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# Random fields of variable order on multifractal domains

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Abstract

We introduce a class of random field models with variable regularity/singularity order on multifractal domains. The derivation of results on atomic decomposition of functions in fractional Sobolev spaces of variable order, and traces of these spaces on multifractal domains plays a key role. The non-triviality of the multifractal spectrum of the measure defining the multifractal domain ensures the non-triviality of the multifractal spectrum of the functions in the trace Sobolev spaces of variable order defined. The generalized reproducing kernel Hilbert space theory is considered to introduce a class of random fields with non-trivial mean-square multifractal spectrum. In the Gaussian case, the random field models studied have multifractal sample paths.

## 1 Introduction

Multifractal models were originally introduced to study strange attractors (Hentschel and Pocaccia, 1983, Halsey et al., 1986) and the spatial distribution of the kinetic energy dissipation rate in fully developed turbulence (Benzi et al., 1984, Frisch and Parisi, 1985, Mandelbrot, 1974). They have been subsequently used in many areas of applied sciences. For example, multifractal models are considered in the characterization of pore systems in rocks (Muller and McCauley, 1992); spatial variability of bioactive marine sediments (Kropp et al., 1994); spatial variability of soil properties (Folorunso et al., 1994, Grout et al., 1998, Kravchenko et al, 1999, Posadas et al., 2001); hydraulic conductivity (Boufadel et al., 2003, Liu and Molz, 1997); scaling of intrinsic permeability (Boufadel et al., 2000); fluctuations of geophysical fields (Mandelbrot, 1989, Pecknold et al., 2001, Riedi, 1998). Topography, earhquake activity and surface gravity over various ranges also provide empirical evidence of multifractality (Lavallé et al., 1993, Lovejoy and Schertzer, 1990, Lovejoy et al., 2001, Marsan and Bean, 1999, Telesca et al., 2003, and Weissel and Pratson, 1994).

In this paper, we introduce a class of random fields with heterogeneous quadratic variation defined on multifractal domains. These random fields are restrictions, in the mean-square sense, to a multifractal domain of random fields defined on $\mathbb{R}^n$ with variable weak-sense (i.e. in terms of test functions) regularity order. Restrictions are also defined in the weak sense, in terms of the trace of the test function spaces involved. This consideration differs from the approaches where warped processes are considered (for example, warped fractional Brownian motion or warped fractional Lévy motion; see Riedi, 1999).

The regularity assumption on the function de…ning the singularity exponent of the variogram allows us to use the theory of fractional Sobolev spaces of variable order (see, for example, Jacob and Leopold, 1993; Kikuchi and Negoro, 1997). Models governed by pseudodi¤erential operators of variable order are introduced from this theory (see Ruiz-Medina et al., 2003, for the Gaussian random …eld case). Their solutions are functions with variable weak-sense singularity/regularity orders. Embeddings between fractional Besov spaces provide the relationship between the concepts of weak-sense singularity/regularity order and Hölder exponent (Leopold, 1999, Triebel, 1997, Samko, 1995). The local variation of the latter characterizes the singularities of a function, which will have a non-trivial singularity spectrum in the case where the singularity exponent presents erratic changes. For this reason, the regularity assumption on the variable order of the fractional Sobolev spaces involved in our approach leads to a trivial singularity spectrum of their functions. Indeed, functions with continuous singularity exponent have a trivial singularity spectrum. This is the case of sample paths of multifractional Brownian motion, where the singularity exponent is assumed to be Hölder continuous (see Benassi et al., 1997, Peltier and Lévy-Vehel, 1995). Alternatively, the concept of generalized multifractional Brownian motion provides an example of continuous Gaussian models which can be multifractal (see Ayache and Lévy-Vehel, 1999, and Ayache, 2000). However, as pointed out in Ayache (2000), 'the problem of constructing a continuous Gaussian process extending, in some sense, the fractional Brownian motion and that has an arbitrary prescribed Hausdor¤ spectrum is still open'. Here, we address this problem in a generalized framework. Speci…cally, we introduce a class of generalized random …elds with their reproducing kernel Hilbert space (RKHS) given by the trace of a fractional Sobolev space of variable order on a multifractal domain. In the ordinary case, random …elds in the class are mean-square Hölder continuous with non-trivial singularity spectra in the second-order moment sense, that is, the associated variogram family is multifractal. The regularity of the function de…ning the singular exponent of the variogram is a¤ected by the erraticity of the multifractal geometry of the domain where it is restricted, ensuring the non-triviality of the singularity spectrum.

In Section 2, some de…nitions and results on fractional Sobolev spaces of variable order, atomic decomposition of functions in Besov spaces of …xed order and multifractal measures are provided. In Section 3, an atomic decomposition of functions in fractional Sobolev spaces of variable order is obtained as an extension of the one given for functions in Besov spaces of …xed order. The de…nition of fractional Sobolev spaces of variable order on multifractal domains is provided in Section 4, considering the weak-sense restriction of their functions via the atomic decomposition derived in the previous section. In Section 5, a class of generalized random …elds of variable fractional regularity/singularity order on multifractal domains is introduced, and their covariance factorization is obtained. Such a covariance factorization leads to the de…nition of a linear …lter which relates generalized random …elds of variable regularity order with white noise on multifractal domains. The isomorphic relationship between the RKHS, defined by the covariance operator, and the trace on a multifractal domain of a fractional Sobolev space of suitable variable order is then derived in Subsection 6.1, where equivalent quasi-norms on the RKHS are defined in terms of the atomic decomposition of its functions. Section 7 provides the spectral properties of the associated covariance operators in terms of the order of the eigenvalues defining their pure point spectra. The local regularity/singularity properties of the functions in the RKHS allow the definition of conditions for the global mean-square Hölder continuity of the class of random fields studied. Under these conditions, the existence and uniqueness of the solution to the linear filter representation derived is proved. Since these functions have a variable local Hölder exponent, the local mean-square Hölder exponent of the associated random fields is obtained, and the non-triviality of its singularity spectrum is yielded by the multifractal geometry of the domain. These problems are addressed in Section 8. Examples of pseudodifferential models of variable order in the class of random fields considered are given in Section 8.1. Finally, Section 9 provides some concluding comments.

# 2 Preliminaries

We first introduce the basic definitions of fractional Sobolev spaces of variable order, the spaces $F^s_{pq}(R^n)$ and $B^s_{pq}(R^n)$ and their atomic decompositions in terms of local means, and multifractal measures.

Let $\delta$ and $\rho$ be real numbers with $0 \leq \delta < \rho \leq 1$; and let $\sigma$ be a real-valued function in $B^{\infty}(R^n)$; the space of all $C^{\infty}$- functions on $R^n$ whose derivatives of all orders are bounded. We say that a function $p(x;\xi) \in B^{\infty}\left(R^n_x \times R^n_\xi\right)$ belongs to $S^{\sigma}_{\rho;\delta}$ if and only if for any multi-indices $\alpha$ and $\beta$ there exists some positive constant $C_{\alpha;\beta}$ such that

$$|D^{\alpha}_{\xi} D^{\beta}_{x} p(x;\xi)| \leq C_{\alpha;\beta} \langle\xi\rangle^{\sigma(x) - \rho|\alpha| + \delta|\beta|}; \tag{1}$$

where $D^{\alpha}_{\xi}$ and $D^{\beta}_{x}$ respectively denote the derivatives with respect to $\xi$ and $x$; and $\langle\xi\rangle = (1 + |\xi|^2)^{1/2}$: The following semi-norm is considered for the elements of $S^{\sigma}_{\rho;\delta}$:

$$|p|^{(\sigma)}_{l} = \max_{|\alpha+\beta| \leq l} \sup_{(x;\xi) \in R^n \times R^n} \left\{ |D^{\alpha}_{\xi} D^{\beta}_{x} p(x;\xi)| \langle\xi\rangle^{-\sigma(x) + \rho|\alpha| - \delta|\beta|} \right\}:$$

**Definition 1** (Kikuchi and Negoro 1995, 1997) For $u \in S(R^n)$; the set of rapidly decreasing Schwartz functions, and $p \in S^{\sigma}_{\rho;\delta}$, let $P : S(R^n) \longrightarrow S(R^n)$ be defined as

$$Pu(x) = (2\pi)^{-n} \int_{R^n} e^{ix\xi} p(x;\xi) \hat{u}(\xi) d\xi; \tag{2}$$

where $\hat{u}(\xi) = \int_{R^n} e^{-ix\xi} u(x) dx$ is the Fourier transform of $u$: We refer to $P = p(x; D_x)$ as a pseudodifferential operator of variable order with symbol $p \in S^{\sigma}_{\rho;\delta}$: The set of all pseudodifferential operators with symbol $p$ of the class $S^{\sigma}_{\rho;\delta}$ is denoted by $S^{\sigma}_{\rho;\delta}$:

A pseudodifferential operator $P \in S^{\sigma}_{\rho,\delta}$ is elliptic if there exist $c > 0$ and $M > 0$ such that

$$|p(x,\xi)| \geq c\langle\xi\rangle^{\sigma(x)}, \quad |\xi| \geq M. \tag{3}$$

Furthermore, $Q \in S^{\infty}_{\rho,\delta} = \bigcup_{m\in\mathbb{R}} S^{m}_{\rho,\delta}$ is said to be a left (resp. right) parametrix of $P$ if there exists $R_L \in S^{-\infty}_{\rho,\delta} = \bigcap_{m\in\mathbb{R}} S^{m}_{\rho,\delta}$ (resp. $R_R \in S^{-\infty}_{\rho,\delta} = \bigcap_{m\in\mathbb{R}} S^{m}_{\rho,\delta}$) such that

$$QP = I + R_L \quad (\text{resp.} \quad PQ = I + R_R),$$

where $I$ denotes the identity operator. A pseudodifferential operator $Q$ is a parametrix of $P$ if $Q$ is simultaneously a left and right parametrix of $P$.

**Definition 2** Let $\sigma$ be a real-valued function in $B^{\infty}(\mathbb{R}^n)$. The Sobolev space of variable order $\sigma$ on $\mathbb{R}^n$ is defined as

$$H^{\sigma(\cdot)}(\mathbb{R}^n) = \left\{ u \in H^{-\infty} = \bigcup_{s\in\mathbb{R}} H^{s}(\mathbb{R}^n) : \langle D_{\cdot}\rangle^{\sigma(\cdot)} u \in L^2(\mathbb{R}^n) \right\}, \tag{4}$$

where

$$\langle D_x\rangle^{\sigma(x)} u = \int_{\mathbb{R}^n} (2\pi)^{-n} \exp(ix\xi)\langle\xi\rangle^{\sigma(x)} \hat{u}(\xi)d\xi, \tag{5}$$

with $\langle\xi\rangle = (1 + |\xi|^2)^{1/2}$, and

$$H^{s}(\mathbb{R}^n) = \left\{ u \in S'(\mathbb{R}^n) : \langle D_x\rangle^{s} u \in L^2(\mathbb{R}^n) \right\}.$$

We write $\underline{\alpha} = \inf_{x\in\mathbb{R}^n} \alpha(x)$ in the following.

**Proposition 1** (Kikuchi and Negoro 1997) The above introduced fractional Sobolev spaces of variable order satisfy the following properties:

(i) If $u \in H^{\sigma(\cdot)}(\mathbb{R}^n)$, then, for $P \in S^{\sigma}_{\rho,\delta}$, $Pu \in L^2(\mathbb{R}^n)$.

(ii) Let $\sigma_1$ and $\sigma_2$ be functions in $B^{\infty}(\mathbb{R}^n)$ with $\sigma_1(x) \geq \sigma_2(x)$, for each $x \in \mathbb{R}^n$. Then, $H^{\sigma_1(\cdot)}(\mathbb{R}^n) \subset H^{\sigma_2(\cdot)}(\mathbb{R}^n)$. In particular, $H^{\sigma(\cdot)}(\mathbb{R}^n) \subset H^{\underline{\sigma}(\cdot)}(\mathbb{R}^n)$.

(iii) $H^{\sigma(\cdot)}(\mathbb{R}^n)$ is a Hilbert space with the inner product

$$\langle u, v\rangle_{H^{\sigma(\cdot)}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \left(\langle D_x\rangle^{\sigma(x)} u\right)(x) \overline{\left(\langle D_x\rangle^{\sigma(x)} v\right)(x)}dx + \int_{\mathbb{R}^n} (\langle D_x\rangle^{\underline{\sigma}} u)(x)\overline{(\langle D_x\rangle^{\underline{\sigma}} v)(x)}dx. \tag{6}$$

Moreover, $S(\mathbb{R}^n)$ is dense in $H^{\sigma(\cdot)}(\mathbb{R}^n)$.

(iv) Let $\sigma$ and $\tau$ be functions in $B^{\infty}(\mathbb{R}^n)$. Suppose that $P \in S^{\sigma}_{\rho,\delta}$. Then, there exist some constant $C > 0$ independent of $P$ and some positive integer $l$ depending only on $\sigma, \tau, \rho, \delta$, and $n$ such that

$$\|Pu\|_{H^{\tau(\cdot)}(\mathbb{R}^n)} \leq C|p|_l^{(\sigma)}\|u\|_{H^{\sigma(\cdot)+\tau(\cdot)}(\mathbb{R}^n)},$$

for $u \in H^{\sigma(\cdot)+\tau(\cdot)}(\mathbb{R}^n)$, which provides the continuity of $P$ from $H^{\sigma(\cdot)+\tau(\cdot)}(\mathbb{R}^n)$ into $H^{\tau(\cdot)}(\mathbb{R}^n)$.

