of which are of order $x^{-½}$, it can be shown that there exist parameters θ_k which have k+2 asymptotes, each of order $x^{-½}$, and at known locations. With the constraint that the density cover unit area, it is then possible to perform calculations concerning equi-balance and autocorrelation, as in the preceding sections, by numerical integration of the associated density. (This proceeds on the assumption that the said density exists in actuality.) By way of mathematical detail supporting this approach, Zeller & Thaler (2001) consider probability transitions of the logistic map when the parameter is not 4, and provide important technical details for some of the problems in estimation mentioned here. Some geometrical features of the Perron-Frobenius operator, which we have used implicity, are presented in a paper by Keller (2000).

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