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FACTORS AFFECTING THE PROCESS OF PROFICIENT MENTAL ADDITION AND SUBTRACTION: CASE STUDIES OF FLEXIBLE AND INFLEXIBLE COMPUTERS

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The relationship between mental computation and number sense is complex: mental computation can facilitate number sense when students are encouraged to be flexible, but flexibility and number sense is neither sufficient nor necessary for accuracy in mental computation. It is possible for familiarity with a strategy to compensate for a lack of number sense and inefficient processes. This study reports on six case studies exploring Year 3 students’ procedures for and understanding of mental addition and subtraction, and understanding of number sense and other cognitive, metacognitive, and affective factors associated with mental computation. The case studies indicate that the mental computation process is composed of four stages in which cognitive, metacognitive and affective factors operate differently for flexible and inflexible computers. The authors propose a model in which the differences between computer types are seen in terms of the application of different knowledges in number facts, numeration, effect of operation on number, and beliefs and metacognition on strategy choice and strategy implementation.
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Literature at national and international levels has supported the inclusion of mental computation in mathematics curricula as a way to assist the development of number sense (e.g., Klein & Beishuizen, 1994; McIntosh, 1998; Reys, Reys, Nohda, & Emori, 1995). International research (e.g., Blöte, Klein, & Beishuizen, 2000; Buzeika, 1999; Hedrén, 1999; Kamii & Dominick, 1998) has contended that mental computation promotes number sense if students are encouraged to formulate their own mental computation strategies.

Accurate mental computation can be the result of successfully applying efficient mental strategies that exhibit number sense (accurate and flexible). Accuracy can also be the result of successfully applying teacher-taught written procedures (accurate and inflexible). Research has found that some students can compute accurately in the head without understanding (e.g., Heirdsfield, 1996; McIntosh & Dole, 2000); that is, high performance in mental computation can be achieved without knowledge of proficient strategies and without accompanying number sense.

The purpose of this paper is to explore the mental addition and subtraction procedure and to develop a model that explains how the knowledge differences between the two types of accurate mental computers (flexible and inflexible) are applied at different stages in the mental computation process. To provide a framework for the study, the literature is summarised with regard to factors associated with mental computation and with mental computation procedures.

Factors associated with mental computation

Blöte, Klein, and Beishuizen (2000) suggested that cognitive, metacognitive, affective, and classroom context factors influence the way students work with numbers. Mclellan (2001) also suggested that mental computation is part of a richly connected web of mental computation and computational estimation for which the child needs a knowledge of number relationships, a facility with basic facts, an understanding of arithmetical operations, the ability to make comparisons between numbers and possession of base-ten place value concepts. (p. 153)

Other research has proposed connections among mental computation and aspects of number sense; in particular, properties of number, operations and numeration (e.g., Kamii, Lewis, & Jones, 1991), and number facts and estimation (e.g., Heirdsfield, 1996; Reys, Bestgen, Rybolt, & Wyatt, 1982); affects (Van der Heijden, 1994); and metacognition (Sowder, 1994). The results of a study investigating Year 4 children’s mental computation, computational estimation, and number facts knowledge (Heirdsfield, 1996) indicated that children who were accurate and flexible in mental computation possessed advanced number facts skills (i.e., they were able to access basic facts using recall, or were able to employ advanced derived facts strategies, e.g., through 10 – 6+9=6+10-1=15). Further, these children were also proficient in computational estimation. In contrast, children who were accurate only (inflexible) were poor in estimation and used count strategies for number facts when the fact could not be recalled.

To be able to manipulate numbers mentally requires an understanding of partitioning of number (e.g., 34 is not only 3 tens and 4 ones in canonical form, but also 2 tens and 14 ones in noncanonical form) and manipulating numbers (e.g., 34 + 21 is 44, 54, 55) (Resnick, 1983). Further, it seems apparent that students need to conceptualise numbers as entities, rather than symbols side by side. That is, they need to comprehend numbers more in terms of the multiplicative nature of our number system (e.g., 100 is 10 tens, 10 is 10 ones), rather than merely as hundreds, tens, and ones place value (Bednarz & Janvier, 1982).

Sowder (1992) argued that an understanding of the effects of operation on number appears to be essential for flexible mental computation, as some of the strategies that good mental computers employ include decomposing and recomposing number to best suit the operations. Further, good mental computers are capable of using mathematical features, such as associative and commutative properties of operations (Reys, 1992; Sowder, 1994).
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Relationships have been posited between mental computation and affects (e.g., Van der Heijden, 1994), where affects include beliefs (with respect to mathematics, self, teaching, and social context), attitudes (including self-efficacy and attribution) and emotions (McLeod, 1992). Beliefs about the nature of mathematics can be manifested in a child’s disposition – mastery orientation or performance orientation (Prawat, 1989). In relation to computation, mastery oriented students would aim for understanding and flexibility, compared with performance orientation students who would aim for accurate completion of a taught algorithm. Here, monitoring, checking, and planning might be evident.