Theorem 1 (Kikuchi and Negoro 1997) Let $P \in S^{\sigma}_{\rho;\delta}$ be elliptic. Then,

$$H^{\sigma(\cdot)}(R^n) = \left\{ u \in H^{-\infty}(R^n) : \ Pu \in L^2(R^n) \right\} \tag{7}$$

as a set. Moreover, the norm $\|u\|_{H^{\sigma(\cdot)}(R^n)}$ is equivalent to the norm

$$\|u\|_{H^{\sigma(\cdot);P}(R^n)} := \left( \|Pu\|^2_{L^2(R^n)} + \|u\|^2_{H^{\underline{\sigma}}(R^n)} \right)^{1/2}. \tag{8}$$

The following results on embeddings and lifting properties for fractional Sobolev spaces of variable order on $L^p(R^n)$ hold (see Jacob and Leopold, 1993).

Theorem 2 Let $1 < p < \infty$ and $j \in N$; and let $\sigma(x) = s + \psi(x)$; with $\psi \in S(R^n)$; satisfying $0 < m' \le \sigma(x) \le m \le 2$; for all $x \in R^n$. Then, the following assertions hold:

(i) The space

$$H_p^{j;\sigma(\cdot)}(R^n) = \left\{ f \in S'(R^n) : \langle D_x \rangle^{j\sigma(x)} f \in L^2(R^n) \right\}$$

is a Banach space and $C_0^{\infty}(R^n)$ is dense in this space.

(ii) For $m'j > n/p$; the embedding of $H_p^{j;\sigma(\cdot)}(R^n)$ into $C^{\infty}(R^n)$ is continuous.

We now define the spaces $F^s_{pq}(R^n)$ and $B^s_{pq}(R^n)$ via a dyadic resolution of unity, which is given in terms of a function $\varphi \in S(R^n)$ with

$$\varphi(x) = 1 \quad \text{if} \quad |x| \le 1; \quad \text{and} \quad \varphi(x) = 0 \quad \text{if} \quad |x| \ge 3/2.$$

We consider a sequence of functions $\varphi_k(x)$ defined by $\varphi_0(x) = \varphi$; $\varphi_1(x) = \varphi\left(\frac{x}{2}\right) - \varphi(x)$ and

$$\varphi_k(x) = \varphi_1(2^{-k+1}x); \quad x \in R^n; \ k \in N.$$

Then,

$$1 = \sum_{k=0}^{\infty} \varphi_k(x); \quad \forall x \in R^n.$$

Note that the inverse Fourier transform of $\varphi_k \hat{f}$ is an entire function on $R^n$ for any $f \in S'(R^n)$; and can be defined pointwise. The spaces $F^s_{pq}(R^n)$ and $B^s_{pq}(R^n)$ can then be defined as follows.

(i) For $s \in R$; $0 < p < \infty$ and $0 < q \le \infty$; the space $F^s_{pq}(R^n)$ is the set of functions $f \in S'(R^n)$ such that

$$\|f|F^s_{pq}(R^n)\|_{\varphi} = \left\| \left( \sum_{j=0}^{\infty} 2^{jsq} |F^{-1}(\varphi_j \hat{f})(\cdot)|^q \right)^{1/q} \Big| L_p(R^n) \right\| < \infty.$$

(ii) For $s \in R$; $0 < p \le \infty$ and $0 < q \le \infty$; the space $B^s_{pq}(R^n)$ is defined as the set of functions $f \in S'(R^n)$ such that

$$\|f|B^s_{pq}(R^n)\|_{\varphi} = \left( \sum_{j=0}^{\infty} 2^{jsq} \left\| F^{-1}(\varphi_j \hat{f})(\cdot)|L_p(R^n) \right\|^q \right)^{1/q} < \infty; \tag{9}$$

with appropriate modification when $q = \infty$.

Let $k^0$ and $k_0$ be two $C^\infty$- functions in $\mathbb{R}^n$ with compact support and

$$\hat{k}_0(0) \neq 0; \quad \hat{k}^0(0) \neq 0. \tag{10}$$

Let

$$k_N(y) = \Delta^N k^0(y) = \left( \sum_{j=1}^{n} \frac{\partial^2}{\partial y_j^2} \right)^N k^0(y); \quad \text{for} \quad N \in \mathbb{N}. \tag{11}$$

For $x \in \mathbb{R}^n$; $t > 0$; $N \in \mathbb{N} - \{0\}$; the local means

$$k_N(t, f)(x) = \int_{\mathbb{R}^n} k_N(y) f(x + ty)\,dy = t^{-n} \int_{\mathbb{R}^n} k_N\left( \frac{y - x}{t} \right) f(y)\,dy \tag{12}$$

are defined for any $f \in S'(\mathbb{R}^n)$. We note that (12) can be reformulated as

$$\hat{k}_N(2^{-\nu}; f)(\xi) = \hat{k}_N(2^{-\nu}\xi)\hat{f}(\xi); \quad \nu \in \mathbb{N}.$$

It follows that

$$1 = \hat{k}_0(\xi) = \sum_{\nu=0}^{\infty} \hat{k}_N\left(2^{-\nu}\xi\right); \ \xi \in \mathbb{R}^n$$

(p.68, Triebel, 1997). The following representation is then obtained for each function $f \in B^s_{pq}(\mathbb{R}^n)$:

$$f = k_0(1; f)(x) + \sum_{\nu=1}^{\infty} k_N(2^{-\nu+1}; f)(x); \tag{13}$$

where convergence is considered in $S'(\mathbb{R}^n)$. For $0 < p \leq \infty$, let $\sigma_p = n\left(\frac{1}{p} - 1\right)_+$; where $(x)_+ = \max\{x; 0\}$. The following characterization in terms of local means hold:

(i) Let $0 < p < \infty$; $0 < q \leq \infty$ and $s \in \mathbb{R}$. Let $N \in \mathbb{N}$ with $2N > \max(s; \sigma_p)$. Then,

$$\|k_0(1; f)|L_p(\mathbb{R}^n)\| + \left\| \left( \sum_{j=1}^{\infty} 2^{jsq} |k_N(2^{-j}; f)(\cdot)|^q \right)^{1/q} \Bigg| L_p(\mathbb{R}^n) \right\|$$

and

$$\|k_0(1; f)|L_p(\mathbb{R}^n)\| + \left\| \left( \int_0^{\infty} t^{-sq} |k_N(t; f)(\cdot)|^q \frac{dt}{t} \right)^{1/q} \Bigg| L_p(\mathbb{R}^n) \right\|$$

are equivalent quasi-norms in $F^s_{pq}(\mathbb{R}^n)$.

(ii) Let $0 < p \leq \infty$; $0 < q \leq \infty$ and $s \in \mathbb{R}^n$. Let $N \in \mathbb{N}$ satisfying $2N > \max(s; \sigma_p)$. Then

$$\|k_0(1; f)|L_p(\mathbb{R}^n)\| + \left( \sum_{j=1}^{\infty} 2^{jsq} \|k_N(2^{-j}; f)|L_p(\mathbb{R}^n)\|^q \right)^{1/q} \tag{14}$$

and

$$\|k_0(1; f)|L_p(\mathbb{R}^n)\| + \left( \int_0^{\infty} t^{-sq} \|k_N(t; f)|L_p(\mathbb{R}^n)\|^q \frac{dt}{t} \right)^{1/q}$$

(with appropriate modification if $q = \infty$) are equivalent quasi-norms in $B^s_{pq}(\mathbb{R}^n)$.

The decomposition of $k_N(2^{-\nu}, f)(x)$ with respect to a lattice in $\mathbb{R}^n$ with side-length $2^{-\nu}$ leads to a series representation of functions in $B^s_{pq}(\mathbb{R}^n)$. Specifically, let $Q_{\nu m}$ be the cube in $\mathbb{R}^n$ with sides parallel to the coordinate axes, centered at $2^{-\nu}m$, with side lengh $2^{-\nu}$, for $m \in \mathbb{Z}^n$ and $\nu \in \mathbb{N} - \{0\}$. Let $Q$ be a cube in $\mathbb{R}^n$ and $r > 0$. Then $rQ$ is the cube in $\mathbb{R}^n$ concentric with $Q$ and with side length $r$ times the side length of $Q$. We consider $\psi \in S(\mathbb{R}^n)$ with compact support and with

$$\sum_{m \in \mathbb{Z}^n} \psi(x - m) = 1, \quad \forall x \in \mathbb{R}^n.$$

Then $\psi_{\nu m}(x) = \psi(2^\nu x - m)$ is a resolution of unity adapted to $2^{-\nu}\mathbb{Z}^n$ and there is a constant $d > 0$ such that the support of $\psi_{\nu m}$ is in $dQ_{\nu m}$, for each $\nu \in \mathbb{N} - \{0\}$ and $m \in \mathbb{Z}^n$. Multiplying the $\nu-$th term in equation (13) with that resolution of unity, the following series representation of $f$ is obtained:

$$f = \sum_{m \in \mathbb{Z}^n} \psi_{0m}(x) k_0(1, f)(x) + \sum_{\nu=1}^{\infty} \sum_{m \in \mathbb{Z}^n} \psi_{\nu m}(x) k_N(2^{-\nu+1}, f)(x). \tag{15}$$

The normalization of the local means on the cubes considered is given in terms of the constants $\lambda_{\nu m}$ defined, for $s > \sigma_p = n\left(\frac{1}{p} - 1\right)_+$, $0 < q \leq \infty$, and $K \in \mathbb{N}$ with $K > s$, by

$$\lambda_{\nu m} = 2^{\nu(s - \frac{n}{p})} \sum_{|\gamma| \leq K} \sup_{y \in dQ_{\nu m}} \left| (D^\gamma k_N)(2^{-\nu+1}, f)(y) \right|,$$

with $k_0$ and $1$ in place of $k_N$ and $2^{-\nu+1}$, respectively, for $\nu = 0$.

The optimal atomic decomposition of $f$ is then given by

$$f = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} a_{\nu m},$$

where convergence is considered in the space $S'(\mathbb{R}^n)$, and

$$a_{\nu m}(x) = \lambda_{\nu m}^{-1} \psi_{\nu m}(x) k_N(2^{-\nu+1}, f)(x); \quad \nu \in \mathbb{N}, \; m \in \mathbb{Z}^n.$$

From the above definition of the atoms $\{a_{\nu m} : m \in \mathbb{Z}^n, \nu \in \mathbb{N}\}$, the following properties are satisfied by the atomic decomposition: For $f \in B^s_{pq}(\mathbb{R}^n)$, $s > \sigma_p = n\left(\frac{1}{p} - 1\right)_+$, $0 < q \leq \infty$, and $K \in \mathbb{N}$ with $K > s$,

(i) $\operatorname{supp} a_{\nu m} \subset dQ_{\nu m}$; $\nu \in \mathbb{N}$, $m \in \mathbb{Z}^n$.

(ii) $|D^\gamma a_{\nu m}(x)| \leq c 2^{-\nu(s - \frac{n}{p}) + \nu|\gamma|}$; $|\gamma| \leq K$, where $c$ is independent of $\nu$, $m$ and $x$. Consequently,

$$\|a_{\nu m} | B^s_{pq}(\mathbb{R}^n)\| = O(1)$$

independently of $\nu$ and $m$.

(iii) From equation (14) (see also Theorem 12.9(ii), p.73, Triebel, 1997), $\|\lambda | b_{pq}\| \asymp \|f | B^s_{pq}(R^n)\|$; where $\lambda = \{\lambda_{\nu m} : \nu \in N;\ m \in Z^n\}$;

$$\|\lambda | b_{pq}\| = \left( \sum_{\nu=0}^{\infty} \left( \sum_{m \in Z^n} |\lambda_{\nu m}|^p \right)^{q/p} \right)^{1/q};$$

and $\asymp$ means that there exist two constants $0 < c_1 \leq c_2$ independent of $f$ such that

$$c_1 \|f | B^s_{pq}(R^n)\| \leq \|\lambda | b_{pq}\| \leq c_2 \|f | B^s_{pq}(R^n)\|.$$

The following result then provides a characterization of the space $B^s_{pq}(R^n)$ in terms of the atomic decomposition:

A function $f \in S'(R^n)$ belongs to the space $B^s_{pq}(R^n)$ if and only if it can be represented as

$$f = \sum_{\nu=0}^{\infty} \sum_{m \in Z^n} \lambda_{\nu m} a_{\nu m}(x);$$

where $a_{\nu m}$ satisfies

$$\begin{aligned} &\operatorname{supp} a_{\nu m} \subset dQ_{\nu m}; \\ &|D^{\alpha} a_{\nu m}(x)| \leq 2^{-\nu(s-n/p)+|\alpha|\nu}; \quad |\alpha| \leq K; \\ &\int_{R^n} x^{\beta} a(x)dx = 0 \quad \text{if} \quad |\beta| \leq L; \end{aligned} \tag{16}$$

where $K \leq (1+|s|)_+$ and $L \leq \max(-1; |\sigma_p - s|)$ are fixed. A similar assertion holds for functions in the space $F^s_{pq}(R^n)$ (see Theorem 13.8, p.75, Triebel, 1997).

We now review some basic definitions and facts on multifractal measures. In the characterization of multifractal measures, the following concepts of Hausdorff dimension, entropy dimension and local dimension play a key role (see, for example, Lau, 1999, Fan et al., 2000). Let $\mu$ be a Borel measure in $R^n$ in the following Definitions 3-5.

**Definition 3** The Hausdorff dimension of $\mu$ is defined as

$$\dim_H \mu = \inf\{\dim F : \mu(F^c) = 0\};$$

where, for a set $F$ of $R^n$, $\dim F$ denotes its Hausdorff dimension; and $F^c$ its complementary set.

