NOISE AND THE FULL COUNTING STATISTICS OF A COULOMB BLOCKADED QUANTUM DOT

by

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Abstract

In this thesis, we consider the noise properties of a Coulomb blockaded quantum dot weakly coupled to two non-interacting leads. We consider two approaches to calculating the density of states of the quantum dot, the first of which uses a functional integral approach which is difficult to extend to consider the noise problem. We show that the second approach also returns the correct result for the density of states and can be extended to calculate the noise power spectrum for the interacting quantum dot. We calculate the Fano factor in the shot noise regime and evaluate numerically the Fano factor as a function of the bias voltage.

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Chapter 1

INTRODUCTION

1.1 Structure of the Thesis

Within this thesis, we will consider the noise properties of a Coulomb blockaded quantum dot weakly coupled to two non-interacting leads. This thesis is split into three main parts, the first of which consists of the first three chapters. In this part, we introduce the system that we will consider, the phenomena in which we are interested and the techniques which we require to tackle the problem. We will begin by introducing quantum dots and the phenomena of the Coulomb blockade. We will then move on to discuss the causes of noise in mesoscopic conductors and the different approaches that are used to model the noise properties. We will conclude this first part of the thesis by introducing Keldysh Green's functions and use the Green's function approach to calculate the noise for the resonant level non-interacting system.

In the second section, we will derive the density of states of the quantum dot using two different approaches. The reason for this is that the first, which is a functional integration approach, cannot be easily extended to consider the two-particle Green's functions which are required to calculate the noise. We will show that the second approach can be extended to consider the two-particle Green's function and use this approach to calculate the noise

power spectrum for the interacting system. We will consider several sensible limits and derive the Fano factor in the shot noise regime.

In the third part, of the thesis we will consider full counting statistics. We will begin by giving a brief explanation of why we are interested in this area of study. We will then develop an effective technique for calculating the full counting statistics of a system using Keldysh Green's functions. We will conclude this chapter by considering some of the recent work in the field.

1.2 Quantum Dots

A quantum dot is a system used to confine electrons to a small region. Quantum dots can vary in size, although they are small enough to be considered as zero-dimensional. This means that they have a diameter less than the Thouless length, L_{Th}. The Thouless length is the length over which an electron will defuse between inelastic collisions which cause dephasing. Quantum dots typically range in size from a few Angstroms to several micrometers and can confine between one and several thousand electrons.

There are several methods of confinement, one of which is to confine the electrons to a metallic island using an insulating material boundary. Alternatively and generally the more popular method of confinement is to use electric fields to restrict electrons to a small region within a semi conductor. A semi-conductor quantum dot can be created in the two-dimensional interface between gallium arsenide (GaAs) and aluminium gallium arsenide (AlGaAs). As the AlGaAs is positively doped due to the aluminium the electrons in the GaAs layer are attracted to the boundary and form a two dimensional layer of electrons. Chromium and gold layers can then be added to form the necessary pattern for the gates. When a negative potential is applied to the gates, a region around them is depleted of elec-

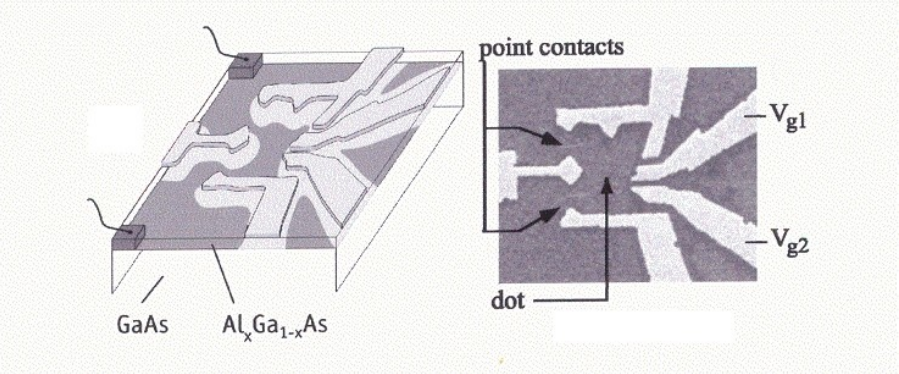

Figure 1.1: A scan of a quantum dot [7]. The point contacts are coupled to the leads which allow electrons in and out of the dot. The voltage bias of the contacts can be controlled to open and close the quantum dot. The electrons are confined to the central region and it's size can be controlled via the gate voltages V_{g1} and V_{g2}.

trons forming a quantum dot. See Fig. (1.1) for a diagram of a semi conductor quantum dot.[1]

By changing the bias on the voltage gates it is possible to confine the electrons in such a way that the quantum dot can be referred to as either open or closed. In an open quantum dot it is easy to tunnel between the dot and the leads. In a closed quantum dot there is only weak coupling between the leads and the dot. As the coupling between the dot and the leads is weakened, i.e the dot is closed, the Coulomb repulsion between the electrons becomes more significant and leads to a charging energy, which restricts the flow of electrons on to the dot. This effect is known as Coulomb blockade.

We can consider a quantum dot in a more rigorous manner, if we consider a dot that is completely isolated from the leads we can write the following Hamiltonian,

$$\hat{H} = \sum_{i,j,\sigma} \varepsilon_{ij} \hat{d}^{\dagger}_{i\sigma} \hat{d}_{j\sigma} + \frac{1}{2} \sum_{i,j,k,l} \sum_{\sigma,\sigma'} u_{ijkl} \hat{d}^{\dagger}_{i\sigma} \hat{d}^{\dagger}_{j\sigma'} \hat{d}_{k\sigma'} \hat{d}_{l\sigma}, \tag{1.2.1}$$

where $\hat{d}_{i\sigma}$ creates a particle in the state $\psi_i(\mathbf{r})$ with spin σ. Transitions between energy levels of the dot are described by ε_{ij} which is spin independent as we assume that the energy states are spin degenerate. The two electron interaction, u_{ijkl} is given by

$$u_{ijkl} = \int d\mathbf{r} \int d\mathbf{r}' \psi_i^*(\mathbf{r}) \psi_j^*(\mathbf{r}') U(\mathbf{r} - \mathbf{r}') \psi_k(\mathbf{r}') \psi_l(\mathbf{r}). \tag{1.2.2}$$

We can simplify this Hamiltonian to the one we will use in later chapters. We will first assume that the energy levels are randomly spaced with mean level spacing Δ, which is small compared with the temperature and all other relevant energy scales. It is then possible to neglect all the off diagonal terms in the interaction, in the limit that the dimensional conductance is large:

$$g \equiv \frac{E_T}{\Delta} \gg 1, \tag{1.2.3}$$

where E_T is the Thouless energy $E_T \sim \frac{v_F}{L}$. However we will not reproduce the proof here as it is far from trivial [2][3]. After neglecting the off diagonal terms, there are only three possible terms which can contribute; the Coulomb interaction which we consider, a cooper interaction which we neglect and the exchange interaction which we also can neglect. The exchange interaction can be neglected as the exchange energy is less than the mean level spacing. The exchange energy is the energy difference between spins which are parallel and anti-parallel. Hence the interaction part of the Hamiltonian can be simplified to

$$H_{int} = \frac{1}{2} E_c \hat{N}^2, \tag{1.2.4}$$

where $E_c = \frac{e^2}{2C}$ is the charging energy, C is the total capacitance of the dot and $\hat{N}$ is the number operator. In terms of the original interaction Eq. (1.2.2) the charging energy is

$$E_c \sim \frac{1}{2} \int \frac{d^2\mathbf{r}}{L^2} U(\mathbf{r}). \tag{1.2.5}$$

The Hamiltonian for the isolated dot thus becomes

$$H = \sum_{i,j} \varepsilon_{ij} \hat{d}_i^\dagger \hat{d}_j + E_c(\hat{N} - N_g)^2, \tag{1.2.6}$$

where we have introduced a gate and are choosing to measure the electrostatic energy of the dot relative to the coupling to the gate, $N_g = CV_g/e$, which we describe in the Hamiltonian as an effective charge. This Hamiltonian appears simple, yet is non-trivial to solve. For an in depth review of quantum dots see [1] and [4].

1.3 Coulomb Blockade

The conductance through a nearly closed quantum dot is suppressed: this effect is referred to as Coulomb blockade. It is due to the Coulomb repulsion between the electrons, this repulsion results in an extra electron entering the dot being energetically unfavourable. To model the effects of the Coulomb blockade, consider an electron tunneling from one of the leads, which we will refer to as the source, to the quantum dot and then from the dot to another lead, which we will refer to as the drain. The source and the drain are modeled as electron reservoirs and the dot is neutrally charged to begin with. The energy required to add an electron to the dot is $\frac{e^2}{2C}$ where C is the capacitance between the dot and the system. This means that the minimum amount of energy required for charge to flow from the source to the dot is $\frac{e^2}{2C}$. In addition, for the electron to tunnel from the dot to the drain, a hole must tunnel from the drain to the dot which requires an amount of energy equal to $\frac{e^2}{2C}$. This results in an energy gap of $\frac{e^2}{C}$ in the energy spectra being formed. [5] This also means that there is a minimum temperature $kT > \frac{e^2}{2C}$ required for electrons to flow.

By altering the gate voltage, U_g, it is possible to change the energy required to add an electron to the dot. Changing U_g will affect the potential between the gate and the source but if the potential difference between the source and the drain is small, the drain, source

and dot can be considered to be at the same potential. The electrostatic potential energy when U_g is applied of a charge Q on the dot is [5]

$$E = QU_g + \frac{Q^2}{2C}, \tag{1.3.1}$$

where the first term represents the attraction to the gate and the second represents the Coulomb repulsion between the electrons. The minimum is when

$$Q = -CU_g, \tag{1.3.2}$$

as the quantum dot that we are considering is nearly closed there is an integer number of electrons on the dot and the charge Q is quantised. This produces the parabolic curves in Fig. (1.2). By altering the gate voltage, it is possible to shift the curve until two of the energy states are degenerate [6]. This will result in the charge being able to fluctuate between these two states even at zero temperature and current can flow. This results in the peaks in conductance, occurring when $CU_g = Q = -(N + \frac{1}{2})e$ periodically every $\frac{e}{C}$ in gate voltage see Fig. (1.3) [5].

From this analysis, we gain the simple picture of the Coulomb staircase. As we increase the gate voltage, U_g, the number of electrons on the quantum dot will increase by one every time we pass through a degeneracy point. This results in the conductance versus gate voltage graph looking like a staircase.

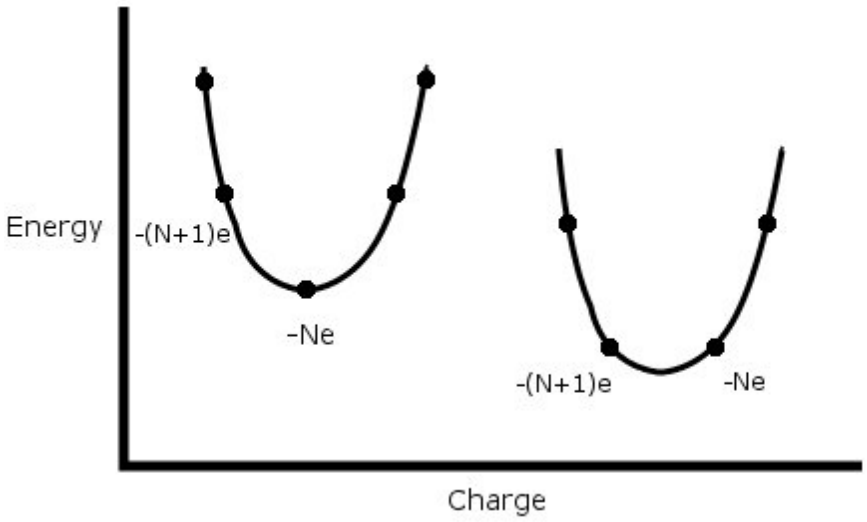

Figure 1.2: The parabolic curves in this diagram correspond to the allowed energy values the electrons may have. Due to the quantisation of charge the electrons can only take values that corresponding to the dots on the curves. The two curves represent two different possible gate voltages. The curve on the right corresponds to a possible choice of gate voltage that makes two of the states degenerate

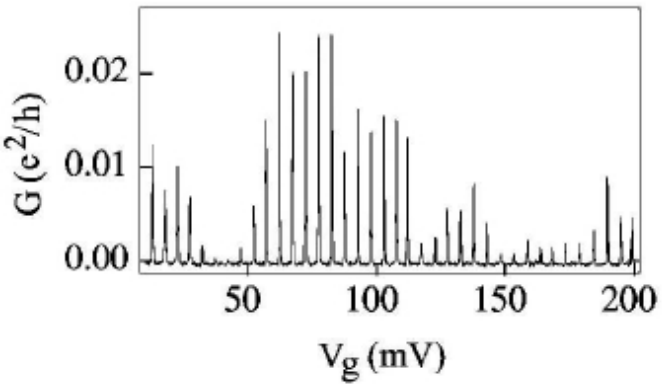

Figure 1.3: The differential conductance G in a quantum dot as a function of the gate voltage V_g. From [7]

1.4 Introduction to Statistics

In this section, we will review some basic statistics that will be useful later on when we consider full counting statistics. We will begin by defining the moment and the generating function and we will use these to define the cumulant.

The definition of the n^{th} raw moment μ (i.e, the moment about zero) of a probability distribution function P(x) is

$$\mu_n = \langle x^n \rangle, \tag{1.4.1}$$

where

$$\langle f(x) \rangle = \sum f(x) P(x). \tag{1.4.2}$$

We will consider the generating function $\chi(\lambda)$, which is sometimes referred to as the characteristic function. It is defined as

$$\chi(\lambda) = \sum_x e^{i\lambda x} P(x) = \langle e^{i\lambda x} \rangle. \tag{1.4.3}$$

If we Taylor expand the exponential term in the above expression, we obtain

$$\chi(\lambda) = \sum_x P(x) + i\lambda \sum_x xP(x) + (i\lambda)^2 \sum_x x^2 P(x) + \dots \tag{1.4.4}$$

Therefore, the generating function can be written in terms of the raw moments as

$$\chi(\lambda) = \sum_{k=0}^{\infty} \frac{(i\lambda)^k}{k!} \mu_k. \tag{1.4.5}$$

It is usually more convenient to calculate the cumulant generating function $\ln \chi(\lambda)$ which is defined as

$$\ln \chi(\lambda) = \sum_{r=1}^{\infty} k_r \frac{(i\lambda)^r}{r!}, \tag{1.4.6}$$

where we have introduced the cumulants k_r. Taking the Maclaurin series of Eq. (1.4.5) we obtain

$$\ln \chi(\lambda) = i\lambda\mu_1 + \frac{1}{2}(i\lambda)^2(\mu_2 - \mu_1^2) + \frac{1}{3!}(i\lambda)^3(2\mu_1^3 - 3\mu_1\mu_2 + \mu_3) + \ldots, \tag{1.4.7}$$

so the cumulants, k_r, are given by

$$\begin{aligned} k_1 &= \mu_1 \\ k_2 &= \mu_2 - \mu_1^2 \\ k_3 &= 2\mu_1^3 - 3\mu_1\mu_2 + \mu_3 \\ &\vdots \end{aligned} \tag{1.4.8}$$

The cumulants give useful information about the system; for example, the first cumulant gives the mean current and the second gives the noise of the system.

1.5 Summary

In this chapter, we have presented an overview of the quantum dot and in particular described the cause of the Coulomb blockade effect which we will be primarily concerned with throughout this thesis. We have also introduced some basic statistics which will be of help when considering the full counting statistics later on. In the next chapter, we will introduce the noise power spectrum; which we will then in later chapters go on to calculate for the Coulomb blockade quantum dot.

Chapter 2

NOISE IN MESOSCOPIC CONDUCTORS

In this chapter, we will introduce and define the noise power spectrum. We will briefly introduce the different approaches used in the literature to calculate the noise. One of the methods we will consider is scattering theory. Using this approach we will derive the Landauer formula for conductance and state the result for the noise power spectrum. We will also go on to define the Fano factor which is a useful way of expressing the noise power spectrum in terms of the noise produced by a poissonian process. We will conclude the chapter by showing the Fano factor result which is obtained for the resonant level double tunnel barrier problem.

2.1 Noise in Mesoscopic Conductors

Noise is the fluctuation in time of a measurement, these fluctuations can be a source of information that cannot be obtained from time averaged results. The noise which we will be concerned with during this thesis, will be the noise in the conductance of mesoscopic systems.

The noise in conductance can be caused by several different processes, not all of which are informative. The most obvious cause of noise is thermal noise which is sometimes referred to as Johnson-Nyquist noise, after the first two physicists to study it in a quantitative way.[8] It is caused by thermal fluctuations and is unavoidable at non zero temperatures: it

also provides no useful information about the system other than the temperature which is usually already known.

At low frequencies (typically below 10kHz) the noise is dominated by the "1/f noise" or "flicker noise" which is caused by the random motion of impurities which produce time-dependent fluctuations in the conductance[8]. However, we will not be considering this type of noise production. Another source of noise in a mesoscopic conductor is shot noise. It is caused by the discreteness of electron charge. It gets its name from the analogy between electrons and the lead pellets in a shotgun shell. This analogy was drawn by Walter Schottky, who in 1918, predicted that there would be two intrinsic sources of time-dependent fluctuations in a vacuum tube; noise from the thermal agitation of the electrons and noise due to the discreteness of electron charge[9]. Noise in a measurement is characterised by its spectral density or power spectrum[10]:

$$2\pi\delta(\omega+\omega')S_{\alpha\beta}(\omega) = 2\langle\Delta\hat{I}_\alpha(\omega)\Delta\hat{I}_\beta(\omega')\rangle, \tag{2.1.1}$$

where $\Delta\hat{I}_\alpha(\omega) = \hat{I}_\alpha(\omega) - \langle\hat{I}_\alpha(\omega)\rangle$ is the frequency dependent fluctuations in the Fourier transformed current operator at a given voltage and temperature of lead α. The triangular brackets indicate taking the ensemble average or equivalently an average over the initial time t_0 of the quantum expectation value. For the two terminal case,

$$S_{11} = S_{22} = -S_{12} = -S_{21} = S, \tag{2.1.2}$$

due to current conservation[26].

Both the thermal and shot noise power do not depend on frequency over a very wide range and therefore both have a white noise power spectrum. It will be shown later that the thermal noise ($V = 0, T \neq 0$) is related to the conductance, G, by the fluctuation-dissipation

theorem [13],

$$S = 4k_B TG, \tag{2.1.3}$$

as long as $\hbar\omega \ll k_B T$. As previously stated, it is apparent that the thermal noise does not provide any new information about the system.

Shot noise, however, is more interesting. It provides information about the temporal correlation of the electrons which you cannot obtain from just measuring the conductance of a system. For some devices the shot noise can be easily predicted. For devices where the transfer of electrons is random and independent of each other, Poisson statistics can be used to describe the transfer of the electrons. Poisson statistics is used to analyze events which have no correlation in time. An example of a device where this is the case is the p-n junction. For devices that obey Poisson statistics the shot noise takes its maximum value,

$$S = 2e\langle I\rangle \equiv S_{Poisson}, \tag{2.1.4}$$

which is proportional to the time averaged current $\langle I\rangle$. In general, electrons in a system are correlated. Even for a model where the electrons are non-interacting, there is a correlation due to the Pauli exclusion principle. This correlation suppresses the shot noise power below $S_{Poisson}$. The shot noise is also reduced by inelastic electron-phonon scattering which averages out the current fluctuations. This is the process that results in macroscopic metals having zero shot noise. However, the shot noise will be non zero on the length scale that we will be considering. This is because the mesoscopic length scales that we are concerned with are small compared with the electron - phonon scattering length.

2.2 Approaches to Calculating the Noise Power Spectrum

Within the literature there are three main approaches to calculating the noise; scattering theory, the rate equation/master equation approach and the Green's function approach. In the scattering theory approach, the system is modeled as leads connected to scattering regions within which the electron can be transmitted or reflected. We will go on to introduce this approach shortly.

In the Green function approach, which we use in later chapters to calculate the noise for the Coulomb blockaded system, the current is expressed via the Heisenberg equation of motion which allows the noise and current to be expressed in terms of Green's functions. The solutions to these Green's functions are then calculated. The third and final approach, which dominates the literature in the field, is the rate equation or master equation approach. In this approach, the system is considered to be classical and the rates of change of charge in the leads are calculated. From these rate equations, the current and the noise can be obtained. We will not however consider this approach in this thesis but there is a large body of work that use this approach [14][15][16][17][18] to calculate the noise in similar regimes to the one in which we consider. However we choose to treat our system using a fully quantum mechanical approach, this is because at the peak of the conductance there are two degenerate interacting levels and it is not clear that these can be treated classically.

2.2.1 Scattering Theory

Scattering theory is a method of expressing the conductance of a system in terms of its scattering properties. This was first discussed by Rolf Landauer in 1957 [19] for two terminals and latter generalised to multiple channels.[20]

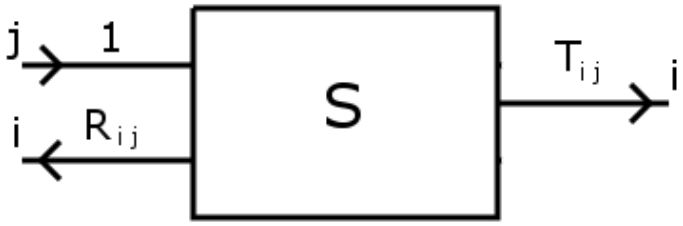

Figure 2.1: A general elastic scattering system. An incoming normalised wave in the j^{th} channel from the left has the probabilities T_{ij} of being transmitted into the i^{th} channel on the right and R_{ij} of being reflected back into the i^{th} channel on the left

2.2.2 The Landauer Formula

We begin by modeling the system as two leads connected by a general elastic scattering system S (see Fig. (2.1)). The leads are ideal and are quantised in the transverse direction with discrete transverse energies ε_i. Each of these energies corresponds to a channel in the lead; this means that the lead will have $N_\perp$ channels at the Fermi energy ε_F. At zero temperature, each channel can be characterised by a wave vector k_i (which relates to velocity by $v_i = \frac{\hbar k_i}{m}$),

$$\epsilon_i + \frac{\hbar k_i^2}{2m} = \epsilon_f \qquad i = 1, \ldots N_\perp, \tag{2.2.1}$$

where $N_\perp = \frac{Ak_f^2}{2\pi}$ for a two dimensional cross section A . For finite temperature, the values of k require a finite thermal width[22]. The incoming channels are fed from electron reservoirs with chemical potentials μ_1, μ_2 and the overall temperature of the system is T. We assume that the difference in chemical potential $\mu_1 - \mu_2$ is small, that the outgoing channels from each reservoir are fed up to a thermal equilibrium population and that the electrons that reach the drain are absorbed there. We also assume that there is no phase relation between electrons in different channels.

Consider an incoming wave from the j^{th} left hand channel. It has probabilities [22]

$$T_{ij} =\mid t_{ij} \mid^2 \tag{2.2.2}$$

of going into the i^{th} channel on the right hand side (R.h.s) of the scattering region and

$$R_{ij} =\mid r_{ij} \mid^2 \tag{2.2.3}$$

of being reflected into the i^{th} channel on the left hand side (L.h.s).

The analogous matrices for waves entering the dot from the right are denoted by primes. These form a $2N_\perp \times 2N_\perp$ scattering matrix S. (It is $2N_\perp$ as $N_\perp$ possible transitions and $N_\perp$ possible reflections.)

$$S = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix}. \tag{2.2.4}$$

Using the fact that the current must be conserved and that time-reversal symmetry holds, we can derive the identities,

$$SS^* = 1 \tag{2.2.5}$$

and

$$S = \widetilde{S}, \tag{2.2.6}$$

where$\widetilde{}$denotes the transpose and * denotes the complex conjugate. The total transmission and reflection probability into the ith channel are [22]

$$T_i = \sum_j T_{ij}, \qquad T'_i = \sum_j T'_{ij} \tag{2.2.7}$$

and

$$R_i = \sum_j R_{ij}, \qquad R'_i = \sum_j R'_{ij}. \tag{2.2.8}$$

As the incoming waves are normalised, we are able to relate the above expressions by the following expressions

$$\sum_i T_i = \sum_i (1 - R_i) \tag{2.2.9}$$

and

$$\sum_i T_i' = \sum_i (1 - R_i'). \tag{2.2.10}$$

Due to current conservation, the S matrix has the restriction $S^\dagger S = 1$. This restriction to the matrix results in two more equalities being produced

$$R_i' + T_i = 1 \qquad \text{and} \qquad R_i + T_i' = 1. \tag{2.2.11}$$

These identities allow us to write the time averaged current on the right hand side of the scattering region as

$$I = \frac{e}{h} \sum_i \int d\varepsilon [f_1(\varepsilon) T_i(\varepsilon) + f_2(\varepsilon) R_i'(\varepsilon) - f_2(\varepsilon)], \tag{2.2.12}$$

where f_1 and f_2 are the Fermi distributions of the two leads, $\alpha = 1(2)$ corresponding to the left (right) lead. The first term inside the integral of Eq. (2.2.12) corresponds to the transmission from the L.h.s, the second term corresponds to the reflection back from the R.h.s and the final term is the normalised wave incoming from the right. By introducing the chemical potential, taking the summation inside the integral and using Eq. (2.2.11), the current can be written as

$$I = \frac{(\mu_1 - \mu_2)e}{h} \int d\varepsilon \left(-\frac{\partial f}{\partial \varepsilon} \right) \sum_i T_i(\varepsilon). \tag{2.2.13}$$

The current on the L.h.s is also equal to Eq. (2.2.13), as the current is conserved. The linear conductance can be defined as,

$$G \equiv \frac{eI}{\mu_1 - \mu_2}, \tag{2.2.14}$$

therefore,

$$G = \frac{e^2}{h} \int d\varepsilon \left(-\frac{\partial f}{\partial \epsilon} \right) \sum_i T_i(E). \tag{2.2.15}$$

In the zero temperature limit, $T \to 0$, everything is evaluated at ε_F and Eq.(2.2.15) becomes

$$\begin{aligned} G &= \frac{e^2}{h} \sum_{ij} T_{ij} \\ &= \frac{e^2}{h} \mathrm{Tr}\ \mathrm{tt}^\dagger. \end{aligned} \tag{2.2.16}$$

This is the Landauer formula for two-terminal conductance[22].

2.3 Shot Noise Power Spectrum

The noise power spectrum can be calculated in a similar manner by introducing the operators $\hat{a}^\dagger_{\alpha,n}(\varepsilon)$ and $\hat{a}_{\alpha,n}(\varepsilon)$ which create and annihilate electrons which are incident on the scattering region with energy ε in the n^{th} transverse channel of lead α. The operators $\hat{b}^\dagger_{\alpha,n}(\varepsilon)$ and $\hat{b}_{\alpha,n}(\varepsilon)$ create and annihilate electrons in the outgoing states. The operators are related by the same scattering matrix S,

$$\begin{pmatrix} \hat{b}_{1,1}(\varepsilon) \\ \vdots \\ \hat{b}_{1,N_1}(\varepsilon) \\ \hat{b}_{2,1}(\varepsilon) \\ \vdots \\ \hat{b}_{2,N_2}(\varepsilon) \end{pmatrix} = S \begin{pmatrix} \hat{a}_{1,1}(\varepsilon) \\ \vdots \\ \hat{a}_{1,N_1}(\varepsilon) \\ \hat{a}_{2,1}(\varepsilon) \\ \vdots \\ \hat{a}_{2,N_2}(\varepsilon) \end{pmatrix}, \tag{2.3.1}$$

as the approach used to calculate the Landauer formula. By deriving an expression for the time dependent current as a function of these creation and annihilation operators and the scattering matrix, it is possible to calculate the shot noise power spectrum[20]. We will not

derive this expression here, as it is a quite long derivation for details see [21][23]. The two terminal case can be written as

$$\begin{aligned} S &= \frac{2e^2}{h}\int d\varepsilon \left\{ [f_1(\varepsilon)(1-f_2(\varepsilon)) + f_2(\varepsilon)(1-f_1(\varepsilon))]\,\mathrm{Tr}\ \mathrm{tt}^\dagger(1-\mathrm{tt}^\dagger) \right. \\ &\quad \left. + [f_1(\varepsilon)(1-f_1(\varepsilon)) + f_2(\varepsilon)(1-f_2(\varepsilon))]\,\mathrm{Tr}\ \mathrm{tt}^\dagger\mathrm{tt}^\dagger \right\}. \end{aligned} \tag{2.3.2}$$

Eq. (2.3.2) allows us to evaluate the noise for a variety of cases. If we assume that eV and k_BT are small, we can neglect the energy dependence of the transmission matrices. Let us first determine the noise in equilibrium, $\mu_1 = \mu_2$, and show that the noise is related to the conductance by the fluctuation - dissipation theorem. In equilibrium, $f_1 = f_2 = f$ and $f(1-f) = -k_BT\frac{df}{d\varepsilon}$ we find that,

$$\begin{aligned} S &= 4k_BT\frac{e^2}{h}\mathrm{Tr}\ \mathrm{tt}^\dagger \\ &= 4k_BTG \end{aligned} \tag{2.3.3}$$

as required.