As proficient mental computers are disposed to making sense of mathematics, they use a variety of strategies in different situations (depending on numbers and context) (Sowder, 1994). Such effortful, reflective and self-regulatory behaviour should involve metacognition. Metacognition can be considered to have three components: metacognitive knowledge (knowledge of own thinking), metacognitive strategies (planning, monitoring, regulating and evaluating), and metacognitive beliefs (perception of own abilities and perception of a particular domain) (Paris & Winograd, 1990).

In summary, research on mental computation and number has proposed connections among mental computation and the following factors: (a) the cognitive area of number sense (particularly number facts, computational estimation, numeration, and effect of operation on number); (b) affective issues, including beliefs, attributions, self-efficacy, and social context (e.g., classroom and home); and (c) metacognitive processes (covering metacognitive knowledge, beliefs, and strategies).

Mental computation strategies

A wide variety of mental addition and subtraction strategies has been identified in the literature (e.g., Beishuizen, 1993; Blöte, Klein, & Beishuizen, 2000; Cooper, Heirdsfield, & Irons, 1996; Reys, Reys, Nohda, & Emori, 1995; Thompson & Smith, 1999). These strategies are summarised in Table 1.

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Description</th>
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<tr>
<td>1010</td>
<td>N10</td>
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<td>u-1010</td>
<td>u-N10</td>
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<tr>
<td>N10C</td>
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The terms 1010 and u-1010 are used for separation strategies in the Dutch literature, N10 and u-N10 are used for the aggregation strategies, and N10C is used for the compensation strategy which is described here as wholistic (e.g., Blöte, Klein, & Beishuizen, 2000). The strategy mental image of pen and paper algorithm is included in the table because of its presence in the literature (Reys, Reys, Nohda, & Emori, 1995). However, most literature considers mental image of pen and paper algorithm to be an inefficient strategy (Carraraer, Carraher, & Schliemann, 1987; Ginsberg, Posner, & Russell, 1981; Hope, 1985; Kamii, 1989; Maier, 1977; Plunkett, 1979; Reys, Reys, Nohda, & Emori, 1995).

In terms of efficiency, Thompson and Smith (1999) classified the strategies so that aggregation and wholistic (Table 1) were the most sophisticated (level 5). Similarly, Heirdsfield and Cooper (1997) argued that separation right to left, separation left to right, aggregation and wholistic represented increasing levels of strategy sophistication.

While it has been posited in the literature that different strategy choice is affected by the semantic structure of word problems (e.g., Riley, Greene, & Heller, 1983; Verschaffel & DeCorte, 1990), Blöte, Klein, and Beishuizen (2000) also found that the number characteristics of problems can affect which strategy is chosen. However, some students do not take into account either semantic structure of the word problem or the number characteristics; they employ a single strategy continuously. As mentioned before, this strategy is usually mental image of pen and paper algorithm. Therefore, the two types of mental computers (flexible and inflexible) seem to respond to different things. Further, the approach they take to mental computation could be quite different from each other. This paper seeks to explain the procedures that the two types of mental computers follow when computing mentally, thus illuminating the differences in the outcomes.
The study

The research consisted of a series of six case studies of students who were accurate computers for mental addition and subtraction. Each case was based on a series of interviews using a variety of instruments designed to probe the students’ mental computation strategies and the other factors identified from the literature.

The analysis of the interviews incorporated three stages. Firstly, each interview for each child was analysed separately. Secondly, relationships across interviews for each child were considered (e.g., whether understanding of noncanonical partitioning of numbers, evident in the numeration interview was used when solving the mental computation tasks). Thirdly, analysis compared commonalities and differences across students. Findings were cumulated across the cases to produce results that reflected the cases combined.

The subjects are now described in detail along with information regarding the Queensland context for mental computation. This is followed by descriptions of the instruments, the procedure followed in the interviews, and a summary of the form of analysis followed.

Subjects and context

The subjects were Year 3 students (approximately 8 years old) from two Independent Schools that served high and middle socio-economic areas in Brisbane, Queensland. The students were selected (from a population of three Year 3 classes, 60 students in all) after participating in a structured mental computation selection interview. As a result of their performance on the selection items, six students were identified as accurate as follows: (a) four of these students employed a variety of efficient mental strategies (e.g., aggregation, wholistic) in solving the mental computation tasks (they were designated as accurate and flexible), and (b) two employed one mental strategy throughout the selection interview, and this strategy was mental image of pen and paper algorithm (designated as accurate and inflexible).