**Definition 4** Let $P_n$ be the partition of $R^n$ into grid boxes $\prod_{i=1}^{n} [2^{-k} j_i; 2^{-k} j_{i+1}]$; with $j_i \in Z$; and let

$$H_n(\mu) = -\sum_{Q \in P_n} \mu(Q) \log \mu(Q).$$

The entropy dimension of $\mu$ is then defined as

$$\dim_e \mu = \lim_{n \to \infty} \frac{H_n(\mu)}{\log 2^n}.$$

The local dimension of $\mu$ is defined, for every point $z$ in the support of $\mu$, as

$$\alpha(z) = \lim_{l \to \infty} \frac{\log\left(\mu\left(Q_{2^{-l}}(z)\right)\right)}{-l \log 2}, \tag{17}$$

where $Q_{2^{-l}}(z)$ is a cube in $R^n$ centered at $z$ with sides parallel to the coordinate axes, and with side length $2^{-l}$. If the above limit does not exist, $\alpha(\cdot)$ is considered to be undefined. The local dimension $\alpha(z)$ of $\mu$ at $z$ defines its local Hölder exponent at this point. The elements

$$\alpha_l(z) = \frac{\log\left(\mu\left(Q_{2^{-l}}(z)\right)\right)}{-l \log 2} \tag{18}$$

of the sequence are referred to as the coarse Hölder exponents of $\mu$ at $z$.

**Remark 1** The coarse Hölder exponents of $\mu$ defined in terms of balls are treated in, for example, Ngai (1997).

As a consequence of the Frostman Lemma (Falconer, 1990) and a theorem of Young (1982), if the local dimension

$$\alpha(z) = \lim_{l \to \infty} \frac{\log\left(\mu\left(Q_{2^{-l}}(z)\right)\right)}{-l \log 2} = \alpha \quad \mu - a.e.,$$

then, $\dim_H \mu = \dim_e \mu = \alpha$.

The following dimension concepts are also considered based on the definitions of lim inf and lim sup in equation (17), which respectively provide the lower and upper dimensions of $\mu$ at $z$:

$$\begin{aligned}
\dim_* \mu &= \operatorname{ess\,inf} \liminf_{l \to \infty} \alpha_l(z); \\
\dim^* \mu &= \operatorname{ess\,sup} \liminf_{l \to \infty} \alpha_l(z); \\
\operatorname{Dim}_* \mu &= \operatorname{ess\,inf} \limsup_{l \to \infty} \alpha_l(z); \\
\operatorname{Dim}^* \mu &= \operatorname{ess\,sup} \limsup_{l \to \infty} \alpha_l(z);
\end{aligned} \tag{19}$$

where $\dim^* \mu$ coincides with the Hausdorff dimension of $\mu$.

**Definition 5** Let $f(\alpha)$ be the Hausdorff dimension of the set $K_\alpha = \{z; \alpha(z) = \alpha\}$, which is non-trivial if the singularity exponent $\alpha(z)$ varies as $z$ varies. We call $f(\alpha)$ the multifractal spectrum (also known as the singularity sprectrum), and refer to $\mu$ as a multifractal measure if $f(\alpha) \neq 0$ for a continuum of $\alpha$.

In connection with the multifractal formalism, the $L^q-$dimension of a measure plays a fundamental role. Specifically, the lower and upper $L^q-$dimensions ($q \neq 1$) of $\mu$ are respectively defined as

$$\begin{aligned}
\underline{\dim}_q &= \liminf_{n \to \infty} \frac{\log \sum_{Q \in P_n} \mu(Q)^q}{(q-1)\log 2^{-n}} \\
\overline{\dim}_q &= \limsup_{n \to \infty} \frac{\log \sum_{Q \in P_n} \mu(Q)^q}{(q-1)\log 2^{-n}}.
\end{aligned} \tag{20}$$

To investigate the multifractal structure of a measure, the function

$$\tau(q) = \liminf_{n\to\infty} \frac{\log \sum_{Q\in P_n} \mu(Q)^q}{\log 2^{-n}}$$

and its Legendre transform

$$\tau^*(\alpha) = \{\alpha q - \tau(q); q \in \mathbb{R}\}$$

are usually considered. The relationship

$$f(\alpha) = \tau^*(\alpha)$$

is known as the multifractal formalism. It was developed as a means to compute the multifractal spectrum $f(\alpha)$ in the physics literature (see Hentschel and Procaccia, 1983, Frisch and Parisi, 1985, Halsey et al., 1986). A rigorous treatment of the multifractal formalism is provided in Lau (1999).

# 3 Atomic decomposition and fractional Sobolev spaces of variable order

In this section, we obtain the atomic decomposition of functions in fractional Sobolev spaces of variable order in terms of local means. The results on equivalent quasi-norms on these spaces will be used to obtain the desired properties of the atomic decomposition.

Note that the wavelet transform is an important example of local means. Positive local Hölder exponents are characterized in terms of the continuous wavelet transform in Jaffard (1997) via the spaces $C^\alpha(x_0)$; $x_0 \in \mathbb{R}^n$; $\alpha > 0$; of functions $f$ for which there exists a polynomial $P$ of degree less than $\alpha$ such that

$$|f(x) - P(x - x_0)| \leq C|x - x_0|^\alpha: \tag{21}$$

The Besov spaces $B^s_{\infty;\infty}(\mathbb{R}^n) = C^s(\mathbb{R}^n)$ are particular cases of $C^\alpha(x_0)$ for $\alpha > 0$ when Equation (21) holds for any $x \in \mathbb{R}^n$ and the constant $C$ is uniform. Negative local Hölder exponents are defined in Jaffard (1997) via

**Definition 6** Suppose that $-n < \alpha \leq 0$: Then, $f \in C^\alpha(x_0)$ if

$$|f(x)| \leq C|x - x_0|^\alpha:$$

For $\alpha \leq -n$; $f \in C^\alpha(x_0)$ if $f \in B^\alpha_{\infty;\infty}$ and if $f$ restricted to $\mathbb{R}^n - \{x_0\}$ is a function satisfying

$$|f(x)| \leq C|x - x_0|^\alpha:$$

Negative local Hölder exponents $\alpha$ can also be obtained from the limiting behaviour

$$\alpha(x_0) = \liminf_{\varepsilon\to 0} h_\varepsilon(x_0)$$

of the coarse Hölder exponents $h_\varepsilon(x_0)$ defined as

$$h_\varepsilon(x_0) = \frac{1}{\log(2\varepsilon)} \log \sup_{|x-x_0|<\varepsilon} |f(x) - f(x_0)|$$

(see Riedi, 2003, where additional definitions of coarse singularity exponents, such as coarse wavelet singularity exponents, are also provided). In our approach, local Hölder exponents and their approximation in terms of coarse Hölder exponents are described in terms of the local behavior of the function defining the variable order of the Sobolev spaces $H^{\sigma(\cdot)}(\mathbb{R}^n)$ considered in the case where $\sigma(\cdot) = n/2 + s(\cdot)$.

In the following proposition, equivalent quasi-norms in the space $H^{\sigma(\cdot)}(\mathbb{R}^n)$ are formulated in terms of the local means of their functions, as defined in equation (12):

**Proposition 2** Let $H^{\sigma(\cdot)}(\mathbb{R}^n)$ be the fractional Sobolev space of variable order given in Definition 1. Then,

$$\|f|H^{\sigma(\cdot)}(\mathbb{R}^n)\|_* = \left[\sum_{j=1}^{\infty}\sum_{m\in\mathbb{Z}^n} \|2^{j\sigma(\cdot)}k_N(2^{-j+1}; f)(\cdot)|L^2(Q_{jm})\|^2 + \sum_{m\in\mathbb{Z}^n} \|k_0(1; f)|L^2(Q_{0m})\|^2\right]^{1/2}$$

and

$$\|f|H^{\sigma(\cdot)}(\mathbb{R}^n)\|_{**} = \left[\sum_{j=1}^{\infty}\sum_{m\in\mathbb{Z}^n} 2^{2(j\inf_{x\in Q_{jm}}[\sigma(x)]-n/2)}\left[\sup_{x\in Q_{jm}}\left|k_N(2^{-j+1}; f)(x)\right|\right]^2 + \sum_{m\in\mathbb{Z}^n}\left[\sup_{x\in Q_{0m}} |k_0(1; f)(x)|\right]^2\right]^{1/2} \tag{22}$$

are equivalent quasi-norms in $H^{\sigma(\cdot)}(\mathbb{R}^n)$.

**Remark 2** Note that for the equivalence between quasi-norms $\|\cdot|H^{\sigma(\cdot)}(\mathbb{R}^n)\|_*$ and $\|f|H^{\sigma(\cdot)}(\mathbb{R}^n)\|_{**}$ the assumption that $\sigma$ is Hölder continuous of order $\gamma > 0$ is sufficient.

**Proof.** The fractional Sobolev spaces of variable order given in Definition 1 can be equivalently defined as

$$H^{\sigma(\cdot)}(\mathbb{R}^n) = \left\{ u \in H^{-\infty} = \bigcup_{s\in\mathbb{R}} H^s(\mathbb{R}^n) : (-\Delta)^{\sigma(\cdot)/2} u \in L^2(\mathbb{R}^n) \right\}. \tag{23}$$

Hence, the norms of these spaces are equivalently given in terms of the $L^2(\mathbb{R}^n)$ norms of the functions $(-\Delta)^{\sigma(\cdot)/2}u$, with $u \in H^{\sigma(\cdot)}(\mathbb{R}^n)$. We consider a resolution of unity, $\{\varphi_j : j \in \mathbb{N}\}$, that is,

$$\sum_{j=0}^{\infty} \varphi_j = 1$$

with $\varphi_j$ having support given by

$$\{\xi \in \mathbb{R}^n : 2^{j-1} \leq |\xi| \leq 2^{j+1}\};$$

for each $j \in N$ (see Triebel, 1992). We then have

$$c_1 \int_{R^n} 2^{j\sigma(\cdot)} \varphi_j(\xi)\hat{f}(\xi)d\xi \le \int_{R^n} |\xi|^{\sigma(\cdot)} \varphi_j(\xi)\hat{f}(\xi)d\xi \le c_2 \int_{R^n} 2^{j\sigma(\cdot)} \varphi_j(\xi)\hat{f}(\xi)d\xi; \qquad (24)$$

with $c_1 = 1/2$ and $c_2 = 2$. Therefore, by definition,

$$\begin{aligned} \|f|H^{\sigma(\cdot)}(R^n)\|^2 &= \int_{R^n} \left| \int_{R^n} \exp\{\langle x,\xi\rangle\} |\xi|^{\sigma(x)} \hat{f}(\xi)d\xi \right|^2 dx \le c_2 \sum_{j=0} \int_{R^n} 2^{2j\sigma(x)} |\varphi_j(f)(x)|^2 dx \\ &\le c_2 \sum_{j=0} \sum_{m\in Z^n} \int_{R^n} |\psi_{jm}(x)|^2 2^{2j\sigma(x)} |\varphi_j(f)(x)|^2 dx \\ &\le \sum_{j=0} \sum_{m\in Z^n} \|2^{j\sigma(\cdot)} \varphi_j(f)|L^2(Q_{jm})\|^2; \end{aligned} \qquad (25)$$

where $\{\psi_{jm} : m \in Z^n\}$ is a resolution of unity adapted to $2^{-j}Z^n$, with $\mathrm{supp}(\psi_{jm}) \subset Q_{jm}$, and $Q_{jm}$ denoting, as before, the cube in $R^n$ with sides parallel to the coordinate axes, centered at $2^{-j}m$, and with side length $2^{-j}$, for all $m \in Z^n$ and $j \ge 0$. Also,

$$\|f|H^{\sigma(\cdot)}(R^n)\|^2 \ge c_1 \int_{R^n} \left| \sum_{j=0} \sum_{m\in Z^n} 2^{j\sigma(x)} \psi_{jm}(x)\varphi_j(f)(x) \right|^2 dx. \qquad (26)$$

In the case where $\varphi_j(\cdot) = \hat{k}_N(2^{-j+1}\cdot)$, for $j \in N - \{0\}$, and $\varphi_0(\cdot) = \hat{k}_0(\cdot)$, with $\hat{k}_N$ and $\hat{k}_0$ representing the Fourier transforms of the functions $k_N$ and $k_0$ in equation (11) whose support is in the unit ball, we have

$$|k_N(2^{-j+1}; f)(x)| \asymp 2^{-jh_j(x)}, \quad \forall j \in N - \{0\}; \qquad (27)$$

where $h_j(x)$ denotes the coarse Hölder exponent corresponding to the ball with center $x$ and radius $2^{-j+1}$ (see, for example, Jaffard, 1997 and Riedi, 2003 for the particular case of the continuous wavelet transform), and $\asymp$ signifies the order of magnitude. Then,

$$\begin{aligned} \|f|H^{\sigma(\cdot)}(R^n)\|^2 &\ge c_1 \sum_{j=0} \sum_{p=0} \sum_{m\in Z^n} \sum_{l\in Z^n} \int_{R^n} 2^{j\sigma(x)} 2^{p\sigma(x)} \psi_{jm}(x)\psi_{pl}(x)\varphi_j(f)(x)\varphi_p(f)(x)dx \\ &\ge c_1 \sum_{j=0} \sum_{m\in Z^n} \|2^{j\sigma(\cdot)} \varphi_j(f)|L^2(Q_{jm})\|^2. \end{aligned} \qquad (28)$$

Thus, the equivalence between $\|\cdot|H^{\sigma(\cdot)}(R^n)\|$ and $\|\cdot|H^{\sigma(\cdot)}(R^n)\|_*$ holds.

