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Mathematical Modelling With Young Learners

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Current research is demonstrating that young children can make significant mathematical and social gains from working authentic modelling problems. This paper argues for the implementation of mathematical modelling activities within the elementary and middle school years. The key features of these activities that make them rich learning experiences for children are explored. Some detailed analyses of how children develop and apply generalizable conceptual systems are then presented. It is argued that analogical and case-based reasoning processes play a powerful role in the construction and application of generalized models.

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INTRODUCTION

Students today are facing a world that is shaped by increasingly complex, dynamic, and powerful systems of information within a knowledge-based economy. As future members of the work force, children need to develop the fundamental components of mathematical modelling—that is, they need to recognise the usefulness of models in today's world, to develop and use models to interpret and explain structurally complex systems, to develop representational fluency, to reason in mathematically diverse ways, and to use sophisticated equipment and resources (English, 2002a; Hmelo, Holton, & Kolodner, 2000; Lesh & Heger, 2001).

Being able to interpret and work with complex systems involves important mathematical processes that are under-utilised in mathematics curricula, such as constructing, explaining, justifying, predicting, conjecturing and representing, together with quantifying, coordinating, organising, and representing data. Dealing with such systems also requires students to be able to work collaboratively on multi-component projects in which planning, monitoring, assessing, and communicating results are essential to success (Lesh & Heger, 2001). The primary school is the educational environment where all children should begin a meaningful development of these modelling processes (Jones, Langrall, Thornton, & Nisbet, 2002). However, as Jones et al. note, even the major periods of reform and enlightenment in primary mathematics do not seem to have given children access to the deep ideas and key processes they need for dealing with complex systems beyond school.

Modelling activities are the ideal vehicle for developing these ideas and processes— yet elementary and middle school students are being denied important modelling opportunities even though research has shown that young children can engage in complex mathematical and scientific investigations, given appropriate teacher support (Diezmann, Watters, & English, 2002; Doerr & English, 2003; Lesh & Lehrer, in press). In this paper I show how children in the elementary and lower-middle grades can make significant mathematical and social gains from working authentic modelling problems. I first contrast traditional classroom modelling with the mathematical modelling experiences children need for today's world. I then consider a number of key features that contribute to rich modelling experiences for children. Some detailed analyses of how children develop and apply generalizable conceptual systems are then presented.

TRADITIONAL MATHEMATICAL MODELLING

Traditionally, mathematical modelling in the elementary grades has focused on arithmetic word (story) problems that are represented with concrete materials and then modelled by more abstract operational rules. Solving these word problems entails a mapping between the structure of the problem situation and the structure of a symbolic mathematical expression (e.g., *Suzie has saved $12. Lillian has saved 3 times this amount. How much has Lillian saved?* can be modelled by the expression, 12x3 = 36). Oftentimes, solving these word problems is not a modelling activity for children, rather, it is one that relies on syntactic cues such as key words or phrases in the problem (e.g., "times," "less," "fewer"). Furthermore, there is usually only one way of interpreting the problems and hence, children engage in limited mathematical thinking. While not denying the importance of these types of problems, they do not address adequately the mathematical knowledge, processes, representational fluency, and social skills that our children need for the 21st century (English, 2002b).

Numerous studies have shown that children who are fed a diet of stereotyped one- or two-step word problems frequently divorce their real-world knowledge from the solution process, that is, they solve the problems without regard for realistic constraints (Greer, 1997; Verschaffel, De Corte, & Borghart, 1997). In standard word problems, questions are presented to which the answer is already known by the one asking them (i.e., the teacher). As Verschaffel et al. (1997) commented, questions are not given so children can obtain information about an authentic problem situation, rather, the questions are designed to give the teacher information about the students. Furthermore, both the students and the teachers are aware of this state of affairs and act accordingly.

MATHEMATICAL MODELLING FOR CHILDREN TODAY

Mathematical modelling is frequently viewed as the construction of a *link* or *bridge* between mathematics as a way of making sense of our physical and social world, and mathematics as a set of abstract, formal structures (Greer, 1997; Mukhopadhyay & Greer, 2001). To foster the mathematical modelling abilities children require for today's world, we need to design activities that display the following features:

- Authentic problem situations;
- Opportunities for model exploration and application;
- Multiple interpretations and approaches;
- Opportunities for social development;
- Multifaceted end products; and
- Opportunities for optimal mathematical development.

