SUPPLEMENTS TO MAXIMAL SUBALGEBRAS OF LIE ALGEBRAS

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Abstract

For a Lie algebra L and a subalgebra M of L we say that a subalgebra U of L is a *supplement* to M in L if $L = M + U$. We investigate those Lie algebras all of whose maximal subalgebras have abelian supplements, those that have nilpotent supplements, those that have nil supplements, and those that have supplements with the property that their derived algebra is inside the maximal subalgebra being supplemented. For the algebras over an algebraically closed field of characteristic zero in the last three of these classes we find complete descriptions; for those in the first class partial results are obtained.
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1 Introduction

Let L be a Lie algebra and let M be a subalgebra of L. We say that a subalgebra U of L is a *supplement* to M in L if $L = M + U$. In similar fashion to [1] we introduce the following conditions:

- (MO) Every maximal subalgebra of L admits a supplement which is one-dimensional; that is every maximal subalgebra of L has codimension one in L.
- (MA) Every maximal subalgebra of L admits an abelian supplement.
- (MD) Every maximal subalgebra M of L admits a supplement whose derived algebra is inside M.
- (MN) Every maximal subalgebra of L admits a nilpotent supplement.
- (MU) Every maximal subalgebra of L admits a supplement every element of which acts nilpotently on L.

We will denote by $\mathcal{MO}$ (respectively, $\mathcal{MA}, \mathcal{MD}, \mathcal{MN}$, and $\mathcal{MU}$) the class of Lie algebras satisfying condition (MO) (respectively, (MA), (MD), (MN), and (MU)). Our objective is to study these classes of algebras. Corresponding classes of groups were studied by Baumeister ([1]), who showed, in particular, that any group in $\mathcal{MA}$ is solvable, and that a group belongs to $\mathcal{MD}$ if and only if it is solvable. Similar problems concerning factorisations of Lie algebras as sums of subalgebras of a certain type have been studied extensively (see, for example, [13], [14], [2], [10], [18], [20], [21], [22] and the references contained therein.)

In section two we collect together a few preliminary results. First the description of the algebras in $\mathcal{MO}$ as derived in [19] is given. Then some straightforward inclusions between these classes of algebras are noted. Next it is shown that all solvable algebras belong to $\mathcal{MD}$, that a completely solvable algebra is in $\mathcal{MU}$ if and only if it is nilpotent, and that all supersolvable and all metabelian algebras are in $\mathcal{MA}$. A relationship is given between decompositions of a nonassociative algebra and corresponding decompositions of the algebra over a finite field extension. The final result here asserts that if $\mathcal{H}$ is a saturated homomorph of Lie algebras, then so is $\mathcal{MH}$.

The third section is concerned with the simple algebras in these classes. It is shown that if the underlying field is algebraically closed of characteristic zero, then A_1 is the only such algebra. We also identify when a maximal parabolic subalgebra has an abelian supplement.

The last section contains the main classification results, describing explicitly the algebras in each of the classes defined above.

Throughout L will denote a finite-dimensional Lie algebra over a field F. If A and B are subalgebras of L for which $L = A + B$ and $A \cap B = 0$ we will write $L = A \oplus B$. The ideals $L^{(k)}$ of the derived series are defined inductively by $L^{(0)} = L$, $L^{(k+1)} = [L^{(k)}, L^{(k)}]$ for $k \geq 0$; we also write L^2 for $L^{(1)}$. We say that L is *completely solvable* if L^2 is nilpotent; L is *metabelian* if L^2 is abelian; and L is *supersolvable* if it has a series $0 = L_0 \subset L_1 \subset \ldots \subset L_n = L$ of ideals of L with $\dim L_i = i$. If A is a subalgebra of L, the *centralizer* of A in L is $C_L(A) = \{x \in L : [x, A] = 0\}$.

2 Preliminary results

The Frattini ideal of L, $\phi(L)$, is the largest ideal of L contained in all maximal subalgebras of L. The algebras in $\mathcal{MO}$ were classified in [19] as follows.