We now prove the equivalence between the quasi-norms $\|\cdot|H^{\sigma(\cdot)}(R^n)\|_*$ and $\|\cdot|H^{\sigma(\cdot)}(R^n)\|_{**}$: From equation (27),

$$\int_{Q_{jm}} 2^{2j\sigma(x)} \left| k_N(2^{-j+1}; f)(x) \right|^2 dx \asymp 2^{-2j\left(\inf_{x\in Q_{jm}}[h_j(x)+n/2-\sigma(x)]\right)} = 2^{-2j\left(\inf_{x\in Q_{jm}}[h_j(x)] - \sup_{x\in Q_{jm}}[\sigma(x)] + n/2\right)}.$$

Hence,

$$\sum_{j=0}^{\infty}\sum_{m\in\mathbb{Z}^n}\left[\left(2^{2j\left(\inf_{x\in Q_{jm}}[\sigma(x)]-n/2\right)}\right)\left(\sup_{x\in Q_{jm}}\left|k_N(2^{-j+1};f)(x)\right|\right)^2\right]$$
$$\leq \sum_{j=0}^{\infty}\sum_{m\in\mathbb{Z}^n}\int_{Q_{jm}} 2^{2j\sigma(x)}\left|k_N(2^{-j+1};f)(x)\right|^2 dx; \tag{29}$$

since, for each $j\in\mathbb{N}$ (with the usual interpretation for $j=0$), the following order holds:

$$\sup_{x\in Q_{jm}}\left|k_N(2^{-j+1};f)(x)\right| \asymp 2^{-j\left(\inf_{x\in Q_{jm}}[h_j(x)]\right)}:$$

For the converse inequality, for each $j\in\mathbb{N}$; and $m\in\mathbb{Z}^n$;

$$\int_{Q_{jm}} 2^{2j\sigma(x)}\left|k_N(2^{-j+1};f)(x)\right|^2 dx \leq 2^{2j\sup_{x\in Q_{jm}}[\sigma(x)]}\|k_N(2^{-j+1};f)|L^2(Q_{jm})\|^2$$
$$\leq \left(2^{2j\sup_{x\in Q_{jm}}[\sigma(x)]}\right)\left(\sup_{x\in Q_{jm}}\left|k_N(2^{-j+1};f)(x)\right|\right)^2 2^{-jn}$$
$$= \left(2^{2j\left(\sup_{x\in Q_{jm}}[\sigma(x)]-\inf_{x\in Q_{jm}}[\sigma(x)]+\inf_{x\in Q_{jm}}[\sigma(x)]-n/2\right)}\right)\left(\sup_{x\in Q_{jm}}\left|k_N(2^{-j+1};f)(x)\right|\right)^2$$
$$\leq C\left(2^{2j\left(\inf_{x\in Q_{jm}}[\sigma(x)]-n/2\right)}\right)\left(\sup_{x\in Q_{jm}}\left|k_N(2^{-j+1};f)(x)\right|\right)^2; \tag{30}$$

where the assumption of Hölder continuity of order $\gamma>0$ on $\sigma$ is sufficient to get the constant $C$ to be uniform with respect to $j$ and $m$. Thus, under this assumption and from equation (30);

$$\|f|H^{\sigma(\cdot)}(\mathbb{R}^n)\|_{*}^2 \leq C\|f|H^{\sigma(\cdot)}(\mathbb{R}^n)\|_{**}^2:$$

■

The atomic decomposition described in Section 2 is now reformulated for functions in the space $H^{\sigma(\cdot)}(\mathbb{R}^n)$ $(p=q=2)$. For each $j\in\mathbb{N}$; and $m\in\mathbb{Z}^n$; we consider

$$\lambda_{jm} = \left(2^{j\left(\inf_{x\in Q_{jm}}[\sigma(x)]-n/2\right)}\right)\sup_{x\in Q_{jm}}\left|k_N(2^{-j+1};f)(x)\right|: \tag{31}$$

Atoms of variable order are then defined, for each $f\in H^{\sigma(\cdot)}(\mathbb{R}^n)$; as

$$a_{jm}(x) = \lambda_{jm}^{-1}\psi_{jm}(x)k_N(2^{-j+1};f)(x); \quad \forall x\in\mathbb{R}^n; \tag{32}$$

for $j\in\mathbb{N}$; and $m\in\mathbb{Z}^n$; where $\psi_{jm}$ is given as in the proof of Proposition 2, and $k_N$ and $k_0$ satisfy suitable moment and regularity conditions according to the values $\inf_{x\in\mathbb{R}^n}[\sigma(x)]$ and $\sup_{x\in\mathbb{R}^n}[\sigma(x)]$:

**Lemma 1** For each $j\in\mathbb{N}$; and $m\in\mathbb{Z}^n$; the atoms $a_{jm}$ defined as in equation (32) satisfies

$$\text{(i)}\quad \operatorname{supp}(a_{jm})\subseteq Q_{jm};$$
$$\text{(ii)}\quad |a_{jm}(x)|\leq C2^{-j(\sigma(x)-n/2)};\quad \forall x\in\mathbb{R}^n;$$
$$\text{(iii)}\quad \|a_{jm}|H^{\sigma(\cdot)}(\mathbb{R}^n)\| = O(1); \tag{33}$$

where the constant $C$ is uniform with respect to $x$, $j$ and $m$.

**Remark 3** For the constant $C$ in (ii) to be uniform with respect to $x$, $m$, and $j$, the Hölder continuity of $\sigma$ of order $\gamma > 2$ is sufficient.

*Proof.* From the definition of $\psi_{jm}$ in Proposition 2 and equation (32), for each $j \in \mathbb{N}$ and $m \in \mathbb{Z}^n$,

$$\mathrm{supp}(a_{jm}) \subseteq Q_{jm}.$$

Inequality (ii) in equation (33) can now be derived:

$$\begin{aligned} |a_{jm}(x)| &= \frac{2^{-j\left(\inf_{x\in Q_{jm}}[\sigma(x)] - n/2\right)} |k_N(2^{-j+1}; f)(x)\psi_{jm}(x)|}{\sup_{x\in Q_{jm}} |k_N(2^{-j+1}; f)(x)|} \\ &\approx \frac{2^{-j\left(\inf_{x\in Q_{jm}}[\sigma(x)] - n/2 + h_j(x)\right)}}{2^{-j\left(\inf_{x\in Q_{jm}}[h_j(x)]\right)}} |\psi_{jm}(x)| \\ &\leq \frac{2^{-j(\sigma(x) - \varepsilon_j(x) - n/2)}}{2^{-j\left(h_j(x) - (\inf_{x\in Q_{jm}}[\sigma(x)] - n/2)\right)}}; \end{aligned} \tag{34}$$

where in the last inequality we have considered that $|\psi(x)| \leq 1$, and $\liminf_{j\to\infty} h_j(x) = \sigma(x) - n/2$, under the continuity of $\sigma$, since $h_j(x) \leq \inf_{x\in B(x, 2^{-j+1})}[\sigma(x) - n/2]$, with $B(x, 2^{-j+1})$ being the ball with center $x$ and radius $2^{-j+1}$. Hence, $h_j(x) = \sigma(x) - n/2 - \varepsilon_j(x)$, with $\lim_{j\to\infty} \varepsilon_j(x) = 0$. Therefore, inequality (34) can be rewritten as

$$\begin{aligned} |a_{jm}(x)| &\leq \frac{2^{-j(\sigma(x) - \varepsilon_j(x) - n/2)}}{2^{-j(-\varepsilon_j(x) + C_1|x - x_0|^{\gamma})}} \\ &\leq \frac{2^{-j(\sigma(x) - n/2)}}{2^{-j\left(C_1[2^{-j}(\sqrt{n}/2)]^{\gamma}\right)}} \\ &\leq C 2^{-j(\sigma(x) - n/2)}, \end{aligned} \tag{35}$$

where we have used

$$\begin{aligned} &\inf_{x\in Q_{jm}} \sigma(x) = \min_{x\in Q_{jm}} \sigma(x) = \sigma(x_0); \quad \text{with } x_0 \in Q_{jm}; \\ &|\sigma(x) - \sigma(x_0)| \leq C_1|x - x_0|^{\gamma}; \\ &|x - x_0| \leq 2^{-j}(\sqrt{n}/2); \quad \text{for } x, x_0 \in Q_{jm}; \quad \text{and} \\ &2^{-j\left(C_1[2^{-j}(\sqrt{n}/2)]^{\gamma}\right)} \to 1; \quad \text{when } j \to \infty. \end{aligned} \tag{36}$$

We finally consider identity (iii). From equation (32),

$$\|a_{jm}|H^{\sigma(\cdot)}(\mathbb{R}^n)\|^2 \simeq \int_{Q_{jm}} 2^{2j\sigma(x)} |a_{jm}(x)|^2 dx.$$

Then, from (ii), we obtain

$$\|a_{jm}|H^{\sigma(\cdot)}(\mathbb{R}^n)\|^2 \leq c_2 \|2^{j\sigma(\cdot)} a_{jm}|L^2(\mathbb{R}^n)\|^2 \leq c_2 C \int_{Q_{jm}} 2^{2j\sigma(x)} 2^{-2j(\sigma(x) - n/2)} dx = c_2 C.$$

∎

**Proposition 3** (i) The following atomic decomposition holds for every function $f$ in $H^{\sigma(\cdot)}$:

$$f(x)=\sum_{j=0}^{\infty}\sum_{m\in\mathbb{Z}^n}\lambda_{jm}a_{jm}(x), \tag{37}$$

with convergence being defined in the sense of Schwartz distributions, where $\lambda_{jm}$ and $a_{jm}$ are as defined in equations (31) and (32) respectively.

(ii) The quasi-norm

$$\inf \|\lambda|l^2\| = \left[\sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^n}|\lambda_{\nu m}|^2\right]^{1/2}$$

is equivalent to the norm in $H^{\sigma(\cdot)}$, where the infimum is taken over all admissible representations (37).

*Proof.* (i) Since

$$\hat{k}_0(\xi)+\sum_{j=0}^{\infty}\hat{k}_N(-2^{-j}\xi)=1,$$

the following spectral identity holds:

$$\hat{f}(\xi)=\hat{k}_0(\xi)\hat{f}(\xi)+\sum_{j=0}^{\infty}\hat{k}_N(-2^{-j}\xi)\hat{f}(\xi),$$

which leads to the weak-sense identity

$$f(x)=k_0(1,f)(x)+\sum_{j=0}^{\infty}k_N(2^{-j},f)(x). \tag{38}$$

Considering now, for each $j\in\mathbb{N}$, $\{\psi_{jm}: m\in\mathbb{Z}^n\}$ to be a resolution of unity adapted to $2^{-j}\mathbb{Z}^n$, equation (38) can be rewritten as

$$\begin{aligned}
f(x) &= \sum_{m\in\mathbb{Z}^n}\psi_{0m}(x)k_0(1,f)(x)+\sum_{j=1}^{\infty}\sum_{m\in\mathbb{Z}^n}\psi_{jm}(x)k_N(2^{-j+1},f)(x)\\
&= \sum_{m\in\mathbb{Z}^n}\lambda_{0m}\lambda_{0m}^{-1}\psi_{0m}(x)k_0(1,f)(x)+\sum_{j=1}^{\infty}\sum_{m\in\mathbb{Z}^n}\lambda_{jm}\lambda_{jm}^{-1}\psi_{jm}(x)k_N(2^{-j+1},f)(x)\\
&= \sum_{j=0}^{\infty}\sum_{m\in\mathbb{Z}^n}\lambda_{jm}a_{jm}(x).
\end{aligned} \tag{39}$$

Assertion (ii) follows from Proposition 2. ∎

# 4 Multifractal domain

In this section, we provide a characterization of fractional Sobolev spaces of variable order on multifractal domains via the atomic decomposition. This characterization allows the interpretation of these spaces as trace spaces of variable order on multifractal domains.

We consider a mulifractal measure $\mu_M$ with support $M$ having singularity exponent $\alpha(z)$ well-defined for every point $z$ in $M$, providing the local dimension of the multifractal domain $M$, and with non-trivial multifractal spectrum. The following two lemmas will be needed in the proof of the fundamental result of this section given in Proposition 4.

**Lemma 2** Let $M$ be a compact multifractal domain defined by a multifractal measure $\mu$ with local exponent $\alpha$ as given in equation (17). Then, $C_0^{\infty}(\mathbb{R}^n)$ restricted to $M$ is dense in $L^2_{\mu_M}(M)$, where $L^2_{\mu_M}(M)$ consists of tempered distributions $f$ on $\mathbb{R}^n$ defined by

$$f(\varphi) = \int_M f(\gamma)\varphi_M(\gamma)\mu_M(d\gamma), \quad \varphi \in S(\mathbb{R}^n), \tag{40}$$

with $\varphi_M$ representing the pointwise restriction of $\varphi$ to the compact set $M$.

**Proof.** By the definition of coarse Hölder exponent of $\mu$ at point $z \in \operatorname{supp}\mu$ (see equation (27)), $\mu$ is a Radon measure. Then, the proof follows from the Approximation Theorem for Radon measures (see Theorem 1.10, p.11, Mattila, 1995). ∎

**Lemma 3** Let $\psi$ be in $C_0^{\infty}(\mathbb{R}^n)$. Then,

$$\|tr_M \psi | H^{\frac{n-\alpha(\cdot)}{2}}(\mathbb{R}^n)\| \leq C\|tr_M \psi | L^2_{\mu_M}(M)\|, \tag{41}$$

where $tr_M$ represents the pointwise restriction of $\psi$ to $M$.