Addition and subtraction mental computation, as defined in this study, is not mentioned in the existing Queensland curriculum document, Years 1 to 10 mathematics teaching, curriculum and assessment guidelines (Department of Education, Queensland, 1987a). Nevertheless, in some of the support documents (e.g., Years 1 to 10 mathematics sourcebooks – Department of Education, Queensland, 1987b, 1988, 1990) specific mental computation strategies are mentioned. However, the mental strategies for two-digit addition that are mentioned in the Year 7 (approximately 12 year olds) sourcebook are taught to Dutch students in second grade (Beishuizen, 1993). Therefore, the students in this study had not been taught mental strategies; they had only learnt the written algorithms for addition and subtraction (2-digit addition and subtraction with regrouping and 3-digit addition and subtraction without regrouping). However, in one of these classrooms, the teacher encouraged the students to at least think about alternative strategies for examples involving nines (e.g., adding 99 is the same as adding 100 and taking 1).

In some of the support documents (e.g., Department of Education, Queensland, 1991), steps for students learning the algorithms are set out. The following figures outline the language, materials and symbolic representations for the addition and subtraction algorithms.

******INSERT FIGURE 1 HERE******

******INSERT FIGURE 2 HERE******

Note that the words “regroup” or “trade” are recommended for regrouping in the addition algorithm, and “exchange” or “change” are used for regrouping in the subtraction algorithm. While “carry” and “borrow – pay back” are never taught, some students seem to pick these terms up, possibly from parents. While the algorithms are taught with some
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understanding, they soon become rote procedures, without much understanding of place values, etc.

**Instruments**

The six students participated in mental computation selection interviews and, then, in indepth interviews that: (a) addressed mental computation strategies, number facts, computational estimation, numeration, and effect of operation on number, and, while doing this, (b) investigated metacognition and affect.

**Selection interviews**

*Mental computation selection interview items.* Firstly, since the focus was on eliciting mental computation strategies, word problems were presented (the question was verbalised by the researcher, while the numbers and pictures were presented on cards). Several researchers (e.g., Carraher, Carraher, & Schliemann, 1987; Cooper, Heirdsfield, & Irons, 1996; Heirdsfield, 1996) found that presentation of word problems rather than number exercises resulted in more mental strategies being used, rather than algorithmic procedures. However, the focus was not on problem solving, but rather computational procedures; therefore, word problems were of the simplest forms, that is, join addition and separate subtraction (Carpenter & Moser, 1983). Further, as money is a context to which most students are accustomed, all problems incorporated whole dollars (Cooper, Heirdsfield, & Irons, 1996, successfully used this format). Secondly, questions were presented where numbers were visible (c.f., Hope & Sherrill, 1987; Reys, Reys, & Hope, 1993; Reys, Reys, Nohda, & Emori, 1995), as the intention was not to place too much demand on short-term memory (Sowder, 1990). Addition tasks were presented so that the larger addend was sometimes on the left of the picture, and sometimes it was on the right. Similarly, in the subtraction tasks, the minuend was sometimes on the left, and sometimes on the right. Finally, one-, two-, and three-digit number combinations, commensurate with the Year 3 syllabus (Department of Education, Queensland, 1991) were presented. Also, some 3-digit regrouping addition and subtraction examples were presented. In summary, word problems of the simplest forms were presented visually, that is, the numbers were visible as whole dollars (see Figure 3).

**********INSERT FIGURE 3 HERE********

**Indepth interviews**

*Number facts test and interview.* Addition and subtraction number facts were assessed in a combined timed written test and interview, when the child explained the solution strategies. The test consisted of eight addition (e.g., 8+7) and eight subtraction number facts (e.g., 13-6) to 20.

*Mental computation indepth interview items.* The mental computation tasks presented in the semi-structured indepth interviews were similar to those presented in the selection interviews. The presentation format was the same as that for the selection interviews (see Figure 1). The number combinations are presented in Table 2.

