INEQUALITIES FOR THE NORM AND THE NUMERICAL RADIUS OF LINEAR OPERATORS IN HILBERT SPACES

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Abstract. In this paper various inequalities between the operator norm and its numerical radius are provided. For this purpose, we employ some classical inequalities for vectors in inner product spaces due to Buzano, Goldstein-Ryff-Clarke, Dragomir-Sándor and the author.

1. Introduction

Let $(H;\langle\cdot,\cdot\rangle)$ be a complex Hilbert space. The *numerical range* of an operator T is the subset of the complex numbers $\mathbb{C}$ given by [10, p. 1]:

$$W(T)=\{\langle Tx,x\rangle,\ x\in H,\ \|x\|=1\}. \tag{1.1}$$

It is well known (see for instance [10]) that:

(i) The numerical range of an operator is convex (the Toeplitz-Hausdorff theorem);
(ii) The spectrum of an operator is contained in the closure of its numerical range;
(iii) T is self-adjoint if and only if W is real.

The *numerical radius* $w(T)$ of an operator T on H is defined by [10, p. 8]:

$$w(T)=\sup\{|\lambda|,\lambda\in W(T)\}=\sup\{|\langle Tx,x\rangle|,\|x\|=1\}. \tag{1.2}$$

It is well known that $w(\cdot)$ is a norm on the Banach algebra $B(H)$ and the following inequality holds true

$$w(T)\le\|T\|\le 2w(T),\ \text{for any } T\in B(H). \tag{1.3}$$

In the previous work [6], the following reverse inequalities have been proved:

$$(0\le)\ \|T\|-w(T)\le\frac{1}{2}\cdot\frac{r^2}{|\lambda|}, \tag{1.4}$$

provided that $\lambda\in\mathbb{C}\setminus\{0\}$, $r>0$ and

$$\|T-\lambda I\|\le r. \tag{1.5}$$

If, in addition $|\lambda|>r$ and (1.5) holds true, then

$$\left(1-\frac{r^2}{|\lambda|^2}\right)^{\frac{1}{2}}\le\frac{w(T)}{\|T\|}\ (\le 1), \tag{1.6}$$

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which provides a refinement of the general inequality

$$\frac{1}{2} \leq \frac{w(T)}{\|T\|}, \tag{1.7}$$

in the case when r and λ satisfy the assumption $r/|\lambda| \leq \sqrt{3}/2$.

With the same assumption on λ and r, i.e. $|\lambda| > r$, we also have the inequality

$$(0 \leq) \|T\|^2 - w^2(T) \leq \frac{2r^2}{|\lambda| + \sqrt{|\lambda|^2 - r^2}} w(T), \tag{1.8}$$

provided (1.5) holds true.

In the same paper, on assuming that $(T^* - \bar{\varphi} I)(\phi I - T)$ is accreative [10, p. 25] (or sufficintly, self-adjoint and nonnegative in the operator order of $B(H)$), where $\varphi, \phi \in \mathbb{C}$, $\phi \neq -\varphi, \varphi$, we have proved the following inequality as well:

$$(0 \leq) \|T\| - w(T) \leq \frac{1}{4} \cdot \frac{|\phi - \varphi|^2}{|\phi + \varphi|}. \tag{1.9}$$

If we assume more, i.e., $\operatorname{Re}(\phi\bar{\varphi}) > 0$ (which implies $\phi \neq -\varphi$), then for T as above, we also have:

$$\frac{2\sqrt{\operatorname{Re}(\phi\bar{\varphi})}}{|\phi + \varphi|} \leq \frac{w(T)}{\|T\|} (\leq 1) \tag{1.10}$$

and

$$(0 \leq) \|T\|^2 - w^2(T) \leq \left[|\phi + \varphi| - 2\sqrt{\operatorname{Re}(\phi\bar{\varphi})}\right] w(T). \tag{1.11}$$

The main aim of this paper is to establish other inequalities between the operator norm and its numerical radius. We employ, amongst others, the Buzano inequality as well as some results for vectors in inner product spaces due to Goldstein-Ryff-Clarke [9], Dragomir-Sándor [7] and Dragomir [5].