**Proof.** For each $j \in \mathbb{N}$, we can consider a finite covering of $M$ by the cubes $Q_{2^{-j}}(z_k)$, $k = 1,\dots,N_j$, centered at $M$, and with side length $2^{-j}$. Let $\varphi_k$, $k = 1,\dots,N_j$, be a smooth resolution of unity in a neighbourhood $\Omega_j$ of $M \cap \operatorname{supp}\psi$ subordinated to $Q_{2^{-j}}(z_k)$, $k = 1,\dots,N_j$. For $\lambda_{jk} = \max_{x \in Q_{2^{-j}}(z_k)} |\psi(x)|$, we have

$$\psi(x) = \sum_{k=1}^{N_j} \psi(x)\varphi_k(x) = \sum_{k=1}^{N_j} \lambda_{jk} 2^{-j\left(\sup_{y\in Q_{2^{-j}}(z_k)} \alpha(y)/2\right)} \left[ 2^{j\left(\sup_{y\in Q_{2^{-j}}(z_k)} \alpha(y)/2\right)} \lambda_{jk}^{-1}\psi(x)\varphi_k(x)\right], \quad x \in \Omega_j. \tag{42}$$

For each $k \in \{1,\dots,N_j\}$, the function $2^{j\left(\sup_{y\in Q_{2^{-j}}(z_k)} \alpha(y)/2\right)} \lambda_{jk}^{-1}\psi(\cdot)\varphi_k(\cdot)$ is a $\left(\frac{n-\alpha(\cdot)}{2}, 2\right)$-atom. From Proposition 3(ii), we then have

$$\|tr_M \psi | H^{\frac{n-\alpha(\cdot)}{2}}(\mathbb{R}^n)\| = \|\psi \sum_{k=1}^{N_j} \varphi_k | H^{\frac{n-\alpha(\cdot)}{2}}(\mathbb{R}^n)\| \leq C' \left[ \sum_{j=1}^{N_j} |\lambda_{jk}|^2 2^{-j\left(\sup_{y\in Q_{2^{-j}}(z_k)} \alpha(y)\right)} \right]^{1/2}.$$

Furthermore,

$$\left[ \sum_{k=1}^{N_j} |\lambda_{jk}|^2 2^{-j\left(\sup_{y\in Q_{2^{-j}}(z_k)} \alpha(y)\right)} \right]^{1/2} \leq C\|\psi | L^2_{\mu_M}(M)\|,$$

for $j \geq j_0$, with $j_0$ satisfying that

$$2^{-j_0\left(\inf_{k\in\{1,\dots,N_{j_0}\}} \left[\sup_{y\in Q_{2^{-j_0}}(z_k)} \alpha(y)\right]\right)} < \frac{\|\psi | L^2_{\mu_M}(M)\|^2}{\sup_{k\in\{1,\dots,N_{j_0}\}} |\lambda_{jk}|^2}.$$

∎

**Proposition 4** Let $M$ be a compact multifractal domain defining the support of a multifractal measure $\mu_M$ with local dimension $\alpha(\cdot)$ defined for every point $z \in M$. Then,

$$\mathrm{tr}_M \left( H^{\frac{n-\alpha(\cdot)}{2}}(\mathbb{R}^n) \right) = L^2_{\mu_M}(M);$$

where $\mathrm{tr}_M$ is used in the weak-sense, that is, by considering the test functions defined by the pointwise restriction to the set $M$ of the functions in the Schwartz space $S(\mathbb{R}^n)$.

*Proof.* We first prove the inclusion

$$\mathrm{tr}_M \left( H^{\frac{n-\alpha(\cdot)}{2}}(\mathbb{R}^n) \right) \subset L^2_{\mu_M}(M).$$

For $\phi \in H^{\frac{n-\alpha(\cdot)}{2}}(\mathbb{R}^n)$, we consider the atomic decomposition of the function $\mathrm{tr}_M \phi$ in the space $H^{\frac{n-\alpha(\cdot)}{2}}(\mathbb{R}^n)$ (see Proposition 3). The following inequalities then hold:

$$\|\mathrm{tr}_M \phi | L^2_{\mu_M}(M)\|^2 \leq \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \int_{M \cap Q_{jm}} |\lambda_{jm} a_{jm}(x)|^2 \mu_M(dx)$$
$$\leq c \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} |\lambda_{jm}|^2 = \|\lambda | l^2\|^2 \sim \|\mathrm{tr}_M \phi | H^{\frac{n-\alpha(\cdot)}{2}}(\mathbb{R}^n)\|^2; \quad (43)$$

where, in the last inequality, the identification of the density associated with $\mu_M$ with an element of the space $H^{-n/2+\alpha(\cdot);M}(\mathbb{R}^n) \subset C^{-n+\alpha(\cdot);M}(\mathbb{R}^n) = B^{-n+\alpha(\cdot);M}_{\infty,\infty}(\mathbb{R}^n)$ is effected. Here,

$$B^{-n+\alpha(\cdot);M}_{p,q}(\mathbb{R}^n) = \{ f \in B^{-n+\alpha(\cdot)}_{p,q}(\mathbb{R}^n) : f(\varphi) = 0 \quad \text{if} \quad \varphi \in S(\mathbb{R}^n);\ \mathrm{tr}_M(\varphi) = 0 \}$$

with $B^{-n+\alpha(\cdot)}_{p,q}(\mathbb{R}^n)$ being the Besov space of variable order $-n + \alpha(\cdot)$ with respect to the $L^p$ and $L^q$ norms (see Triebel, 1997, for the fixed order case). Such an identification comes from an extended concept of capacitary dimension of a measure adapted to the multifractal case, where the local dimension $\alpha$ of $\mu_M$ must be considered, leading to an extended definition of $\alpha(\cdot)$-energy in the form

$$I_{\alpha(\cdot)}(\mu) = \int \int |y - z|^{-\alpha(y)} \mu_M(dy) \mu_M(dz) \quad (44)$$

(see, for example, Falconer, 1985, and Mattila, 1995, for the fractal case). Indeed the order of the space $C^{-n+\alpha(\cdot);M}(\mathbb{R}^n)$ can also be interpreted in terms of an extended version of the concept of distribution dimension to the variable order case in relation to multifractal domains (see Triebel, 1997, for the fractal domain case).

The reverse inclusion follows from the inequality

$$\|\mathrm{tr}_M \phi | H^{\frac{n-\alpha(\cdot)}{2}}(\mathbb{R}^n)\| \leq c \|\mathrm{tr}_M \phi | L^2_{\mu_M}(M)\|; \quad \phi \in C_0^{\infty}(\mathbb{R}^n);$$

derived in Lemma 3. From Lemma 2, $C_0^{\infty}(\mathbb{R}^n)$ restricted to $M$ is dense in $L^2_{\mu_M}(M)$. Thus, for a function $\phi \in L^2_{\mu_M}(M)$, we can consider a series representation of $\phi$ in terms of functions in $C_0^{\infty}(\mathbb{R}^n)$ restricted to $M$. That is,

$$\phi(z) = \sum_{j=0}^{\infty} \phi_j(z), \quad \forall z \in M; \quad \phi_j \in C_0^{\infty}(\mathbb{R}^n).$$

In particular, for each $j \in \mathbb{N} - \{0\}$, we consider

$$\psi_j(z) = k_N(2^{-j+1};\psi)(z); \quad f_0(z) = k_0(1;\psi)(z); \tag{45}$$

for a suitable $k_N$ and $k_0$. We also have that

$$\begin{aligned} \|\mathrm{tr}_M \psi_j | L^2_{\mu_M}(M)\|^2 &= \int_M |\mathrm{tr}_M \psi_j(x)|^2 \mu_M(dx) \leq \sum_{k=1}^{N_j} \int_{Q_{2^{-j}}(z_{jk})} |\varphi_{jk}(x)\mathrm{tr}_M \psi_j(x)|^2 \mu_M(dx) \\ &\leq \sum_{k=1}^{N_j} \|\mathrm{tr}_M \psi_j | L^2_{\mu_M}(Q_{2^{-j}}(z_{jk}))\|^2; \end{aligned} \tag{46}$$

where $\varphi_{jk}$, $k = 1,\dots,N_j$, and $Q_{2^{-j}}(z_{jk})$, $k = 1,\dots,N_j$, are as defined in Lemma 3, for each $j \in \mathbb{N}$. From the particular choice of $\psi_j$, $j \in \mathbb{N}$, in equation (45),

$$\sum_{j=1}^{\infty} \sum_{k=1}^{N_j} \|\mathrm{tr}_M \psi_j | L^2_{\mu_M}(Q_{2^{-j}}(z_{jk}))\|^2 \leq C\|\mathrm{tr}_M(\psi) = \psi | L^2_{\mu_M}(M)\|^2$$

in a similar way to the proof of Proposition 2. Lemma 3 applied to functions $\psi_j$, $j \in \mathbb{N}$, and equations (46) and (??) then lead to the following inequalities:

$$\begin{aligned} \|\mathrm{tr}_M(\psi)|H^{\frac{n-\alpha(\cdot)}{2}}(\mathbb{R}^n)\|^2 &\leq \sum_{j=1}^{\infty} \|\mathrm{tr}_M(\psi_j)|H^{\frac{n-\alpha(\cdot)}{2}}(\mathbb{R}^n)\|^2 \\ &\leq C_1 \sum_{j=1}^{\infty} \|\mathrm{tr}_M(\psi_j)|L^2_{\mu_M}(M)\| \leq C_2\|\mathrm{tr}_M \psi | L^2_{\mu_M}(M)\|^2. \end{aligned} \tag{47}$$

■

**Corollary 1** Under the conditions assumed in the previous proposition, the following identity holds:

$$\mathrm{tr}_M\left(H^{s(\cdot)+\frac{n-\alpha(\cdot)}{2}}(\mathbb{R}^n)\right) = H^{s(\cdot)}(M). \tag{48}$$

**Proof.** We first prove that $\mathrm{tr}_M\left(H^{s(\cdot)+\frac{n-\alpha(\cdot)}{2}}(\mathbb{R}^n)\right) \subset H^{s(\cdot)}(M)$. From Lemma 1 and Propositions 2 and 3, we have

$$\begin{aligned} \|\mathrm{tr}_M f|H^{s(\cdot)}(M)\|^2 &\asymp \int_M |(-\Delta)^{s(z)} f(z)|^2 \mu_M(dz) \\ &= \int_M \left|(-\Delta)^{s(z)} \sum_{j=0}^{\infty}\sum_{m\in\mathbb{Z}^n} \lambda_{jm} a_{jm}(z)\right|^2 \mu_M(dz) \leq \sum_{j=0}^{\infty}\sum_{m\in\mathbb{Z}^n} \int_{M\cap Q_{jm}} 2^{2js(z)} |\lambda_{jm} a_{jm}(z)|^2 \mu_M(dz) \\ &\leq C \sum_{j=1}^{\infty}\sum_{m\in\mathbb{Z}^n} |\lambda_{jm}|^2 \int_{M\cap Q_{jm}} 2^{2js(z)} 2^{-2j\left(s(z)+\frac{n-\alpha(z)}{2}-n/2\right)} \mu_M(dz) + \sum_{m\in\mathbb{Z}^n} \int_{M\cap Q_{0m}} |\lambda_{0m}|^2 \mu_M(dz) \\ &\leq C_1 \sum_{j=0}^{\infty}\sum_{m\in\mathbb{Z}^n} |\lambda_{jm}|^2 \leq C_2\|\mathrm{tr}_M f|H^{s(\cdot)+\frac{n-\alpha(\cdot)}{2}}(\mathbb{R}^n)\|^2; \end{aligned} \tag{49}$$

where we have considered the atomic decomposition of the function $\mathrm{tr}_M f$ in the space $H^{s(\cdot)+\frac{n-\alpha(\cdot)}{2}}(\mathbb{R}^n)$.