Theorem 2.1 *([19, Theorem 1]) Let L be Lie algebra over any field F. Then the following are equivalent:*

(i) $L \in \mathcal{MO}$; and

(ii) $L/\phi(L) = R \oplus S$ where the radical R is supersolvable and ϕ-free, and either $S = 0$ or S is three-dimensional simple with a basis $\{u_{-1}, u_0, u_1\}$ and multiplication $[u_{-1}, u_0] = u_{-1}$, $[u_{-1}, u_1] = u_0$, $[u_0, u_1] = u_1$.

There are some easy relationships between the classes of algebras we have introduced.

Lemma 2.2 *(i) $\mathcal{MO} \subseteq \mathcal{MA} \subseteq \mathcal{MN}$, $\mathcal{MA} \subseteq \mathcal{MD}$ and $\mathcal{MU} \subseteq \mathcal{MN}$.*

(ii) If L is solvable, or F has at least $\dim L - 1$ elements, then $\mathcal{MD} \subseteq \mathcal{MN}$.

(iii) If L is completely solvable then $\mathcal{MN} = \mathcal{MD}$.

Proof. (i) These inclusions are straightforward.
(ii) Suppose that $L \in \mathcal{MD}$, and let M be any maximal subalgebra of L. Then there is a subalgebra B of L such that $L = M + B$ and $B^2 \subseteq M$. Let C be a Cartan subalgebra of B (which, under the stated assumptions, exists, by [4, Theorem 3 and Corollary 1.2]) and let $B = C \oplus B_1$ be the

Fitting decomposition of B relative to $\operatorname{ad} C$. Clearly $B_1 \subseteq B^2 \subseteq M$, whence $L = M + C$ and $L \in \mathcal{MN}$. We have established that $\mathcal{MD} \subseteq \mathcal{MN}$.
(iii) Let L be completely solvable and let M be a maximal subalgebra of L with a nilpotent supplement U. Then $U^2 = \phi(U)$, by [17, Section 5], and $\phi(U) \subseteq \phi(L)$, by [16, Theorem 2], so $U^2 \subseteq \phi(L) \subseteq M$. It follows that $\mathcal{MN} \subseteq \mathcal{MD}$. The reverse inclusion comes from (ii) above. □

We define the *abelian socle* of L, $Asoc\, L$, to be the sum of the minimal abelian ideals of L. Next we consider the solvable algebras in these classes.

Proposition 2.3 *Let L be a Lie algebra over an arbitrary field F.*

(i) If L is solvable then $L \in \mathcal{MD}$.

(ii) If L is completely solvable, then $L \in \mathcal{MU}$ if and only if L is nilpotent.

(iii) If L is supersolvable then $L \in \mathcal{MA}$.

(iv) If L is metabelian (so, in particular, if L is completely solvable and ϕ-free) then $L \in \mathcal{MA}$.

Proof. (i) Let L be solvable and let M be a maximal subalgebra of L. Then there is a $k \geq 0$ such that $L^{(k)} \not\subseteq M$ but $L^{(k+1)} \subseteq M$. Clearly $L = M + L^{(k)}$ and so $L \in \mathcal{MD}$.
(ii) Let L be completely solvable. Suppose that $L \in \mathcal{MU}$, but that L is not nilpotent, and let M be a maximal subalgebra of L with $N \subseteq M$, where N is the nilradical of L. Then there is a nil subalgebra U of L such that $L = M + U$. But now $N + U$ is nilpotent, and is an ideal of L, since $L^2 \subseteq N$, so $U \subseteq N \subseteq M$, a contradiction. The converse is clear.
(iii) If L is supersolvable then $L/\phi(L)$ is supersolvable, and so $L \in \mathcal{MO}$, by Theorem 2.1. Moreover, $\mathcal{MO} \subseteq \mathcal{MA}$, by Lemma 2.2(i).
(iv) Let L be metabelian and suppose that M is a maximal subalgebra of L. If $L^2 \subseteq M$ then M has codimension one in L and so has an abelian supplement. If $L^2 \not\subseteq M$, then L^2 is an abelian supplement to M. That completely solvable ϕ-free algebras lie in this class follows from [17, Theorems 7.3 and 7.4]. □

Not every completely solvable Lie algebra L belongs to $\mathcal{MA}$, as the next example shows.