If we now consider the shot noise at zero-temperature, the terms that contain $f_a(1-f_a)$ vanish and the Fermi functions are just step functions, $f_a(\varepsilon) = 1 - \Theta(\varepsilon - \mu_a)$. Therefore at zero temperature we obtain,

$$S = 2eV\frac{e^2}{h}\mathrm{Tr}\ \mathrm{tt}^\dagger(1-\mathrm{tt}^\dagger) = 2eV\frac{e^2}{h}\sum_{n=1}^{N} T_n(1-T_n). \tag{2.3.4}$$

Eq. (2.3.4) was first derived by Büttiker [20] and is the multi-channel generalisation of the single channel formula derived by G. B. Lesovik [24]. We notice that like the conductance, the noise power is only a function of the transmission eigenvalues. However, unlike the conductance which can be expressed in terms of the transmission probabilities, the shot noise, even for two terminal conductors, cannot. This implies that the carriers from different chan-

nels interfere and must remain indistinguishable.

It is clear from Eq. (2.3.4) that the transmission eigenstates that correspond to $T_n = 0$ and $T_n = 1$ will not contribute to the shot noise. This is because when $T_n = 0$, there are no electrons being transmitted so therefore no noise. When $T_n = 1$ there is complete transmission so the electrons stream will again be noise free. This means that in a Coulomb blockaded system, the plateaus in the Coulomb staircase are noise free as they correspond to the regime where all the channels are either open or closed. Therefore, the shot noise is only generated when moving between the plateaus on the staircase. It is clear that the noise will be suppressed below the Poisson limit given by Eq. (2.1.4). A convenient measure of sub-poissonian shot noise is the Fano factor F.

2.4 Fano Factor

The Fano factor is the ratio of the actual noise S and the Poisson noise $S_{poisson}$ that would be obtained if the noise in the system were created by a Poissonian process [26],

$$F = \frac{S}{S_{poisson}} = \frac{S}{2e\langle I \rangle}. \tag{2.4.1}$$

The Fano factor takes values between zero which corresponds to a noise free system, and one, which corresponds to the noise of the system being Poissonian. In the situation we discuss above, for the two terminal case of an arbitrary scattering region, Eq. (2.3.2), for energy independent transmission the Fano factor is given by

$$F = \frac{\sum_n T_n(1 - T_n)}{\sum_n T_n}. \tag{2.4.2}$$

The Fano factor takes values between zero (all the channels are transparent) and one (Poisson noise) depending on the transmission probabilities of the region under consideration..

Resonant Double Tunnel Barriers

We can now use the scattering approach to derive the Fano factor for a simple model of a quantum dot. We will model the dot as a potential well separated from electron reservoirs by two finite width tunnel barriers. To do this we will require a slightly more general expression for the noise power spectrum. This can be obtained by using the operator approach and is given by,

$$S = \frac{e^2}{\pi} \sum_n \int d\varepsilon \left[T_n(\varepsilon) \left[f_1(\varepsilon) \left(1 - f_1(\varepsilon)\right) + f_2(\varepsilon) \left(1 - f_2(\varepsilon)\right) \right] + T_n(\varepsilon) \left[1 - T_n(\varepsilon)\right] \left(f_1(\varepsilon) - f_2(\varepsilon)\right)^2 \right], \tag{2.4.3}$$

where the transmission probabilities are now energy dependent. For the resonant double barrier problem, the transmission probabilities are given by the Breit-Wigner formula [25],

$$T_n(\varepsilon) = \frac{\Gamma_{1n}\Gamma_{2n}}{\left(\varepsilon - \varepsilon_n\right)^2 + \frac{\Gamma_n^2}{4}}, \tag{2.4.4}$$

where we have introduced the tunneling rates Γ_1,Γ_2 and $\Gamma_n = \Gamma_{1n} + \Gamma_{2n}$. It will be clear later why the transmission probabilities can be expressed as Eq. (2.4.4) when they are calculated using the Keldysh Green's functions, Eq. (3.5.26) in the following chapter. Inserting the tunneling probabilities into the expression for the noise and the equivalent current expression allows us to express the noise as

$$S = 2e^2 \frac{\Gamma_1^2 + \Gamma_2^2}{\left(\Gamma_1 + \Gamma_2\right)^2} I. \tag{2.4.5}$$

Therefore, the Fano factor for the resonant level double barrier problem is given by

$$F = \frac{\Gamma_1^2 + \Gamma_2^2}{\left(\Gamma_1 + \Gamma_2\right)^2}, \tag{2.4.6}$$

in the symmetric case $\Gamma_1 = \Gamma_2$ the Fano factor takes its minimum value of $1/2$ and increases to 1 for very asymmetrical barriers. For a full review of shot noise and the Fano factor see

[26].

2.5 Summary

In this chapter, we have introduced the noise power spectrum and the Fano factor. We have briefly described the scattering matrix approach to calculating the noise and the Fano factor of a system. We obtained the Fano factor for the double barrier resonant level model and showed that it ranges from 1/2 to 1 depending on the symmetry of the coupling of the leads to the dot. The scattering approach considered in this chapter, however, does not take into account the Coulomb blockade phenomena which we are primarily concerned with. To investigate this phenomena, we will use a Keldysh Green's function approach to calculate the noise. In the following chapter, we will introduce the Keldysh Green's function and show how we can use this approach in the resonant level problem to obtain the noise.

Chapter 3

KELDYSH GREEN FUNCTIONS AND FUNCTIONAL INTEGRATION

In this chapter, we will briefly introduce the basic ideas and techniques which we will require for the calculations in the later chapters. We will introduce the Keldysh Green's function technique and derive some basic results. We will then go on to introduce Grassman fields, coherent states and the functional integration representation. We will conclude the chapter by calculating the noise power spectrum of a non-interacting quantum dot as an example of the techniques we have developed.

In order to discuss the statistical properties of a system we will need to consider the correlation functions of the field variables. We will refer to these correlation functions as Green's functions. We will occasionally use the word propagator for various types of Green's functions. Before defining the Green's function, we will briefly introduce the different representations for the wavefunctions and the operators that we will use.

3.1 Interaction Representation

There are three different representations of the wavefunctions and the operators in quantum mechanics that we will consider. They differ in where the time dependence resides, It can

reside in the wavefunction, in the operator or both (in all the representations we use the units $\hbar = 1$).

3.1.1 Schrodinger Representation

The Schrodinger representation assumes the time dependence resides in the wave function $\psi(t)$ and the operators (eg the Hamiltonian) are independent of time,

$$i\frac{\partial}{\partial t}\psi(t) = H\psi(t), \tag{3.1.1}$$

$$\psi(t) = e^{-iHt}\psi(0). \tag{3.1.2}$$

3.1.2 Heisenberg Representation

This is a different way of looking at quantum mechanics that produces exactly the same results. It assumes that the wave functions are time independent and the operators are time dependent,

$$O(t) = e^{iHt}O(0)e^{-iHt}. \tag{3.1.3}$$

When we express the wave functions in the Heisenberg representation, we will use the subscript H.

3.1.3 Interaction Representation

The most useful representation for our purposes will be the interaction representation. In this representation both the wave functions and the operators are time dependent. If we consider the Hamiltonian separated into two parts,

$$H = H_0 + H_i. \tag{3.1.4}$$

H_0 is the free or unperturbed Hamiltonian, which is quadratic in nature and H_i is the interaction Hamiltonian, which is free to take any form. In the interacting representation, the operators contain the time dependence of the free Hamiltonian, but not the interaction,

$$O(t) = e^{iH_0 t} O e^{-iH_0 t}, \tag{3.1.5}$$

and the wave functions have the time dependence:

$$\psi(t) = e^{iH_0 t} e^{-iHt} \psi(0). \tag{3.1.6}$$

It is not possible to combine the exponentials in Eq. (3.1.6) unless

$$[H_0, H_i] = 0, \tag{3.1.7}$$

as

$$e^A e^B = e^{A+B} \quad \textit{iff} \quad [A, B] = 0. \tag{3.1.8}$$

In Eq. (3.1.6), we have introduced an operator which we define as $S(t, 0)$:

$$S(t, 0) = e^{iH_0 t} e^{-iHt}. \tag{3.1.9}$$

This function obeys the differential equation [27]:

$$\frac{\partial S(t, 0)}{\partial t} = -iH_i(t) S(t, 0), \tag{3.1.10}$$

which we can solve to obtain,

$$S(t, 0) = 1 - i \int_0^t dt_1 H_i(t_1) S(t_1, 0). \tag{3.1.11}$$

If this is repeated iteratively, it gives

$$S(t,0) = 1 + \sum \frac{(-i)^n}{n!} \int_0^t dt_1 \ldots \int_0^t dt_n T H_i(t_1) \ldots H_i(t_n) \tag{3.1.12}$$

where we have introduced the time ordering operator T, which is defined as

$$Ta(t_1)b(t_2) = \begin{cases} a(t_1)b(t_2) & \text{if } t_1 > t_2 \\ \mp b(t_2)a(t_1) & \text{if } t_2 > t_1 \end{cases}, \tag{3.1.13}$$

where $\mp$ is for the fermionic/bosonic case. Eq. (3.1.12) can be abbreviated to

$$S(t,0) = T \exp\left[-i \int_0^t dt_1 H_1(t_1)\right], \tag{3.1.14}$$

however, it should always be kept in mind that the exponential form is just shorthand for Eq. (3.1.12).

3.2 Green's Functions

The Green's function can be though of as the inverse of a differential operator, at least for the single particle case [27] [28]. It is possible to write it for the more general many particle and interacting systems. The single particle Green's function can be defined as the solution to

$$\left(\hat{\varepsilon} - \hat{H}\right) G(\mathbf{r}, t; \mathbf{r}', t') = \left[i\partial_t - \frac{\nabla^2}{2m} + \mu \ - H_i\right] G(\mathbf{r}, t; \mathbf{r}', t') = \delta(t - t')\delta(\mathbf{r} - \mathbf{r}'). \tag{3.2.1}$$

If we consider the case $H = H_0$, by Fourier transforming to momentum space, the solution to this can be found to be

$$G_0(\varepsilon, \mathbf{p}) = \frac{1}{\varepsilon - \xi_{\mathbf{p}} + i\delta \mathrm{sgn}(\xi_{\mathbf{p}})} \tag{3.2.2}$$

where $\xi_{\mathbf{p}} = \frac{p^2}{2m} - \mu$ and sgn(α) is $+1$ for positive α and -1 for negative α. This however, is not the most useful form for the more general cases, as not all Green's functions can be expressed as the inverse of a differential operator. In general, the single particle Green's function can be written as

$$G(\mathbf{r}, t; \mathbf{r}', t') = -i\langle T\psi_H(\mathbf{r}, t)\psi_H^\dagger(\mathbf{r}', t')\rangle, \tag{3.2.3}$$

where T is the time ordering operator. It is useful to express the Green's function in the interaction representation. If we assume that the interactions are introduced adiabatically, it is given by

$$G(\mathbf{r}, t; \mathbf{r}', t') = -i\frac{\langle T\psi_0(\mathbf{r}, t)\psi_0^\dagger(\mathbf{r}', t')S(\infty, -\infty)\rangle}{\langle S(\infty, -\infty)\rangle}, \tag{3.2.4}$$

where $\psi_0(\mathbf{r}, t)$ evolves under the free Hamiltonian. The advantage of the interaction representation is that it is possible to Taylor expand $S(\infty, -\infty)$. This leads to a perturbation expansion if the interaction H_i is small.

Many physical properties for the many-body system can be specified using one particle Green's functions. For example, in a system, in an arbitrary state described by the density matrix ρ, the average density at a space time point, is given by

$$n(\mathbf{r}, t) = \mathrm{Tr}\left(\rho\psi_H^\dagger(\mathbf{r}, t)\psi_H(\mathbf{r}, t)\right) \tag{3.2.5}$$

where the quantum fields describing the particles $\psi_H(\mathbf{r}, t)$ are in the Heisenberg picture and Tr denote the trace. The average density can be expressed in terms of the G-lesser Green's function, as

$$n(\mathbf{r}, t) = -G^<(\mathbf{r}, t; \mathbf{r}, t), \tag{3.2.6}$$

where

$$\begin{aligned} G^{<}(\mathbf{r},t;\mathbf{r}',t') &= i\mathrm{Tr}\left(\rho\psi_H^{\dagger}(\mathbf{r}',t')\psi_H(\mathbf{r},t)\right) \\ &\equiv i\langle\psi_H^{\dagger}(\mathbf{r}',t')\psi_H(\mathbf{r},t)\rangle, \end{aligned} \tag{3.2.7}$$

for fermions. It is clear from this that the triangular brackets mean the trace of the operators weighted with respect to the state of the system,

$$\langle\ldots\rangle \equiv \mathrm{Tr}(\rho\ldots). \tag{3.2.8}$$

For the case of a pure state, $G^{<}(\mathbf{r},t;\mathbf{r}',t')$ is the amplitude to remain in the state $|\Psi\rangle$ after removing, at a time t, a particle at position $\mathbf{r}$ and restoring, at a time t', a particle at position $\mathbf{r}'$. In the case of a mixed state, an additional statistical averaging over the distribution of initial states takes place. As well as G-lesser, we shall encounter several other Green's functions and to make the introduction of Keldysh Greens functions easier, we shall define them all here.

The G-greater Green's function,

$$G^{>}(\mathbf{r},t;\mathbf{r}',t') = -i\mathrm{Tr}\left(\rho\psi_H(\mathbf{r},t)\psi_H^{\dagger}(\mathbf{r}',t')\right) = -i\langle\psi_H(\mathbf{r},t)\psi_H^{\dagger}(\mathbf{r}',t')\rangle, \tag{3.2.9}$$

is the amplitude to remain in the state $|\Psi\rangle$ after adding at time t', a particle at position $\mathbf{r}'$ and then removing, at time t, a particle from position $\mathbf{r}$. Using these Green functions it is possible to define the time ordered Greens function Eq. (3.2.3), which we introduced earlier,

$$G(\mathbf{r},t;\mathbf{r}',t') = G^{T}(\mathbf{r},t;\mathbf{r}',t') = -i\langle T\psi_H(\mathbf{r},t)\psi_H^{\dagger}(\mathbf{r}',t')\rangle$$

where the time ordering operator arranges the quantum fields to its right in order of time, largest time value first then decreasing as you move to the right, as either the G-lesser or G-greater Green's function:

$$G(\mathbf{r},t;\mathbf{r}',t') = \begin{cases} G^{<}(\mathbf{r},t;\mathbf{r}',t') & t' > t \\ G^{>}(\mathbf{r},t;\mathbf{r}',t') & t > t' \end{cases} \tag{3.2.10}$$

In Eq. (3.2.10), we have used the minus sign convention when two fermi field operators are interchanged. Later, we shall encounter the anti-time ordered Green's function,

$$G^{\tilde{T}}(\mathbf{r},t;\mathbf{r}',t') = -i < \tilde{T}\psi_H(\mathbf{r},t)\psi_H^{\dagger}(\mathbf{r}',t') > \tag{3.2.11}$$

where $\tilde{T}$ anti-time orders, i.e orders the operators opposite to that of T. We note that with the aid of the step function Θ, the time ordered and anti-time ordered Green functions can be written as

$$G(\mathbf{r},t;\mathbf{r}',t') = \Theta(t-t')G^{>}(\mathbf{r},t;\mathbf{r}',t') + \Theta(t'-t)G^{<}(\mathbf{r},t;\mathbf{r}',t') \tag{3.2.12}$$

and

$$G^{\tilde{T}}(\mathbf{r},t;\mathbf{r}',t') = \Theta(t-t')G^{<}(\mathbf{r},t;\mathbf{r}',t') + \Theta(t'-t)G^{>}(\mathbf{r},t;\mathbf{r}',t'). \tag{3.2.13}$$

The Green's function can also be written in terms of the retarded and advanced Green's functions, G^R and G^A, which again will be useful when we consider the Keldysh approach. They have the following properties:

$$\Re G(\varepsilon,\mathbf{p}) = \Re G^R(\varepsilon,\mathbf{p}) = \Re G^A(\varepsilon,\mathbf{p}) \tag{3.2.14}$$

$$\Im G^R(\varepsilon,\mathbf{p}) = \Im G(\varepsilon,\mathbf{p}) sgn(\xi_{\mathbf{p}}) \tag{3.2.15}$$

$$\Im G^A(\varepsilon,\mathbf{p}) = -\Im G(\varepsilon,\mathbf{p}) sgn(\xi_{\mathbf{p}}) \tag{3.2.16}$$

where $\Re$ indicates the real part and $\Im$ indicates the imaginary part. For the case $H = H_0$, the retarded and advanced Green's functions can be written as

$$G_0^{R/A}(\varepsilon, \mathbf{p}) = \frac{1}{\varepsilon - \xi_{\mathbf{p}} \pm i\delta}. \tag{3.2.17}$$

They can also be formed in terms of the greater and lesser Green's functions. The advanced Green function is

$$G^A(\mathbf{r}, t; \mathbf{r}', t') = -\Theta(t' - t) \left(G^>(\mathbf{r}, t; \mathbf{r}', t') - G^<(\mathbf{r}, t; \mathbf{r}', t')\right) \tag{3.2.18}$$

and the retarded Green's function is

$$G^R(\mathbf{r}, t; \mathbf{r}', t') = \Theta(t - t') \left(G^>(\mathbf{r}, t; \mathbf{r}', t') - G^<(\mathbf{r}, t; \mathbf{r}', t')\right). \tag{3.2.19}$$

The other Green's function that we will encounter is the Keldysh Green's function that can be defined in terms of the greater and lesser Green's functions as

$$G^K(\mathbf{r}, t; \mathbf{r}', t') = G^>(\mathbf{r}, t; \mathbf{r}', t') + G^<(\mathbf{r}, t; \mathbf{r}', t'). \tag{3.2.20}$$

3.3 Keldysh Green's Functions

Throughout this thesis, we will be making use of Keldysh Green's functions. This approach was developed in the 1964 paper by L.V. Keldysh [29]. In addition to this work, parallel work was carried out by Martin [30] and Schwinger [31].

3.3.1 The Time Contour

As opposed to the Matsubara and the zero temperature methods, the Keldysh approach allows us to consider systems that are not in thermal equilibrium. It allows this by consid-

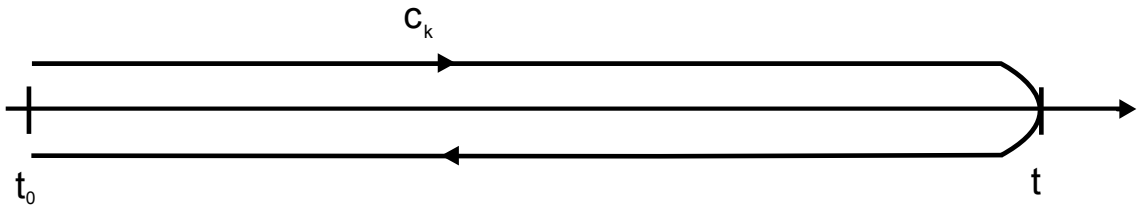

Figure 3.1: The Keldysh time contour c_k, which runs from an initial time t_0 to a time t, which is the $\max\{t, t'\}$, in the standard Keldysh approach we let $t \to \infty$ and $t_0 \to 0. - \infty$.[35]

ering a different time contour to the other approaches that does not require knowledge of the final state of the system. In the zero temperature approach, the time contour runs from $t = -\infty$ to $t = \infty$ where the system is considered to be in the known ground state of the non-interacting Hamiltonian H_0, at $t = -\infty$. The interactions $H - H_0$ are then adiabatically switched on and the system evolves to the ground state of the interacting Hamiltonian H. The interactions are then adiabatically switched off in the distant future, arriving at $t = \infty$ at the state $|\infty\rangle$. The assumptions of this approach is that this state is unique, independent of the switching procedure and is again the ground state of H_0 up to a phase factor. However, out of equilibrium this is not the case when the interactions are turned on and off, the system evolves to an unknown state.

To avoid this, in the Keldysh approach, we consider a time contour that was first suggested by Schwinger [31] which is to take the final state to be exactly the same state as the initial state. To do this, we consider the time contour in figure (3.1) where we once again start at $t = -\infty$ then adiabatically switch on the interactions and let the system evolve in the forward direction to $t = \infty$ and then "unwind" the evolution backwards to $t = -\infty$ to the known initial state. This allows us to bypass the lack of information about the state at $t = \infty$, but does result in the algebraic structure of this approach being more complicated.

We should remind ourselves at this point that, as the Green's function in the Heisenberg

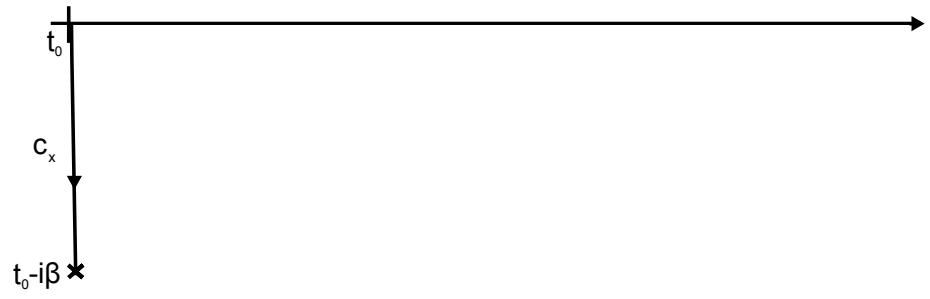

Figure 3.2: The addition imaginary contour c_x.[28][?]

representation is given by

$$G(\mathbf{r},t;\mathbf{r}',t') = -i\mathrm{Tr}\left(\rho T\psi_H^\dagger(\mathbf{r}',t')\psi_H(\mathbf{r},t)\right). \tag{3.3.1}$$

When the system is in equilibrium $\rho = e^{-\beta H}$. This is the Gibbs distribution. If we change to the interaction representation for ψ and its conjugate, we also have to change $e^{-\beta H}$ to the interaction representation. This becomes

$$e^{-\beta H} = e^{-\beta H_0} T \exp\left[-\int_{t_0}^{t_0-i\beta} dt_1 H_i(t_1)\right], \tag{3.3.2}$$

where the time ordering operator T in Eq. (3.3.2) orders along the contour that stretches down into the lower complex plane from t_0 to $t_0 - i\beta$[33]. This results in the extra contour, c_x, in Fig. (3.2) needing to be considered. This can be included with the real time contour to form the interaction contour, c_K, depicted in Fig. (3.3). In the Keldysh formalism, the imaginary leg of the contour is usually ignored and the contour Fig. (3.1) is used. The imaginary part of the contour describes the initial distribution of the system and neglecting this part of the contour corresponds to losing this information [32][34].

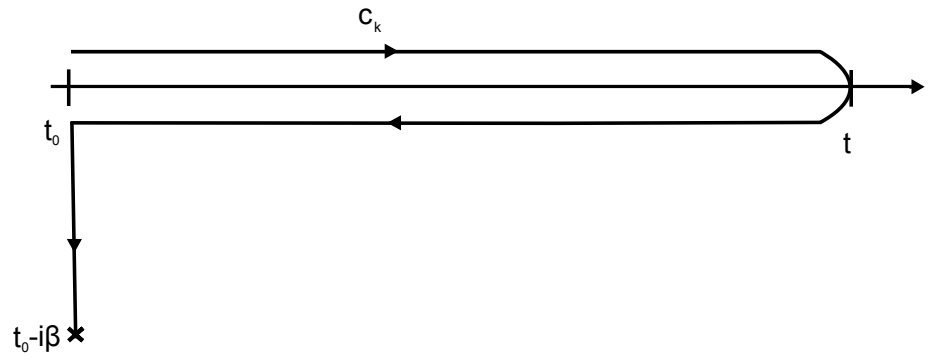

Figure 3.3: The Keldysh interaction time contour c_K.[35]

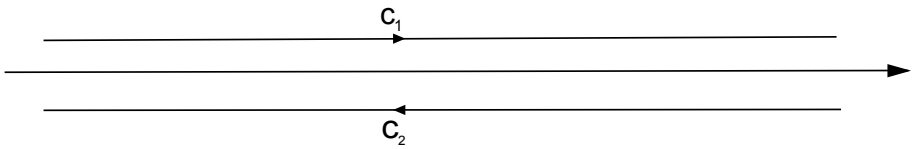

Figure 3.4: The Keldysh contour considered in two parts the upper part, C_1, of the contour runs from $t = -\infty$ to $t = \infty$ and the lower part, C_2, runs from $t = \infty$ back to $t = -\infty$.[35]

3.3.2 The Matrix Structure

The result of using this time contour in the Keldysh approach is that the general expression for the single particle Green's function can be written as

$$G(\mathbf{r}, t; \mathbf{r}', t') = -i\langle T_{c_k} \psi_H(\mathbf{r}, t) \psi_H^\dagger(\mathbf{r}', t') \rangle \tag{3.3.3}$$

where T_{c_k} is the contour time ordering operator which arranges the quantum fields to its right in the order in which they appear on the time contour, as shown in Fig. (3.1). A property of this time contour is that the partition function, Z,

$$Z = 1 \tag{3.3.4}$$

as the contributions from the upper and the lower branches of the contour cancel exactly. In order to derive the matrix structure of the Keldysh Green's functions, we need to consider the two parts of the contour, upper and lower, separately (see Fig. (3.4)). Eq. (3.3.3) is now

split into four separate cases depending on if t and t' reside on the upper, lower or different contours. Labeling the upper contour as 1 and the lower contour as 2, it is possible to write Eq. (3.3.3) as the matrix $\tilde{G}_{ij}$. $i = \{1, 2\}$ and $j = \{1, 2\}$ refer to t and t' respectively residing on the corresponding contour,

$$\tilde{\mathbf{G}} = \begin{pmatrix} \tilde{G}_{11} & \tilde{G}_{12} \\ \tilde{G}_{21} & \tilde{G}_{22} \end{pmatrix}. \tag{3.3.5}$$

For the off diagonal elements $\tilde{G}_{12}$ and $\tilde{G}_{21}$, the time co-ordinates are always on different branches of the time contour. This means that they are always ordered either forward ($\tilde{G}_{12}$) or backwards ($\tilde{G}_{21}$) along the contour and are therefore equivalent to $G^<$ and $G^>$ respectively. For the diagonal elements both t and t' reside on the same branch of the contour so correspond to the time ordered and anti-time ordered Green's functions.