To prove the inclusion $H^{s(\cdot)}(M) \subset H^{s(\cdot)+\frac{n-\alpha(\cdot)}{2}}(\mathbb{R}^n)$; for $f \in H^{s(\cdot)}(M)$; we can consider its decomposition in terms of local means,

$$\mathrm{tr}_M f(z) = f(z) = \sum_{j=0}^{\infty} k_N(2^{-j}; f) + k_0(1; f):$$

As in Lemma 3, for each $j \in \mathbb{N}$; let $Q_{2^{-j}}(z_k)$; $k = 1; \dots; N_j$; be a finite covering of $M$ by cubes centered at $M$; and with side length $2^{-j}$: Let also $\varphi_{jk}$; $k = 1; \dots; N_j$; be a smooth resolution of unity in a neighbourhood $\Omega_j$ of $M \cap \mathrm{supp}\, k_N(2^{-j}; f)$ subordinated to $Q_{2^{-j}}(z_{jk})$; $k = 1; \dots; N_j$; for $j \in \mathbb{N} - \{0\}$; and for $j = 0$; let $\varphi_{0k}$; $k = 1; \dots; N_0$; be a smooth resolution of unity in a neighbourhood $\Omega_0$ of $M \cap \mathrm{supp}\, k_0(1; f)$ subordinated to $Q_1(z_{0k})$; $k = 1; \dots; N_0$; Then,

$$\begin{aligned} k_N(2^{-j+1}; f)(z) &= \sum_{k=1}^{N_j} \varphi_{jk}(z) k_N(2^{-j+1}; f)(z); \quad \forall z \in M; \quad \forall j \in \mathbb{N} - \{0\}; \\ k_0(1; f)(z) &= \sum_{k=1}^{N_0} \varphi_{0k}(z) k_0(1; f)(z); \quad \forall z \in M; \end{aligned} \tag{50}$$

and

$$\begin{aligned} \|\mathrm{tr}_M f \mid H^{s(\cdot)+\frac{n-\alpha(\cdot)}{2}}(\mathbb{R}^n)\|^2 &\leq C \sum_{j=1}^{\infty} \sum_{k=1}^{N_j} \int_{Q_{2^{-j}}(z_{jk})} 2^{2j\left(s(z)+\frac{n-\alpha(z)}{2}\right)} |\varphi_{jk}(z) k_N(2^{-j+1}; f)(z)|^2 dz \\ &\quad + \sum_{k=1}^{N_0} \int_{Q_1(z_{0k})} |\varphi_{0k}(z) k_0(1; f)(z)|^2 dz \\ &\leq C \sum_{j=1}^{\infty} \sum_{k=1}^{N_j} \int_{Q_{2^{-j}}(z_{jk})} 2^{2j\left(s(z)+\frac{n-\alpha(z)}{2}\right)} |k_N(2^{-j+1}; f)(z)|^2 dz \\ &\quad + \sum_{k=1}^{N_0} \int_{Q_1(z_{0k})} |k_0(1; f)(z)|^2 dz \\ &\asymp \sum_{j=1}^{\infty} \sum_{k=1}^{N_j} \int_{Q_{2^{-j}}(z_{jk})} 2^{2js(z)} |k_N(2^{-j+1}; f)(z)|^2 \mu(dz) \\ &\quad + \sum_{k=1}^{N_0} \int_{Q_1(z_{0k})} |k_0(1; f)(z)|^2 \mu(dz) \\ &= \sum_{j=1}^{\infty} \sum_{k=1}^{N_j} \|2^{js(\cdot)} \mathrm{tr}_M k_N(2^{-j+1}; f) \mid L^2_{\mu_M}(Q_{2^{-j}}(z_{jk}))\|^2; \\ &\quad + \sum_{k=1}^{N_0} \|\mathrm{tr}_M k_0(1; f) \mid L^2_{\mu_M}(Q_1(z_{0k}))\|^2 \\ &\asymp \|\mathrm{tr}_M f \mid H^{s(\cdot)}(M)\|^2; \end{aligned} \tag{51}$$

where the first equivalence $\asymp$ is obtained since the order of the space $C^{-n+\alpha(\cdot); M}(\mathbb{R}^n)$ determines the local distribution dimension of $\mu_M$ and a related generalized concept of local energy (see equation (44); and Triebel, 1997, for the fractal case), and the last equivalence $\asymp$ is obtained in a similar way to the first part of Proposition 2. ∎

Corollary 1 provides the relationship between the multifractal geometry and the regularity of functions in fractional Sobolev spaces of variable order. Indeed, the two functions $s(\cdot)$ and $\alpha(\cdot)$ respectively define the order of regularity of functions of variable order restricted to a multifractal domain, and the local dimension of such multifractal domain.

# 5 Reproducing kernel Hilbert spaces of variable order on multifractal domains

In this section, we consider a class of generalized random fields with RKHS isomorphic to a fractional Sobolev space $H^{s(\cdot)}(M)$ of variable order $s(\cdot)$ defined on a multifractal domain $M$; with local dimension $\alpha(\cdot)$: The atomic decomposition of variable order for the functions of $H^{s(\cdot)+\frac{n-\alpha(\cdot)}{2}}(\mathbb{R}^n)$ allows the interpretation of the space $H^{s(\cdot)}(M)$ as the trace of $H^{s(\cdot)+\frac{n-\alpha(\cdot)}{2}}(\mathbb{R}^n)$ on the multifractal domain $M$: Part of the results derived in Ruiz-Medina, Angulo and Anh (2002) can then be extended to multifractal domains using the framework of fractional generalized random fields of variable order developed in Ruiz-Medina, Anh and Angulo (2003). Furthermore, the atomic decomposition of the covariance operator associated with the class of random fields considered also leads to the definition of equivalent quasi-norms on the RKHS, which are also equivalent to the norm defined on the space $H^{s(\cdot)}(M)$:

The following concept of fractional generalized random field of variable order and the pseudoduality condition from Ruiz-Medina, Anh and Angulo (2003) will be used in the introduction of a class of generalized random fields with RKHS isomorphic to the space $H^{s(\cdot)}(M)$: For a complete probability space $(\Omega;\mathcal{A};P)$; $L^2(\Omega;\mathcal{A};P)$ represents the Hilbert space of real-valued zero-mean random variables defined on $(\Omega;\mathcal{A};P)$ with finite second-order moments and with the inner product

$$< X;Y >_{L^2(\Omega)} = E[XY]; \quad X;Y \in L^2(\Omega\ \mathcal{A};P): \tag{52}$$

**Definition 7** Let $\beta(\cdot)$ be a real-valued function in $B^{\infty}(\mathbb{R}^n)$; and let $X_{\beta(\cdot)}$ be defined from $H^{-\beta(\cdot)}(\mathbb{R}^n)$ into $L^2(\Omega;\mathcal{A};P)$: We say that $X_{\beta(\cdot)}$ is a fractional generalized random field of variable order (FGRFVO) $\beta(\cdot)$ if it is linear and continuous, in the mean-square sense, with respect to the norm defined on $H^{-\beta(\cdot)}(\mathbb{R}^n)$:

We consider the Hilbert space $H(X_{\beta(\cdot)})$; which is defined as the closed span in the $L^2(\Omega;\mathcal{A};P)-$ topology of the random components of $X_{\beta(\cdot)}$; with the norm generated by the inner product (52): The covariance function $B_{\beta(\cdot)}$ of an FGRFVO defines a positive, symmetric and continuous operator of variable order $R_{\beta(\cdot)} : H^{-\beta(\cdot)}(\mathbb{R}^n) \longrightarrow H^{\beta(\cdot)}(\mathbb{R}^n)$ by the identity

$$B_{\beta(\cdot)}(f;g) = E\left[X_{\beta(\cdot)}(f)X_{\beta(\cdot)}(g)\right] = R_{\beta(\cdot)}(f)(g) = \left\langle \left[R_{\beta(\cdot)}(f)\right];g\right\rangle_{H^{-\beta(\cdot)}(\mathbb{R}^n)}; \tag{53}$$

for all $f;g \in H^{-\beta(\cdot)}(\mathbb{R}^n)$: We refer to $R_{\beta(\cdot)}$ as the covariance operator of $X_{\beta(\cdot)}$; which generates the RKHS $H(X_{\beta(\cdot)})$ of $X_{\beta(\cdot)}$; constituted by the functions of $H^{\beta(\cdot)}(\mathbb{R}^n)$ defined from the elements of

$H(X_{\beta(\cdot)})$ as follows:

$$\varphi(f) = E\left[X X_{\beta(\cdot)}(f)\right], \quad \forall f \in H^{-\beta(\cdot)}(\mathbb{R}^n), \; X \in H(X_{\beta(\cdot)}). \tag{54}$$

The RKHS $H(X_{\beta(\cdot)})$ is isometric to the dual of the Hilbert space $H(X_{\beta(\cdot)})$; thus, each function in this space defines an element of the dual of $H(X_{\beta(\cdot)})$.

**Definition 8** Let $\beta(\cdot)$ be as given in Definition 7. We say that the generalized random field $\tilde{X}_{\beta(\cdot)} : H^{\beta(\cdot)}(\mathbb{R}^n) \to L^2(\Omega, \mathcal{A}, P)$ is a pseudodual generalized random field of variable order (DGRFVO) for the FGRFVO $X_{\beta(\cdot)}$ if the following conditions hold:
(i) $\tilde{X}_{\beta(\cdot)}$ is linear and continuous in the mean-square sense with respect to $H^{\beta(\cdot)}(\mathbb{R}^n)$;
(ii) the space $H\left(\tilde{X}_{\beta(\cdot)}\right)$ coincides with the space $H(X_{\beta(\cdot)})$; and
(iii) for all $\varphi \in H^{\beta(\cdot)}(\mathbb{R}^n)$ and $g \in H^{-\beta(\cdot)}(\mathbb{R}^n)$, the inner product $\left\langle X_{\beta(\cdot)}(g), \tilde{X}_{\beta(\cdot)}(\varphi)\right\rangle_{H(X_{\beta(\cdot)})}$ is given by

$$\begin{aligned}\left\langle X_{\beta(\cdot)}(g), \tilde{X}_{\beta(\cdot)}(\varphi)\right\rangle_{H(X_{\beta(\cdot)})} &= \tilde{X}_{\beta(\cdot)}(\varphi)\left[X_{\beta(\cdot)}(g)\right] = [(I+R)g](\varphi) \\ &= [(I+R)^{*}\varphi](g),\end{aligned} \tag{55}$$

where $R \in S^{-\infty}_{\rho,\delta} = \bigcap_{m \in \mathbb{R}} S^m_{\rho,\delta}$, for certain $\delta$ and $\rho$ with $0 \leq \delta < \rho \leq 1$. Here, $A^{*}$ denotes the formal adjoint of the operator $A$.

The following Hilbert spaces are also considered:

$$H(\tilde{X}_{\beta(\cdot)}) = \overline{sp}^{L^2(\Omega,\mathcal{A},P)}\left\{\tilde{X}_{\beta(\cdot)}(\varphi) : \varphi \in H^{\beta(\cdot)}(\mathbb{R}^n)\right\}, \tag{56}$$

and the associated RKHS $H(\tilde{X}_{\beta(\cdot)})$, isometric to the dual space of $H(\tilde{X}_{\beta(\cdot)})$, constituted by the functions $f \in H^{-\beta(\cdot)}(\mathbb{R}^n)$ satisfying

$$f(\varphi) = E\left[Y \tilde{X}_{\beta(\varphi)}(f)\right], \text{ for a certain } Y \in H(\tilde{X}_{\beta(\cdot)}). \tag{57}$$

We will apply Corollary 1 for the definition of the restriction of an FGRFVO to a multifractal domain. Specifically, we introduce the restriction of an FGRFVO $X_{\beta(\cdot)}$ defined on $H^{-\beta(\cdot)}(\mathbb{R}^n)$ to a multifractal domain $M$, considering the distributions $H^{-\beta(\cdot),M}(\mathbb{R}^n)$ of variable singularity order $-\beta(\cdot)$ with compact support contained in $M$ as given in the following definition.

**Definition 9** Let $M$ be a compact multifractal domain with local dimension given by the singularity exponent $\alpha(\cdot)$ of $\mu_M$ and let $X_{\beta(\cdot)}$ be an FGRFVO with variable mean-square regularity order $\beta(\cdot)$. For $\beta(\cdot) = s(\cdot) + \frac{n-\alpha(\cdot)}{2}$, the restriction $X^M_{s(\cdot)}$ of $X_{\beta(\cdot)}$ to a compact multifractal domain $M$ is defined as

$$X^M_{s(\cdot)}(f) = X_{\beta(\cdot)}(f), \quad \forall f \in H^{-\beta(\cdot),M}(\mathbb{R}^n).$$

We refer to $X^M_{s(\cdot)}$ as the mean-square trace of $X_{\beta(\cdot)}$ on the compact multifractal domain $M$.

The existence of a pseudodual generalized random field of variable order with support contained in the multifractal domain $M$ allows the definition of a covariance factorization and white-noise representation of variable order on multifractal domains.

Definition 10 For $\beta(\cdot) = s(\cdot) + \frac{n-\alpha(\cdot)}{2}$; we say that the FGRFVO $\tilde{X}^M_{s(\cdot)} : H^{s(\cdot)}(M) \longrightarrow L^2(\Omega;\mathcal{A};P)$; with support contained in the compact multifractal set $M$; is the pseudodual of $X^M_{s(\cdot)}$ if it satisfies the following conditions:

(i) $\tilde{X}^M_{s(\cdot)}$ is continuous in the mean-square sense;

(ii) $H\left(X^M_{s(\cdot)}\right) = H\left(\tilde{X}^M_{s(\cdot)}\right)$; and

(iii) $< X^M_{s(\cdot)}(f); \tilde{X}^M_{s(\cdot)}(\psi) >_{H\left(X^M_{s(\cdot)}\right)} = \int_M (I + R)(f)(z)\psi(z)\mu_M(dz)$; for $f \in H^{-\beta(\cdot);M}(\mathbb{R}^n)$ and $\psi \in H^{s(\cdot)}(M)$; where $\mu_M$ is the multifractal measure with singularity exponent $\alpha(\cdot)$ defining the local dimension of the multifractal compact set $M$; and $R$ is as given in Definition 8.

Note that the spaces $H\left(X^M_{s(\cdot)}\right)$ and $H\left(X^M_{s(\cdot)}\right)$ as well as the spaces $H\left(\tilde{X}^M_{s(\cdot)}\right)$ and $H\left(\tilde{X}^M_{s(\cdot)}\right)$ are as defined before, for an FGRFVO and its pseudodual, considering the spaces $H^{-\beta(\cdot);M}(\mathbb{R}^n)$ and $H^{s(\cdot)}(M)$ instead of the spaces $H^{-\beta(\cdot)}(\mathbb{R}^n)$ and $H^{\beta(\cdot)}(\mathbb{R}^n) = H^{-\beta(\cdot)}(\mathbb{R}^n)$; respectively.