Example **2.1** *Let L be the four-dimensional Lie algebra over the real field with basis e_1, e_2, e_3, e_4 and products $[e_1, e_2] = e_2 - e_3$, $[e_1, e_3] = e_2 + e_3$, $[e_1, e_4] = 2e_4$, $[e_2, e_3] = e_4$, other products being zero.*

Then L is completely solvable, but not ϕ-free, as $\phi(L) = \mathbb{R}e_4$. Also, $M = \mathbb{R}e_1 + \mathbb{R}e_4$ is a maximal subalgebra of L that has no abelian supplement.

We have the following relationship between decompositions of a nonassociative algebra and corresponding decompositions of the algebra over a finite field extension.

Lemma 2.4 *Let A be a nonassociative algebra over a field F with subalgebras A_1 and A_2, and let K be a finite field extension of F, whose degree is not a multiple of the characteristic of F, and so that F is the fixed field of the group $Gal(K/F)$ of F-automorphisms of K. Then $A = A_1 + A_2$ if and only if $\bar{A} = \bar{A_1} + \bar{A_2}$, where $\bar{B} = B \otimes_F K$ for any subalgebra B of A.*

Proof. Clearly $A = A_1 + A_2$ implies that $\bar{A} = \bar{A_1} + \bar{A_2}$. Conversely, let $\bar{A} = \bar{A_1} + \bar{A_2}$, and let $x \in A$. Then $x \otimes 1 = \bar{x_1} + \bar{x_2}$ where $\bar{x_1} \in \bar{A_1}$, $\bar{x_2} \in \bar{A_2}$. Let $\{k_1, \ldots, k_n\}$ be a basis for K over F, $Gal(K/F) = \{\theta_1, \ldots, \theta_n\}$. For each $\theta \in Gal(K/F)$ let U_θ be the semilinear transformation of $\bar{A}$ defined by $U_\theta(\sum a_i \otimes k_i) = \sum a_i \otimes \theta(k_i)$ as in [11, page 295]. Then $U_{\theta_i}(x \otimes 1) = x \otimes 1$, and so $x \otimes 1 = U_{\theta_i}(\bar{x_1}) + U_{\theta_i}(\bar{x_2})$ for all $1 \leq i \leq n$. Hence

$$x \otimes 1 = \frac{1}{n}(U_{\theta_1}(\bar{x_1}) + \ldots + U_{\theta_n}(\bar{x_1})) + \frac{1}{n}(U_{\theta_1}(\bar{x_2}) + \ldots + U_{\theta_n}(\bar{x_2})).$$

Moreover, $\frac{1}{n}(U_{\theta_1}(\bar{x_i}) + \ldots + U_{\theta_n}(\bar{x_i}))$ is fixed by all of the elements of $\{U_\theta : \theta \in Gal(K/F)\}$, and so belongs to A_i for each $i = 1, 2$, whence the result. □

Notice, however, that if L is as in Example 2.1 above then $L \notin \mathcal{MA}$, whereas, considered as an algebra over $\mathbb{C}$, we have $L \in \mathcal{MA}$.

A class $\mathcal{H}$ of finite-dimensional Lie algebras is called a *homomorph* if it contains, along with an algebra L, all epimorphic images of L; it is *saturated* if $L/\phi(L) \in \mathcal{H}$ implies that $L \in \mathcal{H}$. Then we have that if $\mathcal{H}$ is a saturated homomorph, so is $\mathcal{MH}$ (where $\mathcal{MH}$ is the class of Lie algebras all of whose maximal subalgebras have a supplement $U \in \mathcal{H}$.) First we need a lemma.