Authentic Problem Situations

For a number of years, mathematics curriculum documents and mathematics educators have been emphasizing the importance of couching children's problem experiences in situations that are motivating, interesting, and relevant to their world, and where there is a genuine *need* for particular mathematical processes (Boaler, 2002; Kolodner, 1997). Such authentic contexts provide sense-making and experientially real situations for children, rather than simply serve as cover stories for proceduralized and frequently irrelevant tasks.

While the benefits of such experientially real contexts have been well documented (most notably in the Realistic Mathematics Education research, emanating from the Freudenthal Institute; Freudenthal, 1983; Gravemeijer, 1994), there have also been some concerns expressed (Boaler, 2002; Silver, Smith, & Nelson, 1995). A main issue cited is that children are frequently required to both engage with the problem contexts as though they were real and to ignore factors that would pertain to real-life versions of the task (Boaler, 2002). When children situate their reasoning within their own authentic contexts, there are of course several "correct" answers. It has thus been recommended that we need to not only provide real-world contexts, but also "real-world solutions" (Silver et al, 1995, p. 41). Mathematical modelling activities take both aspects into account.

Opportunities for Model Exploration and Application

A program of modelling experiences for children is most effective if it comprises sequences of related activates that enable models to be constructed, explored, and applied. The activities should be structurally related, with discussions and explorations that focus on these structural similarities (Lesh, Cramer, Doerr, Post, & Zawojewski, 2002).

One of several sequences of modelling activates that I have used successfully centers on notions of ranking, and selecting and aggregating ranked quantities. In each context, the children analyze and transform entire data sets or meaningful portions thereof, rather than single data points. The sequence begins with an activity that *elicits* the development of significant mathematical constructs, namely, the Sneaker problem. In this problem, children are asked, “What factors are important to you in buying a pair of sneakers?" In small groups, children generate a list of factors and then determine which factors are most important. This inevitably results in different group rankings of the factors. The teacher then poses the problem of how to create a single set of factors that represents the view of the whole class.

This activity is followed by a *model-exploration* activity or activities, where children can consolidate and refine the conceptual systems they have developed as well as construct powerful representation systems for these systems (e.g., the Weather Problem (Appendix A). In essence, at the end of a model-exploration activity, children should have produced a powerful conceptual tool or model that they can apply to other related problems.

The next activity in the sequence is a *model-application* task (e.g., the Snack Chip Consumer Guide Problem (Appendix B) and the Car Problem (Appendix C), where children deal with a new problem that would have been too difficult without their prior development of a conceptual tool. This new activity requires some adaptation to the tool and involves the children in problem posing as well as problem solving, and information gathering as well as information processing (Lesh et al., 2002).

Multiple Interpretations and Approaches

Hatano (1997) distinguished “understanding through comprehension” from “understanding by schema application.” Schema application occurs when a known solution procedure is applied to a routine problem that usually involves only one interpretation (p. 385). Because the givens, the goal, and the legal solution steps in word problems are usually specified unambiguously, the interpretation process for the solver has been minimalized or eliminated.

Modelling activities for children involve multiple, simultaneous interpretations. With modelling activities, however, the solver has to not only contemplate which of several approaches could be taken in reaching the goal, but must also interpret the goal itself and the accompanying information. Each of these components might be incomplete, ambiguous, or undefined; there might be too much data, or there might be visual representations that are difficult to interpret. For example, notice how Kate, below, is trying to interpret exactly what the first client desires in the Weather Problem.

> Mt. Isa has an extremely high [number of] clear days and that's what they're sort of looking for (first client), but I think you might need a bit more rain because they say "We don't care if there is a lot of rain." That could mean that they want a bit but they don't want all that much. But it could mean two things: That they don't mind and they just don't really care; or that they do want some but they just don't want heaps. It's a bit tricky to decide whether you want a lot of rain or whether you want not too much rain.

In the above situation, children have to interpret the client's comment ("We don't care if there is a lot of rain.") before they can operationalize any actions on any quantities. Through the interpretation process, the quantification of "how much rain" gains meaning for the children.

When presented with information that is open to more than one interpretation, children might make unwarranted assumptions or might impose inappropriate constraints on the products they are to develop (English & Lesh, 2002). This is where the interactions of group members come to the fore as children interpret and re-interpret the problem information.