To summarise the elements of $\tilde{\mathbf{G}}$ are given by

$$\tilde{G}_{11}(\mathbf{r}, t; \mathbf{r}', t') = -i\langle T\psi_H(\mathbf{r}, t)\psi_H^\dagger(\mathbf{r}', t')\rangle \tag{3.3.6}$$

$$\tilde{G}_{22}(\mathbf{r}, t; \mathbf{r}', t') = -i\langle \tilde{T}\psi_H(\mathbf{r}, t)\psi_H^\dagger(\mathbf{r}', t')\rangle \tag{3.3.7}$$

$$\tilde{G}_{12}(\mathbf{r}, t; \mathbf{r}', t') = G^<(\mathbf{r}, t; \mathbf{r}', t') = i\langle \psi_H^\dagger(\mathbf{r}', t')\psi_H(\mathbf{r}, t)\rangle \tag{3.3.8}$$

$$\tilde{G}_{21}(\mathbf{r}, t; \mathbf{r}', t') = G^>(\mathbf{r}, t; \mathbf{r}', t') = i\langle \psi_H(\mathbf{r}, t)\psi_H^\dagger(\mathbf{r}', t')\rangle. \tag{3.3.9}$$

However, this is not the simplest form that the matrix can take, as not all the entries are linearly independent. To simplify the matrix, we can now make the rotation to the Larkin-Ovchinnikov basis [36],

$$\mathbf{G} = \mathbf{L}_0\tau^3\tilde{\mathbf{G}}\mathbf{L}_0^\dagger, \tag{3.3.10}$$

where

$$\mathbf{L}_0 = \frac{1}{\sqrt{2}}\left(\tau^0 - i\tau^2\right) \tag{3.3.11}$$

and the τ's are the Pauli matrices,

$$\tau^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \tau^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \tau^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \tau^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.3.12}$$

The simplified matrix has the form

$$\mathbf{G} = \begin{pmatrix} G^R & G^K \\ 0 & G^A \end{pmatrix}, \tag{3.3.13}$$

for fermions. The new elements of $\mathbf{G}$ relate to the old by

$$\begin{aligned} G^R(t,t') &= \tilde{G}_{11}(t,t') - \tilde{G}_{12}(t,t') = \tilde{G}_{21}(t,t') - \tilde{G}_{22}(t,t'), \\ G^A(t,t') &= \tilde{G}_{11}(t,t') - \tilde{G}_{21}(t,t') = \tilde{G}_{12}(t,t') - \tilde{G}_{22}(t,t') \end{aligned} \tag{3.3.14}$$

and

$$G^K(t,t') = \tilde{G}_{21}(t,t') + \tilde{G}_{12}(t,t') = \tilde{G}_{11}(t,t') + \tilde{G}_{22}(t,t'). \tag{3.3.15}$$

Near thermal equilibrium, the Keldysh component can be written as [35]

$$G^K(\varepsilon,\mathbf{p}) = h(\varepsilon)\left(G^R(\varepsilon,\mathbf{p}) - G^A(\varepsilon,\mathbf{p})\right) \tag{3.3.16}$$

where $h(\varepsilon) = 1 - 2f_\varepsilon = \tanh\left(\frac{1}{2}\beta\varepsilon\right)$ and f_ε is the fermi distribution. It can be verified that this is exact for the equilibrium case. However, away from equilibrium and more generally, the distribution function must be found from the quantum kinetic equation and the Keldysh

component is given by

$$G^K(t,t') = G^R(t,t'')\mathcal{F}(t'',t') - \mathcal{F}(t,t'')G^A(t'',t'), \tag{3.3.17}$$

where integration over the time index t'' is implied and $\mathcal{F}(t,t'')$ is the distribution function.

3.3.3 Langreth Theorem

Within the real time Dyson Equation and in the perturbative expansion of the contour ordered Green's function, we will encounter objects which are integrated over the Keldysh contour. In this section, we will derive the appropriate formula to deal with these situations. In the following section, only the contour ordered time variables are important, so the spacial components and any spin indices shall be suppressed.

If we consider the case where the Hamiltonian contains a time dependent potential V, then the Dyson equation has the form:

$$G(t,t') = G_0(t,t') + \int_{c_k} dt_1 \int_{c_k} dt_2 G_0(t,t_2)\Sigma(t_2,t_1)G(t_1,t') + \int_{c_k} dt_1 G_0(t,t_1)V(t_1)G(t_1,t'), \tag{3.3.18}$$

where Σ is the self energy of the problem. We thus have to solve integrals of the form

$$C(t,t') = \int_{c_k} dt_1 A(t,t_1)B(t_1,t'), \tag{3.3.19}$$

where A and B are functions that have a contour time order dependence. We need to turn the contour time integral into integrations over the real time axis. To do this we need to consider the analytic functions $C^<(t,t')$ and $C^>(t,t')$. We will demonstrate the analytical continuation procedure for the $C^<(t,t')$ case. This means that the contour time t appears earlier than the contour time t'. Exploiting the analyticity of the Keldysh contour, we deform

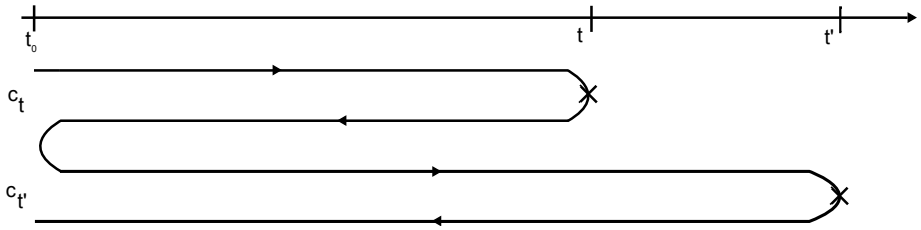

Figure 3.5: Deforming the Keldysh contour into the contour formed from the contours c_t and $c_{t'}$ [37]

the contour into the contour $c_t + c_{t'}$ depicted in Fig. (3.5)[37][28]. Therefore, the expression for $C^<$ becomes

$$\begin{aligned} C^<(t,t') &= \int_{c_t} dt_1 A(t,t_1)B(t_1,t') + \int_{c_{t'}} dt_1 A(t,t_1)B(t_1,t') \\ &= \int_{c_t} dt_1 A(t,t_1)B^<(t_1,t') + \int_{c_{t'}} dt_1 A^<(t,t_1)B(t_1,t'), \end{aligned} \tag{3.3.20}$$

where we have used the fact that on the c_t contour, $t_1 < t'$ and on the $c_{t'}$ contour, we have $t_1 > t$. If we now spilt the contours into upper (forward time) and lower (reverse time) parts, we have

$$\begin{aligned} C^<(t,t') &= \int_{\overrightarrow{c_t}} dt_1 A^>(t,t_1)B^<(t_1,t') + \int_{\overleftarrow{c_t}} dt_1 A^<(t,t_1)B^<(t_1,t') \\ &+ \int_{\overrightarrow{c_{t'}}} dt_1 A^<(t,t_1)B^<(t_1,t') + \int_{\overleftarrow{c_{t'}}} dt_1 A^<(t,t_1)B^>(t_1,t') \end{aligned} \tag{3.3.21}$$

where $\overrightarrow{c_t}$ indicates the upper half of the c_t contour and $\overleftarrow{c_t}$ indicates the lower half of the c_t contour. We can now parameterise the contour in terms of the real time variable and taking the limit $t_0 \to -\infty$, we obtain

$$\begin{aligned} C^<(t,t') &= \int_{-\infty}^{t} dt \left[A^>(t,t_1) - A^<(t,t_1)\right] B^<(t_1,t') \\ &+ \int_{-\infty}^{t'} dt A^<(t,t_1) \left[B^<(t_1,t') - B^>(t_1,t')\right] \end{aligned} \tag{3.3.22}$$

which can be written as

$$\begin{aligned} C^<(t,t') &= \int_{-\infty}^{\infty} dt\Theta(t-t_1)\left[A^>(t,t_1)-A^<(t,t_1)\right]B^<(t_1,t') \\ &+ \int_{-\infty}^{\infty} dt\Theta(t'-t_1)A^<(t,t_1)\left[B^<(t_1,t')-B^>(t_1,t')\right]. \end{aligned} \tag{3.3.23}$$

If we now use the identities for the retarded and advanced Green's functions:

$$\begin{aligned} A^R(t,t') &= \Theta(t-t')\left[A^>(t,t')-A^<(t,t')\right], \\ A^A(t,t') &= \Theta(t'-t)\left[A^>(t,t')-A^<(t,t')\right], \end{aligned} \tag{3.3.24}$$

we obtain the Langreth theorem for the lesser component

$$C^<(t,t') = \int_{c_k} dt_1 A^R(t,t_1)B^<(t_1,t') + \int_{c_k} dt_1 A^<(t,t_1)B^A(t_1,t'). \tag{3.3.25}$$

Analogously, we can show that the greater component is given by

$$C^>(t,t') = \int_{c_k} dt_1 A^R(t,t_1)B^>(t_1,t') + \int_{c_k} dt_1 A^>(t,t_1)B^A(t_1,t'). \tag{3.3.26}$$

3.3.4 Functional Integration

To calculate the density of states and the conductance of the interacting system, we will use functional integration representation. A functional integral is a path integral defined with the overcomplete set of coherent states. As we shall be dealing with Fermions, it is also necessary to introduce Grassmann Algebra, but first it is helpful to define coherent states for the Bosonic case.

Coherent States (Bosons)

A coherent state is defined as the eigenstate of an annihilation operator. For Bosons, the eigenstates are known as bosonic coherent states [38],

$$|\phi\rangle \equiv e^{\left[\sum_i \phi_i \hat{a}_i^\dagger\right]}|0\rangle, \tag{3.3.27}$$

where $|0\rangle$ is the vacuum state and $\phi = \{\phi_i\}$ represents a set of complex numbers. The states are eigenstates in the sense that for all i,

$$\hat{a}_i|\phi\rangle = \phi_i|\phi\rangle. \tag{3.3.28}$$

It is also useful to note here some other properties. By taking the Hermitian conjugate of Eq. (3.3.28), we find

$$\langle\phi|\hat{a}_i^\dagger = \langle\phi|\bar{\phi}_i, \tag{3.3.29}$$

where $\bar{\phi}_i$ is the complex conjugate of ϕ_i and

$$\langle\phi| = \langle 0|e^{\left[\sum_i \bar{\phi}_i \hat{a}_i\right]}. \tag{3.3.30}$$

By Taylor expanding Eq. (3.3.27), it is possible to show that

$$\hat{a}_i^\dagger|\phi\rangle = \partial_{\phi_i}|\phi\rangle. \tag{3.3.31}$$

The overlap between two coherent states is given by

$$\langle\theta|\phi\rangle = e^{\left[\sum_i \bar{\theta}_i \phi_i\right]}. \tag{3.3.32}$$

This implies that the norm is given by

$$\langle\phi|\phi\rangle = e^{\left[\sum_i \bar{\phi}_i \phi_i\right]}. \tag{3.3.33}$$

The coherent states form an overcomplete set of states in Fock space:

$$\int \prod_i \frac{d\bar{\phi}_i d\phi_i}{\pi} e^{\left[-\sum_i \bar{\phi}_i \phi_i\right]} |\phi\rangle\langle\phi| = \hat{\mathbf{1}} \tag{3.3.34}$$

where $d\bar{\phi}_i d\phi_i = d\Re\phi_i d\Im\phi_i$ and $\hat{\mathbf{1}}$ represents the unit operator. With these definitions, we have all that we require to derive the many-body path integral for the bosonic system but, before we proceed, let us first introduce the fermionic version of the coherent state.

Coherent States (Fermions)

As for the bosonic case, the annihilation operators are characterised by a set of coherent states such that for all i [38],

$$\hat{a}_i|\psi\rangle = \psi_i|\psi\rangle \tag{3.3.35}$$

but as we are now dealing with fermions, the operators anticommute $\{\hat{a}_i, \hat{a}_j\} = 0$ for $i \neq j$. This implies that the eigenvalues ψ_i must also anticommute,

$$\psi_i\psi_j = -\psi_j\psi_i. \tag{3.3.36}$$

Clearly, the eigenvalues can not be ordinary numbers. To overcome this we introduce Grassmann numbers that obey the anticommutation relation

$$\{\psi_i, \psi_j\} = 0. \tag{3.3.37}$$

This results in Grassmann numbers having the property that the square of a Grassmann number is zero and hence any function of Grassmann numbers is linear. We can define integration and differentiation as

$$\int d\psi_i = 0, \qquad \int d\psi_i \psi_i = 1 \tag{3.3.38}$$

and

$$\partial_{\psi_i} \psi_j = \delta_{ij}. \tag{3.3.39}$$

In order to be consistent with the anticommutation relation, the differentiation operator ∂_{ψ_i} must be anti-commutative as well, i.e for $i \neq j$, $\partial_{\psi_i} \psi_j \psi_i = -\psi_j$.

By making use of the Grassmann algebra, it is possible for us to define the fermionic coherent state as

$$\hat{a}_i |\psi\rangle = \psi_i |\psi\rangle, \tag{3.3.40}$$

where

$$|\psi\rangle = e^{\left[-\sum_i \psi_i \hat{a}_i^\dagger\right]} |0\rangle. \tag{3.3.41}$$

This expression can be simplified to

$$|\psi\rangle = \prod_i \left(1 - \psi_i \hat{a}_i^\dagger\right) |0\rangle, \tag{3.3.42}$$

where we have Taylor expanded the exponential. This expression is exact, as all higher order terms are zero due to Eq. (3.3.37). It is clear that the properties (3.3.31), (3.3.32) and (3.3.33) carry over to the fermionic case. We associate the a_i with the fermionic operation and replace ϕ_i with ψ_i. There are two major differences between the fermionic and bosonic

cases. The Grassmann variables $\bar{\psi}_i$ appearing in the adjoint

$$\langle\psi| = \langle 0|e^{\left[\sum_i \bar{\psi}_i \hat{a}_i\right]}, \tag{3.3.43}$$

are not related to the ψ_i's of the state $|\psi\rangle$. The ψ_i and $\bar{\psi}_i$ are independent variables. The other major difference is that the Grassmann version of the Gaussian integral

$$\int\int d\bar{\psi} d\psi e^{-\bar{\psi}\psi} = 1, \tag{3.3.44}$$

does not contain the factor of π that the standard Gaussian integral contains (see Appendix A). We will also require the overcompleteness property for the fermions which is given by

$$\int \prod_i d\bar{\psi}_i d\psi_i e^{-\sum_i \bar{\psi}_i \psi_i} |\psi\rangle\langle\psi| = \hat{\mathbf{1}}. \tag{3.3.45}$$

Finally, before we develop functional field integration, it is useful to remind ourselves of some basic results of Gaussian integration which are covered in Appendix A.

To derive the functional integral representation of the Green's function, we will need to make use of the property

$$\mathrm{Tr}\hat{A} = \int D\psi D\bar{\psi} e^{-\bar{\psi}\psi} \langle -\psi|\hat{A}|\psi\rangle \tag{3.3.46}$$

where $D\psi = \prod_i d\psi_i$ and $\bar{\psi}\psi = \sum_i \bar{\psi}_i \psi_i$. This can be easily proved by making use of the properties of coherent states. If we consider the trace of an operator explicitly, as the sum over a complete set of many body states $\{|n\rangle\}$,

$$\mathrm{Tr}\hat{A} = \sum_n \langle n|\hat{A}|n\rangle, \tag{3.3.47}$$

and insert the resolution of unity Eq. (3.3.45)

$$\mathrm{Tr}\hat{A} = \int \prod_i d\bar{\psi}_i^{(0)} d\psi_i^{(0)} e^{-\sum_i \bar{\psi}_i^{(0)}\psi_i^{(0)}} \sum_n \langle n|\hat{A}|\psi^{(0)}\rangle\langle\psi^{(0)}|n\rangle. \tag{3.3.48}$$

Since Grassmann variables are involved in the definition of $|\psi_i^{(0)}\rangle$, we can not simply change the order in which the matrix elements are multiplied

$$\langle n|\hat{A}|\psi^{(0)}\rangle\langle\psi^{(0)}|n\rangle \neq \langle\psi^{(0)}|n\rangle\langle n|\hat{A}|\psi^{(0)}\rangle. \tag{3.3.49}$$

To over come this, we first need to consider $\langle\psi^{(0)}|n\rangle$ and represent $|n\rangle$ as $\hat{a}_n^\dagger\hat{a}_{n-1}^\dagger\ldots\hat{a}_1^\dagger|0\rangle$,

$$\langle\psi^{(0)}|n\rangle = \langle 0|e^{-\sum_i \hat{a}_i\bar{\psi}_i^{(0)}}\hat{a}_n^\dagger\hat{a}_{n-1}^\dagger\ldots\hat{a}_1^\dagger|0\rangle, \tag{3.3.50}$$

by Taylor expanding the exponential, this becomes

$$\begin{aligned}
\langle\psi^{(0)}|n\rangle &= \langle 0|(-\hat{a}_1\bar{\psi}_1^{(0)})\ldots(-\hat{a}_n\bar{\psi}_n^{(0)})\hat{a}_n^\dagger\hat{a}_{n-1}^\dagger\ldots\hat{a}_1^\dagger|0\rangle \\
&= (-1)^n(-1)^{\frac{n(n+1)}{2}}\bar{\psi}_1^{(0)}\bar{\psi}_2^{(0)}\ldots\bar{\psi}_n^{(0)}\langle 0|\hat{a}_1\ldots\hat{a}_n\hat{a}_n^\dagger\ldots\hat{a}_1^\dagger|0\rangle \\
&= (-1)^n(-1)^{\frac{n(n+1)}{2}}\bar{\psi}_1^{(0)}\bar{\psi}_2^{(0)}\ldots\bar{\psi}_n^{(0)}.
\end{aligned} \tag{3.3.51}$$

If we now move the result above to the left of $\hat{A}|\psi^{(0)}\rangle$, we can use the fact that any term of the expansion of both $|\psi^{(0)}\rangle$ and $\hat{A}$ has an even number of anti-commuting terms, so commutes to give

$$\langle n|\hat{A}|\psi^{(0)}\rangle\langle\psi^{(0)}|n\rangle = (-1)^n(-1)^{\frac{n(n+1)}{2}}\langle 0|\hat{a}_1\hat{a}_2\ldots\hat{a}_n\bar{\psi}_1^{(0)}\bar{\psi}_2^{(0)}\ldots\bar{\psi}_n^{(0)}\hat{A}|\psi^{(0)}\rangle. \tag{3.3.52}$$

If we now move all the $\bar{\psi}^{(0)}$'s to the left of the annihilation operators, we get another factor of $(-1)^{n^2}$ and use the fact that $(-1)^{n^2} = (-1)^n$. We can now attribute a minus sign to each

of the Grassmann variables and obtain

$$\langle n|\hat{A}|\psi^{(0)}\rangle\langle\psi^{(0)}|n\rangle = (-1)^n(-1)^{\frac{n(n+1)}{2}}(-\bar{\psi}_1^{(0)})(-\bar{\psi}_2^{(0)})\dots(-\bar{\psi}_n^{(0)})\langle n|\hat{A}|\psi^{(0)}\rangle, \tag{3.3.53}$$

if we now use Eq. (3.3.51) again this becomes Eq. (3.3.46), as required [38].

3.4 Functional Integration Representation of the Green's Function

We will now make use of coherent states to derive the Green's function in the functional integration representation. If we start from the Green's function in the form

$$G(t,t') = -\frac{i}{Z}\sum_n \langle n|T_{c_K}\hat{\psi}_t\hat{\psi}_{t'}^\dagger e^{-i\int_{c_K}dt_1 H(t_1)}|n\rangle, \tag{3.4.1}$$

where we have now suppressed the position indices. We can now use Eq. (3.3.46) and express the Green's function as

$$G(t,t') = -\frac{i}{Z}\int D\psi_0 D\bar{\psi}_0 e^{-\bar{\psi}_0\psi_0}\langle -\psi_0|T_{c_K}\hat{\psi}_t\hat{\psi}_{t'}^\dagger e^{-i\sum_i \delta_i H(\hat{\psi}_i^\dagger,\hat{\psi}_i)}|\psi_0\rangle, \tag{3.4.2}$$

where we have split the time contour into N pieces of width δ_i. We can now define $-\psi_0 = \psi_{N+1}$ and insert the resolution of unity between each of the time slices

$$\begin{aligned} G(t,t') = \; & -\tfrac{i}{Z}\int D^{N+1}\psi D^{N+1}\bar{\psi} e^{-\bar{\psi}_0\psi_0-\sum_{i=1}^N \bar{\psi}_i\psi_i}\langle\psi_{N+1}|e^{-i\delta_N H(\hat{\psi}^\dagger,\hat{\psi})}|\psi_N\rangle\langle\psi_N| \\ & \dots e^{-i\delta_1 H(\hat{\psi}^\dagger,\hat{\psi})}|\psi_1\rangle\langle\psi_1|e^{-i\delta_0 H(\hat{\psi}^\dagger,\hat{\psi})}\hat{\psi}_t\hat{\psi}_{t'}^\dagger|\psi_0\rangle, \end{aligned} \tag{3.4.3}$$

where the 0 and i subscripts refer to the different completeness relations. This expression can be simplified by making use of the properties of coherent states to give [38]

$$G(t,t') = -\frac{i}{Z}\int D\psi D\bar{\psi} e^{iS}\hat{\psi}_t\hat{\psi}^\dagger_{t'}, \tag{3.4.4}$$

where $D\psi D\bar{\psi} = D^{N+1}\psi D^{N+1}\bar{\psi}$ and

$$iS = -\bar{\psi}_0\psi_0 - \sum_{i=1}^{N}\bar{\psi}_i\psi_i + \sum_{i=0}^{N}\left[\bar{\psi}_{i+1}\psi_i - i\delta_i H(\bar{\psi}_{i+1},\psi_i)\right]. \tag{3.4.5}$$

This in turn can be simplified to

$$\sum_{i=0}^{N}\delta_i\left(\bar{\psi}_{i+1}\frac{\psi_i - \psi_{i+1}}{\delta_i} - iH(\bar{\psi}_{i+1},\psi_i)\right), \tag{3.4.6}$$

which can be written in the continuum limit as

$$iS = i\int_{c_k} dt\left(\bar{\psi}(t)i\frac{d\psi(t)}{dt} - H(\bar{\psi}(t),\psi(t))\right). \tag{3.4.7}$$

The continuum expression however, is strictly symbolic and the discrete expression is the only one with any real meaning. It is possible to calculate the partition function in the same way, resulting in

$$Z = \int D\psi D\bar{\psi} e^{iS}. \tag{3.4.8}$$

3.4.1 Useful Results of Functional Integration

We can use the standard definition,

$$\int D\psi D\bar{\psi} e^{-\bar{\psi}_\alpha M_{\alpha\beta}\psi_\beta} = \mathrm{det}\mathbf{M}, \tag{3.4.9}$$

to consider a matrix $\mathbf{M}$ defined as

$$\mathbf{M} = \begin{pmatrix} -1 & 0 & 0 & \cdots & a_{N+1} \\ -a_1 & 1 & 0 & \cdots & 0 \\ 0 & -a_2 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -a_N & 1 \end{pmatrix}, \tag{3.4.10}$$

where $a_i = 1 - i\delta_i\varphi_i$. The index i labels the time, δ_1 is a time segment and φ_i is the Hamiltonian at time t_i. We can compare the matrix with Eq. (3.4.4) to derive the following identities

$$-\int D\psi D\bar{\psi} e^{-\bar{\psi}\mathbf{M}\psi} = -\text{det}\mathbf{M} = 1 + \prod_{i=1}^{N+1} a_i, \tag{3.4.11}$$

and

$$-\int D\psi D\bar{\psi}\psi_m\bar{\psi}_n e^{-\bar{\psi}\mathbf{M}\psi} = \begin{cases} -\prod_{i=n}^{m+1} a_i & \text{if } m > n \\ \prod_{i=1}^{m-1}\prod_{i=n}^{N+1} a_i & \text{if } m < n \end{cases}. \tag{3.4.12}$$

As $a_i = 1 - i\delta_i\varphi_i \approx e^{-i\delta_i\varphi_i}$, Eq. (3.4.11) and Eq. (3.4.12) can be written as

$$\int D\psi D\bar{\psi} e^{-\bar{\psi}\mathbf{M}\psi} = 1 + e^{-i\int\varphi(t)dt} \tag{3.4.13}$$

and

$$\int D\psi D\bar{\psi}\psi(t)\bar{\psi}(t') e^{-\bar{\psi}\mathbf{M}\psi} = \begin{cases} e^{-\int_{t'}^{t}\varphi(t'')dt''} & \text{if } t > t' \\ e^{-\int_{t}^{t_0}\varphi(t'')dt''} e^{-\int_{t_{N+1}}^{t'}\varphi(t'')dt''} & \text{if } t < t' \end{cases}. \tag{3.4.14}$$

If we can not write such a simple form for the matrix, we must use the result

$$\int D\psi D\bar{\psi} e^{-\bar{\psi}\mathbf{M}\psi} = \text{det}\mathbf{M}. \tag{3.4.15}$$

3.5 Noise of a Non-interacting Quantum Dot

In this section, we will calculate the zero frequency noise power spectrum of a non-interacting resonant level quantum dot [39], as an example of using the Keldysh Green's functions approach to calculate the noise. We will begin with a similar system to that which we will consider for the interacting problem later. We will consider two non-interacting leads ($\alpha = 1, 2$) coupled via tunneling contacts to a non-interacting central region modeled as a resonant level. The Hamiltonian for the system is given by

$$H = \varepsilon d^\dagger(t)d(t) + H_0^{leads} + \sum_{\alpha,k}\left[\gamma_{\alpha k}c_k^\dagger(t)d(t) + \text{h.c.}\right]. \tag{3.5.1}$$

The current through the first contact is given by

$$I_1(t) = ei\sum_k\left[\gamma_{1,k}c_k^\dagger(t)d(t) - \gamma_{1,k}^\dagger d^\dagger(t)c_k(t)\right], \tag{3.5.2}$$

where $c_k^\dagger$ is a creation operator for lead 1, d is an annihilation operator for the dot and the tunneling rate from the dot to the lead is $\gamma_{1,k}$. If we now define $\delta I_1(t) = I_1(t) - \langle I_1\rangle$, we can express the noise correlation function as

$$\begin{aligned}
S(t,t') &= \langle\{\delta I_1(t), \delta I_1(t')\}\rangle \\
&= \langle\{I_1(t), I_1(t')\}\rangle - 2\langle I_1\rangle^2 \\
&= (ie)^2\sum_{k,k'}\Big[\gamma_{1,k}\gamma_{1,k'}\langle c_k^\dagger(t)d(t)c_{k'}^\dagger(t')d(t')\rangle - \gamma_{1,k}\gamma_{1,k'}^\dagger\langle c_k^\dagger(t)d(t)d^\dagger(t')c(t')_{k'}\rangle \\
&\quad - \gamma_{1,k}^\dagger\gamma_{1,k'}\langle d^\dagger(t)c(t)_k c_{k'}^\dagger(t')d(t')\rangle + \gamma_{1,k}^\dagger\gamma_{1,k'}^\dagger\langle d^\dagger(t)c(t)_k d^\dagger(t')c_{k'}(t')\rangle\Big] \\
&\quad + \text{h.c.} - 2\langle I_1\rangle^2.
\end{aligned} \tag{3.5.3}$$

The noise power spectrum, Eq. (2.1.1) is defined as the Fourier transform of Eq. (3.5.3) and we will calculate the zero frequency component. We chose to consider just the zero frequency, as this is the component we will be interested in for the interacting system. In

order to evaluate the above expression, we will need to calculate the two-particle Green's functions,

$$\begin{aligned} G_1(t,t') &= i^2\langle Tc_k^\dagger(t)d(t)c_{k'}^\dagger(t')d(t')\rangle, \\ G_2(t,t') &= i^2\langle Tc_k^\dagger(t)d(t)d^\dagger(t')c_{k'}(t')\rangle, \\ G_3(t,t') &= i^2\langle Td^\dagger(t)c_k(t)c_{k'}^\dagger(t')d(t')\rangle, \\ G_4(t,t') &= i^2\langle Td^\dagger(t)c_k(t)d^\dagger(t')c_{k'}(t')\rangle. \end{aligned} \tag{3.5.4}$$

The noise correlator can be written in terms of these Green's functions as

$$\begin{aligned} S(t,t') = e^2\sum_{k,k'}\Big[&\gamma_{1,k}\gamma_{1,k'}G_1^>(t,t') - \gamma_{1,k}\gamma_{1,k'}^\dagger G_2^>(t,t') - \gamma_{1,k}^\dagger\gamma_{1,k'}G_3^>(t,t') \\ &+ \gamma_{1,k}^\dagger\gamma_{1,k'}^\dagger G_4^>(t,t')\Big] + \text{h.c.} - 2\langle I_1\rangle^2. \end{aligned} \tag{3.5.5}$$

To calculate the two particle Green's functions, we will use Wicks theorem to rewrite them in terms of single particle Green's functions. We are allowed to do this as there are no interactions in our system, so the Hamiltonian is quadratic. Therefore, we can express $G_2(t,t')$ as

$$\begin{aligned} G_2(t,t') &= i\langle Tc_{k'}(t')c_k^\dagger(t)\rangle i\langle Td(t)d^\dagger(t')\rangle \\ &- i\langle Td(t)c_k^\dagger(t)\rangle i\langle Tc_{k'}(t')d^\dagger(t')\rangle \end{aligned} \tag{3.5.6}$$

The other two particle Green's functions can be expressed in a similar manor. Using a Dyson equation approach, it is possible to express the mixed Green's functions in terms of the non interacting green's function of the leads and the full Green's function of the dot. The mixed Green's functions come in two forms:

$$F_k(t,t') = i\langle Tc_k(t)d^\dagger(t')\rangle \text{ and } \bar{F}_{k'}(t,t') = i\langle Td(t)c_{k'}^\dagger(t')\rangle \tag{3.5.7}$$

The Dyson series for these expressions are similar, so we will only write one explicitly [39],

$$\begin{aligned} F_k(t,t') &= \int dt_1 \gamma_{1,k} g_{1,k}(t,t_1) G_0(t_1,t') + \int dt_1 dt_2 dt_3 \gamma_{1,k} g_{1,k}(t,t_1) G_0(t_1,t_2) \Sigma(t_2,t_3) G_0(t_3,t') + \dots \\ &= \int dt_1 \gamma_{1,k} g_{1,k}(t,t_1) G(t_1,t') \end{aligned} \tag{3.5.8}$$

and

$$\bar{F}_{k'}(t,t') = \int dt_1 \gamma^{\dagger}_{1,k'} G(t,t_1) g_{1,k'}(t_1,t'), \tag{3.5.9}$$

In Eq. (3.5.8) and Eq. (3.5.9) the zero subscript in the full Green's function of the dot indicates the unperturbed function. The lower case $g_{1,k'}$ is the Green's function of lead 1 and Σ is the mass operator. Inserting these expressions back into the equation for the noise correlator Eq. (3.5.5), we obtain two types of terms; terms where the t_1 and t_2 integrals can be calculated separated and terms where the integrals are intertwined. We will begin by considering the terms of the first kind which we shall refer to as disconnected as their diagrams form two disconnected loops.