Other issues were also addressed during these mental computation interviews. Before the presentation of the tasks, questions addressing affects and self-efficacy were asked of each child. These questions included:

- Are you good at maths?
- Why do you think you are good/not so good?
- Do you ever work out “sums” in your head? Is it useful?
- Do you think it’s important to be able to work things out in your head? Why/why not?
- Do you like solving things in your head?
- Do you think you will be able to solve these?
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While the mental computation tasks were presented, further questions were presented to probe for: (a) number fact knowledge, evidence of understanding of numeration and effect of operation on number; (b) evidence of metacognition, attribution, and beliefs; and (c) access to alternative strategies (with and without help). Questions focused on what Van der Heijden (1994) called structured helping questions:

- Can you think of another way of solving the problem?
- What is __ close to?
- Can you work with this number?
- What can you do now?

If the students successfully accessed an alternative strategy (with or without help), they were asked which strategy was preferred, and why.

Further question focused on the students’ recollections of their experience:

- Why do you think you used that method?
- How do you know if you are correct?
- Are you seeing anything in your head (pictures/numbers)?

Computational estimation tasks. The items (see Figure 4 for an example) used in the computational estimation interviews were closed and open word problems, presented in an interview situation, based on those used by Case and Sowder (1990), Heirdsfield (1996), Rubenstein (1985), and Threadgill-Sowder (1984). Further, to avoid exact calculations, numbers that were too large for easy calculation were included (Heirdsfield, 1996; LeFevre, Greenham, & Waheed, 1993; Reys, Reys, Nohda, Ishida, Yoshikawa, & Shimizu, 1991; Sowder & Wheeler, 1989). As in the mental computation interviews, further questions were presented in order to probe for evidence of understanding of the effect of operation on number. Other questions probed for evidence of metacognition and beliefs. These questions included:

- Do you know what it means to estimate? What does it mean?
- Do you ever estimate? When?
- Do you think it’s useful to be able to estimate? Why?

Tasks for numeration. Here, the tasks investigated understanding of place value, renaming and regrouping, and multiplicativity. Tasks were based on those of Bednarz and Janvier (1982), Resnick and Omanson (1987), Ross (1990), and Sierink and Watson (1991). They included the understanding of canonical and noncanonical representations of number, and understanding of the multiplicative structure of the number system.

Tasks for the effect of operation on number. To investigate understanding of the effect of operation on number, examples aimed at investigating properties or principles of whole number operations (e.g., subtraction is the inverse of addition; effects of changing the addend, subtrahend, and minuend) were drawn from McIntosh, Reys, and Reys (1992), Sowder (1992), and Sowder and Wheeler (1987). Understanding the effect of changing the addend and subtrahend seemed to be particularly pertinent for the employment of the wholistic strategies in both addition and subtraction (e.g., 246+99=246+100-1; 265-99=265-100+1).

Tasks for metacognition and affective factors. Specific instruments were not developed for metacognition and affects in the same way as they were for the other interviews. What was designed was a checklist of types of comments that would be noted as indicating metacognitive knowledge and affective traits, and specific follow-up questions to use when responses were inadequate and lacked clarity. These would be used throughout the interviews, but not necessarily all together at a particular instance. However, the specific questions were also used at the end of the interviews if responses across the interviews were inadequate with respect to metacognitive and affective factors.
The comments made regularly by students that provided indication, for instance, of their self confidence; beliefs about mathematics; and planning and monitoring, included statements such as “I don’t like this”, “I can’t do this”, “I like these”, and “No, that won’t work”, because most of these factors were observed in previous interviews (e.g., mental computation, number facts). In addition to those questions mentioned previously, other questions that addressed the students’ perceptions of number facts were presented (e.g., “Are you good at number facts?” “Is it important to know number facts?” “Why?”)

To complement the data on beliefs and self-efficacy all students completed a modified version of the Student Preference Survey (SPS) (McIntosh, 1996). The students were to indicate if they would compute various addition and subtraction examples mentally (these were matched examples in the mental computation interviews to confirm whether they could compute the examples mentally).

**Interview procedures**

All 60 students from the three classes participated in videotaped selection interviews, where they were withdrawn from class and interviewed in a quiet room. The students were presented with the mental computation tasks, asked to solve each example and then explain their strategy. From the students’ responses, six students were selected on the basis of flexibility and accuracy in mental computation for further indepth study in a series of longer semi-structured interviews.

These six students were then withdrawn from class and participated in a series of videotaped semi-structured clinical interviews in a quiet room in the school. As stated before, these interviews probed understanding of mental computation, number facts, computational estimation, numeration, and effect of operation on number; and investigated metacognition, and affects. The order for the interviews is shown in Figure 5.

*******INSERT FIGURE 5 HERE******

**Analysis**

**Selection interviews**

The mental computation strategies employed by each child for each question were documented, using the scheme in Table 1. A record was kept of the accuracy of these responses for each student, and the degree of flexibility. As a result of this analysis, the six accurate students (4 flexible and 2 inflexible) were selected for indepth interviews.