The pseudoduality condition on multifractal sets, introduced in Definition 10, allows the definition of a bounded parametrix on $M$ for the following operators:

$$J_M : H\left(X^M_{s(\cdot)}\right) \longrightarrow H\left(X^M_{s(\cdot)}\right) \subset H^{s(\cdot)}(M); \quad \text{and} \tag{58}$$
$$J'_M : H\left(\tilde{X}^M_{s(\cdot)}\right) \longrightarrow H\left(\tilde{X}^M_{s(\cdot)}\right) \subset H^{-\beta(\cdot);M}(\mathbb{R}^n); \tag{59}$$

respectively defined as

$$X \longrightarrow J_M[X] = \varphi_X \quad \text{with} \quad \varphi_X(f) = EX\,X^M_{s(\cdot)}(f); \quad f \in H^{-\beta(\cdot);M}(\mathbb{R}^n); \quad \text{and}$$
$$Y \longrightarrow J'_M[Y] = g_Y \quad \text{with} \quad g_Y(\psi) = EY\,\tilde{X}^M_{s(\cdot)}(\psi); \quad \psi \in H^{s(\cdot)}(M):$$

Specifically, the pseudoduality condition means that

$$\begin{aligned} J_M\left(\tilde{X}^M_{s(\cdot)}(\psi)\right) &= (I + R)^*(\psi); \quad \forall\psi \in H^{s(\cdot)}(M); \quad \text{and} \\ J'_M\left(X^M_{s(\cdot)}(f)\right) &= I + R)(f); \quad \forall f \in H^{-\beta(\cdot);M}(\mathbb{R}^n): \end{aligned} \tag{60}$$

Furthermore, from the definition of RKHS,

$$\begin{aligned} J_M^{-1}R_{X^M_{s(\cdot)}}(f) &= X^M_{s(\cdot)}(f); \quad \forall f \in H^{-\beta(\cdot);M}(\mathbb{R}^n); \quad \text{and} \\ (J'_M)^{-1}R_{\tilde{X}^M_{s(\cdot)}}(\psi) &= \tilde{X}^M_{s(\cdot)}(\psi); \quad \forall\psi \in H^{s(\cdot)}(M): \end{aligned} \tag{61}$$

Equations (60) and (61) lead to the identities

$$\begin{aligned} J'_M J_M^{-1}R_{X^M_{s(\cdot)}}(f) &= (I + R)(f); \quad \forall f \in H^{-\beta(\cdot);M}(\mathbb{R}^n); \quad \text{and} \\ J_M(J'_M)^{-1}R_{\tilde{X}^M_{s(\cdot)}}(\psi) &= (I + R)^*(\psi); \quad \forall\psi \in H^{s(\cdot)}(M): \end{aligned} \tag{62}$$

From the pseudoduality condition, we also have

$$J_M^{-1}(I+R)^*(\psi) = [J_M^0]^*(\psi);\quad \forall \psi \in H^{s(\cdot)}(M);\quad \text{and}$$

$$J_M\,[J_M^0]^* = (I+R)^*. \tag{63}$$

Similarly, we have

$$J_M^0\, J_M^* = (I+R). \tag{64}$$

From equations (62); (63) and (64); we obtain

$$\begin{aligned} R_{X^M_{s(\cdot)}} &= J_M J_M^*;\quad \text{and} \\ R_{\widetilde{X}^M_{s(\cdot)}} &= J_M^0 [J_M^0]^*. \end{aligned} \tag{65}$$

The RKHS of $X^M_{s(\cdot)}$ coincides, as a set of functions, with the space $H^{s(\cdot)}(M)$; since by definition $H\left(X^M_{s(\cdot)}\right) \subseteq H^{s(\cdot)}(M) \subseteq (I+R)(H^{s(\cdot)}(M))$ and from the pseudoduality condition

$$J_M\left(\widetilde{X}^M_{s(\cdot)}(\psi)\right) = (I+R)^*(\psi);\quad \forall \psi \in H^{s(\cdot)}(M). \tag{66}$$

Thus, $(I+R)^*(H^{s(\cdot)}(M)) \subseteq H\left(X^M_{s(\cdot)}\right)$. The spaces $H\left(X^M_{s(\cdot)}\right)$ and $H^{s(\cdot)}(M)$ are also isomorphically related by the identity operator (see Proposition 6 below). Similarly, an isomorphic relationship can also be established between $H\left(\widetilde{X}^M_{s(\cdot)}\right)$ and $H^{-\beta(\cdot);M}(\mathbb{R}^n)$ from the identity

$$J_M^0\left(X^M_{s(\cdot)}(f)\right) = (I+R)(f);\quad \forall f \in H^{-\beta(\cdot);M}(\mathbb{R}^n)$$

and the ellipticity of $J_M^*$; which is obtained from the mean-square continuity of $\widetilde{X}^M_{s(\cdot)}$.

## 6 Variable order white-noise filter representation on multifractal domains

A linear filter relating the FGRFVOs $X^M_{s(\cdot)}$ and $\widetilde{X}^M_{s(\cdot)}$ to white noise is obtained in terms of the operators of variable order $J_M$ and $J_M^0$ from the identities derived in the previous section.

The space $L^2_{\mu_M}(M)$ defines the RKHS of a white-noise process $\varepsilon_{\mu_M}$ on a multifractal domain $M$ whose elements are given by equation (40). The following relationships then hold between the Hilbert spaces of random variables and the RKHSs associated with $\varepsilon_{\mu_M}$; $X^M_{s(\cdot)}$ and $\widetilde{X}^M_{s(\cdot)}$:

$$\begin{aligned} H(X^M_{s(\cdot)}) &\underset{J_M}{\simeq} H(X^M_{s(\cdot)}) \underset{-}{\cong} H^{s(\cdot)}(M) \underset{[\langle D_x\rangle^{-s(\cdot)}]_{\mu_M}}{\simeq} L^2_{\mu_M}(M) \underset{J_0}{\simeq} H(\varepsilon_{\mu_M});\quad \text{and} \\ H(\widetilde{X}^M_{s(\cdot)}) &\underset{J_M^0}{\simeq} H(\widetilde{X}^M_{s(\cdot)}) \underset{\phi}{\cong} H^{-\beta(\cdot);M}(\mathbb{R}^n) \underset{[\langle D_x\rangle^{-\beta(\cdot)}]_M}{\simeq} L^2_{\mu_M}(M) \underset{J_0}{\simeq} H(\varepsilon_{\mu_M}); \end{aligned} \tag{67}$$

where

$$J_0 X(g) = E[X\varepsilon_{\mu_M}(g)];\quad \forall g \in L^2_{\mu_M}(M) \text{ and } X \in H(\varepsilon_{\mu_M}).$$

The following result provides the linear filters relating the FGRFVOs $X_{s(\cdot)}$ and $\widetilde{X}_{s(\cdot)}$ with white noise.

**Proposition 5** Under the pseudoduality condition given in Definition 10, the following white-noise linear filter representations on the multifractal domain $M$ hold:

$$X^M_{s(\cdot)} L_M f \underset{m.s.}{=} \varepsilon_{\mu_M}((I+R)f); \quad \forall f \in L^2_{\mu_M}(M); \quad \text{and} \tag{68}$$

$$\widetilde{X}^M_{s(\cdot)} \widetilde{L}_M g \underset{m.s.}{=} \varepsilon_{\mu_M}((I+R)^* g); \quad \forall g \in L^2_{\mu_M}(M); \tag{69}$$

where

$$L_M = J^0_M J_0^{-1}; \quad \text{and} \tag{70}$$

$$\widetilde{L}_M = J_M J_0^{-1}; \tag{71}$$

with $J_0$ representing the isometric isomorphisms between the spaces $H(\varepsilon_{\mu_M})$ and $H(\varepsilon_{\mu_M}) = L^2_{\mu_M}(M)$.

**Proof.** The operators $J^0_M$ and $J_M$ can be composed with the operator $J_0^{-1}$ from the identifications given in equation (67).

From condition (iii) of Definition 8, for all $f, g \in L^2_{\mu_M}(M)$, the following identities hold:

$$\begin{aligned} &\left\langle X^M_{s(\cdot)}\left([(I+R)]^{-1} J^0_M J_0^{-1} f\right), X^M_{s(\cdot)}\left([(I+R)]^{-1} J^0_M J_0^{-1} g\right)\right\rangle_{H(X_{s(\cdot)})} \\ &= \left\langle J_0^{-1} f, J_0^{-1} g\right\rangle_{H(\varepsilon_{\mu_M})} = \langle \varepsilon_{\mu_M}(f), \varepsilon_{\mu_M}(g)\rangle_{H(\varepsilon_{\mu_M})}. \end{aligned} \tag{72}$$

From equation (72),

$$X^M_{s(\cdot)}\left([(I+R)]^{-1} J^0_M J_0^{-1} f\right) \underset{m.s.}{=} \varepsilon_{\mu_M}(f); \quad \forall f \in L^2_{\mu_M}(M).$$

That is,

$$X^M_{s(\cdot)}(g) \underset{m.s.}{=} \varepsilon_{\mu_M}\left(J_0 [J^0_M]^{-1}(I+R)(g)\right); \quad \forall g \in H^{-\beta(\cdot),M}(\mathbb{R}^n).$$

Hence,

$$X^M_{s(\cdot)}\left(J^0_M J_0^{-1} f\right) \underset{m.s.}{=} \varepsilon_M((I+R)f); \quad \forall f \in L^2_{\mu_M}(M).$$

Equation (69) can be obtained in a similar way from condition (iii) of Definition 8. ■

## 6.1 Atomic decomposition of the reproducing kernel Hilbert space

The operators of variable order $R_{\widetilde{X}^M_{s(\cdot)}}$ and $R_{X_{s(\cdot)}}$ respectively define the inner products in the RKHSs $H(X^M_{s(\cdot)})$ and $H(\widetilde{X}^M_{s(\cdot)})$. The representation of functions in spaces $H^{s(\cdot)}(M)$ and $H^{-\beta(\cdot),M}(\mathbb{R}^n)$ in terms of the weak-sense trace on $M$ of local means (see Corollary 1, equation (51)) leads to the definition of equivalent quasi-norms in the spaces $H(X^M_{s(\cdot)})$ and $H(\widetilde{X}^M_{s(\cdot)})$ in a similar way to Proposition 2.

As before, let $Q_{2^{-j}}(z_{jk})$, $k = 1, \dots, N_j$, be a finite covering of $M$ by cubes centered at $M$, and with side length $2^{-j}$, for $j \in \mathbb{N}$. For each $\phi \in H^{s(\cdot)}(M)$, let $\varphi_{jk}$, $k = 1, \dots, N_j$, be a smooth resolution of unity in a neighbourhood $\Lambda_j$ of $M \cap \operatorname{supp} k_N(2^{-j};\varphi)$ subordinated to $Q_{2^{-j}}(z_{jk})$; $k = 1,\dots,N_j$; for $j \in \mathbb{N} - \{0\}$; and let $\phi_{0k}$; $k = 1,\dots,N_0$; be a smooth resolution of unity in a neighbourhood $\Lambda_0$ of $M \cap \operatorname{supp} k_0(1;\varphi)$ subordinated to $Q_1(z_{0k})$; $k = 1,\dots,N_0$; with $k_N$ and $k_0$ satisfying appropriate regularity and moment conditions. We similarly consider the weak-sense restriction to the compact multifractal set $M$ of the local means involved in the series representation of functions in $H^{-\beta(\cdot),M}(\mathbb{R}^n)$:

Proposition 6 Under the pseudoduality condition given in Definitionf 10, the following assertions hold:

(i) The norm generated by the covariance operator $R_{\widetilde{X}^M_{s(\cdot)}}$ and the quotient norm on the space $H^{s(\cdot)}(M)$ are equivalent. The quasi-norms given by the infimum over all admissible series representations of

$$\|\varphi|H(X^M_{s(\cdot)})\|_{>} = \left[\sum_{j=1}^{\infty}\sum_{k=1}^{N_j}\|L^*_M k_N(2^{-j+1};\varphi)|L^2_{\mu_M}(Q_{2^{-j}}(z_{jk}))\|^2 + \sum_{k=1}^{N_0}\|L^*_M k_0(1;\varphi)|L^2_{\mu_M}(Q_1(z_{0k}))\|^2\right]^{1/2}; \quad \varphi \in H(X^M_{s(\cdot)});$$

$$\|\varphi|H(X^M_{s(\cdot)})\|_{*} = \left[\sum_{j=1}^{\infty}\sum_{k=1}^{N_j}\|2^{js(\cdot)}k_N(2^{-j+1};\varphi)|L^2_{\mu_M}(Q_{2^{-j}}(z_{jk}))\|^2 + \sum_{k=1}^{N_0}\|k_0(1;\varphi)|L^2_{\mu_M}(Q_1(z_{0k}))\|^2\right]^{1/2}; \quad \varphi \in H(X^M_{s(\cdot)}); \tag{73}$$

are equivalent in the RKHS $H(X^M_{s(\cdot)})$; where $L_M$ is as given in equation (70):

(ii) The norm generated by the covariance operator $R_{X^M_{s(\cdot)}}$ and the dual quotient norm on the space $H^{-\beta(\cdot),M}(\mathbb{R}^n) = [H^{s(\cdot)}(M)]^*$ are equivalent. The quasi-norms given by the infimum over all admissible series representations of

$$\|f|H(\widetilde{X}^M_{s(\cdot)})\|_{>} = \left[\sum_{j=1}^{\infty}\sum_{k=1}^{N_j}\|\widetilde{L}^*_M k_N(2^{-j+1};f)|L^2_{\mu_M}(Q_{2^{-j}}(z_{jk})\|^2 + \sum_{k=1}^{N_0}\|\widetilde{L}^*_M k_0(1;f)|L^2_{\mu_M}(Q_1(z_{0k}))\|^2\right]^{1/2}; \quad f \in H(\widetilde{X}^M_{s(\cdot)});$$

$$\|f|H(\widetilde{X}^M_{s(\cdot)})\|_{*} = \left[\sum_{j=1}^{\infty}\sum_{k=1}^{N_j}\|2^{-js(\cdot)}k_N(2^{-j+1};f)|L^2_{\mu_M}(Q_{2^{-j}}(z_{jk})\|^2 + \sum_{k=1}^{N_0}\|k_0(1;f)|L^2_{\mu_M}(Q_1(z_{0k}))\|^2\right]^{1/2}; \quad f \in H(\widetilde{X}^M_{s(\cdot)});$$

$$\sim \left[\sum_{j=1}^{\infty}\sum_{k=1}^{N_j}\|2^{-j\beta(\cdot)}k_N(2^{-j+1};f)|L^2(Q_{2^{-j}}(z_{jk})\|^2 + \sum_{k=1}^{N_0}\|k_0(1;f)|L^2(Q_1(z_{0k}))\|^2\right]^{1/2}; \quad f \in H(\widetilde{X}^M_{s(\cdot)}); \tag{74}$$

are equivalent in the RKHS $H(\widetilde{X}^M_{s(\cdot)})$; where $\widetilde{L}_M$ is as given in equation (71); and, as before, $\beta(\cdot) = s(\cdot) + \frac{n-\alpha(\cdot)}{2}$.