2. The Results

The following result may be stated as well:

Theorem 1. *Let $(H; \langle \cdot, \cdot \rangle)$ be a Hilbert space and $T : H \to H$ a bounded linear operator on H. Then*

$$w^2(T) \leq \frac{1}{2}\left[w\left(T^2\right) + \|T\|^2\right]. \tag{2.1}$$

The constant $\frac{1}{2}$ is best possible in (2.1).

Proof. We need the following refinement of Schwarz's inequality obtained by the author in 1985 [2, Theorem 2] (see also [7] and [5]):

$$\|a\| \|b\| \geq |\langle a, b\rangle - \langle a, e\rangle \langle e, b\rangle| + |\langle a, e\rangle \langle e, b\rangle| \geq |\langle a, b\rangle|, \tag{2.2}$$

provided a, b, e are vectors in H and $\|e\| = 1$.

Observing that

$$|\langle a, b\rangle - \langle a, e\rangle \langle e, b\rangle| \geq |\langle a, e\rangle \langle e, b\rangle| - |\langle a, b\rangle|,$$

hence by the first inequality in (2.2) we deduce

$$\frac{1}{2}(\|a\| \|b\| + |\langle a, b\rangle|) \geq |\langle a, e\rangle \langle e, b\rangle|. \tag{2.3}$$

This inequality was obtained in a different way earlier by M.L. Buzano in [1].

Now, choose in (2.3), $e = x$, $\|x\| = 1$, $a = Tx$ and $b = T^*x$ to get

$$\frac{1}{2}\left(\|Tx\|\,\|T^*x\| + \left|\left\langle T^2x, x\right\rangle\right|\right) \geq |\langle Tx, x\rangle|^2 \tag{2.4}$$

for any $x \in H$, $\|x\| = 1$.

Taking the supremum in (2.4) over $x \in H$, $\|x\| = 1$, we deduce the desired inequality (2.1).

Now, if we assume that (2.1) holds with a constant $C > 0$, *i.e.*,

$$w^2(T) \leq C\left[w\left(T^2\right) + \|T\|^2\right] \tag{2.5}$$

for any $T \in B(H)$, then if we choose T a normal operator and use the fact that for normal operators we have $w(T) = \|T\|$ and $w\left(T^2\right) = \left\|T^2\right\| = \|T\|^2$, then by (2.5) we deduce that $2C \geq 1$ which proves the sharpnes of the constant. ∎

Remark 1. *From the above result (2.1) we obviously have*

$$w(T) \leq \left\{\frac{1}{2}\left[w\left(T^2\right) + \|T\|^2\right]\right\}^{1/2} \leq \left\{\frac{1}{2}\left(\left\|T^2\right\| + \|T\|^2\right)\right\}^{1/2} \leq \|T\| \tag{2.6}$$

and

$$w(T) \leq \left\{\frac{1}{2}\left[w\left(T^2\right) + \|T\|^2\right]\right\}^{1/2} \leq \left\{\frac{1}{2}\left(w^2(T) + \|T\|^2\right)\right\}^{1/2} \leq \|T\|, \tag{2.7}$$

that provide refinements for the first inequality in (1.3).

The following result may be stated.

Theorem 2. *Let $T : H \to H$ be a bounded linear operator on the Hilbert space H and $\lambda \in \mathbb{C}\setminus\{0\}$. If $\|T\| \leq |\lambda|$, then*

$$\|T\|^{2r} + |\lambda|^{2r} \leq 2\|T\|^{r-1}|\lambda|^r w(T) + r^2|\lambda|^{2r-2}\|T - \lambda I\|^2, \tag{2.8}$$

where $r \geq 1$.