3.5.1 Disconnected Terms

The disconnected part of the noise is

$$\begin{aligned} S_{\text{dis}}(t,t') &= e^2 \sum_{k,k'} |\gamma_k|^2 \gamma_{k'}|^2 \int dt_1 \int dt_2 \, [G(t,t_1) g_{1,k}(t_1,t) G(t',t_2) g_{1,k'}(t_2,t') \\ &- G(t,t_2) g_{1,k}(t_1,t) g_{1,k'}(t',t_2) G(t_2,t') - g_{1,k}(t,t_1) G(t_1,t) G(t',t_2) g_{1,k'}(t_2,t') \\ &+ g_{1,k}(t,t_1) G(t_1,t) g_{1,k'}(t',t_2) G(t_2,t')] + \text{ h.c.} \end{aligned} \tag{3.5.10}$$

As we can separate the integrals in the above expression, lets begin by examining the first t_1 integral in Eq. (3.5.10): $\int dt_1 G(t,t_1) g_{1,k}(t_1,t)$. At first it appears that there is some ambiguity to the time ordering of this term and that we may be unable to use the Langreth

theorem [27][28]. However, if we go back to the origin of this term in Eq. (3.5.4) we see that the time t in the lead Green's function, $g_{1,k}$ must be greater than that of the time t in the Green's function of the dot, G, because otherwise the $c_k^\dagger(t)$ would not be to the left of the of $d(t)$. Therefore, we need to consider the lesser component which by the Langreth theorem, Eq. (3.3.25), is

$$\left[\int dt_1 G(t,t_1)g_{1,k}(t_1,t)\right]^< = \int dt_1 \left[G^R(t,t_1)g^<_{1,k}(t_1,t) + G^<(t,t_1)g^A_{1,k}(t_1,t)\right] \quad (3.5.11)$$

which is the same form as the Green's functions that appear in the average current. In fact using the Langreth theorem on all the expressions in both the t_1 and t_2 integrals in Eq. (3.5.10) we obtain,

$$\begin{aligned} S_{\rm dis}(t,t') &= e^2 \sum_{k,k'} |\gamma_k|^2\gamma_{k'}|^2 \left[G^<_{nk}(t,t) - G^<_{kn}(t,t)\right]\left[G^<_{nk'}(t',t') - G^<_{k'n}(t',t')\right] \\ &= 2\langle I_1\rangle^2 \end{aligned} \quad (3.5.12)$$

where

$$G^<_{nk}(t,t) = \left[\int dt_1 G(t,t_1)g_{1,k}(t_1,t)\right]^< \text{ and } G^<_{kn}(t,t) = \left[\int dt_1 g_{1,k}(t,t_1)G(t_1,t)\right]^<. \quad (3.5.13)$$

Therefore, the disconnected terms cancel the current squared terms exactly.

3.5.2 Connected Terms

The remaining terms which, can be represented in terms of ring diagrams, are

$$\begin{aligned} S(t,t') &= e^2 \sum_k |\gamma_k|^2 \left[g_{1,k}(t',t)G(t,t') + g_{1,k}(t,t')G(t',t)\right] + \sum k,k'|\gamma_k|^2\gamma_{k'}|^2 \int dt_1 \int dt_2 \\ &\times \left[-G(t,t_2)g_{1,k'}(t_2,t')G(t',t_1)g_{1,k}(t_1,t) + G(t,t')g_{1,k'}(t',t_1)G(t_1,t_2)g_{1,k}(t_2,t)\right. \\ &+ \left. g_{1,k}(t,t_1)G(t_1,t_2)g_{1,k'}(t_2,t')G(t',t) - g_{1,k}(t,t_1)G(t_1,t')g_{1,k'}(t',t_2)G(t_2,t)\right] + \text{ h.c.} \end{aligned}$$

(3.5.14)

We will now only consider the situation where $t > t'$. We have chosen this case as the noise correlator only depends on the difference between the two times and this case corresponds to the difference being positive. The first two terms in Eq. (3.5.14) are only products of two Green's functions, so we can easily obtain the result,

$$[g_{1,k}(t',t)G(t,t') + g_{1,k}(t,t')G(t',t)]_{t>t'} = g^{<}_{1,k}(t',t)G^{>}(t,t') + g^{>}_{1,k}(t,t')G^{<}(t',t). \quad (3.5.15)$$

To calculate the required components from the other terms in Eq. (3.5.14), we need to consider the Keldysh contour. As we have defined $t > t'$, t must be after t' on the contour, therefore we will place t' on the upper branch of the contour and t on the lower branch. However, we still have two variables t_1 and t_2 which are integrated over the whole Keldysh contour. To calculate these terms, we need to split the integrals into two halfs; the integral over the upper half of the contour from minus to plus infinity and the integral over the lower half from infinity to minus infinity. As this produces four double integrals for each of the remaining terms in Eq. (3.5.14), we will not list all 32 terms here as once they are expressed in this form, we can now easily determine the required time ordering of the Green's functions.

As single particle Green's fuctions only depend on the difference between their variables, $G(t,t') = G(t-t')$, we now make the change of variables:

$$t - t' = \tau, \qquad t - t_1 = \tau_1, \qquad t' - t_2 = \tau_2 \quad (3.5.16)$$

and perform the Fourier transform,

$$S(\omega) = \int_{-\infty}^{\infty} d\tau e^{i\omega\tau} S(\tau). \quad (3.5.17)$$

As the time dependence of all the terms in the noise correlator have only two different forms,

$$\begin{aligned}\Pi_1(\tau) &= A(-\tau)B(\tau) \\ \Pi_2(\tau) &= \int_{-\infty}^{\infty} d\tau_1 \int_{-\infty}^{\infty} d\tau_2 A(\tau_1)B(-\tau_1+\tau)C(\tau_2)D(-\tau_2-\tau),\end{aligned} \tag{3.5.18}$$

the transform results in terms which have the frequency dependence,

$$\begin{aligned}\Pi_1(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega_1 A(\omega_1)B(\omega+\omega_1) \\ \Pi_2(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega_1 A(\omega+\omega_1)B(\omega+\omega_1)C(\omega_1)D(\omega_1).\end{aligned} \tag{3.5.19}$$

As we are only concerned with the zero frequency noise we can simplify the expressions further by taking the $\omega \to 0$ limit,

$$\begin{aligned}\Pi_1(0) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega A(\omega)B(\omega) \\ \Pi_2(0) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega A(\omega)B(\omega)C(\omega)D(\omega).\end{aligned} \tag{3.5.20}$$

Using the definitions for the retarded and advanced Green's functions, Eqs. (3.3.14), and substituting in the expressions for the Green's functions of the leads,

$$\begin{aligned}\sum_k |\gamma_{\alpha k}|^2 g^<_{\alpha,k}(\omega) &= i\Gamma_\alpha f_\alpha(\omega), \\ \sum_k |\gamma_{\alpha k}|^2 g^>_{\alpha,k}(\omega) &= -i\Gamma_\alpha\left[1-f_\alpha(\omega)\right]\end{aligned} \tag{3.5.21}$$

where $\Gamma_\alpha = 2\pi\sum_k |\gamma_{\alpha k}|^2\delta(\omega-\omega_k)$, into the expression for the noise we obtain

$$S(0) = S_1 + S_2. \tag{3.5.22}$$

$S_{1,2}$ are the expressions arising from the gG and $ggGG$ terms respectively and are given by

$$S_1 = 2e^2 i\Gamma_1 \int \frac{d\omega}{2\pi} \left\{ f_1(\omega) \left[G^R(\omega) - G^A(\omega) \right] + \left[2f_1 - 1 \right] G^<(\omega) \right\} \tag{3.5.23}$$

and

$$\begin{aligned} S_2 &= 2e^2 \int \frac{d\omega}{2\pi} \left\{ i f_1(\omega)\Gamma_1 \left[G^R(\omega) - G^A(\omega) \right] + i\Gamma_1 \left[2f_1(\omega) - 1 \right] G^<(\omega) \right. \\ &+ f_1(\omega)\Gamma_1\Gamma_1 \left[G^R(\omega) - G^A(\omega) \right] \left[G^R(\omega) - G^A(\omega) \right] \\ &+ \Gamma_1\Gamma_1 \left[2f_1(\omega) - 1 \right] G^<(\omega) \left[G^R(\omega) - G^A(\omega) \right] \\ &- \Gamma_1\Gamma_1 f_1(\omega) \left[1 - f_1(\omega) \right] \left[G^A(\omega)G^A(\omega) + G^R(\omega)G^R(\omega) \right] \\ &+ \left. \Gamma_1\Gamma_1 G^<(\omega) \left[G^R(\omega) - G^A(\omega) \right] + \Gamma_1\Gamma_1 G^<(\omega)G^<(\omega) \right\}. \end{aligned} \tag{3.5.24}$$

We now use the following relations for the resonant level Green's functions [39],

$$\begin{aligned} G^<(\omega) &= iG^R(\omega) \left[f_1(\omega)\Gamma_1 + f_2(\omega)\Gamma_2 \right] G^A(\omega), \\ G^R(\omega) - G^A(\omega) &= -iG^R(\omega) \left[\Gamma_1 + \Gamma_2 \right] G^A(\omega), \\ G^A(\omega)G^A(\omega) + G^R(\omega)G^R(\omega) &= \left[G^R(\omega) - G^A(\omega) \right] \left[G^R(\omega) - G^A(\omega) \right] + 2G^R(\omega)G^A(\omega) \end{aligned} \tag{3.5.25}$$

and the definition of the transmission coeffcient [39],

$$T(\omega) = \Gamma_1\Gamma_2 G^R(\omega)G^A(\omega). \tag{3.5.26}$$

We can express the zero frequency noise power spectrum as

$$\begin{aligned} S(0) &= 2e^2 \int \frac{d\omega}{2\pi} \left\{ \left[f_1(\omega)(1 - f_1(\omega)) + f_2(\omega)(1 - f_2(\omega)) \right] T(\omega) \right. \\ &+ \left. \left[f_1(\omega) - f_2(\omega) \right]^2 T(\omega) \left[1 - T(\omega) \right] \right\}. \end{aligned} \tag{3.5.27}$$

Eq. (3.5.27) is an important result that agrees with the results obtained from the scattering theory approach, which we discussed in the previous chapter [26]. The first term is the thermal noise and at zero temperature, it vanishes. The second term is the nonequilibrium term and it vanishes at zero bias.

3.6 Summary

Within this chapter, we have introduced several techniques and basic results that we will make use of in the following chapters. We have introduced Green's functions, in particular the Keldysh Green's function and it's associated contours. We have also reviewed functional integration and derived several basic results. We finished the chapter with an example of using the Keldysh Green function approach to calculate the non-interacting noise of a quantum dot. We chose this example due to the similarities it has to the interacting result that we calculate later and to show the Green's function approach produces the same $T(1-T)$ result as the scattering theory approach we discussed in Chapter 2. In the following chapter, we will consider two different approaches to deriving the density of states for the Coulomb blockaded quantum dot.

Chapter 4

THE DENSITY OF STATES AND CONDUCTANCE OF A QUANTUM DOT

In this chapter, we will derive two different approaches to calculating the tunneling density of states and the conductance of a quantum dot weakly coupled to non-interacting leads. In both approaches, we consider the quantum dot to be zero dimensional and will primarily be concerned with the Coulomb blockade effects. We will always consider the charging energy, E_c to be the largest energy scale. We will begin by considering a functional integration approach which we are unable to extend to calculate the noise power spectrum. We will then turn our attention to an alternative derivation of the density of states to show that the same result is obtained. We will then extend this approach in the next chapter to calculate the noise power spectrum.

4.1 Functional Integration Approach

In this section, we will summarise a technique developed by Sedlmayr *et al.*[40] to derive the Green's function for an isolated zero dimensional quantum dot. We will then make use of this Green's function to derive the tunneling density of states and conductance of a dot weakly coupled to non-interacting leads.

4.1.1 Green's Function for an Isolated Quantum Dot

We start from the standard universal Hamiltonian for a zero-dimensional system [3] keeping only the charging term:

$$\hat{H} = \hat{H}_0 + \frac{E_c}{2}\left(\hat{N} - N_g\right)^2. \tag{4.1.1}$$

$\hat{H}_0$ is the hamiltonian for a tightly confined non-interacting electron system in a random potential,

$$\hat{H}_0 = \sum_n \psi_n^\dagger \varepsilon_n \psi_n, \tag{4.1.2}$$

where n labels the energy levels of the dot. $\hat{N} = \bar{\psi}\psi$ is the number of electron on the dot, eN_g is the neutralising background charge which is proportional to a gate voltage and E_c is the charging energy, which is dependent on the effective capacitance of the system, $E_c = e^2/C$. We wish to define the Green's function on the full interacting Keldysh contour. To do this we start from the functional integral form, which can be written as

$$iG_n(t,t') = \frac{1}{\mathcal{Z}} \int D\psi D\bar{\psi} \psi_n(t) \bar{\psi}_n(t') e^{i\int_{c_K} dt \left[\sum_k \bar{\psi}_k(t) i\partial_t \psi_k(t) - H\right]}. \tag{4.1.3}$$

If we now perform the Hubbard-Stratonovich transformation on the Green's function (see appendix B) we find

$$iG_n(t,t') = \frac{1}{\mathcal{Z}} \int D\phi e^{iS(\phi)} \int D\psi D\bar{\psi} \psi_n(t) \bar{\psi}_n(t') e^{iS(\bar{\psi},\psi,\phi)}, \tag{4.1.4}$$

$$\mathcal{Z} = \int D\phi e^{iS(\phi)} \int D\psi D\bar{\psi} e^{iS(\bar{\psi},\psi,\phi)}, \tag{4.1.5}$$

where the new actions are defined as

$$iS(\bar{\psi},\psi,\phi) = i\sum_k \int_{c_K} dt\bar{\psi}_k(t)\left[i\partial_t - \xi_k - i\phi(t)\right]\psi_k(t) \tag{4.1.6}$$

and

$$iS(\phi) = -i\frac{1}{2E_c}\int_{c_K} dt\phi^2(t). \tag{4.1.7}$$

The Fermionic integrals are now Gaussian and can be calculated using the identities Eq. (3.4.13) and Eq. (3.4.14) to obtain an expression for the Green's function in terms of just the Bosonic field:

$$iG_n(t,t') = \frac{\mathrm{sgn}(t,t')}{Z}\int D\phi e^{iS(\phi)} e^{\int_{c_k tt'} dt[\phi(t)-i\xi_n]} \prod_{k\neq n}\left[1 + e^{\int_{c_k} dt[\phi(t)-i\xi_k]}\right] \tag{4.1.8}$$

where

$$Z = \int D\phi e^{iS(\phi)} \prod_k \left[1 + e^{\int_{c_K} dt[\phi(t)-i\xi_k]}\right]. \tag{4.1.9}$$

The sgn function is defined on the contour in Fig. (3.3) and equals 1 (or -1) when t precedes (or goes after) t'. We have also introduced the contour:

$$\int_{c_k tt'} dt = \begin{cases} \int_{t'}^{t} dt & \text{if } t > t' \text{ on the contour } c_K \text{ and} \\ \int_{c_K} dt - \int_{t}^{t'} dt & \text{if } t < t' \text{ on the contour } c_K. \end{cases} \tag{4.1.10}$$

We can now use the canonical ensemble to rewrite the Bosonic integrals. If we define $\phi_0 = \int_{c_K} dt\phi(t)$, then Eq. (4.1.8) can be written as

$$iG_n(t,t') = \frac{\mathrm{sgn}(t,t')}{Z}\int D\phi e^{iS(\phi)} e^{\int_{c_k tt'} dt[\phi(t)-i\xi_n]} \Xi_n(\phi_0) \tag{4.1.11}$$

where

$$Z = \int D\phi e^{iS(\phi)} \Xi(\phi_0). \tag{4.1.12}$$

We have chosen at this point to include the background charge, N_g, in the chemical potential. We have introduced the grand canonical partition function, $\Xi(\phi_0)$, and the grand canonical partition function with the n^{th} level removed, $\Xi_n(\phi)$ where the energy levels in both have been shifted due to the charging effects of the dot:

$$\Xi(\phi_0) = \prod_k \left[1 + e^{-\beta\xi_k + \phi_0}\right], \qquad \Xi_n(\phi_0) = \frac{\Xi(\phi_0)}{1 + e^{-\beta\xi_n + \phi_0}}. \tag{4.1.13}$$

We can now expand the grand canonical partition functions in Eq. (4.1.11) in terms of the canonical partition functions:

$$\begin{aligned} \Xi(\phi_0) &= \sum_{N=0}^{\infty} Z_N e^{(\beta\mu+\phi_0)N}, & Z_N &= \oint \frac{d\theta}{2\pi} e^{-iN\theta} \prod_k \left[1 + e^{-\beta\xi_k + i\theta}\right], \\ \Xi_n(\phi_0) &= \sum_{N=0}^{\infty} Z_N(\varepsilon_n) e^{(\beta\mu+\phi_0)N}, & Z_N(\varepsilon_n) &= \oint \frac{d\theta}{2\pi} e^{-iN\theta} \prod_{k\neq n} \left[1 + e^{-\beta\xi_k + i\theta}\right]. \end{aligned} \tag{4.1.14}$$

Z_N is the canonical partition function for N particles and $Z_N(\varepsilon_n)$ is the partition function without any N-particle states which contain the single particle level ε_n. This can be formally defined as

$$\frac{Z_N(\varepsilon_n)}{Z_N} = \frac{\mathrm{Tr}_N\left(\psi_n \psi_n^\dagger e^{-\beta\hat{H}_0}\right)}{\mathrm{Tr}_N\left(e^{-\beta\hat{H}_0}\right)} = 1 - F_N(\varepsilon_n) \tag{4.1.15}$$

where $F_N(\varepsilon_n)$ is the canonical distribution function for the system of N noninteracting electrons containing ε_n . As $F_N(\varepsilon)$ is a canonical distribution, the charging energy is constant and does not contribute. Substituting the expression for $Z_N(\varepsilon_n)$ into the Green's function, we can easily perform the Gaussian integral over the Bosonic fields. Thus, the Fourier

transform of the greater Green's function is given by:

$$G^{>}(\varepsilon) \;=\; -\frac{2\pi i}{Z}\sum_{n}\sum_{N=0}^{\infty} e^{-\beta E_N} Z_N(\varepsilon-\Omega_N)\delta(\varepsilon-\varepsilon_n-\Omega_N), \tag{4.1.16}$$

where

$$Z=\sum_{N=0}^{\infty} e^{-\beta E_N} Z_N. \tag{4.1.17}$$

The energy levels E_N and the difference in energy between consecutive levels Ω_N are defined as

$$E_N=\frac{E_c}{2}\left(N-N_g\right)^2-\mu N \qquad \text{and} \qquad \Omega_N=E_{N+1}-E_N=E_c\left(N-N_g+\frac{1}{2}\right). \tag{4.1.18}$$

If we average over disorder, by substituting the mean tunneling density of states of non-interacting electrons, ν_0, for the sum over the delta function, $\sum_n \delta\left(\varepsilon-\varepsilon_n-\Omega_N\right)$. In doing so, we assume that the tunneling density of states is smooth in any realisation of disorder which is valid when the temperature, T, is much greater than the mean level spacing, Δ. Hence, we obtain

$$G^{>}(\varepsilon) \;=\; -\frac{2\pi i\nu_0}{Z}\sum_{N=0}^{\infty} e^{-\beta E_N}\left[1-F_N(\varepsilon-\Omega_N)\right], \tag{4.1.19}$$

where we have used Eq. (4.1.15) to rewrite our Green's function in terms of the canonical distribution function. We have also used the fact that Z_N is a smooth function, on a scale Δ/T, to cancel it from both the numerator and the denominator. If we limit ourselves to considering the situation were there are many electrons occupying the dot $N\gg 1$, the canonical distribution function $F_N(\varepsilon-\Omega_N)$ is approximately the same as the grand canonical Fermi function $f(\varepsilon-\Omega_N)$ with a chemical potential of order $N\Delta$ which we can ignore as it is small compared with Ω_N. Therefore, the greater and lesser Green's functions are given

by:

$$\begin{aligned} G^{>}(\varepsilon) &= -\frac{2\pi i\nu_0}{Z}\sum_{N=0}^{\infty} e^{-\beta E_N}\left[1-f(\varepsilon-\Omega_N)\right], \\ G^{<}(\varepsilon) &= \frac{2\pi i\nu_0}{Z}\sum_{N=0}^{\infty} e^{-\beta E_N} f(\varepsilon-\Omega_{N-1}). \end{aligned} \tag{4.1.20}$$

4.1.2 Density of States

Starting from the standard formula for the tunneling density of states,

$$\nu(\varepsilon) = \frac{i}{2\pi}\left[G^{>}(\varepsilon) - G^{<}(\varepsilon)\right], \tag{4.1.21}$$

and inserting the Green's function expressions Eqs. (4.1.20), we find the tunneling density of states to be:

$$\nu(\varepsilon) = \frac{\nu_0}{Z}\sum_{N} e^{-\beta E_N}\left[f(\varepsilon-\Omega_{N-1}) + 1 - f(\varepsilon-\Omega_N)\right]. \tag{4.1.22}$$

If we now only keep the leading order terms in the summation, we need to maximise the coefficients $e^{-\beta\Omega_N}$ and $e^{-\beta E_N}$. It can be clearly seen that the terms that we need to keep are the terms for N closest to $N_g + 1/2$. These are of order E_{N_0} and E_{N_0+1} where N_0 corresponds to the maximal term. We can ignore the $N_0 - 1$ term as it is exponentially suppressed. Therefore, we obtain

$$\frac{\nu(\varepsilon)}{\nu_0} = \frac{U(\varepsilon-\Omega_N) + e^{-\beta\Omega_N}U(\varepsilon-\Omega_{N+1})}{1+e^{-\beta\Omega_N}}, \tag{4.1.23}$$

where

$$U(\varepsilon-\Omega_N) = f(\varepsilon-\Omega_{N-1}) + 1 - f(\varepsilon-\omega_N). \tag{4.1.24}$$

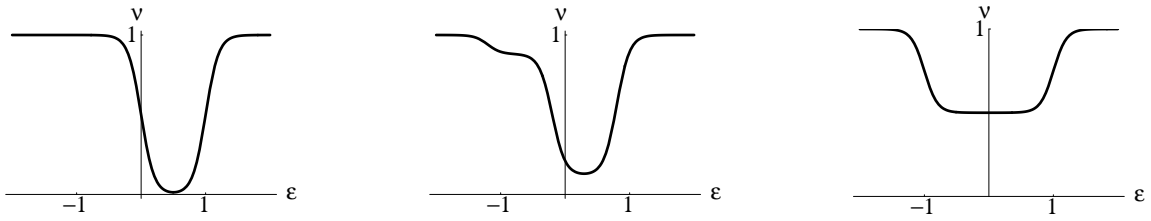

Figure 4.1: The tunneling density of states (in the units of ν_0) as a function of energy (measured in E_c)[40]: (a) in the valley, (b) through an intermediate region, and (c) at the peak.

Away from the degeneracy point which corresponds to the valley of the Coulomb blockade, one of the terms in Eq. (4.1.23) is exponentially suppressed and the tunneling density of states has a gap. At the degeneracy point, $\Omega_N = 0$ which corresponds to the peak of the Coulomb blockade, the tunneling density of states remains finite for all energy values but has a half gap at $|\varepsilon| < E_C$ (see Fig. (4.1))[40].