**Indepth interviews**

When analysing the number facts tests, three aspects were investigated: speed, accuracy, and strategy use. Strategies used by individuals in the number facts interviews were compared with those used in the mental computation interviews. Also, individual’s errors in the number facts test were compared with errors found in the mental computation interviews.

For the indepth mental computation interview, responses were analysed for strategy choice, flexibility, accuracy, access to alternative strategies, number facts knowledge, computational estimation, understanding of the effect of operation on number, numeration, affects, and evidence of metacognition. Analysis of the interviews investigating these individual factors was also undertaken, with the intention of exploring connections with mental computation.

Each student’s responses were summarised and these summaries were combined for each of the accurate computation types, accurate and flexible and accurate and inflexible, so that comparisons could be made between the two types of mental computers. From this comparison, factors associated with mental computation were identified as present in each of the types. The mental computation procedures were separated into steps, and flowcharts were formulated for each type to propose how the factors acted across and within the steps.
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Results

To lay the foundation for the flow charts to explain the differences between the two types of mental computers, the factors associated with mental computation identified for both types of students are described. This is followed by proposals in the form of flow charts for how these factors act in each of the steps for accurate and flexible students and then for accurate and inflexible students.

General findings

The factors associated with mental computation that accurate and flexible and accurate and inflexible students exhibited are described in detail in Heirdsfield and Cooper (2002). These factors are summarised here as a prelude to the models which consider the influence of the factors on each step in the mental computation procedure.

Although both groups of students were accurate, they used different mental computation strategies form each other. The flexible students employed strategies including separation (left to right, right to left and cumulative sum/difference) and wholistic. In contrast, the inflexible students employed mental image of pen and paper algorithm throughout the selection interview. However, in the indepth interview, they also employed other strategies; for instance, one student used aggregation left to right (with scaffolding) and the other used wholistic and separation right to left (with scaffolding and then later for other examples). Although this latter strategy is similar to mental image of pen and paper algorithm, and is sometimes indistinguishable, it is not exactly the same. When the student employed mental image of pen and paper algorithm, she used such terms as “answer column”, “carried the one”, and “moving the ... and changing it into a ...”. When she employed separation right to left, she used place value terms and referred to regrouping. It is interesting to note that the other inflexible student stated that she always viewed the examples as if they were set out on paper, no matter what strategy she accessed. Thus, although these two students were identified as being inflexible, they showed some flexibility in the indepth interview.

Only the inflexible students reported “seeing” numbers in their head, when calculating. They imagined the numbers moving, one under the other, to represent the algorithm. None of the flexible students reported “seeing” numbers while calculating; although one flexible student stated that he “saw” MAB (Multibase arithmetic blocks – see Figures 1 and 2), and this helped him “work things out in my head”.

For all students, fast and accurate number facts supported accuracy in mental computation (all students scored 100% on the number facts test, and all students completed the test quickly). This would make sense, as fast and accurate recall of number facts from long term memory should result in less load on working memory when more complex calculations are involved (as in mental computation of two- and three-digit addition and subtraction). This suggests that fast and accurate number facts are essential knowledge for accuracy in mental addition and subtraction.

In contrast, flexibility in mental computation was supported by number fact strategies. Students who were flexible in mental computation employed efficient number facts strategies (derived facts strategies) in the number facts test. Further, some of the number facts strategies were applied to mental computation strategies (e.g., 9+7: add 1 to 9, take 1 from 7, so 10+6=16; c.f., 148+99 is the same as 147+100). On the other hand, the inflexible students did not possess efficient number fact strategies. They resorted to count if the number fact was not known by recall, particularly for the interim calculations in the mental computation tasks.

Efficient mental strategies (e.g., wholistic and aggregation – see Table 1) required good numeration understanding; although, some numeration understanding was also required for procedural understanding of mental image of pen and paper algorithm (canonical and noncanonical understanding). In general, the more aspects of numeration understanding present, the greater the access to a variety of high-level mental strategies. One aspect of
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numeration that was not mentioned in the literature is labelled here as proximity of number. In order to access wholistic compensation for say, 234-99, students had to firstly recognise that 99 is close to 100, and that 100 is an appropriate number to use, rather than, say, 98. The inflexible students were scaffolded to make this choice, although they did not always proceed with the calculation using wholistic compensation. In fact, one inflexible student refused to use 100, and stayed with her first strategy of mental image of pen and paper algorithm. The other inflexible student was not able to access wholistic compensation even with scaffolding, as she did not recognise that 100 is close to 99. Instead, she chose 90 as being close to 99, and employed aggregation (234-90=144, 144-9=135), which, although was more efficient than her original choice of mental image of pen and paper algorithm, was not as efficient as wholistic compensation.