Proof. We use the following inequality for vectors in inner product spaces due to Goldstein, Ryff and Clarke [9]:

$$\|a\|^{2r} + \|b\|^{2r} - 2\|a\|^r\|b\|^r\frac{\operatorname{Re}\langle a, b\rangle}{\|a\|\,\|b\|} \leq \begin{cases} r^2\|a\|^{2r-2}\|a-b\|^2 & \text{if } r \geq 1, \\ \|b\|^{2r-2}\|a-b\|^2 & \text{if } r < 1, \end{cases} \tag{2.9}$$

provided $r \in \mathbb{R}$ and $a, b \in H$ with $\|a\| \geq \|b\|$.

Now, let $x \in H$ with $\|x\| = 1$. From the hypothesis of the theorem, we have that $\|Tx\| \leq |\lambda|\,\|x\|$ and applying (2.9) for the choices $a = \lambda x$, $\|x\| = 1$, $b = Tx$, we get

$$\|Tx\|^{2r} + |\lambda|^{2r} - 2\|Tx\|^{r-1}|\lambda|^r|\langle Tx, x\rangle|\,\mu \leq r^2|\lambda|^{2r-2}\|Tx - \lambda x\|^2 \tag{2.10}$$

for any $x \in H$, $\|x\| = 1$ and $r \geq 1$.

Taking the supremum in (2.10) over $x \in H$, $\|x\| = 1$, we deduce the desired inequality (2.8). ∎

The following result may be stated as well:

Theorem 3. *Let $T : H \to H$ be a bounded linear operator on the Hilbert space $(H, \langle\cdot,\cdot\rangle)$. Then for any $\alpha \in [0,1]$ and $t \in \mathbb{R}$ one has the inequality:*

$$\|T\|^2 \le \left[(1-\alpha)^2 + \alpha^2\right] w^2(T) + \alpha \|T - tI\|^2 + (1-\alpha)\|T - itI\|^2 . \tag{2.11}$$

Proof. We use the following inequality obtained by the author in [5]:

$$\left[\alpha \|tb - a\|^2 + (1-\alpha)\|itb - a\|^2\right]\|b\|^2 \ge \|a\|^2\|b\|^2 - \left[(1-\alpha)\operatorname{Im}\langle a,b\rangle + \alpha \operatorname{Re}\langle a,b\rangle\right]^2 (\ge 0)$$

to get:

$$\begin{aligned} \|a\|^2\|b\|^2 &\le \left[(1-\alpha)\operatorname{Im}\langle a,b\rangle + \alpha \operatorname{Re}\langle a,b\rangle\right]^2 \\ &\quad + \left[\alpha\|tb-a\|^2 + (1-\alpha)\|itb-a\|^2\right]\|b\|^2 \\ &\le \left[(1-\alpha)^2+\alpha^2\right]|\langle a,b\rangle|^2 \\ &\quad + \left[\alpha\|tb-a\|^2 + (1-\alpha)\|itb-a\|^2\right]\|b\|^2 \end{aligned} \tag{2.12}$$

for any $a, b \in H$, $\alpha \in [0,1]$ and $t \in \mathbb{R}$.

Choosing in (2.12) $a = Tx$, $b = x$, $x \in H$, $\|x\| = 1$, we get

$$\|Tx\|^2 \le \left[(1-\alpha)^2+\alpha^2\right]|\langle Tx,x\rangle|^2 + \alpha\|tx - Tx\|^2 + (1-\alpha)\|itx - Tx\|^2 . \tag{2.13}$$

Finally, taking the supremum over $x \in H$, $\|x\| = 1$ in (2.13), we deduce the desired result. ∎

The following particular cases may be of interest.

Corollary 1. *For any T a bounded linear operator on H, one has:*

$$(0 \le)\ \|T\|^2 - w^2(T) \le \begin{cases} \inf_{t\in\mathbb{R}} \|T - tI\|^2 \\ \\ \inf_{t\in\mathbb{R}} \|T - itI\|^2 \end{cases} \tag{2.14}$$

and

$$\|T\|^2 \le \frac{1}{2} w^2(T) + \frac{1}{2}\inf_{t\in\mathbb{R}}\left[\|T - tI\|^2 + \|T - itI\|^2\right]. \tag{2.15}$$

Remark 2. *The inequality (2.14) can in fact be improved taking into account that for any $a, b \in H$, $b \neq 0$, (see for instance [3]) the bound*

$$\inf_{\lambda\in\mathbb{C}} \|a - \lambda b\|^2 = \frac{\|a\|^2\|b\|^2 - |\langle a,b\rangle|^2}{\|b\|^2}$$

actually implies that

$$\|a\|^2\|b\|^2 - |\langle a,b\rangle|^2 \le \|b\|^2\|a - \lambda b\|^2 \tag{2.16}$$

for any $a, b \in H$ and $\lambda \in \mathbb{C}$.