4.1.3 Current Through a Dot

We can also consider the dot connected to two non-interacting leads via point contacts using the functional integration approach. The Hamiltonian for this system is given by

$$\hat{H} = \hat{H}_0 + \frac{E_c}{2}\left(\hat{N} - N_g\right)^2 + \hat{H}_t, \tag{4.1.25}$$

where we have now also included the leads in $\hat{H}_0$ and introduced the coupling term,

$$\hat{H}_t = \sum_{\alpha,k,n} \gamma_{\alpha k n} c^\dagger_{\alpha k}(t) d_n(t) + h.c., \tag{4.1.26}$$

between the leads and the dot. $c^\dagger_{\alpha k}$ is the creation operator with momentum k on lead $\alpha = 1, 2$ and d_n is an annihilation operator on level n of the dot. The parameter $\gamma_{\alpha k n}$ is the tunneling rate from dot to lead. Using the Heisenberg equation of motion, the current is

defined as

$$I = \dot{Q} = ei[\hat{H}, \hat{N}] = ei[\hat{H}_t, \hat{N}], \tag{4.1.27}$$

which after calculating the commutator is given by

$$I = ie \sum_{\alpha,k,n} \langle \gamma_{\alpha kn} c^{\dagger}_{\alpha k}(t) d_n(t) - \gamma^{*}_{\alpha kn} d^{\dagger}_n(t) c_{\alpha k}(t) \rangle. \tag{4.1.28}$$

As we would like to work in the functional integral representation, we can rewrite the current using a source field, J as

$$I = e \sum_{\alpha,k,n} \left. \frac{\partial \ln Z[J]}{\partial J_{\alpha kn}(t)} \right|_{J=0} \tag{4.1.29}$$

with

$$Z(J) = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp\left\{ iS_0 + iS_t + i \sum_{\alpha,k,n} \left(\gamma_{\alpha kn} \psi_n(t) \bar{\psi}_{\alpha k}(t) - \gamma^{*}_{\alpha kn} \psi_{\alpha k}(t) \bar{\psi}_n(t) \right) J_{\alpha kn}(t) \right\}, \tag{4.1.30}$$

where

$$iS_0 = i \int_c dt \left(\sum_n \psi_n(t) i\partial_t \psi_n(t) - H \right) + i \sum_{\alpha,k} \int_c dt \bar{\psi}_{\alpha k}(t) \left(i\partial_t - \xi_{\alpha k} \right) \psi_{\alpha k}(t) \tag{4.1.31}$$

$$iS_t = i \sum_{\alpha,k,n} \int_c dt \gamma_{\alpha,k,n} \bar{\psi}_{\alpha k}(t) \psi_n(t) + \gamma^{*}_{\alpha kn} \bar{\psi}_n(t) \psi_{\alpha k}(t). \tag{4.1.32}$$

H is the Hamiltonian of the dot given by Eq. (4.1.1) and $\xi_{\alpha k}$ are the dispersion relations for the leads. $\mathcal{D}\psi\mathcal{D}\bar{\psi}$ are the integrals over the fields for both the leads and the dot. The

current now takes the form

$$I = \frac{ie}{Z[0]} \sum_{\alpha,k,n} \int \mathcal{D}\psi\mathcal{D}\bar{\psi} \left(\gamma_{\alpha kn}\psi_n(t)\bar{\psi}_{\alpha k}(t) - \gamma^*_{\alpha kn}\psi_{\alpha k}(t)\bar{\psi}_n(t)\right) e^{iS_0+iS_t}, \tag{4.1.33}$$

which is equivalent to Eq. (4.1.28). To be able to perform the field integrals, we have to complete the square and make the action Gaussian. This can be done by shifting the fields in the following way

$$\bar{\psi}_{\alpha k}(t) \rightarrow \bar{\psi}_{\alpha,k}(t) - \sum_{n'} \int_K dt' \gamma^*_{\alpha kn} \bar{\psi}_{n'}(t') G_0(t',t), \tag{4.1.34}$$

$$\psi_{\alpha k}(t) \rightarrow \psi_{\alpha k}(t) - \sum_{n'} \int_K dt' \gamma_{\alpha kn'} G_0(t,t') \psi_{n'}(t'). \tag{4.1.35}$$

Inserting these into the action in Eq. (4.1.31) and Eq. (4.1.32), we can integrate out the leads. The current can now be written as the Green's function of the dot coupled to two mass operators due to the leads:

$$I = e \sum_{\alpha,n,n'} \int_K dt' \left(iG_{nn'}(t,t')i\Sigma_{\alpha n'n}(t',t) - i\Sigma_{\alpha nn'}(t,t')iG_{n'n}(t',t)\right). \tag{4.1.36}$$

The mass operators are given by

$$i\Sigma_{\alpha nn'}(t,t') = \sum_k \gamma_{\alpha kn}\gamma^*_{\alpha kn'} iG_{\alpha k}(t,t'). \tag{4.1.37}$$

and the green's function of the dot is

$$G_{n,n'}(t,t') = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\psi_n(t)\bar{\psi}_{n'}(t') e^{\left(iS_0 - i\sum_{\alpha,n,n'}\int_K dtdt'\bar{\psi}_n(t)\Sigma_{\alpha nn'}(t,t')\psi_{n'}(t')\right)}. \tag{4.1.38}$$

If we now neglect the "tail" term ($t_0 \rightarrow t_0 - i\beta$), contour c_x, from the Keldysh contour and extend the contour to plus and minus infinity, we can rearrange the current expression to

give

$$I = e \sum_{\alpha,n,n'} \int_{-\infty}^{\infty} dt' \left(iG^{>}_{nn'}(t,t') i\Sigma^{<}_{\alpha n' n}(t',t) - i\Sigma^{>}_{\alpha nn'}(t,t') iG^{<}_{n'n}(t',t) \right). \tag{4.1.39}$$

Using the Green function identities, Eqs. (3.3.14) and Eq. (3.3.15), we can rewrite this as

$$I = \frac{e}{2} \sum_{\alpha,n,n'} \int_{-\infty}^{\infty} dt' \left(iG^{K}_{nn'}(t-t') \Delta\Sigma_{\alpha n' n}(t'-t) - i\Sigma^{K}_{\alpha n' n}(t-t') \Delta G(t'-t) \right), \tag{4.1.40}$$

where $\Delta G = iG^R - iG^A$.

Dyson's Equations

We now can use Eq. (4.1.38) to calculate the Dyson's equations. If we define $\Sigma = \Sigma_1 + \Sigma_2$ where 1 and 2 refer to the lead index and denote the Green's function for the quantum dot not coupled to the leads, Eqs. (4.1.20) as g. We find $G^{-1} = g^{-1} - \Sigma$ which gives

$$iG^{R/A}_{nn'}(\varepsilon) = ig^{R/A}_{nn'}(\varepsilon) - ig^{R/A}_{nl}(\varepsilon) i\Sigma^{R/A}_{lm}(\varepsilon) iG^{R/A}_{mn'}(\varepsilon) \tag{4.1.41}$$

for the retarded and advanced Green's functions. The Keldysh component is given by

$$\begin{aligned} iG^{K}_{nn'}(\varepsilon) &= ig^{K}_{nn'}(\varepsilon) - ig^{R}_{nl}(\varepsilon) i\Sigma^{R}_{lm}(\varepsilon) iG^{K}_{mn'}(\varepsilon) - ig^{R}_{nl}(\varepsilon) i\Sigma^{K}_{lm}(\varepsilon) iG^{A}_{mn'}(\varepsilon) \\ &- ig^{K}_{nl}(\varepsilon) i\Sigma^{A}_{lm}(\varepsilon) iG^{A}_{mn'}(\varepsilon), \end{aligned} \tag{4.1.42}$$

where the dummy indices m and l are summed over. If we now assume that the probability to tunnel to different levels is uncorrelated and Σ_{nm} is diagonal, we can easily solve Dyson's equations. Using the Dyson's equations as well as

$$\sum_n g^{R/A}_n(\varepsilon) = \mp i\nu(\varepsilon) \tag{4.1.43}$$

and

$$i\Sigma_{nm}^{R/A} = \pm\frac{1}{2}\Gamma\delta_{nm}, \tag{4.1.44}$$

where $\Gamma = \Gamma_1 + \Gamma_2$ and $\Gamma_\alpha = 2\pi\nu_\alpha|t|^2$, we find that the current to the lowest order in Γ is

$$I = \frac{e}{2}\sum_\alpha \int \frac{d\varepsilon}{2\pi}\Gamma_\alpha \left[h(\varepsilon) - h_\alpha(\varepsilon)\right] 2\pi\nu(\varepsilon). \tag{4.1.45}$$

The distribution function of the dot, $h(\varepsilon) = 1 - 2f(\varepsilon)$, can be found from the quantum kinetic equation. Alternatively, it can be found by balancing the currents through the dot. As we are in the steady state and we are not allowing charge to acuminate on the dot, the current through the left contact has to equal the current through the right, due to current conservation. In other words $I_1 = I_2$ where $I = I_1 + I_2$ which results in the distribution function being given by

$$f(\varepsilon) = \frac{\Gamma_1 f_1(\varepsilon) + \Gamma_2 f_2(\varepsilon)}{\Gamma_1 + \Gamma_2}, \tag{4.1.46}$$

where f_α is the fermi function for the lead α with the chemical potential given by $\mu_\alpha = \mu - eV_\alpha$. Thus, we find the current to be given by

$$I = \frac{e}{2}\int d\varepsilon \frac{\Gamma_1\Gamma_2}{\Gamma_1 + \Gamma_2}\nu(\varepsilon)\left[h_1(\varepsilon) - h_2(\varepsilon)\right]. \tag{4.1.47}$$

This is the Landauer formula Eq. (2.2.15) that we introduced earlier written in a different form. If we rewrite Eq. (4.1.40) using the identity

$$\Delta G = G^R\Delta\Sigma G^A = G^R G^A\left[\Gamma_1 + \Gamma_2\right], \tag{4.1.48}$$

we obtain

$$I = \frac{e}{2}\int \frac{d\varepsilon}{2\pi}\left[h_1(\varepsilon) - h_2(\varepsilon)\right] T(\varepsilon) \tag{4.1.49}$$

where $T(\varepsilon)$ is given by Eq. (3.5.26). If we now insert the expression for the density of states in the Coulomb blockade regime ($E_c\beta \gg 1$) Eq. (4.1.23), we find that to linear order in the bias voltage $V = V_1 - V_2$,

$$\begin{aligned} I &= \frac{e^2 V \nu_0}{2}\frac{\Gamma_1\Gamma_2}{\Gamma_1+\Gamma_2}\int dx \frac{\mathrm{sech}^2(x)}{\left[1+e^{-\beta\Omega_N}\right]}\left[1 + \frac{1}{2}\tanh\left(x + \frac{\beta}{2}\left[\Omega_N - E_c\right]\right) - \frac{1}{2}\tanh\left(x + \frac{\beta\Omega_N}{2}\right)\right. \\ &+ \left. e^{\beta\Omega_N}\left\{1 - \frac{1}{2}\tanh\left(x + \frac{\beta}{2}\left[\Omega_N + E_c\right]\right) + \frac{1}{2}\tanh\left(x + \frac{\beta\Omega_N}{2}\right)\right\}\right]. \end{aligned} \tag{4.1.50}$$

This leads to the linear conductance, for small Ω_N, being given by

$$G = \frac{dI}{dV} = \frac{e^2\nu_0}{2}\frac{\Gamma_1\Gamma_2}{\Gamma_1+\Gamma_2}\frac{\beta\Omega_N}{\sinh\left(\beta\Omega_N\right)}. \tag{4.1.51}$$

This is the classic result for a peak in the conductance [3],[1],[40]. However, it is worth noting that the correct expression for the density of states is required to describe the conductance.

4.2 Alternative Derivation of the Green's function

In the previous section, we calculated the Green's function using a functional integration approach which allowed us to calculate the conductance. However, the functional integration approach is not ideal for the noise calculation. We have attempted to calculate the noise using this approach but with no success. The main problem is that the noise requires the calculation of two-particle Green's functions which in the functional integration approach would take the form,

$$iG_{nm}(t,t',t_1,t_2) = \frac{1}{\mathcal{Z}}\int D\psi D\bar{\psi}\psi_n(t)\bar{\psi}_n(t')\psi_n(t_1)\bar{\psi}_n(t_2)e^{i\int_{c_K}dt\left[\sum_k \bar{\psi}_k(t)i\partial_t\psi_k(t)-H\right]}. \quad (4.2.1)$$

When we attempt the functional integration approach outlined in the previous section, there is only one Bosonic field that is introduced from the Hubbard-Stratonovich transformation which links the time dependence of the terms. After the Hubbard-Stratonovich transformation the Fermionic action is quadratic and Wick's theorem can be used to obtain expressions for which the Fermionic integrals can be performed. The resulting Bosonic integral now however depends on all four of the time variables due to the time dependence of the operators $\psi_n(t)$. Therefore, when this is coupled with the two legs of the Keldysh contour there are sixteen possible time orderings, this results in a large number of integrals to perform. However, I am sure that it is possible to over come this problem but we chose to derive a different approach to obtaining the Green's functions of the quantum dot.

In this section, we will derive a new approach to obtain the Green's function for our system, showing that the same expressions for the tunneling density of states and conductance can be obtained.

It will be useful at this point to clearly define the hierarchy of the energy scales that we are considering in this problem. We are interested in the Coulomb blockade regime. This is

when the charging energy, E_c, is the dominating energy scale of the system. The temperature, $1/\beta$, should however remain much greater than the mean level spacing, Δ, which in turn should be much greater than the tunnelling rate, γ,

$$\gamma \ll \Delta \ll 1/\beta \ll E_c. \tag{4.2.2}$$

We will put no restriction on the energy scale of the bias voltage, eV, across the leads. However, we are primarily interested in the shot noise which is dominant when the bias voltage is much greater than the temperature $eV\beta >> 1$[26].

In the new approach we will begin by considering the quantum dot coupled via tunnelling contacts to two non-interacting leads. We will begin from the resonant level approximation. This is the assumption that the tunnelling between the leads and the dot is dominated by resonant level processes, which is a reasonable assumption for this problem as $\Gamma \ll \Delta$. As the quantum dot we are considering is zero-dimensional and the energy levels are quantised we can write the Green's function of the quantum dot as the sum over n of the Green's function for the n^{th} level of the dot,

$$G(\varepsilon) = \sum_n G_n(\varepsilon). \tag{4.2.3}$$

The tunnelling is dominated by the resonant processes because the energy levels of the dot are quantised and the width of these levels (Γ) is much smaller than the distance between the levels (Δ) this allows us to treat each level n with energy ε_n as a resonant level. For the resonant level problem when an electron from a lead is incident on the potential barrier that separates the lead from the dot with an energy which isn't coincident with one of the levels of the dot then its transmission coefficient is very small. However, when the electrons energy coincides with that of a level, resonance occurs and its transmission coefficient is greatly increased.

4.2.1 Calculation of the Green's function

We will start from the definition of the Green's function written in the Heisenberg representation as a trace,

$$G_n(t,t') = \frac{\mathrm{Tr}\left(e^{-\beta H} d_n(t) d_n^\dagger(t)\right)}{\mathrm{Tr}\left(e^{-\beta H}\right)}. \tag{4.2.4}$$

We will now use the resonant level approximation to simplify the Hamiltonian and limit the coupling between the leads and the dot to just the n^{th} level,

$$H_T = \sum_{\alpha,k} \left[\gamma_{\alpha n} d_n^\dagger(t) c_{\alpha k}(t) + h.c.\right]. \tag{4.2.5}$$

This is equivalent to ignoring the terms from the Hamiltonian which couple the leads to any other level of the dot,

$$H_{T'} = \sum_{\alpha,k,m\neq n} \left[\gamma_{\alpha m} d_m^\dagger(t) c_{\alpha k}(t) + h.c.\right]. \tag{4.2.6}$$

We now can write the simplified Hamiltonian in a form, where we have singled out the n^{th} level,

$$H = H_0 + E_N + H_n + H_0^{leads} \tag{4.2.7}$$

where

$$H_0 = \sum_{m\neq n} \varepsilon_m d_m^\dagger(t) d_m(t), \tag{4.2.8}$$

$$E_N = \frac{E_c}{2}\left(\hat{N} - N_g\right)^2, \tag{4.2.9}$$

$$H_n = \varepsilon_n d_n^\dagger(t) d_n(t) + \Omega(N) d_n^\dagger(t) d_n(t) + \sum_{\alpha,k} \left[\gamma_{\alpha n} d_n^\dagger(t) c_{\alpha k}(t) + h.c.\right]. \tag{4.2.10}$$

$\hat{N}$ is now no longer the total number of electrons confined to the dot, it is the number of electrons on the dot, excluding the occupancy of the n^{th} level. We also assume that there is no independent chemical potential on the dot, the role of the chemical potential is instead performed by the gate voltage which determines the occupancy of the dot. We are now able

to use the Hamiltonian in this slightly peculiar form to derive the Green's function of the dot. The first step is to separate out the trace over the E_N term, we are able to do this as it commutes with the rest of the Hamiltonian. As the trace over the E_N term only depends on the total number of electrons confined to the dot, not on their configuration we can write the Green's function as

$$G_n(t,t') = \frac{1}{\mathcal{Z}} \sum_N e^{-\beta E_N} \mathrm{Tr} \left(e^{-\beta \left[H_0 + H_0^{leads} + H_n \right]} d_n(t) d_n^\dagger(t) \right). \tag{4.2.11}$$

The next step is to split up the trace to consider the n^{th} level separately we are again able to do this as H_0 commutes with the other terms in the Hamiltonian. This gives

$$G_n(t,t') = \frac{1}{\mathcal{Z}} \sum_N e^{-\beta E_N} \mathrm{Tr} \left(e^{-\beta H_0} \right) \mathrm{Tr} \left(e^{-\beta \left[H_0^{leads} + H_n \right]} d_n(t) d_n^\dagger(t) \right). \tag{4.2.12}$$

The first trace in the equation above is a trace over everything but the n^{th} level and is equivalent to the canonical partition function with the n^{th} level removed, Z_N. The second trace is just over the n^{th} level and is nearly the resonant level Green's function, $G_{RL}(t,t')$ with a shift in energy $\varepsilon_n = \varepsilon_n + \Omega(N)$. Multiplying and dividing by the resonant level partition function (Z_{RL}) allows us to express the Green's function it terms of the resonant level Green's function,

$$G_n(t,t') = \frac{1}{\mathcal{Z}} \sum_N e^{-\beta E_N} Z_N G_{RL}(t,t') Z_{RL}. \tag{4.2.13}$$

The partition function can be obtained using the same technique and is given by

$$\mathcal{Z} = \sum_N e^{-\beta E_N} Z_N Z_{RL}. \tag{4.2.14}$$

4.2.2 Tunneling Density of states

Using the newly derived Green's function Eq. (4.2.13), we will now derive the tunneling density of states (TDoS) and compare it with the result from Sedlmayr et al.[40]. we re-derived earlier, Eq. (4.1.23), to check that we obtain the correct expression for the Green's function. We will begin with the retarded Green's function

$$G^R(\varepsilon) = \sum_n G^R_n(\varepsilon), \tag{4.2.15}$$

and write the resonant level partition function explicitly

$$Z_{RL} = 1 + e^{-\beta(\varepsilon_n + \Omega(N))}. \tag{4.2.16}$$

Substituting this expression into the Green's function, we can now express the Green's function in two parts depending on the occupancy of the n^{th} level,

$$G^R(\varepsilon) = \sum_n \frac{1}{\mathcal{Z}} \sum_N Z_N \left(e^{-\beta E_N} G^R_{RL}(\varepsilon, \Omega_N) + e^{-\beta(E_N + \varepsilon_n + \Omega(N))} G^R_{RL}(\varepsilon, \Omega_N) \right) \tag{4.2.17}$$

where it is useful to remember that

$$E_N + \Omega(N) = E_{N+1}, \tag{4.2.18}$$

so the Green's function becomes

$$G^R(\varepsilon) = \sum_n \frac{1}{\mathcal{Z}} \sum_N Z_N \left[\left(e^{-\beta E_N} \right) G^R_{RL}(\varepsilon, \Omega_N) \right]. \tag{4.2.19}$$

If we now make a shift of variables,$N + 1 \to N$ in the second of these terms, we obtain

$$G^R(\varepsilon) = \sum_n \frac{1}{\mathcal{Z}} \sum_N Z_N \left[e^{-\beta E_N} G^R_{RL}(\varepsilon, \Omega_N) + e^{-\beta(E_N + \varepsilon_n)} G^R_{RL}(\varepsilon, \Omega_{N-1}) \right]. \tag{4.2.20}$$

We can carry out the same procedure on the partition function,

$$\mathcal{Z} = \sum_N Z_N \left(e^{-\beta E_N} + e^{-\beta[\varepsilon_n + E_{N+1}]} \right). \tag{4.2.21}$$

If we now tune the gate voltage (N_g) such that we are close to the peak of conductance, we can truncate the summation over N to the two terms closest to $N_g + 1/2$, as all the other terms will be exponentially suppressed. If we also assume that $N \gg 1$, we can assume that the partition functions Z_N an Z_{N+1} are approximately equal. This allows us to consider just two terms, from the summation over N, which correspond to there being N or $N+1$ electrons upon the dot. This means in total there will be four terms in the Green's function expression. We obtain two expressions for each situation, we have N or $N+1$ electrons on the dot, with the n^{th} level begin either occupied or unoccupied. We can now write the Green's function as

$$\begin{aligned} G^R(\varepsilon) &= \sum_n \frac{1}{\mathcal{Z}} \left\{ e^{-\beta\varepsilon_n} G^R_{RL}(\varepsilon, \Omega(N-1)) + \left[1 + e^{-\beta(\varepsilon_n + \Omega_N)}\right] G^R_{RL}(\varepsilon, \Omega(N)) \right. \\ &+ \left. e^{-\beta\Omega_N} G^R_{RL}(\varepsilon, \Omega(N+1)) \right\} \end{aligned} \tag{4.2.22}$$

where the partition function is

$$\mathcal{Z} = \left(1 + e^{-\beta\varepsilon_n}\right)\left(1 + e^{-\beta\Omega_N}\right). \tag{4.2.23}$$

The resonant level retarded Green's function is given by

$$G^R_{RL}(\varepsilon, \Omega_N) = \frac{1}{\varepsilon - \varepsilon_n - \Omega_N + i\frac{\Gamma_n}{2}}, \tag{4.2.24}$$

which upon substitution gives

$$G^R(\varepsilon) = \sum_n \frac{1}{\mathcal{Z}} \left\{ \frac{e^{-\beta\varepsilon_n}}{\varepsilon - \varepsilon_n - \Omega_{N-1} + i\frac{\Gamma_n}{2}} + \frac{\left[1 + e^{-\beta(\varepsilon_n + \Omega_N)}\right]}{\varepsilon - \varepsilon_n - \Omega_N + i\frac{\Gamma_n}{2}} \right.$$

$$+ \quad \frac{e^{-\beta\Omega_N}}{\varepsilon - \varepsilon_n - \Omega_{N+1} + i\frac{\Gamma_n}{2}} \Bigg\}. \tag{4.2.25}$$

Using this Green's function and the advanced Green's function which can be calculated in the same manner we can calculate the tunneling density of states ($\nu(\varepsilon)$) starting from the standard formula [41],

$$\nu(\varepsilon) = \frac{i}{2\pi}\left[G^R(\varepsilon) - G^A(\varepsilon)\right]. \tag{4.2.26}$$

Inserting the Green's function expressions into Eq. (4.2.26) gives

$$\begin{aligned}\nu(\varepsilon) &= \frac{1}{\pi}\sum_n \frac{1}{\mathcal{Z}}\Bigg\{\frac{\frac{\Gamma_n}{2}e^{-\beta\varepsilon_n}}{\left[\varepsilon - \varepsilon_n - \Omega_{N'-1}\right]^2 + \left(\frac{\Gamma_n}{2}\right)^2} + \frac{\frac{\Gamma_n}{2}\left[1 + e^{-\beta[\varepsilon_n + \Omega_{N'}]}\right]}{\left[\varepsilon - \varepsilon_n - \Omega_{N'}\right]^2 + \left(\frac{\Gamma_n}{2}\right)^2} \\ &+ \quad \frac{\frac{\Gamma_n}{2}e^{-\beta\Omega_{N'}}}{\left[\varepsilon - \varepsilon_n - \Omega_{N'+1}\right]^2 + \left(\frac{\Gamma_n}{2}\right)^2}\Bigg\}.\end{aligned} \tag{4.2.27}$$

As Γ_n is the smallest energy scale in our system ($\Gamma_n \ll \Delta \ll T \ll E_c$), we can make the approximation that the Lorentzian function is a delta like function of width Γ,

$$\frac{1}{\pi}\frac{\frac{\Gamma_n}{2}}{\left[\varepsilon - \varepsilon_n - \Omega_{N'}\right]^2 + \left(\frac{\Gamma_n}{2}\right)^2} = \delta_\Gamma(\varepsilon - \varepsilon_n - \Omega_{N'}). \tag{4.2.28}$$

We now average over disorder by substituting the mean tunnelling density of states (TDoS) of noninteracting electrons, ν_0 for the sum over the delta like functions, with the assumption that the TDoS is smooth in any realisation of disorder. This is valid when the mean level spacing, Δ, is much smaller than the temperature $1/\beta$. Therefore, we can now write the TDoS as

$$\frac{\nu(\varepsilon)}{\nu_0} = \frac{1}{1 + e^{-\beta\Omega_{N'}}}\left\{\frac{e^{-\beta\left[\varepsilon - \Omega_{N'-1}\right]}}{1 + e^{-\beta\left[\varepsilon - \Omega_{N'-1}\right]}} + \frac{1 + e^{-\beta[\varepsilon - \Omega_{N'} + \Omega_{N'}]}}{1 + e^{-\beta[\varepsilon - \Omega_{N'}]}} + \frac{e^{-\beta\Omega_{N'}}}{1 + e^{-\beta\left[\varepsilon - \Omega_{N'+1}\right]}}\right\}. \tag{4.2.29}$$

The above expression can be rewritten in terms of Fermi functions as

$$\frac{\nu(\varepsilon)}{\nu_0} = \frac{U(\varepsilon - \Omega_{N'}) + e^{-\beta\Omega_{N'}}U(\varepsilon - \Omega_{N'+1})}{1 + e^{-\beta\Omega_{N'}}}, \tag{4.2.30}$$

where we have defined $U(\varepsilon - \Omega_{N'}) = f(\varepsilon - \Omega_{N'-1}) + 1 - f(\varepsilon - \Omega_{N'})$. This is the correct expression for the density of states [40] and is identical to the expression obtained using the functional integral approach in the previous chapter. In the following chapter, we will use this same approach to calculate the two-particle Green's function which will then allow us to calculate the noise power spectrum for the Coulomb blockaded quantum dot.

4.3 Summary

In this chapter, we have reviewed a functional integration approach to calculating the Green's function for the Coulomb blockaded dot. Using the Green's function, we then derive the tunnelling density of states and the conductance of our system. This approach, however, presents difficulties when one tries to extend the approach to calculating the two-particle Green's functions which is required for the noise calculation. Therefore, we have derived a secondary approach for calculating the Green's function of the dot. In this approach, we write the Green's function as a trace and by extracting the term responsible for charging energy, express the Green's function as a summation over resonant level Green's functions. We then go on to show that the newly derived Green's function produces the expected result for the tunnelling density of states Eq. (4.1.23). In the next chapter, we will calculate the two-particle Green's function and use this to obtain an expression for the noise power spectrum.

Chapter 5

THE NOISE POWER SPECTRUM OF A QUANTUM DOT

In this chapter, we will derive the two-particle Green's function and use this to derive an expression for the noise power spectrum. We will then consider some sensible limits of the noise power spectrum and derive the Fano factor in the shot noise regime and calculate numerically the Fano factor as a function of the bias voltage.