An analysis of specific numeration understanding that is necessary for mental computation strategies was reported in Heirdsfield and Cooper (Table 2, 2002). These understandings (and others) are summarised below:

- 246+199 (wholistic compensation)
  - 199 is close to 200 (proximity of number)
  - 246+200=446 (extension of number fact strategy of through 10)
  - 446-1=445 (extension of number fact strategy of through 10; effect of changing the addend; multiplicativity – take a one, not 100)

- 246+199 (mental image of pen and paper algorithm)
  - 9+6=15, 15=1 ten and 5 ones; 9+4+1=14, 14 tens = 1 hundred and 4 tens; 1+2+1 = 4, 4 hundreds (canonical and noncanonical understanding)

Understanding of the effect of operation on number with respect to changing the addend and subtrahend affected the ability to employ some wholistic strategies. Students who were flexible exhibited these number and operation understandings and, therefore, employed high-level strategies (e.g., wholistic compensation). These students also recognised proximity of one number to another; for instance, “199 is close to 200” or “99 is close to 100” to use wholistic compensation to solve addition (246+199) and subtraction (234-99) examples. It appeared that both numeration and effect of operation on number understandings were required for successful employment of wholistic mental strategies. In contrast, inflexible students lacked understanding of the effect of operation on number, and their numeration understanding was diminished.

In contrast to the findings of Reys, Bestgen, Rybolt, and Wyatt (1982), computational estimation did not support mental computation. Even proficient mental computers did not exhibit proficiency in computational estimation. One reason could be the students were too young to have developed estimation strategies (c.f., Case & Sowder, 1990). Heirdsfield (1996) found that some Year 4 students had developed some appropriate estimation strategies, but these strategies were probably developed outside the classroom. It is possible that students in Year 3 have not reached the vectorial stage, which Case and Sowder (1990) suggested was the earliest developmental stage at which true computational estimation could occur.

Although metacognition did not feature strongly in this study, there appeared to be differences in the metacognitive processes of the two groups of students. The accurate and flexible students showed more evidence of monitoring and checking (metacognitive strategies) than the accurate and inflexible students. The flexible students were heard to utter monitoring and checking statements, such as, “No, it wouldn’t be …”, “No, that can’t be right…”. The inflexible students showed no evidence of checking their solutions or monitoring the reasonableness of their answers. On the other hand, both groups were able to verbalise their metacognitive beliefs (perceptions of their abilities – accurate perceptions – they believed that they could complete the tasks and they could do so).
One last factor that distinguished the two types of accurate computers from each other was beliefs with regard to self and teaching. The flexible students were confident in the use of their own self-developed strategies (e.g., “I do it this way in class.” “The teacher doesn’t know I do it this way.”) and held ideas about the place of the teacher in the student’s learning (e.g., “I do it this way, even though I know it annoys Miss …”). In contrast, the inflexible students were confident in the teacher-taught algorithms (e.g., “I always do it this way – I’m used to it.”) and held high expectations for the place of the teacher in the learning process (e.g., “I always wait for the teacher to tell me if I’m right or wrong.”).

Summary
In summary, as described in Heirdsfield and Cooper (2002), the proficient accurate and flexible students appeared to use an integrated understanding of number facts (speed, accuracy, and efficient number facts strategies when facts could not be recalled automatically), numeration, and the effect of operation on number. They also exhibited some metacognitive strategies and accurate metacognitive beliefs, and held strong beliefs about their own strategies.

However, accuracy could still be achieved with less knowledge and fewer connections between knowledge (e.g., less number fact knowledge with regard to number fact strategies, less numeration understanding, and little understanding of the effect of operation on number). The accurate and inflexible students compensated for these deficiencies by employing teacher-taught strategies in which strong beliefs were held. Fast and accurate number facts and some numeration understanding supported these procedures. Although these students did not exhibit metacognitive strategies, they did exhibit metacognitive beliefs. Finally, the inflexible students stated that they “saw” numbers in their head.

The mental computation steps for accurate and flexible students
As was mentioned previously, the students who were accurate and flexible in mental computation possessed well-integrated knowledge bases. These students were fast and accurate with their number facts, and used efficient number facts strategies when facts were not known by recall. Also, number facts strategies were extended to efficient mental computation strategies (e.g., the derived fact strategy, through 10 [6+9: 6+10-1] became wholistic for 246+99: 246+100-1). Good numeration understanding (particularly canonical, noncanonical, multiplicative, and proximity of number) and some understanding of the effect of operation on number supported efficient mental strategies. Further, the combined understandings of numeration (particularly proximity of number) and the effect of operation on number appeared to be essential for employment of the mental computation strategy, wholistic compensation.