Proof. (i) From the mean-square continuity of $X^M_{s(\cdot)}$ and $\widetilde{X}^M_{s(\cdot)}$; and hence, the continuity and ellipticity of the operators $J^0_M$ and $J_M$ defining the respective operators $L_M$ and $\widetilde{L}_M$ in equations (70) and (71); and from the covariance factorizations in equation (65); the equivalence between the norm defined by $R_{\widetilde{X}^M_{s(\cdot)}}$ on the space $H(X^M_{s(\cdot)})$ and the quotient norm defined on the space $H^{s(\cdot)}(M)$ from Corollary 1 follows.

The equivalence between the norm generated by $R_{\widetilde{X}^M_{s(\cdot)}}$ and the quasi-norm $\inf \|\cdot|H(X^M_{s(\cdot)})\|_>$ follows from the identity

$$R_{\widetilde{X}^M_{s(\cdot)}} = L_M L^*_M$$

and the weak-sense series representation

$$L^*_M(\psi)(\cdot) = \sum_{j=0}^{\infty}\sum_{k=1}^{N_j} \varphi_{jk}(\cdot) L^*_M k_N(2^{-j+1};\psi)(\cdot);$$

in a similar way to the first part of the proof of Proposition 2.

The equivalence between $\left[\inf \|\cdot|H(X^M_{s(\cdot)})\|_>\right]$ and $\left[\inf \|\cdot|H(X^M_{s(\cdot)})\|_*\right]$ is obtained from the continuity and ellipticity of the operator $L^*_M$; from the definition of the equivalent quasi-norm $\left[\inf \|\cdot|H^{s(\cdot)}(M)\|_*\right]$ on $H^{s(\cdot)}(M)$ in a similar way to Proposition 2 and equation (51) of Corollary 1.

(ii) The proof also follows from the bicontinuity of $R_{X^M_{s(\cdot)}}$ and the factorization

$$R_{X^M_{s(\cdot)}} = \widetilde{L}_M \widetilde{L}^*_M;$$

considering the corresponding atomic representation of $\widetilde{L}^*_M$ in terms of suitable functions $k_N$; $k_0$; and $\varphi_{jk}$; $k = 1, \ldots, N_j$; $j \in \mathbb{N}$; in a similar way to the first part of the proof of Proposition 2. ■

# 7 Spectral properties

Examples of multifractional differential models on compact multifractal domains can be constructed as restrictions to such domains of pseudodifferential operators of variable order in the class described in Ruiz-Medina, Anh and Angulo (2003). The compactness of the restrictions comes from the compactness of the fractal domain. Their continuous spectra are then empty and their pure point spectra are constituted by eigenvalues with asymptotic variable order depending on functions $s(\cdot)$ and $\alpha(\cdot)$.

**Lemma 4** Let $0 \leq s_1(\cdot) < s_2(\cdot) < \infty$. The embedding of $H^{s_2(\cdot)}(M)$ into $H^{s_1(\cdot)}(M)$ is compact, and for the associated entropy numbers $\{e_k\}_{k\in\mathbb{N}}$ the following approximation holds:

$$e_k\left(id : H^{s_2(\cdot)}(M) \longrightarrow H^{s_1(\cdot)}(M)\right) \asymp k^{-\frac{s_2(\cdot)-s_1(\cdot)}{\alpha(\cdot)}}; \quad k \in \mathbb{N}.$$

Proof. The proof follows from the atomic decomposition of variable order of the spaces $H^{\beta(\cdot)}(\mathbb{R}^n)$ given in Proposition 3, in a similar way to Proposition 20.5 and Theorem 20.6 of Triebel (1997), considering, in this case, variable weighted $l_2-$spaces, $l_2\left(2^{js(\cdot)} l_2^{N_j^{\overline{\alpha}}}\right)$; with $\overline{\alpha} = \max_{z\in M} \alpha(z)$; that is, the linear spaces of sequences

$$\underline{x} = \left\{ x_{j,k};\ k = 1, \dots, N_j^{\overline{\alpha}};\ j \in \mathbb{N} \right\}$$

with the quasi-norm

$$\left\| \underline{x} \mid l_2\left(2^{js(\cdot)} l_2^{N_j^{\overline{\alpha}}}\right) \right\| = \left[ \sum_{j=0}^{\infty} \sum_{k=1}^{N_j^{\overline{\alpha}}} 2^{2j\underline{s}_{Q_{2^{-j}}(z_k)}} |x_{j,k}|^2 \right]^{1/2};$$

where

$$c_1 \leq N_j^{\overline{\alpha}} 2^{-j\overline{\alpha}} \leq c_2;$$

for certain positive constants $c_1$ and $c_2$; and $\underline{s}_{Q_{2^{-j}}(z_k)} = \min_{z\in Q_{2^{-j}}(z_k)} s(z)$; with $Q_{2^{-j}}(z_{jk})$; $k = 1, \dots, N_j$; being a finite covering of $M$ by cubes centered at $M$; and with side length $2^{-j}$; for each $j \in \mathbb{N}$. ■

Proposition 7 Under the pseudoduality condition on multifractal domains and in the case where the function $s(\cdot)$ is positive,

(i) the covariance operator $R_{X^M_{s(\cdot)}}$ of $X^M_{s(\cdot)}$ is compact, and the point spectrum $\left\{ \lambda_k\left(R_{X^M_{s(\cdot)}}\right) \right\}_{k\in\mathbb{N}}$ of $R_{X^M_{s(\cdot)}}$ satisfies

$$ck^{\frac{2s(\cdot)}{\alpha(\cdot)}} \leq \lambda_k\left(R_{X^M_{s(\cdot)}}\right) \leq Ck^{\frac{2s(\cdot)}{\alpha(\cdot)}}; \quad k \in \mathbb{N}; \tag{75}$$

(ii) if

$$\inf_{z\in M} s(z) = \min_{z\in M} s(z) = \underline{s} > \sup_{z\in M} \alpha(z)/2 = \max_{z\in M} \alpha(z)/2 = \overline{\alpha}/2;$$

then the operator $R_{X^M_{s(\cdot)}}$ is in the trace class.

Proof. (i) Equation (75) is obtained from Proposition 6 and Lemma 4, in a similar way to Theorems 25.2 of Triebel (1997). The compactness of $R_{X^M_{s(\cdot)}}$ is obtained from equations (63)-(65) and (75).

(ii) From equation (75), since $\underline{s} > \overline{\alpha}/2$,

$$\sum_{k\in\mathbb{N}} \left| \lambda_k\left(R_{X^M_{s(\cdot)}}\right) \right| \leq C \sum_{k\in\mathbb{N}} k^{-2s(\cdot)/\alpha(\cdot)} \leq \sum_{k\in\mathbb{N}} k^{-2\underline{s}/\overline{\alpha}} < \infty;$$

which means that $R_{X^M_{s(\cdot)}}$ is in the trace class. ■

# 8 Hölder continuity and multifractal formalism

In this section we consider the case where the functions of the RKHS $H(X^M_{s(\cdot)})$ are Hölder continuous. Specifically, we consider the case where

$$\inf_{z \in M} s(z) > \sup_{z \in M} \alpha(z)/2. \tag{76}$$

In this case, the linear filter representation (68) admits a mean-square Hölder continuous ordinary solution. The local mean-square Hölder continuity of this solution follows from the variable fractional order of its variogram (see Proposition 8).

We also study the local Hölder spectrum of the functions in the RKHS $H(X^M_{s(\cdot)})$; that is, the mean-square local Hölder spectrum of the ordinary solution to (68); in terms of the function $s(\cdot) - \alpha(\cdot)/2$; with the associated multifractal (dimension) spectrum given by

$$d(\rho) = \dim_H \{z \in M : \alpha(z) = 2(s(z) - \rho)\}, \quad \rho \geq 0, \tag{77}$$

where $\dim_H$ denotes the Hausdorff dimension of a set.

In the Gaussian case, the mean-square Hölder continuity can be related to the sample-path Hölder continuity (see Adler, 1981). Hence, Gaussian multifractal models with non-trivial multifractal spectrum can be introduced from this framework.

**Proposition 8** Under the pseudoduality condition and assuming that condition (76) holds. Then, there exists a unique mean-square Hölder continuous ordinary random field $X^M_{s(\cdot)}$ satisfying

$$L^*_M X^M_{s(\cdot)}(z) \underset{m.s.}{=} \varepsilon_{\mathbf{M}}(z), \quad \forall z \in M. \tag{78}$$

Its local quadratic variation is defined by the function $s(\cdot) - \alpha(\cdot)/2$.

**Proof.** From Proposition 7(i), the covariance operator $R_{X^{\mathbf{M}}_{\mathbf{s}(\cdot)}}$ admits a weak-sense integral representation in terms of a kernel in the space $H^{2s(\cdot)}(M \times M)$ in the case where $s$ is a positive function, which can be assumed without loss of generality since $s$ is bounded, and we can consider the positive function $s(z) - \inf_{z \in M} s(z)$. From Proposition 7(ii), under condition (76); this weak-sense representation of $R_{X^{\mathbf{M}}_{\mathbf{s}(\cdot)}}$ becomes a strong-sense representation, in terms of a continuous kernel with local Hölder exponent defined by the function $2s(\cdot) - \alpha(\cdot)$. The existence and uniqueness of a mean-square Hölder continuous ordinary solution with variogram having local variable order $2s(\cdot) - \alpha(\cdot)$ then follows. ■

From the above proposition, the mean-square Hölder spectrum (singularity spectrum in the second-order moment sense) of the random field $X^M_{s(\cdot)}$ is defined as the function $d(\cdot)$ given in equation (77).

## 8.1 Examples

We consider here a few examples of multifractional models in the class introduced in this paper.

Example 1

$$\sum_{|\sigma(\cdot)| \leq [s(\cdot)]^{-}} a^{M}_{\sigma(\cdot)}(\cdot)\langle D_{\cdot}\rangle^{s_{+}(\cdot)}\langle D_{\cdot}\rangle^{\sigma(\cdot)} \cdot \mathrm{tr}_{M} X_{s(\cdot)} \underset{m.s.}{=} \varepsilon_{\mu_M}; \tag{79}$$

where $a^{M}_{\sigma(\cdot)}(\cdot)$ denotes a distribution with compact support $M$; $s(\cdot) = [s(\cdot)]^{-} + s_{+}(\cdot) \in B^{\infty}(\mathbb{R}^n)$; $\sigma(\cdot)$ is an interger-valued function, and $\mathrm{tr}_M$ denotes the trace operator on $M$ in the sense given by Corollary 1. Here, the multifractional operators involved are defined as in equation (5):

Example 2

$$(-\Delta)^{\gamma(\cdot)/2}\mathrm{tr}_{M} X_{\gamma(\cdot)} \underset{m.s.}{=} \varepsilon_{\mu_M}; \tag{80}$$

where $(-\Delta)^{\gamma(\cdot)/2}$ is the negative Laplacian of variable order $\gamma(\cdot)/2$; with $\gamma \in B^{\infty}(\mathbb{R}^n)$.

Example 3

$$\langle D_{\cdot}\rangle^{s(\cdot)}(-\Delta)^{\gamma(\cdot)}\mathrm{tr}_{M} X_{\gamma(\cdot)+s(\cdot)} \underset{m.s.}{=} \varepsilon_{\mu_M}; \tag{81}$$

where the multifractional operators of variable order involved are as given in Examples 2 and 3.

Example 4

$$Q_{q(\cdot)}(A)\mathrm{tr}_{M} X_{-((q(\cdot)-p(\cdot))s(\cdot))} = P_{p(\cdot)}(A)\varepsilon_{\mu_M}; \tag{82}$$

where $Q/P$ is an elliptic rational function with variable order $q(\cdot)/2 - p(\cdot)/2$ of an elliptic self-adjoint differential operator $A$ of variable order $s(\cdot)$:

# 9 Conclusion

In this paper, an extension to the variable order and multifractal domains of the results on atomic decompositions and traces on fractal domains of functions of fractional Sobolev spaces is provided. In this extension, the local dimension of the multifractal domain $M$ affects the variable local regularity order of the functions involved. Since we assume that the multifractal measure $\mu_M$ with support $M$ has a non-trivial multifractal spectrum, the functions in the spaces defined by the trace on $M$ of fractional Sobolev spaces of variable order have a non-trivial multifractal spectrum. The consideration of these spaces as RKHSs of the class of generalized random fields considered allows the definition, in the second-order moment sense, of a class of ordinary random fields with non-trivial mean-square multifractal spectrum, and, in the Gaussian case, with non-trivial sample-path multifractal spectrum. Thus, a class of Gaussian multifractal random fields is introduced in this paper.

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