A NOTE ON HARDY-TYPE INEQUALITIES

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Abstract. We use a theorem of Cartlidge and the technique of Redheffer’s “recurrent inequalities” to give some results on inequalities related to Hardy’s inequality.

1. Introduction

Suppose throughout that \( p \neq 0, 1/p + 1/q = 1 \). Let \( l^p \) be the Banach space of all complex sequences \( a = (a_n)_{n \geq 1} \) with norm

\[
||a|| := (\sum_{n=1}^{\infty} |a_n|^p)^{1/p} < \infty.
\]

The celebrated Hardy’s inequality [10], Theorem 326 asserts that for \( p > 1 \),

\[
\sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{k=1}^{n} a_k \right)^p \leq \left( \frac{p}{p-1} \right)^p \sum_{k=1}^{\infty} |a_k|^p.
\]  (1.1)

Among the many papers appeared providing new proofs, generalizations and sharpenings of [1.1], we refer the reader to the work of G. Bennett [2]-[6] for his study of factorable matrices. Hardy’s inequality can be regarded as a special case of the following inequality:

\[
\sum_{j=1}^{\infty} \left| \sum_{k=1}^{\infty} c_{j,k} a_k \right|^p \leq U \sum_{k=1}^{\infty} |a_k|^p,
\]

in which \( C = (c_{j,k}) \) and the parameter \( p \) are assumed fixed(\( p > 1 \)), and the estimate is to hold for all real sequences \( a \). The \( \ell^p \) operator norm of \( C \) is then defined as the \( p \)-th root of the smallest value of the constant \( U \):

\[
||C||_{\ell^p,p} = U^{1/p}.
\]

Hardy’s inequality thus asserts that the Cesàro matrix operator \( C \), given by \( c_{j,k} = \frac{1}{j}, k \leq j \) and 0 otherwise, is bounded on \( \ell^p \) and has norm \( \leq p/(p-1) \). (The norm is in fact \( p/(p-1) \).)

We say a matrix \( A \) is a summability matrix if its entries satisfy: \( a_{j,k} \geq 0, a_{j,k} = 0 \) for \( k > j \) and \( \sum_{k=1}^{j} a_{j,k} = 1 \). We say a summability matrix \( A \) is a weighted mean matrix if its entries satisfy:

\[
a_{j,k} = \lambda_k/\Lambda_j, \quad 1 \leq k \leq j; \Lambda_j = \sum_{i=1}^{j} \lambda_i.
\]  (1.2)

We refer to the n-tuple \((a_{n1}, a_{n2}, \ldots, a_{nn})\) as the n-th row of a summability matrix \( A \) and then have the following result of Bennett [6], Theorem 1.14 for the \( \ell^p \) operator norm of \( A \).

**Theorem 1.1.** Let \( p > 1 \) be fixed and suppose \( A \) is a summability matrix. If the rows of \( A \) are decreasing, then \( ||A||_{\ell^p,p} \geq p/(p-1) \). If the rows of \( A \) are increasing, then \( ||A||_{\ell^p,p} \leq p/(p-1) \).

The above theorem, when applied to weighted mean matrixes, gives the following inequality([6], Corollary 4.10).
Theorem 1.2. If $0 < \lambda_1 \leq \lambda_2 \leq \cdots$ and $0 < p < 1$, then
\begin{equation}
(1.3) \quad \sum_{n=1}^{\infty} \left( \frac{1}{\sum_{i=1}^{n} \lambda_i} \right)^{1/p} \leq \left( \frac{1}{1-p} \right)^{1/p} \sum_{n=1}^{\infty} a_n,
\end{equation}
whenever $a$ is a sequence of non-negative terms.

Even though the constant in the above theorem is best possible, some improvement may be possible with specific choices of the $\lambda_i$'s. For examples, the following two inequalities were claimed to hold by Bennett[5], page 40-41; see also [6], page 407):
\begin{align}
(1.4) & \quad \sum_{n=1}^{\infty} \left( \frac{1}{n^p} \sum_{i=1}^{n} (i^n - (i-1)^n) a_i \right)^p \leq \left( \frac{\alpha p}{\alpha p - 1} \right)^p \sum_{n=1}^{\infty} |a_n|^p, \\
(1.5) & \quad \sum_{n=1}^{\infty} \left( \sum_{i=1}^{n} i^{\alpha-1} a_i \right)^p \leq \left( \frac{\alpha p}{\alpha p - 1} \right)^p \sum_{n=1}^{\infty} |a_n|^p,
\end{align}
whenever $\alpha > 0, p > 1, \alpha p > 1$.