5.1 The Noise Power Spectrum

The noise power spectrum is defined as the Fourier transform of the noise correlator [26],

$$S_{\alpha\beta}(\omega) = \int d\omega e^{i\omega t} S_{\alpha\beta}(t,t') = \langle\{\delta I_\alpha(t), \delta I_\beta(t')\}\rangle \tag{5.1.1}$$

where $\delta I_\alpha(t) = I_\alpha(t) - \langle I_\alpha(t)\rangle$. The noise correlator can equivalently be written in the form

$$S_{\alpha\beta}(t,t') = \langle\{I_\alpha(t), I_\beta(t')\}\rangle - 2\langle I(t)\rangle^2. \tag{5.1.2}$$

It can be clearly shown, by using the definition of the current

$$I_\alpha(t) = \frac{ei}{\hbar}\sum_{k,n}\left[\gamma_{\alpha kn} c^\dagger_{\alpha k}(t) d_n(t) - \gamma^\dagger_{\alpha kn} d^\dagger_n(t) c_{\alpha k}(t)\right], \tag{5.1.3}$$

that, as we showed for the non-interacting case earlier, the noise correlator can be written in terms of Green's functions as

$$\begin{aligned} S(t,t') &= \left(\frac{e}{\hbar}\right)^2 \sum_{n,m,k,k'} \gamma_{kn}\gamma_{k'm} G_1^{>}(t,t') - \gamma_{kn}\gamma_{k'm}^{\dagger} G_2^{>}(t,t') - \gamma_{kn}^{\dagger}\gamma_{k'm} G_3^{>}(t,t') \\ &+ \gamma_{kn}^{\dagger}\gamma_{k'm}^{\dagger} G_4^{>}(t,t') + h.c. - 2\langle I(t)\rangle^2. \end{aligned} \tag{5.1.4}$$

The Green's functions are given by

$$\begin{aligned} G_{1,nm}(t,t') &= i^2\langle T c_k^{\dagger}(t) d_n(t) c_{k'}^{\dagger}(t') d_m(t')\rangle, \\ G_{2,nm}(t,t') &= i^2\langle T c_k^{\dagger}(t) d_n(t) d_m^{\dagger}(t') c_{k'}(t')\rangle, \\ G_{3,nm}(t,t') &= i^2\langle T d_n^{\dagger}(t) c_k(t) c_{k'}^{\dagger}(t') d_m(t')\rangle, \\ G_{4,nm}(t,t') &= i^2\langle T d_n^{\dagger}(t) c_k(t) d_m^{\dagger}(t') c_{k'}(t')\rangle. \end{aligned} \tag{5.1.5}$$

Therefore, the noise calculation requires the evaluation of the two-particle Green's functions Eqs. (5.1.5). As for the resonant level example we considered earlier, to calculate these Green's functions we are required to calculate Green's functions of the form

$$G_{nm}(t,t',t_1,t_2) = i^2\left\langle T_{c_k} d_n(t) d_n^{\dagger}(t') d_m(t_1) d_m^{\dagger}(t_2)\right\rangle. \tag{5.1.6}$$

We will however now assume that the resonant level tunneling is dominant, as $\Gamma \ll \Delta$. This allows us to approximate the Green's functions by their diagonal elements, [27]

$$\sum_{n,m} G_{nm}(t,t') \approx \sum_{n} G_{nn}(t,t'). \tag{5.1.7}$$

5.1.1 Two-particle Green's Function

As we have made the assumption that we only need consider the diagonal components of the Green's functions, we are required to calculate expressions of this form,

$$G_{nn}(t,t') = \frac{\mathrm{Tr}\left(e^{-\beta H}c_k^\dagger(t)d_n(t)c_{k'}^\dagger(t')d_n(t')\right)}{\mathrm{Tr}\left(e^{-\beta H}\right)}. \tag{5.1.8}$$

It is clear that we can use the same method for calculating this Green's function expression as we used in the previous chapter. If we once again single out the n^{th} level of the Hamiltonian, we can follow the same method as for the single particle Green's function. This is because the only differences between the two expressions are the extra creation and annihilation operators in the two-particle case. This allows us to express the Green's function Eq. (5.1.8) in the form

$$G_{nn}(\varepsilon) = \frac{1}{\mathcal{Z}}\sum_N e^{-\beta E_N} Z_N G_{RL}(\varepsilon,\Omega_N) Z_{RL}, \tag{5.1.9}$$

where

$$\mathcal{Z} = \sum_N e^{-\beta E_N} Z_N Z_{RL} \tag{5.1.10}$$

and $G_{RL}(\varepsilon,\Omega_N)$ is the corresponding two particle resonant level Green's function with the shift in energy, $\varepsilon_n = \varepsilon_n + \Omega_N$. Z_{RL} is the resonant level partition function.

5.1.2 The Shot Noise Power Spectrum

Using the Green's functions, we can now easily derive an expression for the noise. If we insert the Green's functions in the form of Eq. (5.1.9) back into the noise power spectrum Eq. (5.1.22), we are able to arrange the terms such that we can write the noise in terms of

the resonant level noise expression as

$$S(\varepsilon) = \sum_n \frac{1}{\mathcal{Z}} \sum_N e^{-\beta E_N} S_{RL}(\varepsilon, \Omega_N) Z_{RL} Z_N. \tag{5.1.11}$$

We will now insert the resonant level partition function,

$$Z_{RL} = 1 + e^{-\beta(\varepsilon_n + \Omega_N)}, \tag{5.1.12}$$

and tune the gate voltage, N_g, such that we are close to the peak of conductance. This means that we can truncate the sum over N to the two terms closest to $N_g + 1/2$, as all other terms will be exponentially suppressed. We will once again consider the limit $N \gg 1$ and therefore assume that $Z_{N+1} \approx Z_N$. We can now write the zero frequency noise as

$$S(0) = \sum_n \frac{1}{\mathcal{Z}} \left\{ e^{-\beta\varepsilon_n} S_{nRL}(0, \Omega_{N-1}) + \left(1 + e^{\varepsilon_n + \Omega_N}\right) S_{nRL}(0, \Omega_N) + e^{-\beta\Omega_N} S_{nRL}(0, \Omega_{N+1}) \right\}. \tag{5.1.13}$$

Inserting the partition function and rewriting the exponential terms as Fermi functions, we obtain

$$\begin{aligned} S(0) &= \sum_n \frac{1}{1 + e^{-\beta\Omega_{N'}}} \left\{ f(\varepsilon_n) S_{nRL}(0, \Omega_{N-1}) + \left(1 - f(\varepsilon_n) + e^{-\beta\Omega_N} f(\varepsilon_n)\right) S_{nRL}(0, \Omega_N) \right. \\ &+ \left. e^{-\beta\Omega_N} \left(1 - f(\varepsilon_n)\right) S_{nRL}(0, \Omega_{N+1}) \right\}. \end{aligned} \tag{5.1.14}$$

As we showed in Eq. (3.5.27), the zero frequency resonant level noise power spectrum is given by

$$\begin{aligned} S_{nRL}(0, \Omega_N) &= \frac{2e^2}{\hbar^2} \int \frac{d\varepsilon}{2\pi} \left\{ f_1(\varepsilon) \left[1 - f_1(\varepsilon)\right] + f_2(\varepsilon) \left[1 - f_2(\varepsilon)\right] \right\} T(\varepsilon, \Omega_N) \\ &+ \left[f_1(\varepsilon) - f_2(\varepsilon)\right]^2 T_n(\varepsilon, \Omega_N) \left[1 - T_n(\varepsilon, \Omega_N)\right]. \end{aligned} \tag{5.1.15}$$

The transmission probabilities, $T_n(\varepsilon, \Omega_N)$, are given by

$$\begin{aligned} T_n(\varepsilon, \Omega_{N'}) &= \Gamma_1\Gamma_2 G^R_{nRL}(\varepsilon, \Omega_{N'}) G^A_{nRL}(\varepsilon, \Omega_{N'}) \\ &= \frac{\Gamma_1\Gamma_2}{\Gamma_1+\Gamma_2} 2\pi\nu_{nRL}(\varepsilon, \Omega_{N'}) \end{aligned} \tag{5.1.16}$$

where the resonant level density of states is a Lorentzian which we approximate as being delta like;

$$\nu_{nRL}(\varepsilon, \Omega_{N'}) = \frac{1}{\pi}\frac{\Gamma/2}{\left(\varepsilon - \varepsilon_n - \Omega_{N'}\right)^2 + \left(\Gamma/2\right)^2} = \delta_\Gamma\left(\varepsilon - \varepsilon_n - \Omega_{N'}\right). \tag{5.1.17}$$

The summation over the delta like terms is the bare density of states,

$$\sum_n \delta_\Gamma\left(\varepsilon - \varepsilon_n - \Omega_{N'}\right) = \nu_0. \tag{5.1.18}$$

The transmission squared term, $T^2_n(\varepsilon, \Omega_{N'})$, can also be written as a delta like term as

$$\begin{aligned} T_n(\varepsilon, \Omega_{N'})^2 &= \left(\Gamma_1\Gamma_2\right)^2 \left(G^R_{nRL}(\varepsilon, \Omega_{N'}) G^A_{nRL}(\varepsilon, \Omega_{N'})\right)^2 \\ &= \left[\frac{\Gamma_1\Gamma_2}{\Gamma_1+\Gamma_2}\right]^2 (2\pi)^2 \nu^2_{nRL}(\varepsilon, \Omega_{N'}), \end{aligned} \tag{5.1.19}$$

where the summation over the resonant level density of states squared is given by

$$\begin{aligned} \sum_n \nu^2_{nRL}(\varepsilon, \Omega_{N'}) &= \sum_n \frac{1}{\pi^2}\frac{(\Gamma/2)^2}{\left[\left(\varepsilon - \varepsilon_n - \Omega_{N'}\right)^2 + \left(\Gamma/2\right)^2\right]^2} \\ &= \frac{1}{\pi^2}\left(\frac{\Gamma}{2}\right)^2 \sum_n \frac{1}{\left[\left(\varepsilon - \varepsilon_n - \Omega_{N'}\right)^2 + \left(\Gamma/2\right)^2\right]^2}. \end{aligned} \tag{5.1.20}$$

We can express this in terms of the Lorentzian by considering the sum above as a differential with respect to $(\Gamma/2)^2$,

$$\sum_n \nu^2_{nRL}(\varepsilon, \Omega_{N'}) = = -\frac{1}{\pi}\left(\frac{\Gamma}{2}\right)^2 \frac{d}{d\left(\frac{\Gamma}{2}\right)^2} \sum_n \frac{1}{\pi}\frac{1}{\left(\varepsilon - \varepsilon_n - \Omega_{N'}\right)^2 + \left(\Gamma/2\right)^2}$$

$$
\begin{aligned}
&= -\frac{1}{\pi}\left(\frac{\Gamma}{2}\right)^2 \frac{d}{d\left(\frac{\Gamma}{2}\right)^2}\frac{2}{\Gamma}\nu_0 \\
&= \frac{1}{\pi\Gamma}\nu_0. \qquad (5.1.21)
\end{aligned}
$$

If we now insert these expressions into the noise power spectrum, we obtain

$$
\begin{aligned}
S(0) &= 2\frac{e^2}{\hbar^2}\int \frac{d\varepsilon}{2\pi}\frac{\nu(\varepsilon)}{\nu_0}\left\{\left[f_1(\varepsilon)\left[1-f_1(\varepsilon)\right]+f_2(\varepsilon)\left[1-f_2(\varepsilon)\right]+\left[f_1(\varepsilon)-f_2(\varepsilon)\right]^2\right]T_1\right. \\
&- \left.\left[f_1(\varepsilon)-f_2(\varepsilon)\right]^2 T_2\right\}, \qquad (5.1.22)
\end{aligned}
$$

where the density of states, $\nu(\varepsilon)$, is given by Eq. (4.2.30). The constants T_1 and T_2 are given by

$$
\begin{aligned}
T_1 &= \frac{\Gamma_1\Gamma_2}{\Gamma_1+\Gamma_2}2\pi\nu_0, \\
T_2 &= \frac{\left(\Gamma_1\Gamma_2\right)^2}{\left(\Gamma_1+\Gamma_2\right)^3}4\pi\nu_0. \qquad (5.1.23)
\end{aligned}
$$

The noise expression Eq. (5.1.22) can be solved exactly. However, the expression that is obtained is very messy and little is to be gained by doing the tedious and longwinded calculation.

We will now consider several sensible limits to the noise power spectrum. This will enable us to compare the relevant current calculations to obtain the Fano factor in the shot noise regime.

5.2 Limits of the Noise Expression

We will now consider some sensible limits of the noise power spectrum Eq. (5.1.22). We will begin by calculating the noise in the zero bias regime and by comparing it to the conductance expression, we will show that we obtain the fluctuation dissipation theorem, as expected.

We will then go on to consider the shot noise regime. This is the limit where the voltage difference between the leads is much greater than the temperature. In this limit, we will derive a noise expression and by comparing it to the average current expression, we will obtain the Fano factor.

5.2.1 Zero Bias Voltage

In this section, we will consider the zero bias regime and calculate the thermal noise in the system. We will show that the noise is given by the usual fluctuation dissipation expression. We will begin by considering the current in the form of Eq. (4.1.47),

$$I = \frac{e}{2}\int d\varepsilon \frac{\Gamma_1\Gamma_2}{\Gamma_1+\Gamma_2}\nu(\varepsilon)\left[h_1(\varepsilon)-h_2(\varepsilon)\right]. \tag{5.2.1}$$

From this, we can derive an expression for the linear conductance in terms of the integral of the density of states. If we rewrite this expression in terms of Fermi functions of the leads, we obtain

$$\begin{aligned} I &= e\frac{\Gamma_1\Gamma_2}{\Gamma_1+\Gamma_2}\int d\varepsilon\nu(\varepsilon)\left[f_2(\varepsilon)-f_1(\varepsilon)\right] \\ &= e\frac{\Gamma_1\Gamma_2}{\Gamma_1+\Gamma_2}\int d\varepsilon\nu(\varepsilon)eV\left(-\frac{df}{d\varepsilon}\right), \end{aligned} \tag{5.2.2}$$

where we have expanded the Fermi functions to linear order in the applied voltage. If we now make the substitution $x = \tanh(\frac{\beta\varepsilon}{2})$, we can write the current as

$$I = \frac{e^2V}{2}\frac{\Gamma_1\Gamma_2}{\Gamma_1+\Gamma_2}\int_{-1}^{1} dx\nu(x), \tag{5.2.3}$$

and therefore the linear conductance, G, is given by

$$G = \frac{dI}{dV} = \frac{e^2}{2}\frac{\Gamma_1\Gamma_2}{\Gamma_1+\Gamma_2}\int_{-1}^{1} dx\nu(x). \tag{5.2.4}$$

If we now consider the noise expression at zero bias voltage, the Fermi functions for the leads are identical and a lot of the terms cancel, to leave

$$S(0) = 4e^2 \frac{\Gamma_1 \Gamma_2}{\Gamma_1 + \Gamma_2} \int d\varepsilon \nu(\varepsilon) f(\varepsilon) \left[1 - f(\varepsilon)\right]. \tag{5.2.5}$$

Now we can rewrite the Fermi functions as $f(\varepsilon) = 1/2 - 1/2 \tanh(\varepsilon)$ and use the substitution $x = \tanh(\frac{\beta\varepsilon}{2})$, to express the noise as,

$$S(0) = \frac{2e^2}{\beta} \frac{\Gamma_1 \Gamma_2}{\Gamma_1 + \Gamma_2} \int_{-1}^{1} dx \nu(x). \tag{5.2.6}$$

Therefore, we can write the noise in terms of the linear conductance as

$$S(0) = \frac{4G}{\beta}, \tag{5.2.7}$$

and the thermal noise is just given by the fluctuation dissipation theorem.

5.2.2 The Shot Noise Regime

In this section, we will consider the shot noise regime, We will derive expressions for the current and the noise in the the large voltage limit, $eV \gg 1/\beta$, using these results to derive the Fano factor.

We will begin by deriving the expression for the average current. If we again start from Eq. (4.1.47), we can write the current as

$$I = e \int d\varepsilon \frac{\Gamma_1 \Gamma_2}{\Gamma_1 + \Gamma_2} \nu(\varepsilon) \left[f_2(\varepsilon) - f_1(\varepsilon)\right]. \tag{5.2.8}$$

If we write the Fermi functions in terms of tanh functions, make the substitution $x = \tanh(\frac{\beta\varepsilon}{2})$ and use the tanh identity,

$$\tanh(A \pm B) = \frac{\tanh(A) \pm \tanh(B)}{1 \pm \tanh(A)\tanh(B)}, \tag{5.2.9}$$

we can express the average current as

$$I = \frac{2e}{\beta}\frac{\Gamma_1\Gamma_2}{\Gamma_1+\Gamma_2}\int_{-1}^{1} dx\nu(x)\frac{A}{1-A^2x^2}. \tag{5.2.10}$$

We have introduced the shorthand notation $A = \tanh(\frac{\beta eV}{4})$. We will now derive the noise power spectrum and show that we can write it as a function of the same integral. We will start from the noise expression Eq. (5.1.22) and using the same substitution as the current case, we obtain

$$S(0) = \frac{2e^2}{\beta}\frac{\Gamma_1\Gamma_2}{\Gamma_1+\Gamma_2}\int_{-1}^{1} dx\nu(x)\left[\frac{1+A^2}{1-A^2x^2} - \frac{\Gamma_1\Gamma_2}{(\Gamma_1+\Gamma_2)^2}4A^2\frac{1-x^2}{(1-A^2x^2)^2}\right]. \tag{5.2.11}$$

Both the current Eq. (5.2.10) and the noise Eq. (5.2.11) above are general expressions. We now consider the large voltage limit $eV \gg 1/\beta$. We can clearly see that this implies $A = \tanh(\frac{\beta eV}{4}) = 1$ in terms of the notation we have introduced. Therefore, we can express the current as

$$I = 2\frac{2e}{\beta}\frac{\Gamma_1\Gamma_2}{\Gamma_1+\Gamma_2}\int_{-1}^{1} dx\nu(x)\frac{1}{1-x^2}, \tag{5.2.12}$$

and the noise power spectrum can be written as a function of the noise;

$$S(0) = 2eI\left[1 - 2\frac{\Gamma_1\Gamma_2}{(\Gamma_1+\Gamma_2)^2}\right]. \tag{5.2.13}$$

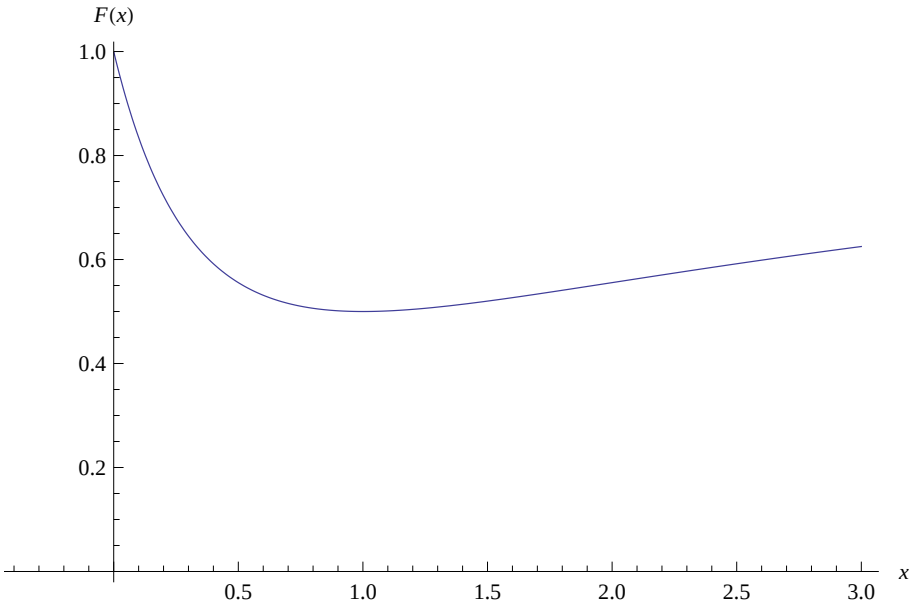

Figure 5.1: The Fano factor as a function of the asymmetry in the coupling to the leads, $x = \frac{\Gamma_2}{\Gamma_1}$. The Fano factor takes it's minimum value of 1/2 when the coupling to the leads is symmetric and tends to 1 if one of the leads is isolated from the system.

Using the definition of the Fano factor Eq. (2.4.1),

$$F = \frac{S(0)}{2eI} = 1 - 2\frac{\Gamma_1\Gamma_2}{(\Gamma_1+\Gamma_2)^2} = \frac{\Gamma_1^2+\Gamma_2^2}{(\Gamma_1+\Gamma_2)^2}. \tag{5.2.14}$$

A plot of the Fano factor can be seen in Fig. (5.1), In this figure, we have written Γ_2 as a function of Γ_1, $\Gamma_2 = x\Gamma_1$, We can do this without any loss of generality. In doing so we see that the Fano factor is now a function of the ratio of the Γ's and ranges from a half to one. The minimum value for the Fano factor of a half corresponds to the tunnelling rates being equal, $\Gamma_1 = \Gamma_2$, and means that the noise is suppressed to half of what would be measured if the noise were produced by a Poissonian process. This result, however, is exactly the same expression as that which is obtained if you consider the double tunnel barrier problem without taking into account the Coulomb interaction between the electrons [26]. It is also in complete agrement with the classical master equation approach [14][16][18] at the peak of conductance, $\Omega_N = 1$.

Using the method that we have developed, we do not need to restrict ourselves to these limits and it is possible to calculate the noise power spectrum analytically in the vicinity of the peak of conductance for voltage values between these limits. We choose not to express the noise and current calculations explicitly in this region as the expressions are extremely long winded and no more physical understanding can be interpreted from them. We will calculate these expressions numerically to show the Fano factor as a function of the bias voltage.

5.3 The Noise and Fano Factor as a Function of the Applied Voltage

We showed in the previous section that we can express the generic noise power spectrum and the current expressions as Eq. (5.2.11) and Eq. (5.2.10). The integrals can be solved analytically to obtain expressions for the noise and current. However, we do not need to do this to see that the Fano factor does not have the same dependence on the coupling to the leads for all values of eV. It is clear from the integral form of the noise power spectrum Eq. (5.2.11) that as the voltage changes the dependence on Γ will also change.

In Fig. (5.2), we have plotted the Fano factor for symmetric coupling, $\Gamma_1 = \Gamma_2$, as a function of bias voltage. In the limit $\beta eV \gg 1$, we obtain the limit of $1/2$ for the Fano factor. As the voltage tends to zero the Fano factor tends to infinity. This is due to the current tending to zero linearly and the thermal noise being constant at zero voltage. If we define our Fano factor as the ratio of excess noise instead of the total noise with the poissonian noise we can avoid this effect which otherwise dominates the small voltage regime. The excess noise is defined as the total noise of the system minus the zero voltage contribution,

$$S_{ex}(\omega, eV) = S(\omega, eV) - S(\omega, 0). \tag{5.3.1}$$

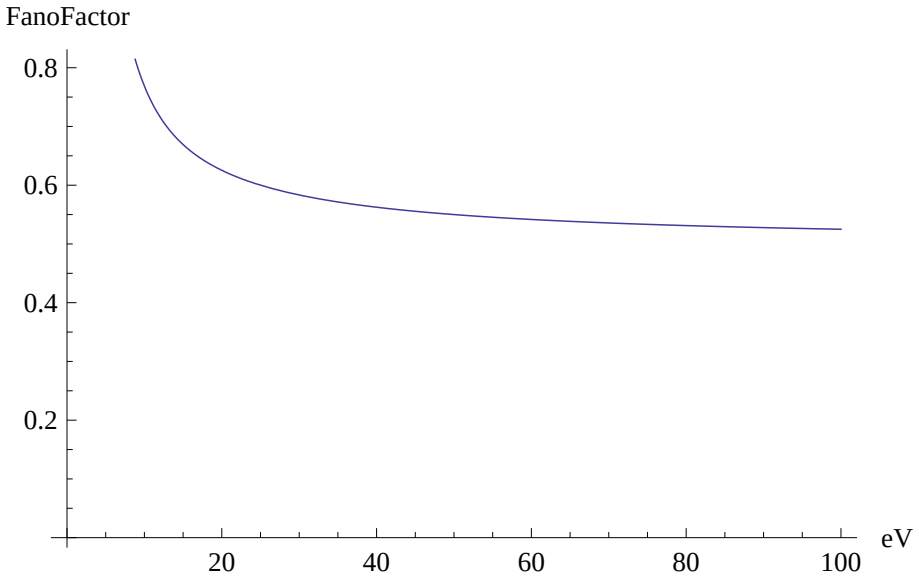

Figure 5.2: The Fano factor of the noise as a function of the bias voltage, for the symmetric case, $\Gamma_1 = \Gamma_2$, in units of temperature β. The Fano factor tends to $\frac{1}{2}$ for large eV.

The Fano factor of the excess noise is considered in Fig. (5.3). We plot the Fano factor for both the symmetric case, $\Gamma_1 = \Gamma_2$ and an asymmetric situation, $\Gamma_2 = 10\Gamma_1$. For both cases, in the large voltage limit, we obtain the required shot noise result. For small voltages, the Fano factor of the excess noise is linear. This is due to the excess noise being quadratic and the current being linear in nature.

5.4 Application of the Noise Power Spectrum Result

In the previous sections we have calculated several sensible limits of our noise power spectrum. We showed that in the zero voltage limit the expected fluctuation dissipation result is obtained. We have also calculated the noise in the shot noise regime and shown that the same Fano factor is obtained as that which is obtained from a fully classical approach to calculating the noise at the peak of conductance. We can conclude from this that it is reasonable in this limit to model the system classically, this however, was not clear from the outset as at the peak of conductance there are two interacting degenerate energy levels.

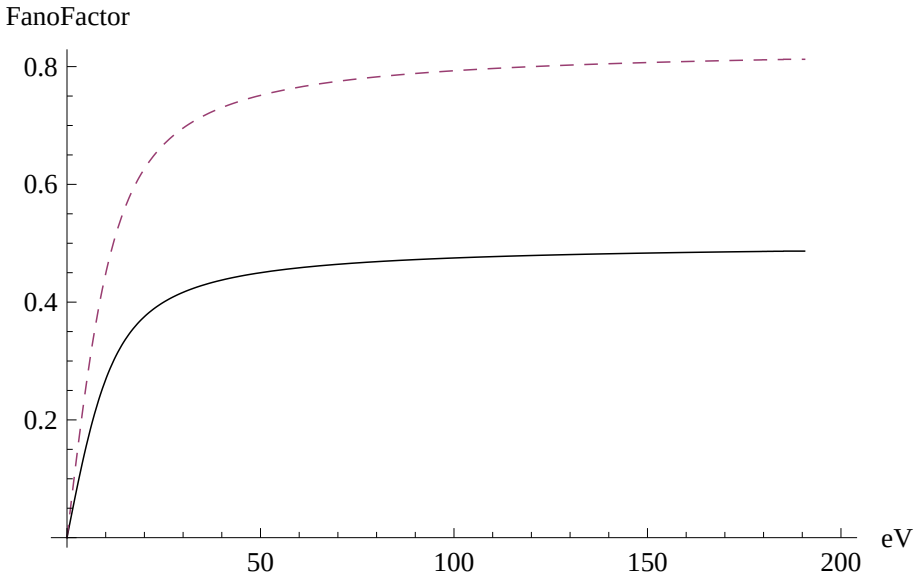

Figure 5.3: The Fano factor of the excess noise as a function of the bias voltage, in units of temperature β. We consider two situations; the symmetric case, $\Gamma_1 = \Gamma_2$ (solid line) and the asymmetric case,$\Gamma_1 = 10\Gamma_2$. The Fano factor tends to the shot noise result for both cases for large eV. The Fano factor is linear in nature for small eV.

The expressions we have calculated however are not just limited to these regimes and this is the advantage of the approach we have developed within this thesis.

Experimentally it would be interesting to study the noise properties of a quantum dot in the energy regime that we consider within this thesis. Initially it would be sensible to check the limits we have talked about in detail, but the expressions we have calculated hold for all values of the applied voltage and it would be very interesting to experimentally investigate the noise power in the small voltage regime were we predict that the excess noise would be linear in nature. It would also be interesting to investigate experimentally the effect on the noise of moving away from the peak of conductance. It is possible to calculate the noise expression for this situation as long as we remain close to the peak, as in the valleys of conductance of co-tunnelling processes are dominate, which are not considered in the approach.

5.5 Summary

In this chapter, we have derived an expression for the noise power spectrum of a Coulomb blockaded quantum dot. In doing so, we have made use of the techniques that we developed in the previous chapter to derive the diagonal two-particle Green's function. We then calculated both the zero voltage and shot noise limits. In the zero voltage limit, we showed that our approach recovers the standard fluctuation dissipation theorem, as expected. In the shot noise limit, we also recover the standard result for the non-interacting double tunnel barrier problem for the Fano factor. This is in agreement with the classical master equation approach [18]. We also study the quantum dot in the regime where the noise isn't dominated by the shot noise and plot the Fano factor as a function of the bias voltage. In the following chapter, we will move on to introduce full counting statistics.