Opportunities for Social Development

The communication processes inherent in these modelling activities play an important role in children's social, as well as mathematical, developments (Zawojewski, Lesh, & English, 2002; English, 2002b). Modelling activities are specifically designed for small-group work where children are required to develop explicitly sharable products that are subject to scrutiny by others. This means the children have a shared responsibility to ensure their models meet the desired criteria and that what they produce is informative and user-friendly.

Numerous questions, conjectures, arguments, revisions, and resolutions arise as children develop and assess their models, and communicate their models to a wider audience. Although their discussions during model construction can be off task at times, children nevertheless develop powerful skills of argumentation in which they challenge one another's assumptions, ask for justification of ideas, and present counter-arguments.

Multifaceted End Products

Modelling problems for children call for multifaceted products, in contrast to the solutions required by standard types of problems they meet in class. The modelling problems I have used with younger children present them with a number of criteria that have to be met in producing their final model. In the Weather Problem, for example, the requirements of the two clients serve to guide children's development of a model for determining which is the best city to locate in. These criteria not only guide model development but also model *assessment*, both during and following model construction. That is, children can progressively assess their intermediate products, identify any deficiencies, and then revise and refine their models. Or, if several alternative models are being considered at the same time, then the children are able to assess the strengths and weaknesses of each.

Children's final models are usually expressed using various representational media, including written and oral reports, computer-based representations, and paper-based diagrams or graphs (Lesh & Lehrer, in press). The use of a range of media, in particular the computer-based forms, is especially beneficial to younger children because they engage in *purposeful* learning. Indeed, *representational fluency* has been shown to be at the heart of an understanding of many of the key constructs in elementary mathematics and science (Goldin, 2002; Lesh & Heger, 2001) and working flexibly with different representational forms is an increasingly important skill in the workplace.

Opportunities for Optimal Mathematical Development

As children work such problems, they engage in important mathematical processes including describing, analyzing, coordinating, explaining, constructing, reasoning critically, and mathematizing objects, relations, patterns, or rules pertaining to the modelled system (Lesh, Hoover, Hole, Kelly, & Post, 2000; Gravemeijer, 1997). Some of these processes are evident in the following excerpt where a group of sixth-graders is deciding on factors to consider in developing their snack chip consumer guide. After the group had discussed a number of factors (with a strong emphasis on flavor), the teacher asked, "But if you were choosing to buy one of those two products [pointing to two different packets of snack chips], what are the things—forget about the individual—like if it's barbeque or not—what other things would you consider? One child responded as follows:

> Well, I think you should consider.... you should consider how much is in the bag, if you can get that bag...you should also consider...they might be the same price...I'm not saying they're the same. Pretend that was 150 grams, and that was 230 grams and that was a bit bigger. That might be 150 grams and $7 and this 230 grams and $4. That might be l30 and might only be $1. And you have to think: What would you rather buy: two of these which is about as much as that one. Because you've got to think "If I'm going to buy that, I pay around the same amount but I don't get as much, but I pay only a little bit more to buy two of these and get more than that.

In the above situation the child is comparing the two products in terms of their (estimated) mass and cost, with the aim of determining which item is better value for money. In doing so, she engages in informal proportional reasoning.

GENERALIZABLE CONCEPTUAL SYSTEMS

In this section, I consider one of the key goals of mathematical modelling for children, namely, children's development of generalized conceptual systems. A key criterion in designing modelling problems for children is that the tasks should have the potential to elicit mathematically significant constructs that ultimately become generalizable and re-usable (Doerr & English, 2003). Children are observed to progress through a number of learning cycles on their way towards producing a generalized model (Doerr & English, 2003; Lesh & Lehrer, in press).

When children are presented with modelling problems of the type addressed here, they first of all have to interpret the problem and draw upon their existing contextual and mathematical knowledge in doing so. By contextual, I mean knowledge of previously experienced situations in related settings. Sometimes, children will discuss upfront how they should interpret a given problem—through their own perspective or through the perspective of the characters in the problem. For example, in the Car Problem, some children argued that they should interpret the problem only in terms of what Carl and his mother wanted, while others stated, “It helps if you also think about what *you* would do.”

At other times, children's contextual knowledge can be all consuming, taking the children away from the goal of the task. In interpreting the Weather Problem, for example, one group of children spent considerable time discussing their interpretations of “dry” weather:

Tom: Clear doesn't mean they're hot.
Anne: No, I'm saying dry.
Olli: Clear is basically dry……
Tom: Clear days don't mean dry days or do they?
Anne: They DO mean dry days. If it's clear it's not going to be wet, is it?
Tom: Yeah, but it's not going to be dry. That means when you walk on ground it will be dust.