Lemma 2.5 *Let L be a Lie algebra over any field F and let M be a maximal subalgebra of L with a supplement U. Then M has a supplement W with $\phi(L) \cap W \subseteq \phi(W)$.*

Proof. Choose W to be a subalgebra of U that is minimal with respect to $U = \phi(L) \cap U + W$. Then $L = M + U = M + \phi(L) \cap U + W = M + W$ and $\phi(L) \cap W = (\phi(L) \cap U) \cap W \subseteq \phi(W)$, by [17, Lemma 7.1]. □

Proposition 2.6 *Let $\mathcal{H}$ be a saturated homomorph of Lie algebras. Then $L \in \mathcal{MH}$ if and only if $L/\phi(L) \in \mathcal{MH}$; that is $\mathcal{MH}$ is also a saturated homomorph.*

Proof. Suppose first that $L \in \mathcal{MH}$ and let $M/\phi(L)$ be maximal subalgebra of $L/\phi(L)$. Then M is a maximal subalgebra of L and so there is a subalgebra $U \in \mathcal{H}$ such that $L = M + U$. But now

$$\frac{L}{\phi(L)} = \frac{M}{\phi(L)} + \frac{U + \phi(L)}{\phi(L)} \text{ and } \frac{U + \phi(L)}{\phi(L)} \cong \frac{U}{U \cap \phi(L)} \in \mathcal{H},$$

whence $L/\phi(L) \in \mathcal{MH}$.

So suppose now that $L/\phi(L) \in \mathcal{MH}$ and let M be a maximal subalgebra of L. Then there is a subalgebra $U/\phi(L) \in \mathcal{H}$ such that $L/\phi(L) = M/\phi(L) + U/\phi(L)$, and so $L = M + U$. Thus there is a subalgebra W of U with $U = \phi(L) + W$, $L = M + W$ and $\phi(L) \cap W \subseteq \phi(W)$, by Lemma 2.5. Moreover,

$$\frac{W}{\phi(L) \cap W} \cong \frac{\phi(L) + W}{\phi(L)} = \frac{U}{\phi(L)} \in \mathcal{H}, \text{ so } \frac{W}{\phi(W)} \cong \frac{W/(\phi(L) \cap W)}{\phi(W)/(\phi(L) \cap W)} \in \mathcal{H}.$$

It follows that $W \in \mathcal{H}$ and therefore $L \in \mathcal{MH}$.

It is easy to see that $\mathcal{MH}$ is a homomorph. □

3 Simple algebras

Our first objective in this section is to establish the following result.

Theorem 3.1 *The only simple Lie algebra over an algebraically closed field of characteristic zero belonging to $\mathcal{MA}$, or to $\mathcal{MU}$, is A_1.*

First let us recall some facts about maximal subalgebras of simple Lie algebras L over an algebraically closed fields F of characteristic zero. They fall into three types:

(I) reducible maximal subalgebras, which are described in [9, Theorems 1.1, 1.2, page 252];

(II) irreducible non-simple maximal subalgebras, which are described in [9, Theorems 1.3, 1.4, page 253]; and

(III) irreducible simple maximal subalgebras, which are described in [9, Theorem 1.5, page 252].

A subalgebra B of a semisimple Lie algebra L is called *regular* in L if we can choose a basis for B in such a way that any vector of this basis is either a root vector of L corresponding to some Cartan subalgebra C of L, or otherwise belongs to C; B is an *R-subalgebra* of L if it is contained in a regular subalgebra of L, and is an *S-subalgebra* otherwise (see [8, page 158]). We say that B is *parabolic* in L if it contains a Borel subalgebra of L; it is *reductive* in L if the representation $x \mapsto \mathrm{ad}_L\, x$ of B is semisimple.

Then a further way of describing the maximal subalgebras of L is that they are either

(a) parabolic, all of which are regular and of type (I); or

(b) reductive, which further subdivide into:

  (i) regular reductive subalgebras, which are of type (I), and are semisimple of maximal rank; and

  (ii) S-subalgebras, which are semisimple and are mostly of type (II) or type (III) (entirely so in the case of A_n, B_n and C_n).