There was evidence of the flexible students possessing metacognitive strategies (e.g., monitoring, reflecting, regulating, and evaluation). Beliefs in self seemed to be associated with a belief about the place of the teacher in the student’s learning; for instance, confidence in self-initiated strategies (c.f., teacher-taught strategies) supported flexibility in mental computation.

To the question, “Do you see anything in your head when you are calculating?”, one flexible student stated that he “saw” MAB in his head. It would be expected that numbers would be represented in some visual form, yet no student reported this. Certainly, the students did not appear to be manipulating symbols in their head, but they did not report “seeing” numbers in any form.

To illustrate a specific case, consider the following example that was presented to the students: 246+199. One of the accurate and flexible students selected the efficient wholistic compensation mental strategy (246+199=246+200-1). Analysis of her responses showed that she used understandings of the following factors: numeration (in particular, proximity of number and multiplicativity), effect of operation on number (effect of changing the addend),
and number facts (speed and accuracy, at a minimum; and more importantly, the extension of the number fact strategy, through 10). The student held accurate beliefs about her ability to perform the calculation, and exhibited metacognition when choosing the strategy. The student also used all these understandings for implementing the strategy. Metacognitive strategies were also used for monitoring the progress of the calculation and evaluating the answer in the final stage of checking the solution.

Summary

In summary, the factors affecting the steps of the mental computation process of proficient mental computers (accurate and flexible) are presented in Figure 6. As the figure shows, the effect of the factors are interrelated and complex. This flowchart is a general representation, and is dependent on the tasks presented to the students, that is, different number combinations and operations required different aspects of the flowchart to be drawn upon. The ability to choose an efficient mental strategy was supported by a broad numeration understanding (canonical, noncanonical, multiplicativity, and proximity of number), number facts (particularly number facts strategies), metacognition (beliefs and strategies), understanding the effect of operation on number, and strong beliefs about their own strategies. Numeration understanding, number facts, metacognition, and the effect of operation on number understanding were involved again when the accurate and flexible students implemented their strategies in the third step – implementing the strategy. Finally, metacognitive strategies supported the final stage, checking.

********INSERT FIGURE 6 HERE**********

The mental computation steps for accurate and inflexible students

The students who were accurate, but did not use alternative and efficient mental strategies (they used mental image of pen and paper algorithm), were similar to students who were both accurate and flexible in only a few factors. Comparing the inflexible students with the flexible students, it was evident that the inflexible students had more limited and less connected knowledge bases than those of the flexible students.

It was evident that fast and accurate number facts supported accuracy in mental computation. However, number facts strategies did not seem to be important, as one inflexible student used count as a backup strategy in both the number facts test and for interim calculations in mental computation. Although the other inflexible student possessed some efficient derived facts strategies, she did not employ them in mental computation. Instead, she also resorted to count when number facts could not be recalled.

Numeration was not well understood; however, there did seem to be some threshold knowledge of canonical and noncanonical numeration. Canonical and noncanonical understanding contributed to successful employment of pen and paper algorithms. Knowledge of multiplicativity was not evident; for instance, when one inflexible student was scaffolded to use wholistic for 265-99, she proceeded 265-100, but then did not know whether to add a one or one hundred. Although one of the inflexible students recognised proximity of number (e.g., 99 is close to 100), she mostly refused to use the information to access the efficient strategy of wholistic. The other student did not recognise the concept at all; in fact, she accessed aggregation when scaffolded by the question, “What is 99 close to?” She changed 99 to 90. It is posited that the absence of knowledge of proximity of number and knowledge of the effect of operation on number resulted in students’ not employing the mental strategy wholistic compensation.

Strong beliefs in teacher-taught strategies and teacher feedback contributed towards selection of the teacher-taught strategies for mental computation. Although these students did not demonstrate metacognitive strategies, they did hold accurate perceptions of their ability to perform the tasks (metacognitive beliefs). The students might have felt no need to check their solutions, as they believed so strongly in the accuracy of the procedure they used.
Finally, the inflexible students reported “seeing” the algorithm in their heads and this might have supported manipulation and storage of the calculations.

For the specific example of 246+199, the accurate and inflexible students applied the automatic strategy, *mental image of pen and paper algorithm*, used fast and accurate number facts, used canonical numeration understanding, and possessed the metacognitive belief of accurate perception of ability to complete the task.