After discussing various alternative interpretations and negotiating meanings of expressions, children normally return to the problem criteria to redirect their efforts (cf., Wyndhamn & Saljo, 1997). Children usually cycle through a number of processes in constructing their models. The processes include sharing, describing, explaining, and justifying their ideas, and rejecting or revising intermediate models depending on how well these meet the problem criteria. As children progress through these learning cycles, they select relevant quantities, create meaningful representations, and define operations that might lead to new quantities (Doerr & English, 2003; Kolodner, 1997).

The reporting-back process, where groups share their models with their class, also provides important opportunities for learning. For example, when Roberta’s group was describing their model for the weather problem, there were some inherent difficulties in the model. Roberta took control and modified their system to make it generalizable. She had realized that they needed to apply some operations that would mathematize their actions.

Roberta: Before when Lyndie said nobody knows if Hobart actually is the best….I put 1st, 2nd, 3rd and things for yearly rainfall and then I did it for cities below 15 [degrees], and I added those… and the lowest one would be first and the highest would be last. For example, the yearly rainfall for Hobart came 7th and days below 15 came 1st so I'd add 1 and 7, and that would be pretty low so that would be 8 so that would be pretty low so that would be first….
Teacher: So you only added up two categories?
Roberta: Yes, yearly rainfall and days below 15, because they were the ones that were important. Alice Springs got 13 because rain was 9 and days below 15 was 4.

Cairns got 9 because the days below 15 were 8 and rainfall was first…. (she continued to describe her procedure).

Teacher: So then how did you decide which was the most suitable city for the first client? Which one did you decide again?

When the teacher asked Roberta to explain how she would use her system to address the first client's needs, Roberta had in the meantime decided to refine it further. She now incorporated a third factor, namely, days above 30 degrees Celsius. The revised system generated negative numbers, which Roberta handled easily even though the class had not been taught these.

Roberta: Hobart. Oh, I just added this part now. I just realised… I thought a way might be to take away the days above 30.

Teacher: Yes. But Hobart has the same number as Sydney and Canberra, so how would you decide that Hobart was more suitable than Sydney or Canberra?

Roberta: (Referring to Hobart) For the days above, take away 9 because that was what it was, so it would be below zero. Sydney......average for that was 8…. And Melbourne take away 7. And Canberra take away 6…..In the end Hobart would win because it's less.

Teacher: So you're saying Hobart is the first choice based on three criteria. And Sydney is your second choice?

Roberta: Yes, because it's zero. And 3rd would be Melbourne, and 4th would be Canberra.

A class member then asked Roberta which client her system would be used for, to which she replied, "It would be for any client who wanted cold weather along with snow."

Roberta had effectively generated a model that was generalizable. Her class had developed their own term for a generalized system, namely, a "universal" model. For example, in reference to the Snack Chip Consumer Guide activity, Isaac explained that a universal system "would work for every type of snack chip, not just the ones we looked at."

Children's development of generalized systems is, of course, a major goal of mathematics education. The use of sequences of modelling tasks provides opportunities for this development, when accompanied by teacher-initiated class discussions on the structural links between the problems. How children apply their generalized models to new modelling problems is an issue that requires further attention.

APPLYING GENERALIZED CONCEPTUAL SYSTEMS

It is proposed that *analogical* and *case-based reasoning* processes facilitate children's application of generalized models. Case-based reasoning involves reasoning by analogy, where the structural similarities between a known situation and a new situation are identified and utilized (English, 1999). While case-based reasoning has been explored in other domains, such as science education (Kolodner, 1997, Kardos, 2002), it has received little attention in mathematics education.

Case-based reasoning involves reasoning that makes use of previous experiences or cases (Kolodner et al., in press). The cases are fundamentally analogs that represent personally experienced problem situations and include a rich representation of the problem situation, the ways in which the situation was handled, and the outcomes of resolving the situation (Kolodner, 1997).