Proof of Theorem 3.1. For each class of simple Lie algebras L we shall show that there is a maximal subalgebra M such that for any abelian subalgebra A (respectively, nil subalgebra U) of maximal dimension in L, $\dim(M + A) < \dim L$ (respectively, $\dim(M + U) < \dim L$). This is the content of the Table 1 below, whose entries we will explain next.

For each class of algebra we list in column 2 a possible choice of maximal subalgebra M to meet our claim. In some cases two subalgebras are given: the top one is sufficient to rule out the possibility of an abelian supplement, but not of a nil supplement. Of course the lower one of the two would suffice on its own, but the top one is a more straightforward example and so is listed for interest.

Many maximal subalgebras can be found from Dynkin's trick of removing a node from the extended Dynkin diagram: this gives regular reductive subalgebras. The top entry for the algebras of types B, C or D can be found in this way (and remembering that $B_1 = A_1$ and $D_2 = A_1 \oplus A_1$); they can also be found in [8, Table 12, page 150; page 232]. The lower entries for C_n and B_2 follow from the fact that all of the three-dimensional S-subalgebras of B_n and C_n are maximal, except for $A_1^{28} \subset G_2 \subset B_3$ (whereas

simple algebra L	maximal subalgebra M	$\dim L$	$\dim M$	$\operatorname{rank} L$	$\dim A$ $\dim U$
A_2	A_1	8	3	2	2 3
A_{2n} $(n \geq 2)$	B_n	$4n^2 + 4n$	$2n^2 + n$	$2n$	$n^2 + n$ $2n^2 + n$
A_{2n+1} $(n \geq 1)$	D_{n+1}	$4n^2 + 8n + 3$	$2n^2 + 3n + 1$	$2n + 1$	$n^2 + 2n + 1$ $2n^2 + 3n + 1$
B_2	$A_1 \oplus A_1$ A_1^{10}	10	6 3	2	3 4
B_3	$A_1 \oplus A_1 \oplus A_1$	21	9	3	5 9
B_{2n} $(n \geq 2)$	$B_n \oplus D_n$	$8n^2 + 2n$	$4n^2$	$2n$	$2n^2 - n + 1$ $4n^2$
B_{2n+1} $(n \geq 2)$	$B_n \oplus D_{n+1}$	$8n^2 + 10n + 3$	$4n^2 + 4n + 1$	$2n + 1$	$2n^2 + n + 1$ $4n^2 + 4n + 1$
C_{2n} $(n \geq 2)$	$C_n \oplus C_n$ A_1	$8n^2 + 2n$	$4n^2 + 2n$ 3	$2n$	$2n^2 + n$ $4n^2$
C_{2n+1} $(n \geq 1)$	$C_n \oplus C_{n+1}$ A_1	$8n^2 + 10n + 3$	$4n^2 + 6n + 3$ 3	$2n + 1$	$2n^2 + 3n + 1$ $4n^2 + 4n + 1$
D_{2n} $(n \geq 2)$	$D_n \oplus D_n$	$8n^2 - 2n$	$4n^2 - 2n$	$2n$	$2n^2 - n$ $4n^2 - 2n$
D_{2n+1} $(n \geq 2)$	$D_n \oplus D_{n+1}$	$8n^2 + 6n + 3$	$4n^2 + 2n + 1$	$2n + 1$	$2n^2 + n$ $4n^2 + 2n + 1$
E_6	A_2^9	78	8	6	16 36
E_7	A_1^{231}, A_1^{399}	133	3	7	27 63
E_8	A_1^{520}, A_1^{760}, A_1^{1240}	248	3	8	36 120
F_4	A_1^{156}	52	3	4	9 24
G_2	A_1^{28}	14	3	2	3 6

Table 1: Maximal Subalgebras

all three-dimensional S-subalgebras of A_n and D_n are non-maximal except for $A_1^4 \subset A_2$.) The superfixes indicate the index of the embedding as defined in [8].