Now if in (2.16) we choose $a = Tx$, $b = x$, $x \in H$, $\|x\| = 1$, then we obtain

$$\|Tx\|^2 - |\langle Tx,x\rangle|^2 \le \|Tx - \lambda x\|^2 \tag{2.17}$$

for any $\lambda \in \mathbb{C}$*, which, by taking the supremum over* $x \in H$*,* $\|x\| = 1$*, implies that*

$$(0 \leq) \|T\|^2 - w^2(T) \leq \inf_{\lambda \in \mathbb{C}} \|T - \lambda I\|^2 . \tag{2.18}$$

Remark 3. *If we take* $a = x$*,* $b = Tx$ *in (2.16), then we obtain*

$$\|Tx\|^2 \leq |\langle Tx, x\rangle|^2 + \|Tx\|^2 \|x - \mu Tx\|^2 \tag{2.19}$$

for any $x \in H$*,* $\|x\| = 1$ *and* $\mu \in \mathbb{C}$*. Now, if we take the supremum over* $x \in H$*,* $\|x\| = 1$ *in (2.19), then we get*

$$(0 \leq) \|T\|^2 - w^2(T) \leq \|T\|^2 \inf_{\mu \in \mathbb{C}} \|I - \mu T\|^2 . \tag{2.20}$$

From a different view point we may state:

Theorem 4. *Let* $T : H \to H$ *be a bounded linear operator on* H*. If* $p \geq 2$*, then:*

$$\|T\|^p + |\lambda|^p \leq \frac{1}{2}\left(\|T + \lambda I\|^p + \|T - \lambda I\|^p\right), \tag{2.21}$$

for any $\lambda \in \mathbb{C}$*.*

Proof. We use the following inequality obtained by Dragomir and Sándor in [7]:

$$\|a + b\|^p + \|a - b\|^p \geq 2\left(\|a\|^p + \|b\|^p\right) \text{ for any } a, b \in H \text{ and } p \geq 2. \tag{2.22}$$

Now, if we choose $a = Tx$, $b = \lambda x$, then we get

$$\|Tx + \lambda x\|^p + \|Tx - \lambda x\|^p \geq 2\left(\|Tx\|^p + |\lambda|^p\right) \tag{2.23}$$

for any $x \in H$, $\|x\| = 1$.

Taking the supremum in (2.23) over $x \in H$, $\|x\| = 1$, we get the desired result (2.21). ∎

Remark 4. *For* $p = 2$*, we have the simpler result:*

$$\|T\|^2 + |\lambda|^2 \leq \frac{1}{2}\left(\|T + \lambda I\|^2 + \|T - \lambda I\|^2\right) \tag{2.24}$$

for any $\lambda \in \mathbb{C}$*. This can easily be obtained from the parallelogram identity as well.*

Theorem 5. *Let* $T : H \to H$ *be a bounded linear operator on* H*. Then:*

$$\|T + \lambda I\|^2 \leq \|T\|^2 + 2|\lambda| w(T) + |\lambda|^2 \tag{2.25}$$

and

$$\|T + \lambda I\|^2 \leq \left\|T^*T + |\lambda|^2 I\right\| + 2|\lambda| w(T), \tag{2.26}$$

for any $\lambda \in \mathbb{C}$*.*

Proof. Observe that, for $\lambda \in \mathbb{C}$,

$$\begin{aligned} \|Tx + \lambda x\|^2 &= \|Tx\|^2 + |\lambda|^2 + 2\operatorname{Re}\left\langle \bar{\lambda} \langle Tx, x\rangle\right\rangle \\ &\leq \|Tx\|^2 + |\lambda|^2 + 2|\lambda| |\langle Tx, x\rangle| . \end{aligned} \tag{2.27}$$

Taking the supremum over $x \in H$, $\|x\| = 1$, we deduce the desired inequality (2.25).