We haven’t seen the proofs of Bennett but find the following unpublished result of J. Cartlidge[7] is very helpful to treat the above two inequalities. We don’t have access to his thesis either, so here we quote the one in [2](p. 416):

Theorem 1.3. Let $1 < p < \infty$ be fixed. Let $A$ be a weighted mean matrix given by (1.2). If
\begin{equation}
(1.6) \quad L = \sup_n \left( \frac{\Lambda_{n+1}}{\Lambda_{n+1}} \right) < p,
\end{equation}
then $\|A\|_{p,p} \leq p/(p-L)$.

We will apply the above theorem to prove (1.4)-(1.5) for $\alpha \geq 2, p > 1, \alpha p > 1$ in section 3.

Suppose $a_n \geq 0$, by a change of variables $a_n \to a_n^{1/p}$ and let $p \to \infty$, (1.1) gives the well-known Carleman’s inequality:
\begin{equation}
\sum_{n=1}^{\infty} \left( \sum_{k=1}^{n} a_k \right)^{\frac{1}{n}} \leq e \sum_{n=1}^{\infty} a_n.
\end{equation}

We refer the reader to the survey article [13] and the references therein for an account of Carleman’s inequality. Among the various generalizations of Carleman’s inequality, we mention a result of E. Love, who proved for $\alpha > 0, \lambda_i = i^\alpha - (i-1)^\alpha$,
\begin{equation}
(1.7) \quad \sum_{n=1}^{\infty} \left( \prod_{i=1}^{n} a_i^\alpha (i-1)^\alpha \right)^{1/n^\alpha} \leq e^{\frac{1}{\alpha}} \sum_{n=1}^{\infty} a_n,
\end{equation}
and the constant $e^{\frac{1}{\alpha}}$ is best possible. We note here after a change of variables $a_n \to a_n^{1/p}$, (1.7) corresponds to the limiting case $p \to \infty$ of (1.4).

R. Redheffer gave a remarkable proof of Hardy’s inequality in [14] by developing the method of “recurrerent inequalities”. His method also works for Carleman’s inequality. Another proof of Carleman’s inequality was given by him in [15] and his result has been generalized by H. Alzer[1] and most recently by J. Pečarić and K. Stolarsky[13], who proved for $b_n > 0, N \geq 1, G_n = (\prod_{i=1}^{n} a_i)^{1/n},$
\begin{equation}
\sum_{n=1}^{N} \Lambda_n (b_n - 1) G_n + \Lambda_N G_N \leq \sum_{n=1}^{N} \Lambda_n G_n b_n^{\lambda_n/\lambda_n}.
\end{equation}

In this paper, we will use Redheffer’s method to give a weighted version of his treatment of Hardy’s and Carleman’s inequalities. As we shall see, our result for $1 < p < \infty$ is less satisfactory than that of Cartlidge’s while for the limiting case the result is almost the same as his.

From now on we will assume $a_n \geq 0$ for $n \geq 1$ and any infinite sum converges.
2. Lemmas

Lemma 2.1. Let $\Lambda_k = \sum_{i=1}^{k} \lambda_i$, $\lambda_i > 0$ and $S_n = \sum_{i=1}^{n} \lambda_i a_i$. Let $0 < p < 1$ be fixed and let $(\mu_n)_{n \geq 1}, (\eta_n)_{n \geq 1}$ be two sequences of real numbers such that $\mu_i \leq \eta_i$ for $0 < p < 1$ and $\mu_i \geq \eta_i$ for $p < 0$, then for $n \geq 2$,

\[
(2.1) \quad \sum_{i=2}^{n-1} \left[ \mu_i - (\mu_{i+1}^{q} - \eta_{i+1}^{q})^{1/q} \right] S_i^{1/p} + \mu_n S_n^{1/p} \leq (\mu_2^{q} - \eta_2^{q})^{1/q} a_1^{1/p} + \sum_{i=2}^{n} \eta_i \lambda_i^{1/p} a_i^{1/p}.
\]