The three-dimensional representation of A_1 gives an embedding of A_1 in A_2 under which it is a maximal S-subalgebra of that algebra. There are natural embeddings of B_n in $A_{2n}(n \geq 2)$ and D_{n+1} in $A_{2n+1}(n \geq 1)$ (also in [9, Table 5, page 366]) under which these are maximal S-subalgebras of those algebras. Finally the maximal S-subalgebras listed for each of the exceptional simple Lie algebras are taken from [8, Table 39, page 233].

In the final column of the table the upper number, α (respectively, lower number γ), is the maximal possible dimension of an abelian (respectively, nil subalgebra). The values for α were determined by Malcev in [12], or are given in [6, Table 1]. The value for γ is computed as $\frac{1}{2}(\dim L - \operatorname{rank} L)$ (see [7]).

The above result can be extended to cover $\mathcal{MN}$ as follows.

Corollary 3.2 *The only simple Lie algebra over an algebraically closed field of characteristic zero belonging to $\mathcal{MN}$ is A_1.*

Proof. Suppose that $L \ncong A_1$ and let M be a maximal subalgebra of L from Table 1 (the lower entry if two are given) and suppose that $L = M + N$ where N is a maximal nilpotent subalgebra of L. Then $N = C_L(N)$ and so N is algebraic. It follows that $N = T \oplus U$ where T is a toral subalgebra and U is the nilradical of N. Since N is nilpotent we must have that $[T, U] = 0$.

Now it can be seen from Table 1 that N must have the same dimension as a Borel subalgebra of L. In fact, in view of [7], it must be equal to a Borel subalgebra B of L in which T is a Cartan subalgebra of L and U is the nilradical of B. Since $[T, U] = 0$ this is impossible. □

All of the subalgebras in Table 1, of course, are reductive. The maximal parabolic subalgebras generally are too large to yield to dimension arguments of the above kind. However, we can determine which of them have abelian supplements.

Let L be a simple Lie algebra over an algebraically closed field of characteristic zero, H a Cartan subalgebra of L and Γ, Γ^+, Σ, respectively, the systems of roots, positive roots and simple roots of L with respect to H. Then $L = H + \Sigma_{\alpha \in \Gamma} V_\alpha$, where V_α is spanned by a unique element e_α. Let $\Sigma_1 \subseteq \Sigma$ be a non-empty subsystem of Σ, and put $\Delta_1 = \{\gamma \in \Gamma : \gamma = \Sigma_{\alpha \in \Sigma \setminus \Sigma_1} m_\alpha \alpha\}$ and $\Delta_2^+ = (\Gamma \setminus \Delta_1) \cap \Gamma^+$.

Then every parabolic subalgebra of L is conjugate to a standard parabolic subalgebra of the form $P = H + \Sigma_{\alpha\in\Delta_1} V_\alpha + \Sigma_{\alpha\in\Delta_2^+} V_\alpha = R \oplus U$, where $R = H + \Sigma_{\alpha\in\Delta_1} V_\alpha$ is its reductive summand and the ideal $N = \Sigma_{\alpha\in\Delta_2^+} V_\alpha$ is its nilradical. It is clear that every maximal parabolic subalgebra has a nil supplement, namely the nilradical, $N^o = \Sigma_{-\alpha\in\Delta_2^+} V_\alpha$, of the opposite parabolic subalgebra of L. When they have an abelian supplement is given by the next result, where we use the Bourbaki numbering of roots (see [5]).