Chapter 6

FULL COUNTING STATISTICS

Within this chapter, we will introduce full counting statistics. We will begin the chapter with a brief explanation of why we are interested in this area of study. We will then go on to develop one of the more effective techniques for calculating the full counting statistics which uses Keldysh Green's functions. Whilst we develop this method, we will use the example of a tunneling junction to explain the steps required to formulate this approach. We will finally review some recent work in the field.

Counting Statistics is a method that was first considered in the field of quantum optics. It is understood that photon counting is essentially a many-particle statistics problem. A photon detector counts the number n of photons that reach it within a given time by absorbing them. If this process is repeated many times, one can obtain the counting distribution $P(n)$.

The difference, when considering electrical noise measurements, is that electrons in a circuit cannot be absorbed. To obtain the distribution $P(Q)$ of the transferred charge Q, the system needs to be coupled to a measuring device. This can be achieved in several ways. The current I can be coupled to the electromagnetic field and the magnetic field generated by I can then be detected in a current meter. Alternatively, the voltage drop over a resistor can be measured by a voltage meter.

Moments of the transferred charge

$$Q = \int_0^\tau dt I(t), \tag{6.0.1}$$

where τ is the detection time, are related to the current correlations by

$$< Q^n > = < \left[\int_0^\tau dt I(t) \right]^n > \tag{6.0.2}$$

$$= \prod_{i=1}^{n} \int_0^\tau dt_1 \ldots \int_0^\tau dt_n < I(t_1) \ldots I(t_n) > . \tag{6.0.3}$$

This relation can be rewritten to provide the cumulants by Fourier transforming to the frequency domain. The frequency dependent current is given by

$$I(\omega) = \int dt e^{i\omega t} I(t) \tag{6.0.4}$$

for a long detection time τ. In the low frequency limit $\omega \to 0$, the relation takes the form

$$<< I(\omega_1) \ldots I(\omega_n) >> = \frac{2\pi}{\tau} \delta \left(\sum_{k=1}^{n} \omega_k \right) << Q^n >> \tag{6.0.5}$$

where $<< \ldots >>$ indicates the cumulants [42]. These relations indicate that the moments of $P(Q)$ can be obtained by calculating the correlations of the current fluctuations.

6.1 Electron Gas at Zero Temperature

For electron counting statistics, we need to consider a closed electric circuit. To do this, we need to consider two electron reservoirs connected by a conductor in which the electrons scatter. The simplest form this can take is a potential barrier at zero temperature. In this case, the scattering is described by a single transmission probability T, in an energy interval

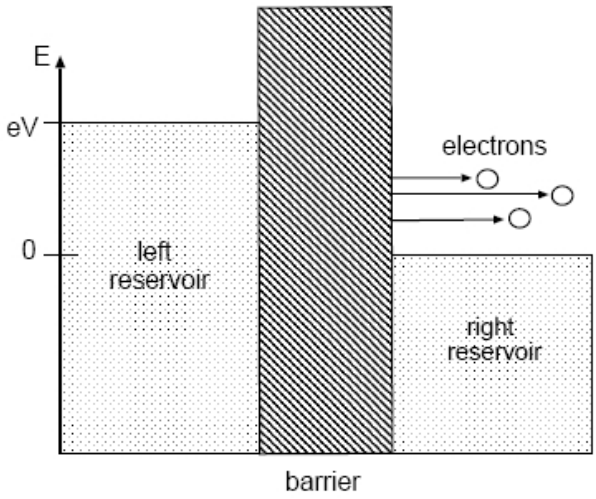

Figure 6.1: An illustration of electrons tunneling through a potential barrier between two electron reservoirs at zero temperature with voltage difference V. The current due to the electron states below the Fermi energy $E = 0$ cancel each other pairwise and therefore need not be considered. [42]

$\delta E = eV$ above the fermi level, see Fig. (6.1). In this case, particle transfer is a Bernoulli process [44]. $N = \frac{\tau eV}{h}$ particles try to pass the barrier independently in a time τ and each succeeds with a probability T. The number of transmitted particles n for a given number of trials N has binomial statistics and the distribution is given by

$$P(n) = \begin{pmatrix} N \\ n \end{pmatrix} T^n (1-T)^{N-n} \tag{6.1.1}$$

where

$$\begin{pmatrix} N \\ n \end{pmatrix} = \frac{N!}{(N-n)!n!}. \tag{6.1.2}$$

The cumulant generating function is given by

$$\ln \chi(\lambda) = N \ln \left[1 + T(e^{-i\lambda} - 1)\right]. \tag{6.1.3}$$

The first few cumulants are

$$<< n^2 >>= NT(1-T) \tag{6.1.4}$$

$$<< n^3 >>= NT(1-T)(1-2T). \tag{6.1.5}$$

If the transmission probability is $T \ll 1$, the counting statistics will have a Poisson distribution,

$$P(n) = \frac{< n >^n}{n!} e^{-<n>}, \tag{6.1.6}$$

where $< n >= NT = \frac{\tau eVT}{h}$ is the mean number of transferred particles.

The Poisson distribution has the cumulant generating function

$$\ln \chi(\lambda) =< n > \left(e^{-i\lambda} - 1\right) \tag{6.1.7}$$

where all the cummulants are equal to $< n >$.

6.2 Microscopic Theory and the Keldysh Approach

To consider more complicated systems, we need to consider a more rigorous method of obtaining the counting statistics. The first attempt to do this in the literature [43] was a straight forward calculation of the expectation values $< [\int_0^\tau dt I(t)]^m >$. However, for $m \geq 2$ this leads to unphysical results. The full counting statistics obtained in this manner for non-interacting electrons suggests that the charge carriers have a fraction of the electron charge. It is clear when we look at the higher moments that the expression is incorrect. We obtain $\langle Q_1 \dots Q_2 \rangle = \int dt_1 \dots dt_n \langle I(t_1) \dots I(t_n) \rangle$ which doesn't contain any time ordering. As the current operators $I(t)$ do not commute at different times, this expression is clearly not well defined.

The problem was first solved by L.S. Levitov and G.B. Lesovik [44]. For a full review, see Ref. [12] [45]. An alternative method can be found in Ref. [46] which produces the same expression for the full counting statistics. As we have already mentioned that in a realistic noise measurement, for a example a point contact, the current fluctuations are not detected directly. Instead, the measurement is performed on the electromagnetic fluctuations and induced by the current fluctuations. The conversion from fluctuations due to fermions to fluctuations due to bosons is important, as it allows amplification without compromising the noise statistics. The measurement scheme that they develop is to consider a spin 1/2 placed near an electron system and magnetically coupled to the current. As the electrons move through the system they will cause the spin to precess and, by measuring the angle of precession, the number of transmitted charges can be calculated.

It has been shown by Yu. Nazarov and M. Kindermann [49] that this approach can be generalised to arbitary quantum variables. In particular, the details of the measuring device is not relevant as long as the device is passive. This means that the back action of the measuring device on the system is negligible. The method which we chose to follow is the refined version of the original approach [48] which involves a slight modification of the usual Keldysh approach. A good introduction to this approach can be found in [50].

6.2.1 Transformation of the Hamiltonian

This approach can be developed quite generally. However, we will use the example of the tunnelling junction to outline the method. We will characterise the tunnelling junction by the tunnelling amplitude, γ, which we will assume is energy-independent. As we showed earlier, the transport statistics for this system will be binomial [44]. Despite this, we will develop a method which can be easily generalised to apply to interacting systems [51]. The

Hamiltonian for the junction is

$$H = H_1 + H_2 + H_T + H_I, \tag{6.2.1}$$

where H_1 and H_2 describe the leads 1 and 2 respectively. We will model the leads as non-interacting with an applied voltage $eV = \mu_1 - \mu_2$. The tunnelling term is

$$H_T = \gamma \left[\psi_1^\dagger(0)\psi_2(0) + \psi_2^\dagger(0)\psi_1(0) \right], \tag{6.2.2}$$

which describes the tunnelling barrier at $x = 0$. The particle number on the left hand side of the barrier is $N_1 = \int dx \psi_1^\dagger \psi_1$. The current from left to right is given by

$$I = -\dot{N}_1 = i\gamma \left[\psi_1^\dagger(0)\psi_2(0) - \psi_2^\dagger(0)\psi_1(0) \right]. \tag{6.2.3}$$

The remaining term in the Hamiltonian H_I has been included because, in order to calculate the full counting statistics the device for measuring, the current has to be be included in the Hamiltonian. Irrespective of the measuring device, this term takes the form $H_I = \lambda(t)I/2$ where $\lambda(t)$ is the time dependent coupling between the counting device and the system. It is possible to remove this term from the Hamiltonian. The result of doing so is that the counting fields will multiply the operators. The unitary transformation which is suggested is

$$U = e^{-i\lambda(t)N_L/2}. \tag{6.2.4}$$

This is explicitly time dependent[50].

We will now briefly explain how to perform a time dependent unitary transformation.

6.2.2 Unitary transformation

The transformation of an arbitrary operator ψ is given by

$$\psi' = U\psi U^\dagger, \tag{6.2.5}$$

where we are denoting transformed operators with primes. We would like to transform all the operators in this manner. However, we need to be careful about how we transform the Hamiltonian. If we take the naive solution for the new Hamiltonian H', the Heisenberg equation,

$$\frac{dO}{dt} = \dot{O} = i\left[H, O\right], \tag{6.2.6}$$

will no longer be valid for the transformed quantities. Instead, for the transformed operator O', we have the Heisenberg equation

$$\dot{O}' = \dot{U}OU^\dagger + U\dot{O}U^\dagger + UO\dot{U}^\dagger, \tag{6.2.7}$$

which with the identity $UU^\dagger = 1$, we can write as

$$\dot{O}' = \dot{U}U^\dagger O' + O'U\dot{U}^\dagger + i\left[H', O'\right]. \tag{6.2.8}$$

Using the identities $\dot{U}U^\dagger = U^\dagger\dot{U}$ and $U\dot{U}^\dagger = -U^\dagger\dot{U}$, we obtain

$$\dot{O}' = \left[U^\dagger\dot{U}, O'\right] + i\left[H', O'\right]. \tag{6.2.9}$$

Therefore, for the Heisenberg equation to hold for the transformed operators we require the Hamiltonian,

$$H^* = UHU^\dagger - iU^\dagger\dot{U}. \tag{6.2.10}$$

Therefore, whenever we use time dependent unitary transformations, we acquire an additional term in our Hamiltonian. It is this additional term which we will use to remove the H_I term from the Hamiltonian.

6.2.3 Applying the Unitary Transformation

We would now like to apply the transformation Eq. (6.2.4) to the Hamiltonian Eq. (6.2.1)[50]. To do so, we will use the general transformation rule,

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]] + \ldots, \tag{6.2.11}$$

for the operator ψ_1 which obeys the anti-commutation rule $\{\psi_1(x), \psi_1^\dagger(y)\} = \delta(x-y)$. This leads to

$$\begin{aligned} U\psi_1(x)U^\dagger &= \psi_1(x) - i\frac{\lambda(t)}{2}\int dy \left[\psi_1^\dagger(y)\psi_1(y), \psi_1(x)\right] + \ldots \\ &= \left[1 + i\frac{\lambda(t)}{2} + \ldots\right]\psi_1(x) \\ &= e^{i\frac{\lambda(t)}{2}}\psi_1(x). \end{aligned} \tag{6.2.12}$$

Before we go on to transform the Hamiltonian, it is useful to "integrate by parts" the extra term in the Hamiltonian. By this, we mean we make the following change [50]

$$H_I = \frac{\lambda(t)I}{2} = -\frac{\lambda(t)\dot{N}_1}{2} = \frac{\dot{\lambda}(t)N_1}{2}. \tag{6.2.13}$$

It is not obvious from the first glance that this change is allowed but if the system were written using the path integral formalism, the action would be written as a integral over t, which we can integrate by parts to obtain the expression on the right above. This requires that the boundary terms vanish which means that $\lambda(-\infty) = \lambda(\infty) = 0$. As we require $\lambda \neq 0$, $\lambda(t)$ has to have explicit time dependence. We assume that this time dependence is

given by

$$\lambda(t) = \begin{cases} 0, & \text{for } t < 0 \\ \lambda, & \text{for } t < \tau \\ 0, & \text{for } t > \tau, \end{cases} \tag{6.2.14}$$

where τ is measuring time. If we now apply the transformation to the Hamiltonian, it is clear that H_I, H_1 and H_2 will all commute with U. Therefore, the only affected part of the Hamiltonian will be the tunnelling term H_T. The transformed Hamiltonian, H_λ, is given by

$$H_\lambda = H_1 + H_2 + \gamma \left[e^{-i\frac{\lambda(t)}{2}} \psi_1^\dagger(0)\psi_2(0) + e^{i\frac{\lambda(t)}{2}} \psi_2^\dagger(0)\psi_1(0) \right] + H_I - iU^\dagger \dot{U}. \tag{6.2.15}$$

Using the definition of the transformation U, we note that $iU^\dagger \dot{U} = \dot{\lambda}(t)N_1/2$, therefore we can cancel the last two terms of the Hamiltonian as planned. We are now left with the transformed Hamiltonian

$$H_\lambda = H_1 + H_2 + T^\lambda, \tag{6.2.16}$$

where the interaction term has become

$$T^\lambda = \gamma \left[e^{-i\frac{\lambda(t)}{2}} \psi_1^\dagger(0)\psi_2(0) + e^{i\frac{\lambda(t)}{2}} \psi_2^\dagger(0)\psi_1(0) \right]. \tag{6.2.17}$$

Within this section, we have introduced a unitary transformation. This demonstrates that the only effect of including the measuring device in the system is a time dependent transformation of the field operators. We have introduced the function $\lambda(t)$ which is usually referred to in the literature as the counting field, as it can be interpreted as counting the charges. This agrees with the time dependence that we chose for the counting field, as we switch the field on for the measurement time τ and then switch the field back off.

6.3 The Cumulant Generating Function

Within this section, we will derive an expression for the cumulant generating function. By definition Eq. (1.4.3), the generating function is given by

$$\chi(\lambda) = \langle e^{i\lambda Q(\tau)} \rangle, \tag{6.3.1}$$

where τ is the measuring time and $Q(\tau)$ is the number of charges transferred in this time. We can write $Q(\tau)$ in terms of the particle number operator for the tunnelling juction, $Q(t) = N_1(0) - N_1(t) = N_0 - N_1(t)$, where $N_1(t)$ is the number of electrons in lead 1 at time t. We can assume that $N_0 = 0$, as we are only concerned with the difference in electron number. Using this, we can rewrite Eq. (6.3.1) in the Heisenberg representation as

$$\chi(\lambda) = \langle e^{iH\tau} e^{-i\lambda N_1} e^{iH\tau} \rangle, \tag{6.3.2}$$

where H is the original Hamiltonian. Using the fact that $e^{-i\lambda N_1/2} e^{i\lambda N_1/2} = 1$ this can be expressed as

$$\chi(\lambda) = \langle e^{-i\lambda N_1/2} e^{i\lambda N_1/2} e^{iH\tau} e^{-i\lambda N_1/2} e^{-i\lambda N_1/2} e^{iH\tau} e^{i\lambda N_1/2} e^{-i\lambda N_1/2} \rangle. \tag{6.3.3}$$

If we now use the unitary transformation, we can rewrite the expression as

$$\chi(\lambda) = \langle e^{-i\lambda N_1/2} e^{iH_{-\lambda}\tau} e^{iH_\lambda \tau} e^{-i\lambda N_1/2} \rangle, \tag{6.3.4}$$

where H_λ is the transformed Hamiltonian given by Eq. (6.2.16). If we now assume that at time $t = 0$, we are in an eigenstate of N_1, $N_1|0\rangle = N_0|0 >= 0$, we can write the generating function as

$$\chi(\lambda) = \langle e^{iH_{-\lambda}\tau} e^{iH_\lambda \tau} \rangle. \tag{6.3.5}$$

If we now define a time dependent Hamiltonian such that it has different λ's on the different branches of the Keldysh contour, in particular if we define $\lambda(t)$ on the Keldysh contour as

$$\lambda(t) = \begin{cases} \lambda & \text{for } 0 < t < \tau \text{ and } t \text{ on the upper half contour, } C_1 \\ -\lambda & \text{for } 0 < t < \tau \text{ and } t \text{ on the lower half contour, } C_2 \\ 0 & \text{otherwise,} \end{cases} \tag{6.3.6}$$

we can write the generating function as

$$\chi(\lambda) = \langle T_{c_k} e^{-i \int_{c_k} dt H_{\lambda(t)}} \rangle. \tag{6.3.7}$$

If we change to the interaction representation, the generating function becomes

$$\chi(\lambda) = \langle T_{c_k} e^{-i \int_{c_k} dt T^{\lambda(t)}} \rangle, \tag{6.3.8}$$

where we have taken the unperturbed part of the Hamiltonian to be $H_0 = H_1 + H_2$.[52]

6.4 Green's functions

In this section, we will apply the Green function formalism to our example of the tunnelling junction. We will begin by first generalising the generating function Eq. (6.3.8). We will do this by letting λ_1 and λ_2 be arbitrary functions, respectively on the upper branch, C_1 and the lower branch C_2 of the Keldysh contour, such that

$$\chi(\lambda_1, \lambda_2) = \langle T_{c_k} e^{-i \int_{c_k} dt T^{\lambda(t)}} \rangle. \tag{6.4.1}$$

We can recover the original expression by taking $\chi(\lambda, -\lambda)$ at the end. It has been argued for a similar problem [53] that, for large measuring times τ, the above expression can be

described in terms of an effective action, $U(t, \lambda_1, \lambda_2)$,

$$\chi(\lambda_1, \lambda_2) = e^{-i \int_0^\tau dt U(t, \lambda_1, \lambda_2)}. \tag{6.4.2}$$

If we now take the derivative of this expression with respect to λ_1, we obtain

$$\frac{\partial}{\partial \lambda_1} e^{-i \int_0^\tau dt U(t, \lambda_1, \lambda_2)} = -i \chi(\lambda_1, \lambda_2) \int_0^\tau dt \frac{\partial U(t, \lambda_1, \lambda_2)}{\partial \lambda_1}. \tag{6.4.3}$$

We will use this result shortly but first let us consider the differential of Eq. (6.4.1) with to respect λ_1,

$$\frac{\partial}{\partial \lambda_1} \left\langle T_{c_k} e^{-i \int_{c_k} dt T^{\lambda(t)}} \right\rangle = \left\langle T_{c_k} \frac{\partial}{\partial \lambda_1} e^{-i \int_{c_k} dt T^{\lambda(t)}} \right\rangle, \tag{6.4.4}$$

where we have used the Hellmann-Feynman theorem which states that the order of the differentiating and averaging maybe switched in this case. In brief the proof of this theorem proceeds through the application of the chain rule for the time derivative of the wavefunction [55] This expression can be simplified further to give

$$\begin{aligned} \frac{\partial}{\partial \lambda_1} \left\langle T_{c_k} e^{-i \int_{c_k} dt T^{\lambda(t)}} \right\rangle &= \left\langle T_{c_k} \left(-i \int_0^\tau dt \frac{\partial T^{\lambda_1}}{\partial \lambda_1} \right) e^{-i \int_{c_k} dt' T^{\lambda(t')}} \right\rangle \\ &= -i \int_0^\tau dt \left\langle T_{c_k} \frac{\partial T^{\lambda_1}}{\partial \lambda_1} e^{-i \int_{c_k} dt' T^{\lambda(t')}} \right\rangle. \end{aligned} \tag{6.4.5}$$

If we combine this expression with that of Eq. (6.4.3), we obtain

$$\frac{\partial U(t, \lambda_1, \lambda_2)}{\partial \lambda_1} = \frac{1}{\chi(\lambda_1, \lambda_2)} \left\langle T_{c_k} \frac{\partial T^{\lambda_1}}{\partial \lambda_1} e^{-i \int_{c_k} dt' T^{\lambda(t')}} \right\rangle, \tag{6.4.6}$$

which holds for the integrands as the expectation value will be time independent. If we now define the "λ expectation value" as

$$\frac{1}{\chi(\lambda_1,\lambda_2)}\left\langle T_{c_k}\frac{\partial T^{\lambda_1}}{\partial\lambda_1}e^{-i\int_{c_k}dt' T^{\lambda(t')}}\right\rangle = \left\langle T_{c_k}\frac{\partial T^{\lambda_1}}{\partial\lambda_1}\right\rangle_\lambda, \tag{6.4.7}$$

it takes the same form as the interaction representation expectation value.

We now have all the building blocks required to calculate the full counting statistics. If we calculate the expectation value of the derivative of our modified interaction part of the Hamiltonian T^λ, we obtain the derivative of the effective action U using Eq. (6.4.6). If we then integrate this expression, we obtain U which leads to the generating function via Eq. (6.4.2) and the substitution $\lambda_1 = -\lambda_2 = \lambda$.

The important task which still remains is the calculation of

$$\left\langle T_{c_k}\frac{\partial T^{\lambda_1}}{\partial\lambda_1}\right\rangle_\lambda, \tag{6.4.8}$$

we can solve this for our example by using Eq. (6.2.17) to obtain

$$\left\langle T_{c_k}\frac{\partial T^{\lambda_1}}{\partial\lambda_1}\right\rangle_\lambda = -i\frac{\gamma}{2}e^{-i\frac{\lambda_1}{2}}\left\langle T_{c_k}\psi_1^\dagger(t)\psi_2(t)\right\rangle_\lambda + i\frac{\gamma}{2}e^{i\frac{\lambda_1}{2}}\left\langle T_{c_k}\psi_2^\dagger(t)\psi_1(t)\right\rangle_\lambda. \tag{6.4.9}$$

These expressions are correlation functions and, as t is on the upper branch of the contour, correspond to time ordered Green's functions,

$$\left\langle T_{c_k}\psi_1^\dagger(t)\psi_2(t)\right\rangle_\lambda = -iG_{21}^T(t,t) \text{ and } \left\langle T_{c_k}\psi_2^\dagger(t)\psi_1(t)\right\rangle_\lambda = -iG_{12}^T(t,t). \tag{6.4.10}$$

Generally, we can write

$$G_{\alpha\beta}(t,t') = -i\left\langle T_{c_k}\psi_\alpha(t)\psi_\beta^\dagger(t')\right\rangle. \tag{6.4.11}$$

Using this representation has the obvious advantage that we are able to use the well established Green's function formulism to calculate the expressions. In the next section, we will introduce several Dyson's equations that will allow us to calculate the Green's functions.

6.5 Dyson's Equations

To calculate the Green's functions, we shall derive several Dyson's equations It is clear to see from the expansion of the Green's functions that they are given by

$$G_{12}(t,t') = \gamma \int_{c_k} dt'' e^{-i\lambda(t'')/2} g_1(t,t'') G_{22}(t'',t'), \tag{6.5.1}$$

and

$$G_{21}(t,t') = \gamma \int_{c_k} dt'' e^{i\lambda(t'')/2} g_2(t,t'') G_{11}(t'',t'), \tag{6.5.2}$$

where g_α is the unperturbed Green's function of lead α which does not contain any tunnelling. However, we still have the Green's functions G_{11} and G_{22} to solve. If we split the Keldysh contour into the upper and lower parts and Fourier transform, we can write the Dyson's equations for the required components

$$\begin{aligned} G_{12}^T(\varepsilon) &= \gamma \left(e^{-i\frac{\lambda_1}{2}} g_1^T(\varepsilon) G_{22}^T(\varepsilon) - e^{-i\frac{\lambda_2}{2}} g_1^>(\varepsilon) G_{22}^<(\varepsilon) \right), & (6.5.3) \\ G_{21}^T(\varepsilon) &= \gamma \left(e^{i\frac{\lambda_1}{2}} g_2^T(\varepsilon) G_{11}^T(\varepsilon) - e^{i\frac{\lambda_2}{2}} g_2^>(\varepsilon) G_{11}^<(\varepsilon) \right). & (6.5.4) \end{aligned}$$

To calculate the $G_{\alpha\alpha}$ Green's functions we use another pair of Dyson's equations

$$\begin{aligned} G_{11}(t,t') &= g_1(t,t') + \gamma^2 \int_{c_k}\int_{c_k} dt''dt''' e^{-i\frac{\lambda(t'')}{2}} e^{i\frac{\lambda(t''')}{2}} G_{11}(t,t'') g_2(t'',t''') g_1(t''',t'), & (6.5.5) \\ G_{22}(t,t') &= g_2(t,t') + \gamma^2 \int_{c_k}\int_{c_k} dt''dt''' e^{i\frac{\lambda(t'')}{2}} e^{-i\frac{\lambda(t''')}{2}} G_{22}(t,t'') g_1(t'',t''') g_2(t''',t'). \end{aligned}$$

(6.5.6)

If we now split the Keldysh contour into upper and lower parts and Fourier transform, we can write the above expressions as

$$G^{ij}_{\alpha\alpha}(\varepsilon) = g^{ij}_{\alpha}(\varepsilon) + \gamma^2 \sum_{k,l=\{1,2\}} a_{kl} e^{-im\lambda_k} e^{im\lambda_l} G^{ik}_{\alpha\alpha}(\varepsilon) g^{kl}_{\bar{\alpha}}(\varepsilon) g^{lj}_{\alpha}(\varepsilon) \tag{6.5.7}$$

where we have introduced a lot of new notation. Firstly, the ij in the Green's function $G^{ij}_{\alpha\alpha}(t,t')$ denotes the contour $1,2$ which the t,t' reside respectively. We have also introduced the matrix,

$$a_k l = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \tag{6.5.8}$$

We have also introduced the vector $m = (1,-1)$ for $\alpha = (1,2)$. Finally, we have introduced $\bar{\alpha} = (2,1)$ for $\alpha = (1,2)$. The reason for writing the Green's function in this compressed form is that it allows us to introduce the Dyson equation in terms of the following matrices;

$$\mathbf{G}_{\alpha\alpha}(\varepsilon) = \begin{pmatrix} G^{11}_{\alpha\alpha}(\varepsilon) & G^{12}_{\alpha\alpha}(\varepsilon) \\ G^{21}_{\alpha\alpha}(\varepsilon) & G^{22}_{\alpha\alpha}(\varepsilon) \end{pmatrix}, \tag{6.5.9}$$

and

$$\mathbf{\Sigma}_{\alpha}(\varepsilon) = \gamma^2 \begin{pmatrix} g^{11}_{\alpha}(\varepsilon) & -e^{im\lambda} g^{12}_{\alpha}(\varepsilon) \\ -e^{-im\lambda} g^{21}_{\alpha}(\varepsilon) & g^{22}_{\alpha}(\varepsilon) \end{pmatrix}, \tag{6.5.10}$$

where $\lambda = \frac{1}{2}[\lambda_1 - \lambda_2]$. If we also use the matrix $\mathbf{g}_{\alpha}(\varepsilon)$ for the unperturbed Green's functions, we can write the Dyson's equation as

$$\mathbf{G}_{\alpha\alpha}(\varepsilon) = \mathbf{g}_{\alpha}(\varepsilon) + \mathbf{G}_{\alpha\alpha}(\varepsilon)\mathbf{\Sigma}_{\bar{\alpha}}(\varepsilon)\mathbf{g}_{\alpha}(\varepsilon). \tag{6.5.11}$$

Rearranging this equation, we obtain

$$\mathbf{G}_{\alpha\alpha}(\varepsilon) = \mathbf{g}_{\alpha}(\varepsilon)\left(\mathbf{1} - \mathbf{\Sigma}_{\bar{\alpha}}(\varepsilon)\mathbf{g}_{\alpha}(\varepsilon)\right)^{-1}. \tag{6.5.12}$$

We can solve this equation using the standard definitions for the unperturbed Green's functions, to obtain

$$\mathbf{G}_{\alpha\alpha}(\varepsilon) = \frac{1}{\det\mathbf{A}}\mathbf{B}_{\alpha}, \tag{6.5.13}$$

where the determinant and matrix are given by

$$\det\mathbf{A} = (1+\Gamma)^2 + 4\Gamma\left[f_1(\varepsilon)\left(1-f_2(\varepsilon)\right)\left(e^{i\lambda}-1\right) + f_2(\varepsilon)\left(1-f_1(\varepsilon)\right)\left(e^{-i\lambda}-1\right)\right], \tag{6.5.14}$$

and

$$\mathbf{B}_{\alpha} = -i\pi\nu_0\begin{pmatrix} 2f_{\alpha}(\varepsilon) - 1 + \Gamma\left(2f_{\bar{\alpha}}(\varepsilon)-1\right) & 2f_{\alpha}(\varepsilon) + 2\Gamma e^{-im\lambda}f_{\bar{\alpha}}(\varepsilon) \\ 2\left(f_{\alpha}(\varepsilon)-1\right) + 2\Gamma e^{im\lambda}\left(f_{\bar{\alpha}}(\varepsilon)-1\right) & 2f_{\alpha}(\varepsilon) - 1 + \Gamma\left(2f_{\bar{\alpha}}(\varepsilon)-1\right)\end{pmatrix}. \tag{6.5.15}$$

The density states of both the leads is assumed to be constant, ν_0, and we have used a slightly different form for $\Gamma = (\pi\nu_0\gamma)^2$. We now have everything we need to calculate the cumulant generating function for the tunneling junction. If we begin from Eq. (6.4.6) and use Eq. (6.4.9), we can write

$$\begin{aligned}\frac{\partial U(\lambda_1,\lambda_2)}{\partial\lambda_1} &= \left\langle T_{c_k}\frac{\partial T^{\lambda_1}}{\partial\lambda_1}\right\rangle_{\lambda} \\ &= -\frac{\gamma^2}{2}\int\frac{d\varepsilon}{2\pi}\left(g_2^{11}(\varepsilon)G_{11}^{11}(\varepsilon) - g_1^{11}(\varepsilon)G_{22}^{11}(\varepsilon) + e^{i\lambda}g_1^{12}(\varepsilon)G_{22}^{21}(\varepsilon) - e^{-i\lambda}g_2^{12}(\varepsilon)G_{11}^{21}(\varepsilon)\right).\end{aligned} \tag{6.5.16}$$

Inserting the solution into this expression we obtain

$$\frac{\partial U(\lambda_1,\lambda_2)}{\partial \lambda_1} = \frac{i}{2}\int \frac{d\varepsilon}{2\pi}\frac{\partial_\lambda \det\mathbf{A}(\varepsilon,\lambda)}{\det\mathbf{A}(\varepsilon,\lambda)} \tag{6.5.17}$$

$$= \frac{i}{2}\int \frac{d\varepsilon}{2\pi}\partial_\lambda \ln\det\mathbf{A}(\varepsilon,\lambda). \tag{6.5.18}$$

We can now integrate this expression, to obtain $U(\lambda_1,\lambda_2)$, change the λ variables and then use Eq. (6.4.2) to obtain the cumulant generating function

$$\ln\chi(\lambda) = \tau\int\frac{d\varepsilon}{2\pi}\ln\left[1+T\left\{f_1(\varepsilon)\left(1-f_2(\varepsilon)\right)\left(e^{i\lambda}-1\right)+f_2(\varepsilon)\left(1-f_1(\varepsilon)\right)\left(e^{-i\lambda}-1\right)\right\}\right] \tag{6.5.19}$$

where $T = 4\Gamma/\left(1+\Gamma\right)^2$ and can be thought of as a transmission coefficient. This obeys binomial statistics as we expected. This approach is extremely effective and can be applied to much more complicated systems.