Now, since for $\|x\| = 1$, we have

$$\|Tx\|^2 + |\lambda|^2 \|x\|^2 = \langle Tx, Tx\rangle + |\lambda|^2 \langle Ix, x\rangle = \left\langle \left(T^*T + |\lambda|^2 I\right) x, x \right\rangle,$$

then by (2.27) we obtain:

$$\|Tx + \lambda x\|^2 \leq \left\langle \left(T^*T + |\lambda|^2 I\right) x, x \right\rangle + 2|\lambda| |\langle Tx, x\rangle| . \tag{2.28}$$

Taking the supremum in (2.28) and noticing that $T^*T + |\lambda|^2 I$ is a self-adjoint operator, we get the desired inequality (2.26). ∎

Theorem 6. *Let $T : H \to H$ be a bounded linear operator on H and $\alpha, \beta \in \mathbb{C}$. Then*

$$\|\alpha T + \beta T^*\|^2 \leq \left\| |\alpha|^2 T^*T + |\beta|^2 TT^* \right\| + 2|\alpha\beta| w(T^2) \tag{2.29}$$

and

$$\left\| |\alpha|^2 T^*T + |\beta|^2 TT^* \right\| \leq \|\alpha T - \beta T^*\|^2 + 2|\alpha\beta| w(T^2). \tag{2.30}$$

Proof. Observe that for $x \in H$, $\|x\| = 1$, we have:

$$\begin{aligned} \|\alpha Tx + \beta T^*x\|^2 &= |\alpha|^2 \|Tx\|^2 + |\beta|^2 \|T^*x\|^2 + 2\operatorname{Re}\left[\alpha\bar{\beta}\langle Tx, T^*x\rangle\right] \\ &\leq |\alpha|^2 \langle Tx, Tx\rangle + |\beta|^2 \langle T^*x, T^*x\rangle + 2|\alpha\beta| \left|\langle T^2x, x\rangle\right| \\ &= \left\langle \left(|\alpha|^2 T^*T + |\beta|^2 TT^*\right)x, x\right\rangle + 2|\alpha\beta| \left|\langle T^2x, x\rangle\right|. \end{aligned}$$

Taking the supremum over $x \in H$, $\|x\| = 1$, we get

$$\|\alpha T + \beta T^*\|^2 \leq w\left(|\alpha|^2 T^*T + |\beta|^2 TT^*\right) + 2|\alpha\beta| w(T^2)$$

and since $|\alpha|^2 T^*T + |\beta|^2 TT^*$ is self-adjoint, hence $w\left(|\alpha|^2 T^*T + |\beta|^2 TT^*\right) = \left\| |\alpha|^2 T^*T + |\beta|^2 TT^* \right\|$ and the inequality (2.29) is obtained.

Similarly,

$$\begin{aligned} \|\alpha Tx - \beta T^*x\|^2 &= \left\langle \left(|\alpha|^2 T^*T + |\beta|^2 TT^*\right)x, x\right\rangle - 2\operatorname{Re}\left[\alpha\bar{\beta}\langle T^2x, x\rangle\right] \\ &\geq \left\langle \left(|\alpha|^2 T^*T + |\beta|^2 TT^*\right)x, x\right\rangle - 2|\alpha\beta| \left|\langle T^2x, x\rangle\right| \end{aligned}$$

giving that

$$\|\alpha Tx - \beta T^*x\|^2 + 2|\alpha\beta| \left|\langle T^2x, x\rangle\right| \geq \left\langle \left(|\alpha|^2 T^*T + |\beta|^2 TT^*\right)x, x\right\rangle \tag{2.31}$$

for any $x \in H$, $\|x\| = 1$. Taking the supremum over $x \in H$, $\|x\| = 1$ in (2.31) we deduce (2.30). ∎

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