**Summary**

In summary, the factors affecting the mental computation process of the inaccurate students are presented in Figure 7. As the figure shows, although these students did not possess a complexity of contributing factors, recourse to a well learnt automatic strategy resulted in accuracy. In other words, applying an automatic strategy compensated for limited knowledge.

For the accurate and inflexible students, the automatic strategy, *mental image of pen and paper algorithm* was selected. The strategy did not depend on the number or operation presented in the tasks. Then, certain factors were essential for successful implementation of the strategy, but not necessary for the initial “choice” of strategy. These factors were fast and accurate number facts, canonical and noncanonical numeration understanding, possession of metacognitive beliefs (accurate perception of their ability to complete the task), and beliefs about the teacher-taught procedures (see Figure 7). Finally, there was little evidence of the inflexible students checking their solutions, possibly because of strong beliefs in the success of the procedure they used.

***INSERT FIGURE 7 HERE**********

**Discussion and conclusions**

All students were able to recognise the numbers and operation (the first step in the flowchart) involved in each problem. It was firstly at the second step that differences were noted. The flexible students were able to choose efficient mental strategies. The ability to choose was supported by a broad numeration understanding (canonical, noncanonical, multiplicativity, and proximity of number), number facts knowledge (particularly number facts strategies), metacognition (strategies and beliefs), effect of operation on number understanding, and strong beliefs in their own strategies. When implementing the strategy, most of these factors were involved again. Finally, metacognitive strategies supported the final stage, checking. Therefore, the factors affecting the mental computation process of proficient mental computers were interrelated and complex (see Figure 6).

If, on the other hand, as in the case of accurate and inflexible students, an automatic strategy was selected (and this strategy was *mental image of pen and paper algorithm*), certain factors were essential for successful implementation of the strategy, but not necessary for the initial “choice” of strategy. These factors were fast and accurate number facts, canonical and noncanonical numeration understanding, possession of metacognitive beliefs (accurate perception of their ability to complete the task), and beliefs about teacher-taught strategies (see Figure 7). Further, “seeing” the algorithm might have supported the implementation of the mental strategy. So, although these students did not possess a complexity of contributing factors, recourse to a well learnt automatic strategy resulted in accuracy. In other words, applying an automatic strategy compensated for limited knowledge. Finally, there was little evidence of the inflexible students checking their solutions.

As Blöte, Klein, and Beishuizen (2000) suggested, the number characteristics of the tasks affected the strategies the students chose. This was only the case for the flexible students, though. The inflexible students resorted to an automatic strategy. This was partly explained by the “blind faith” in the teacher-taught written algorithms, and partly explained
by the diminished knowledge that could not support more efficient mental strategies. These students did not exhibit number sense and yet were accurate (c.f., Heirdsfield, 1996; McIntosh & Dole, 2000). In contrast, the flexible students exhibited number sense (c.f., Klein & Beishuizen, 1994; McIntosh, 1998; Reys, Reys, Nohda, & Emori, 1995), and rather than following teacher-taught procedures, held strong beliefs in their own self-developed and efficient strategies.

The authors here agree with the contention that meaningful mental computation promotes the development of number sense (e.g., Blöte, Klein, & Beishuizen, 2000; Buzeika, 1999; Hedrén, 1999; Kamii & Dominick, 1998; Mclellan, 2001). If we are to encourage students to develop their own efficient mental computation strategies, it appears that we need to intervene in the second step of the flowchart – beliefs. Teachers need to focus on promoting students’ thinking, rather than teaching them written procedures that do not support the development of number sense, and to expect students to use their self-developed strategies. That way, the students will value their strategies, rather than resort to teacher-taught procedures without thought for the number combinations or context. Certainly, students should be encouraged to develop and adopt more efficient strategies that reflect understanding. This can be promoted through classroom discussions and promoting metacognitive learning.

Beliefs in self-developed strategies and metacognitive strategies were absent from the flowchart that represented the process of the inflexible mental computer, so too were various understandings. Further research is required to investigate the best ways of developing these understandings to support efficient mental computation. Will these understandings develop when efficient mental strategies are encouraged? Do these understandings need to be explicitly addressed, so that efficient mental strategies can build on these understandings?

One might question why students need to be flexible, considering that some can still achieve a correct result without flexibility. The position that the authors (and Blöte, Klein, & Beishuizen, 2000) take is students “encounter difficult problems for which they have to create new procedures or modify old ones” (p. 244).

References


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*Figure 1.* Addition algorithm. (Source: Department of Education, 1991, p. 89)