Proposition 3.3 *Let L be a simple Lie algebra over an algebraically closed field F of characteristic zero, and let P be a standard maximal parabolic subalgebra of L. Then P has an abelian supplement in L if and only if $L \cong A_n$ and $\Sigma_1 = \{\alpha_i\}$, B_n and $\Sigma_1 = \{\alpha_1\}, C_n$ and $\Sigma_1 = \{\alpha_n\}$, D_n and $\Sigma_1 = \{\alpha_1\}, \{\alpha_{n-1}\}$ or $\{\alpha_n\}$, E_6 and $\Sigma_1 = \{\alpha_1\}, \{\alpha_6\}$, or E_7 and $\Sigma_1 = \{\alpha_7\}$.*

Proof. We have that $L = P \oplus N^o$ where N_o is the nilradical of the opposite parabolic subalgebra to P. Suppose that N^o is not abelian, but that $L = P + A$, where A is abelian. We can embed A in a Borel subalgebra B of L. Then B is conjugate to a Borel subalgebra B^o containing N^o as its nilradical. Let A^o be the image of A under the conjugating automorphism. Let $\alpha(B^o)$, respectively $\beta(B^o)$, be the maximal possible dimension of an abelian subalgebra, respectively ideal, of B^o. Then $\alpha(B^o) = \beta(B^o)$, by [6, Proposition 2.5], so $\dim A^o \leq \alpha(B^o) = \beta(B^o) < \dim N^o$ if N^o is not abelian. It follows that P has an abelian supplement precisely when N^o is abelian. But this occurs exactly in the situations given in the result (see [15], or [3, Table 1, page 24]). □

4 Main results

First we need to extend the results of the previous section to semisimple Lie algebras.

Lemma 4.1 *Let L be a semisimple Lie algebra over an algebraically closed field F of characteristic zero. Then $L \in \mathcal{MN} = \mathcal{MA} = \mathcal{MD}$ if and only if $L \cong A_1$.*

Proof. Suppose that $L \in \mathcal{MN}$ is semisimple. Clearly every simple summand of L is isomorphic to A_1, by Corollary 3.2. Suppose that $L = S \oplus \bar{S}$, where $S \cong A_1$, $\bar{S}$ is an isomorphic copy of S with $[S, \bar{S}] = 0$, and denote the image of $s \in S$ in $\bar{S}$ by $\bar{s}$. Let $D = \{s + \bar{s} : s \in S\}$, which is easily seen to be a maximal subalgebra of L, and suppose that $L = D + N$, where N is a nilpotent subalgebra of L. Then $\dim N \geq 3$.

Clearly $L \neq S + N$, so $S \cap N \neq \{0\}$. Similarly, $\bar{S} \cap N \neq \{0\}$. Let $s \in S \cap N$, $\bar{x} \in \bar{S} \cap N$ and let $n = u + \bar{v} \in N$, where $u \in S$, $\bar{v} \in \bar{S}$. Then $[s, u] = [s, n] \in S \cap N$, so $[s, n] = \lambda s$ for some $\lambda \in F$, since S has no two-dimensional nilpotent subalgebras. As N is nilpotent, $\lambda = 0$. But now $Fs + Fu$ is an abelian subalgebra of S, and so $u = \mu s$ for some $\mu \in F$. In similar manner $\bar{v} = \nu \bar{x}$ for some $\nu \in F$. But this means that $\dim N \leq 2$, a contradiction. It follows that L is simple and $L \cong A_1$. The same conclusion follows if $L \in \mathcal{MA}$ or $L \in \mathcal{MD}$, by Lemma 2.2.

The converse is easily checked. □

Theorem 4.2 *Let L be a Lie algebra over an algebraically closed field of characteristic zero with solvable radical R. Then $L \in \mathcal{MD} = \mathcal{MN}$ if and only if L is solvable or $L/R \cong A_1$.*

Proof. Suppose first that $L \in \mathcal{MN}$ and L is not solvable. Then $L/R \cong A_1$ by Lemma 4.1. If $L \in \mathcal{MD}$ the same conclusion follows from Lemma 2.2.