The models that we have been addressing may be regarded as cases that serve as a basis for reasoning about new problems. For children to make effective use of these cases, they need to reason analogically. That is, children need to identify and match the structural or relational correspondences between their known cases and the new problem. They then need to know how to make any necessary modifications to their existing case in order to accommodate additional features of the new problem (cf. English, 1997; Kolodner, 1997). In working the Car Problem, the final problem in the present sequence, children were observed to make use of the models they had developed in the previous problems as cases for this new problem:

Michelle: We can just use a process of elimination.
Group: No, we need to consider all the features.
Roberta: We'll use a rating system. We can use that rating system (the one they had used on the previous problems); we could rate the features that they consider important and we can do this for Carl and for his Mother. I'll write down all the features (she drew a table and started to list all the cars down the left-hand side.) We'll do our old rating system.
Michelle: I don't know what you mean.
Roberta: Well, we got all the features that we considered important for the chips and this time, we're looking at the features that Carl and his mother think are important, and then we'll rate them, like 1 to 10.

Roberta was mapping the key components of the Snack Chip Consumer Guide problem, namely, the selection of product features and the ranking of these features, onto the Car Problem. In drawing a table to assist them, the children had to decide which features they were going to include to define the criteria *safe*, *reliable*, and *fun*. They used a vote-of-hands to decide this. These features were recorded across the top of their table. The group then completed the respective cells in the table, checking those cars that displayed the particular features. However, the children subsequently returned to the features they had chosen and argued whether all the features should be included; they also questioned their interpretations of some of the features (e.g., whether alloy wheels are a safety feature or a *cool* feature). As a consequence, they removed some of the columns in their table.

While Roberta continued to stress the need to use their old rating system, other members of her group wanted to use a process of elimination where they would delete those cars that did not meet most of the features desired by Carl and his mother. After much debate, the group accepted that Roberta's system would be more effective. The system was applied with some modification, however. The children ranked the quantitative features (mileage and gas consumption) and did a frequency count for the remaining features (antilock brakes, airbags, air-conditioning, sunroof, alloy wheels, and power windows).

The group then applied their system, recording their actions in two additional columns of their table (e.g., in one column they recorded partial results such as C = 2+7 = 9 to indicate that the Nissan Silva had 2 qualitative features that Carl desired and its mileage had a ranking of 7). In the other column, they recorded the final result for the Nissan Silva (13) by adding the previous result (9) to the car's ranking for gas consumption.

Applying a known model or case to the solution of a new problem is not a simple process for children. It is, in fact, a multifaceted activity that requires children to be able to:

(a) Construct models that comprise the necessary *structural elements* to enable them to reason analogically with these models;
(b) *Know* to *look for* related structures in dealing with problems;
(c) Know *when* and *how* to utilize their existing models in solving new problems. In this vein, Kolodner, 1997, stressed the importance of students being able to *anticipate* situations in which a case might be applied; and
(d) Make any *necessary modifications* to an existing model in applying it to a new problem.

Teacher-guided discussions are important in helping children move beyond just thinking *about* their models to thinking *with* them, that is, making their models "explicit objects of thought" (Lesh et al., 2002). These discussions can be included in sessions where children share their models with the class to receive constructive peer feedback, or students might provide written critiques of other student models (after first critiquing their own). In my current research, children submit a critique on our project website (www.ourmathmodels.com). The website enables classes of Australian children to share their ideas with classes in other countries (at present, USA).

CONCLUDING POINTS

It is imperative that we take children beyond the traditional classroom experiences, where problem solving rarely extends their thinking or mathematical abilities. We need to implement worthwhile modelling experiences in the elementary and middle school years if we are to make mathematical modelling a way of life for our students. As this paper has argued, younger learners, irrespective of their class achievement levels, can successfully complete modelling problems of the type presented here.

Mathematical modelling activities for children should build on their existing understandings and should engage them in thought-provoking, multifaceted problems that involve small group participation. Such activities should be set within authentic contexts that allow for multiple interpretations and approaches. As children work these activities, they engage in important mathematical processes such as describing, analyzing, coordinating, explaining, constructing, and reasoning critically as they mathematize objects, relations, patterns, or rules.

Modelling activities not only provide opportunities for optimal mathematical development, but they also facilitate children's social development. As children collaborate on constructing a model that meets given criteria, they raise numerous questions and conjectures, engage in argumentation, and learn how to resolve issues of disagreement. In doing so, children see different points of view and ways of thinking, which helps them to become more flexible in their own patterns of thinking.

The importance of providing children with numerous opportunities for model exploration and application has been stressed in this paper, with sequences of modelling activities recommended. Completion of these activities facilitates children's development of generalizable conceptual systems, where they move beyond just thinking about their models to thinking with them.