In the next section, we will briefly explain how to modify the procedure outlined above to apply to a non-interacting quantum dot. We will then review some of the recent work that has been carried out which uses this method and other approaches to calculate the full counting statistics of coulomb blockade systems and quantum dots.

6.6 Application to the Quantum dot

The method derived in the previous section can easily be applied to the non-interacting dot. The main difference with the example of the tunnel junction is that we have to introduce an additional counting field. This is because we would require a counting field for each contact. If we start from the Hamiltonian

$$H = H_1 + H_2 + H_d + H_T + H_I, \tag{6.6.1}$$

where H_d is the Hamiltonian of the dot

$$H_d = \sum_n \varepsilon_n d_n^\dagger d_n, \tag{6.6.2}$$

H_α is the Hamiltonian of the lead α,

$$H_\alpha = \sum_k \varepsilon_{\alpha k}, \tag{6.6.3}$$

and the tunnelling term is given by

$$H_T = \sum_{\alpha,k,n} \left[\gamma_\alpha d_n^\dagger \psi_{\alpha k} + \text{h.c}\right]. \tag{6.6.4}$$

The extra term in the Hamiltonian due to the measuring device has two terms in this system, as we are counting the charges at two points, and is given by $H_I = \sum_\alpha \lambda_\alpha(t) I/2$ where $\lambda_\alpha(t)$ is the time dependent coupling to lead α. We can perform both the unitary transformations simultaneously to remove these terms from the Hamiltonian. This is because the number operators N_1 and N_2 commute. This procedure is identical to the tunneling junction case and results in the new tunneling term being given by

$$T^\lambda = \sum_{\alpha,k,n} \left[\gamma_\alpha e^{i\lambda_\alpha/2} d_n^\dagger \psi_{\alpha k} + \text{h.c}\right], \tag{6.6.5}$$

where the transformed Hamiltonian is

$$H_\lambda = H_1 + H_2 + H_d + T^\lambda. \tag{6.6.6}$$

Due to the fact that we have two measuring devices we have to define the current in a slightly different form as

$$\langle I_{12}\rangle = \frac{1}{2}\left[\langle I_1\rangle - \langle I_2\rangle\right] \tag{6.6.7}$$

where 1/2 has been included to avoid double counting. The cumulant generating function can be generalised slightly, as shown below:

$$\ln \chi(\lambda_1, \lambda_2) = \left\langle e^{-i\sum_\alpha \lambda_\alpha N_\alpha} \right\rangle . \tag{6.6.8}$$

Once again, we can use the effective potential U to derive an expression for the cumulant generating function and the problem is reduced to solving

$$\left\langle T_{c_k} \frac{\partial T^\lambda}{\partial \lambda_\alpha^1} \right\rangle_\lambda = \sum_{nk} \left\{ \frac{\gamma_\alpha}{2} e^{-i\frac{\lambda_\alpha^1}{2}} G_{\alpha k n}(t,t) - \frac{\gamma_\alpha^\dagger}{2} e^{i\frac{\lambda_\alpha^1}{2}} G_{n\alpha k}(t,t) \right\}, \tag{6.6.9}$$

where we have had to change notation slightly. The superscript on the λ now indicates the half contour on which the λ resides and the subscript indicates the lead it is associated with. We have introduced two new Green's functions

$$G_{\alpha k n}(t,t') = i \left\langle T_{c_k} d_n^\dagger(t') \psi_{\alpha k}(t) \right\rangle_\lambda \tag{6.6.10}$$

and

$$G_{n\alpha k}(t,t') = i \left\langle T_{c_k} \psi_{\alpha k}^\dagger(t') d_n(t) \right\rangle_\lambda . \tag{6.6.11}$$

Therefore, the noninteracting quantum dot is reduced to the calculation of these Green's functions. In fact, the interacting system is given by the same calculation but with the added complication of having the Coulomb term in the Hamiltonian. We will not calculate these Green's functions at this point, as it is quite a long process but is similar to the tunnelling junction case, solving the Dyson's equations and inserting the known results. We would like to suggest as future work the possible extension of this approach to calculate the Coulomb blockaded quantum dot. We believe this will be an interesting area of study, as it would allow access to the higher order cumulants which may contain a more pronounced difference

from the non-interacting results, due to the effects of the electron-electron interactions[54].

6.7 Recent Work on Full Counting Statistics in the Coulomb Blockaded Regime

Within this final section, we will briefly review a few of the recent results in full counting statistics of Coulomb blockaded dots. For reviews of full counting statistics see [12], [47]. A lot of work has recently been done in this area by Schmidt *et al.* [56][57]. In [56], they have modeled a quantum dot using the Anderson impurity model. They use the same approach as we outlined above and used a perturbative approach to calculate the Green's functions. In this paper, however, they concentrate on a different limit to the one in which we are interested. They consider Γ to be large and to perturb U (which is the equivalent to E_c). They discover in this limit that to the second order terms emerge in the cumulant generating function which can be interpreted as co-tunneling.

In a second paper by the same authors [57], they model the dot using a Kondo model. In doing so they go to a fixed point that corresponds to $U \to \infty$. At this point, they find that the system is governed by binomial statistics. Away from the fixed point for large U, they find that the system has the same qualitative behaviour as in the case for small U. In both papers, they are interested in the effect of spin and use a perturbative approach.

Another important approach to the full counting statistics of Coulomb blockaded system is the master equation approach developed by D. A. Bagrets and Yu. V. Nazarov [58]. In this paper, they develop a method to evaluate the full counting statistics using a master equation approach. This approach can be readily applied to the Coulomb blockaded quantum dot, as shown by the large body of work which calculates the noise using this approach [18] [17] [**?**]. They discover that for two and three lead quantum dots in the Coulomb blockaded regime

the Coulomb interaction suppresses the relative probabilities of large current fluctuations compared with the non-interacting case. This approach is very successful but can only be applied to systems which can be described classically via the master equation.

6.8 Summary

In summary, in this chapter, we have introduced the topic of full counting statistics. With the aid of the example of a tunneling junction, we have shown how to derive a Green's function approach to calculating the cumulant generating function, which allows the calculation of all cumulants. We then went on to show how this approach can be applied to calculate the generating function for a quantum dot and we showed that the problem is just reduced to calculating the Green's functions Eq. (6.6.10) and Eq. (6.6.11). We have concluded the chapter by reviewing some recent work on full counting statistics of quantum dots in the Coulomb blockaded regime.

Chapter 7

CONCLUSION

7.1 Discussion of the Coulomb Blockaded Quantum Dot

In this thesis, we have considered the Coulomb blockaded quantum dot. We began by introducing the phenomena of the Coulomb blockade and how this affects the closed quantum dot. We then moved on to discuss the different causes of noise that impact on the current measurements and the regimes in which they are dominant. We introduced the three approaches to calculating the noise power spectrum in mesoscopic conductors. These are the scattering theory approach, the Green function approach and the master equation approach.

In the scattering theory approach, we consider the system to be modeled as a scattering region connected to ideal leads fed from electron reservoirs. This approach allows us to express the noise properties of the system in terms of a product of transmission probabilities and we obtain the famous expression first derived by Lesovik [24], Eq. (2.3.4). We also define the Fano factor, which is a way to express the noise as a ratio of the Poisson noise.

The master equation approach is a classical approach in which the rates of change of charge for the leads are calculated. Using these rate equations, the noise is calculated. This ap-

proach is well studied and nearly all limits of the Coulomb blockaded noise are studied in this manner. We chose however to consider a full quantum mechanical approach to study the Coulomb blockaded dot. This is because at the peak of conductance, there are two degenerate interacting levels and it is not clear if it is reasonable to treat them using a classical approach.

We have used the Green function approach to calculate the noise power spectrum to the Coulomb blockaded. We first introduced the Keldysh Green function and its contour which can be used to study systems out of equilibrium. We also introduce the functional integration representation for the Green's function and, using this representation, re-derive the density of states for the quantum dot obtained by Sedlmayr *et al.* [40]. This approach, however, is not ideal to extend to calculate the noise power spectrum, due to a large number of terms being produced, due to the Bosonic field introduced during the Hubbard-Stratonovich transformation. We therefore derive a different Green's function approach to calculate the density of states of the quantum dot. This approach involved treating the n^{th} level separately to express the Green's function for the dot as a summation over resonant level Green's functions coupled to a background charge, Eq. (4.2.13).

We extend this approach by calculating the single particle Green's function to calculate the two-particle Green's function. Using these two particle Green's functions, we obtain integral expressions for the current Eq. (5.2.10) and noise power spectrum Eq. (5.2.11). We calculate these expressions for zero bias voltage and show that we obtain the fluctuation dissipation theorem. In the shot noise regime, the regime where the noise is dominated by the contribution due to the discreteness of the charge carriers, we show that the Fano factor is given by Eq. (5.2.14), which is the same result as that which is obtained for the non-interacting double barrier problem. This result is in complete agreement with the master equation approach [18] and we can conclude that at the peak of conductance we can treat

the noise power spectrum classically. The results that we obtain are not only valid in the large voltage limit of the shot noise regime. Using our method, we are able to consider all voltages less the the charging energy, E_c. We also note that, from the integral form of the noise power spectrum, Eq. (5.2.11), the Γ dependence of the Fano factor is not constant and changes as a function of voltage.

In the final chapter of the thesis, we have discussed the full counting statistics. We derive how the full counting statistics can be calculated using a Keldysh Green's function approach. We discuss how this approach can be applied to a quantum dot and some of the recent work that has taken place in this field. We believe that extending the approach that we have used to calculate the noise power spectrum to the full counting statistics would be an interesting problem for consideration.

Appendix A

GAUSSIAN INTEGRATION

In this appendix, we will remind ourselves of some basic results of Gaussian integration. For a more complete review see Reference [38]. The starting point for Gaussian integration is the identity

$$\int_{-\infty}^{\infty} dx e^{-\frac{a}{2}x^2} = \sqrt{\frac{2\pi}{a}} \qquad \Re a > 0. \tag{A.0.1}$$

We will also require various generalisations of this expression. Firstly

$$\int_{-\infty}^{\infty} dx e^{-\frac{a}{2}x^2} x^2 = \sqrt{\frac{2\pi}{a^3}} \tag{A.0.2}$$

which is obtained by differentiating Eq. (A.0.1). We will also encounter integrals where the exponent is not purely quadratic the generalisation in this case is given by

$$\int_{-\infty}^{\infty} dx e^{-\frac{a}{2}+bx} = \sqrt{\frac{2\pi}{a}} e^{\frac{b^2}{2a}}. \tag{A.0.3}$$

We must also consider the generalisation to consider complex arguments. In this case Eq. (4.2.13) is generalised to

$$\int d(\bar{z}, z) e^{-\bar{z}wz} = \frac{\pi}{w} \qquad \Re w > 0 \tag{A.0.4}$$

where $\bar{z}$ represents the complex conjugate of z and $\int d(\bar{z}, z) \equiv \int_{-\infty}^{\infty}$ where $z = x + iy$. It is also possible to consider linear components in the exponent in the complex case. This

generalises Eq. (A.0.4) to

$$\int d(\bar{z},z)e^{-\bar{z}wz+\bar{u}z+\bar{z}v} = \frac{\pi}{w}e^{\frac{\bar{u}v}{w}} \tag{A.0.5}$$

where $\bar{u}$ and v may be independent complex numbers.

A.1 Gaussian Integration in N-Dimensions

The next situation that we would like to consider is the generalisation to N-dimensional integration. To simplify notation, we will consider the real and complex cases separately although this is not strictly necessary.

A.1.1 The Real Case

The N-dimensional generalisation of Eq. (G1) is given by

$$\int d\mathbf{v}e^{-\frac{1}{2}\mathbf{v}^T\mathbf{A}\mathbf{v}} = (2\pi)^{\frac{N}{2}}\det\mathbf{A}^{-\frac{1}{2}} \tag{A.1.1}$$

where $\mathbf{A}$ is a positive definite real symmetric N-dimensional matrix and $\mathbf{v}$ is a N-component real vector. If we add a linear term into the exponent of Eq. (A.1.3), we obtain

$$\int d\mathbf{v}e^{-\frac{1}{2}\mathbf{v}^T\mathbf{A}\mathbf{v}+\mathbf{j}^T.\mathbf{v}} = (2\pi)^{\frac{N}{2}}\det\mathbf{A}^{-\frac{1}{2}}e^{\frac{1}{2}\mathbf{j}^T\mathbf{A}^{-1}\mathbf{j}} \tag{A.1.2}$$

where $\mathbf{j}$ is an arbitrary N-component vector. Eq. (A.1.2) is very useful as we can use it to generate other identities. If we apply the differential operation $\partial^2_{j_m j_n}|_{\mathbf{j}=0}$ to both sides of Eq. (A.1.2), we obtain the identity

$$\int d\mathbf{v}e^{-\frac{1}{2}\mathbf{v}^T\mathbf{A}\mathbf{v}}v_m v_n = (2\pi)^{\frac{N}{2}}\det\mathbf{A}^{-\frac{1}{2}}A^{-1}_{mn} \tag{A.1.3}$$

which can be compactly written as

$$< v_m v_n >= A^{-1}_{mn} \tag{A.1.4}$$

where we have used the shorthand notation

$$< \ldots >= (2\pi)^{-\frac{N}{2}} \det \mathbf{A}^{\frac{1}{2}} \int d\mathbf{v} e^{-\frac{1}{2}\mathbf{v}^T \mathbf{A}\mathbf{v}} (\ldots) . \tag{A.1.5}$$

It is possible to iterate this differential operation, for example if we differentiate four times we obtain

$$< v_m v_n v_q v_p >= A^{-1}_{mn} A^{-1}_{qp} + A^{-1}_{mq} A^{-1}_{mp} + A^{-1}_{mp} A^{-1}_{nq}. \tag{A.1.6}$$

This generalises to expectation values of arbitrary order

$$< v_{i_1}, v_{i_2} \ldots v_{i_{2n}} >= \sum_{\substack{\text{all possible} \\ \text{pairings of} \{x_1, \ldots x_{2n}\}}} A^{-1}_{i_{k_1} i_{k_2}} \ldots A^{-1}_{i_{k_{2n-1}} i_{k_{2n}}} \tag{A.1.7}$$

A.1.2 The Complex Case

The results for the N-dimensional Gaussian integrals above are easily extended to the complex case. The complex version of Eq. (A.1.3) is give by

$$\int d(\mathbf{v}^\dagger, \mathbf{v}) e^{-\mathbf{v}^\dagger \mathbf{A}\mathbf{v}} = \pi^N \det \mathbf{A}^{-1} \tag{A.1.8}$$

where $\mathbf{v}$ is a complex N- component vector and $d(\mathbf{v}^\dagger, \mathbf{v}) = \prod_{i=1}^{N} d\Re v_i d\Im v_i$ and $\mathbf{A}$ is a complex matrix. When linear contributions are included in the exponent, Eq. (A.1.8) becomes

$$\int d(\mathbf{v}^\dagger, \mathbf{v}) e^{-\mathbf{v}^\dagger \mathbf{A}\mathbf{v} + \mathbf{w}^\dagger . \mathbf{v} + \mathbf{v}^\dagger . \mathbf{w}'} = \pi^N \det \mathbf{A}^{-1} e^{\mathbf{w}^\dagger \mathbf{A}^{-1} \mathbf{w}'}. \tag{A.1.9}$$

where $\mathbf{w}$ and $\mathbf{w}'$ can be independent complex vectors. If we differentiate the integral using the operation $\partial^2_{w_m,w'_n}|_{\mathbf{w}=\mathbf{w}'=0}$, we obtain the expression

$$< \bar{v}_m v_n >= A^{-1}_{nm} \tag{A.1.10}$$

where

$$< \ldots >= (\pi)^{-N} \det \mathbf{A} \int d(\mathbf{v}^\dagger, \mathbf{v}) e^{-\mathbf{v}^\dagger \mathbf{A} \mathbf{v}} (\ldots) . \tag{A.1.11}$$

Once again, it is possible to iterate this operation to obtain

$$< \bar{v}_{i_1} \bar{v}_{i_2} \ldots \bar{v}_{i_n} v_{j_1} v_{j_2} \ldots v_{j_n} >= \sum_P A^{-1}_{j_1 P_1} \ldots A^{-1}_{j_n P_n} \tag{A.1.12}$$

where $\sum_P$ represents the summation over all the permutations of N integers. This is the mathematical identity underlying Wicks theorem.

Appendix B

THE HUBBARD-STRATONOVICH TRANSFORMATION

The Hubbard-Stratonovich Transformation can be used to transform quartic terms of the action in functional integration into quadratic terms. The expense of doing so, however, is the addition of a new field which requires integrating. Using the transformation, we would like to cancel terms in the action of the form,

$$e^{-\frac{i}{2}\mathrm{Tr}[\bar{\psi}\bar{\psi}V\psi\psi]}. \tag{B.0.1}$$

To do so, the following identity is suggested:

$$1 = \int \frac{d\phi}{\mathcal{N}} e^{-\frac{i}{2}\mathrm{Tr}[\phi V^{-1}\phi]}. \tag{B.0.2}$$

If ψ is a fermionic field, then we require ϕ to be bosoninc and obey periodic boundary conditions on the relevant time contour. If we insert the identity into Eq. (B.0.1) and introduce the shift the variables in the bosonic field $\phi \to \phi - \bar{\psi}\psi V$, we obtain

$$\begin{aligned} e^{-\frac{i}{2}\mathrm{Tr}[\bar{\psi}\bar{\psi}V\psi\psi]} &= \int \frac{d\phi}{\mathcal{N}} e^{-\frac{i}{2}\mathrm{Tr}\left[\left(\phi-\bar{\psi}\psi V\right)V^{-1}\left(\phi-\bar{\psi}\psi V\right)\right]} e^{-\frac{i}{2}\mathrm{Tr}\left[\bar{\psi}\bar{\psi}V\psi\psi\right]} \\ &= \int \frac{d\phi}{\mathcal{N}} e^{-\frac{i}{2}\mathrm{Tr}\left[\phi V^{-1}\phi\right]+i\mathrm{Tr}\left[\bar{\psi}\phi\psi\right]}. \end{aligned} \tag{B.0.3}$$

List of references

[1] Y. Alhassid, Rev. Mod. Phys. **72**, 895 (2000)

[2] I. Kurland, I. Aleiner, B. Altshuler, Phys. Rev. B **62**, 14866 (2000)

[3] I. L. Aleiner, P. W. Brouwer, L. I. Glazman, Phys. Rep. **358**, 309 (2002)

[4] L. Kouwenhoven, C. Marcus, Physics World **11**, 35 (1998)

[5] M.A. Kastner, Physics Today **46** 1 (1993)

[6] L.I. Glazman, R.I. Shekhter, J. Phys.: Condens. Matter **1** 5811 (1989)

[7] J. Folk, S.Patel, S. Godjin, A. Huibers, S. Cronenwett and C. Marcus, Phys. Rev. Lett. **76**, 1699 (1996)

[8] C. Beenakker, C. Schönenberger, Physics Today **56**, No. 5, 37 (2003)

[9] W. Schottky, Ann. Phys. (Leipzig) **57**, 541 (1918)

[10] M.J.M. de Jong, C.W.J. Beenakker, Physica A **230** 219 (1996)

[11] H. B Callen and T. W. Welton, Phys. Rev. **83**, 34 (1951)

[12] L.S. Levitov, *Quantum Noise*, edited by Yu. V. Nazarov and Ya. M. Blanter, NATO Science Series II Vol. 97, Kluwer (2003)

[13] H. B Callen and T. W. Welton, Phys. Rev. **83**, 34 (1951)

[14] E. V. Sukhorukov, G. Burkhard, D. Loss Phys. Rev. B **63**, 125315 (2001)

[15] A. N Korotkov, Phys. Rev. B **49**, 10 318 (1994)

[16] U. Hanke *et al.* Phys. Rev. B **48**, 17 209 (1993)

[17] S. Hershfield *et al.* Phys. Rev. B **47**, 1967 (1993)

[18] S. Hershfield *et al.* Phys. Rev. B **46**, 7061 (1992)

[19] R. Landauer, IBM J. Res. Dev. **1**, 233 (1957)

[20] M. Büttiker, Phys. Rev. Lett. **65**, 2901 (1990)

[21] M. Büttiker J. Math. Phys. **37**, (10) 4793 (1996)

[22] Imry Y., *Introduction to mesoscopic physics*, Oxford University Press, (1997)

[23] M. Büttiker, Phys. Rev. B. **46**, 12485 (1992)

[24] G. B. Lesovik, JETP Lett. **49**, 592 (1989)

[25] A.D Stone, P.A. Lee Phys. Rev. Lett. **54**, 1196 (1985)

[26] Ya. M. Blanter, M. Büttiker Phys. Rep. **336**, 1-166 (2000)

[27] G. D. Mahan, *Many-Particle Physics* (Plenum Press, New York, 1990).

[28] J. Rammer *Quantum Field Theory of Non-equilibrium States* (Cambridge University Press, New York, 2007)

[29] L. V. Keldysh, Sov. Phys. JETP **20**, 1018 (1965)

[30] P. Martin, J. Schwinger, Phys Rev. **115**, 1342 (1959)

[31] J. Schwinger, J. Math. Phsys. **2**, 407 (1961)

[32] A. G. Hall, J. Phys. A, Vol. **8**,No. 2, 214-224 (1975)

[33] A. Kamenev, A. Levchencko, Adv. Phys. **58**, 197 (2009)

[34] Y. A. Kukharenkov, S. G. Tikhodeev, Soviet Phys.JETP, Vol. **56**, no. 4, pp. 831-838, 1982.

[35] J. Rammer, H. Smith, Rev. Mod. Phys. **58**, 323 (1986)

[36] A. Larkin, Y. Ovchinnikov, Sov. Phys. JETP **41**, 960 (1975)

[37] D. C. Langreth, *Linear and Nonlinear Electron Transport in Solids*, edited by J. T. Devreese and E. van Doren, NATO Advanced Study Institute Series B, Vol. 17, New York/ London: Plenum (1976)

[38] A. Altland, B. Simons, *Condensed Matter Field Theory*, Cambridge University Press (2006)

[39] H. J. W. Haug and A.-P. Jauho, *Quantum Kinetics in Transport and Optics of Semiconductors* Springer-Verlag, Berlin, (2008).

[40] N. Sedlmayr, I. V. Yurkevich and I. V. Lerner, Europhys. Lett. **76**, 109 (2006)

[41] Y. Meir, N. S. Wingreen, Phys. Rev. Lett. **68**, 2512 (1992)

[42] M. Kindermann, Ph. D. thesis, Leiden University (2003)

[43] L.S. Levitov, G. B. Lesovik, JETP Lett. **55**, 555 (1992)

[44] L.S. Levitov, G. B. Lesovik, JETP Lett. **58**, 230 (1993)

[45] L. s. Levitov, H. Lee, G. B. Lesovik, J. Math. Phys. **37** 4845 (1996)

[46] M. Kindermann, Yu. V. Nazarov, *Quantum Noise*, edited by Yu. V. Nazarov and Ya. M. Blanter, NATO Science Series II Vol. 97, Kluwer (2003)

[47] W. Belzig, Yu.V. Nazarov, Phys. Rev. Lett. **87**, 197006 (2001)

[48] Levitov L., Lee H., Lesovik G., J. Math. Phys. **37**, 4845 (1996)

[49] Yu. Nazarov, M. Kindermann, Eur. Phys. J. B. **35**, 413 (2003)

[50] T. Schmidt, PhD Thesis (Albert-Ludwigs-Universität, Freiburg, 2007).

[51] Gogolin A.O., Komnik A., Phys. Rev. B **73**, 195301 (2006)

[52] Levitov L.S., Reznikov M., Phys. Rev. B **70**, 115305 (2004)

[53] Hamann D.R., Phys. Rev. Lett. **26**, 1030 (1971)

[54] L. S. Levitov, M. Reznikov Phys. Rev. B **70** 115305 (2004)

[55] R. P. Feynman, Phys. Rev. **56** 340 (1939)

[56] T.L. Schmidt , A. Komnik , A.O. Gogolin, Phys. Rev. B 76 (2007)

[57] T. L. Schmidt, A. O. Gogolin and A. Komnik, Phys Rev B **75**, 235105 (2007).

[58] D. A. Bagrets, Yu. V. Nazarov, Phys. Rev. B **67**, 085316 (2003)