Suppose now that $L = R$ or $L/R \cong A_1$, and let M be a maximal subalgebra of L. If $R \subseteq M$ then M has a supplement whose derived algebra is inside M, by Lemma 4.1. So suppose that $R \not\subseteq M$. Then there is a $k \geq 0$ such that $R^{(k)} \not\subseteq M$ but $R^{(k+1)} \subseteq M$. Clearly $L = M + R^{(k)}$ and again M has a supplement whose derived subalgebra is inside M. It follows that $L \in \mathcal{MD}$. Lemma 2.2(ii) also implies that $L \in \mathcal{MN}$. □

Notice that Lemma 4.1 and Proposition 2.3 imply that if L is a semisimple or solvable Lie algebra over an algebraically closed field of characteristic zero then $L \in \mathcal{MA}$ if and only if $L \in \mathcal{MN}$. However, it is not the case that $\mathcal{MA} = \mathcal{MN}$, as the following example shows.

Example **4.1** *Let L be the six-dimensional Lie algebra over the complex field with basis e, f, h, x_0, x_1, x_2 and products $[e, h] = 2e$, $[f, h] = -2f$, $[e, f] = h$, $[x_0, h] = x_0$, $[x_1, h] = -x_1$, $[x_0, f] = x_1$, $[x_1, e] = -x_0$, $[x_0, x_1] = x_2$, other products being zero.*

Clearly, $L = R \oplus S$, where $R = \mathbb{C}x_0 + \mathbb{C}x_1 + \mathbb{C}x_2$ is nilpotent and $S = \mathbb{C}e + \mathbb{C}f + \mathbb{C}h \cong A_1$. Then $M = \mathbb{C}e + \mathbb{C}f + \mathbb{C}h + \mathbb{C}x_2$ is a maximal subalgebra of L that has no abelian supplement in L.

Theorem 4.3 *Let L be a Lie algebra over an algebraically closed field of characteristic zero with nilradical N. Then $L \in \mathcal{M}\mathcal{U}$ if and only if L is nilpotent or $L/N \cong A_1$.*

Proof. Let $L \in \mathcal{M}U$. Then $L \in \mathcal{M}N$, by Lemma 2.2, and so L is solvable or $L/R \cong A_1$, by Theorem 4.2. Suppose that L is solvable but not nilpotent, and let M be a maximal subalgebra of L with $N \subseteq M$. Then there is a nil subalgebra U of L such that $L = M + U$. But now $N + U$ is nilpotent, and is an ideal of L, since $L^2 \subseteq N$, so $U \subseteq N \subseteq M$, a contradiction.

So suppose now that $L/R \cong A_1$ and R is not nilpotent. Clearly $L/\phi(L) \in \mathcal{M}U$, so assume that L is ϕ-free. Then $L = Asoc\,L \oplus (C \oplus S)$, where C is abelian and acts semisimply on $Asoc\,L$, $S \cong A_1$, and $[S, C] = 0$, by [17, Theorem 7.5]. Let M be a maximal subalgebra of L with $Asoc\,L + S \subseteq M$. There is a nil subalgebra U such that $L = M + U$. Since $L^2 \subseteq M$, M is an ideal of L and there is a $u \in U$ such that $L = M + Fu$. Let $u = a + c + s$, where $a \in Asoc\,L$, $c \in C$, $s \in S$. It is easy to see that since $\text{ad}\,(a + c + s)$ acts nilpotently on S, then s must be a nil element of S. But now $\text{ad}\,(a+c+s)|_{Asoc\,L} = \text{ad}\,(c+s)|_{Asoc\,L}$ is nilpotent. As c acts semisimply on $Asoc\,L$ it follows that $c \in C_L(Asoc\,L) \subseteq Asoc\,L$. But then $u \in M$, a contradiction. Hence R is nilpotent.

Suppose conversely that L is nilpotent or $L/N \cong A_1$. If the former holds that then, clearly, $L \in \mathcal{M}U$. So suppose the latter holds and let M be a maximal subalgebra of L. If $N \not\subseteq M$ then $L = M + N$ and N is nil. If $N \subseteq M$ then $M = N + M \cap A_1$. Also $M \cap A_1$ is a maximal subalgebra of A_1 and there is a nil element of A_1, e say, such that $L = M + Fe$. But then e acts nilpotently on L. It follows that $L \in \mathcal{M}U$. □

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