Analogical and case-based reasoning have been proposed as key processes in children's application of generalized models. To effectively apply these reasoning processes, children's models must comprise the structural elements that enable an existing model—which serves as a form of analog or case—to be mapped onto a new, similarly structured problem situation. To facilitate this mapping process, children need to anticipate situations in which their models might be applicable, and know when and how to utilize these models. Finally, children need to be able to make any necessary modifications to their existing models to accommodate the new situation. These processes require specific attention in the classroom through whole-class discussions. As this paper has illustrated, modelling activities for children develop important mathematical ideas and processes that would be left largely untapped in more traditional classroom activities. It is thus imperative that we introduce young children, and their teachers, to the world of mathematical modelling.

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APPENDIX A

The Weather Problem (Doerr & English, 2003)
The Global Travel Agency is interested in starting a re-location service to help advise people who are moving to a new area. The travel agency needs your help to develop an advising system for choosing places for their clients to live. The clients are primarily interested in the climate: how much rain, how cold it gets, how hot it gets, and if the days are sunny or cloudy. Each of these factors, however, is not of the same importance to every client.

Two potential clients have sent the following letters to the agency describing their preferences and asking for the agency's advice on the best places for them to live. The agency also has gathered some information on the nine cities listed below.

1. Develop a rating system for comparing the climates in different places. Be sure your system will really help the agency evaluate places, even those not listed below.
2. Write two letters for the travel agency with a recommendation for each of the clients. You should put the cities into three groups: the best cities, the second best cities, and the worst cities. This way the client will know which cities to consider living in and which cities to avoid.
3. You should explain to the travel agency how your rating system works and why it is a good one.

Dear Global Travel:
My wife and I are retiring in several months and would like to relocate in a warm and sunny area. We don't care if there is a lot of rain and we definitely don't want to be too cold. What are some cities we should consider living in?

Sincerely,
Mr & Mrs Johnson

Dear Global Travel:
I am looking for some new job opportunities in my field of computer programming. I am quite confident that I will be able to find a job anywhere. I really like all kinds of outdoor sports, especially bushwalking! So I would like to move to a city that has good weather and doesn't get too hot. Where should I consider living?

Sincerely,
Donna Smith

Climatic Information

City	Clear Days	Days below 15°	Days above 30°	Average yearly rainfall (mm)
Sydney	85	12	15	1220.4
Alice Springs	195	40	169	274.5
Hobart	36	184	6	516.3
Cairns	71	0	185	2222.4
Melbourne	45	55	30	661
Darwin	85	0	328	1534.7
Mount Isa	178	4	237	386.4
Canberra	84	157	36	633.1
Perth	114	10	58	863.7

APPENDIX B

The Snack Chip Consumer Guide Problem (English, 2002a)
Students are presented with an introductory article on consumer guides, with questions to answer about the article. They are then given the following problem, with various packets of snack chips provided for them.

In this investigation, you will be developing a consumer guide to help people determine which type of snack chip is the best to buy. It is your decision what to focus on in your consumer guide. Your consumer guide must help people in choosing any snack chip, not just the ones you use in this activity.

As a whole class, brainstorm some factors or criteria that you might consider when you are trying to work out which chip is the best to buy. Think about what we could mean by *best*. Next, in your groups, discuss the following.

1. Describe the nature or type of factors that the whole class brainstormed. What type of information does each factor give you?
2. How might you categorize the factors?
3. How might you rate the factors to help the consumer determine which packet of snack chips they should buy?

4. Make sure that your guide can be used with any type of snack chip, not just the ones you have on your desk at present. Write clear instructions for the consumer on how to use your guide to compare different kinds of snack chips.
5. Finally, prepare a short report for your class members explaining why the system you developed for your consumer guide is a good one.

APPENDIX C

The Car Problem (developed by Helen Doerr and Lyn English)
Carl and his mother have been out shopping for cars. Carl wants a car that will be fun to drive around in, gets good gas mileage, but doesn't cost too much. But Carl's mother, who is going to help pay for the car, wants him to have a car that is reliable and safe. Your job is to create a list for Carl and a list for his mother showing which cars are the best. Then they will have to decide which one to buy! (Students are given a table of data comprising 9 different cars with their properties listed. These include: year of manufacture, cost, color, mileage, liters per 100 km of city driving, specific features, and body style.