Proof. This is essentially due to R. Redheffer [14]. We note for $k \geq 2$,

\[
(2.2) \quad \mu_k S_k^{1/p} - \eta_k^{1/p} a_k^{1/p} = S_{k-1}^{1/p} (\mu_k (1 + t)^{1/p} - \eta t^{1/p}) \leq (\mu_k^{q} - \eta_k^{q})^{1/q} S_{k-1}^{1/p},
\]

with $t = \lambda_k a_k / S_{k-1}$ (compare this with the one on page 686 of [14]). The lemma then follows by adding (2.2) for $2 \leq k \leq n$ together. \hfill \Box

Lemma 2.2. Let $\Lambda_k = \sum_{i=1}^{k} \lambda_i$, $\lambda_i > 0$ and $G_k = (\prod_{i=1}^{k} a_i^{1/\lambda_i})^{1/\Lambda_k}$, then for $\mu_i > 0, n \geq 2$,

\[
(2.3) \quad G_1 + \sum_{i=2}^{n-1} \left( \frac{\Lambda_i}{\lambda_i} - \frac{\Lambda_i}{\lambda_{i+1}} \right) G_i + \frac{\Lambda_n \mu_n}{\lambda_n} G_n \leq (1 + \frac{\Lambda_1}{\lambda_2}) a_1 + \sum_{i=2}^{n} \frac{\Lambda_i}{\lambda_i} a_i.
\]

Proof. This is essentially due to R. Redheffer [14]. We note for $k \geq 2, \mu > 0, \eta > 0$,

\[
\mu G_k - \eta a_k = G_{k-1} (\mu t - \eta t^{1/\lambda_k}) \leq G_{k-1} (\frac{\Lambda_{k-1}}{\lambda_k})^{\frac{\Lambda_k}{\lambda_k}} (\frac{\mu \lambda_k}{\lambda_k})^{\frac{\lambda_k}{\lambda_k}},
\]

where $t^{\lambda_k} = a_k / G_{k-1}$ (compare this with the one on page 686 of [14]). By setting $\mu_k \Lambda_k / \lambda_k = \mu, \eta_k = \eta = \mu \Lambda_k / \lambda_k$, we get

\[
(2.4) \quad \frac{\Lambda_k \mu_k}{\lambda_k} G_k - a_k \mu_k^{\lambda_k} / \lambda_k \leq \frac{\Lambda_{k-1}}{\lambda_k} G_{k-1}.
\]

The lemma then follows by adding (2.4) for $2 \leq k \leq n$ and $G_1 = a_1$ together. \hfill \Box

Lemma 2.3. Let $f(x) \in C^3[a, b]$ and $f'''(x) \geq 0$ for $x \in [a, b]$. Then

\[
(2.5) \quad f(b) - f(a) \geq f'(\frac{a + b}{2})(b - a).
\]

Proof. By Taylor’s expansion,

\[
f(b) = f(a + \frac{b}{2}) + f'(a + \frac{b}{2})(b - a + \frac{b}{2}) + f''(\eta_1)(b - a)^2/4,
\]

\[
f(a) = f(a + \frac{b}{2}) + f'(a + \frac{b}{2})(a - a + \frac{b}{2}) + f''(\eta_2)(a - b)^2/4,
\]

where $a < \eta_2 < (a + b)/2 < \eta_1 < b$. The lemma then follows by noticing $f'''(x) \geq 0$ for $x \in [a, b]$. \hfill \Box

Lemma 2.4. If $s \geq 1$, then

\[
(2.6) \quad \sum_{i=1}^{n} i^s \geq \frac{s}{s+1} \frac{n^s(n+1)^s}{(n+1)^s - n^s}.
\]

Proof. This is a result of V. Levin and S. Stečkin, see Lemma 2 on page 18 in [11]. \hfill \Box
3. Applications of Cartlidge’s Theorem

We say a weighted mean matrix $A$ given by (1.2) is generated by a logarithmico-exponential function if for all sufficiently large $n$, $\lambda_n := l(n)$, where $l(x)$ is a positive logarithmico-exponential function and a logarithmico-exponential function on $[x_0, \infty]$ is defined by Hardy\cite{9} as a real valued function defined by a finite combination of ordinary algebraic symbols (viz, $+, -, \times, \div, \sqrt{\cdot}$) and the functional symbols $\log(\cdot)$ and $e^{(\cdot)}$, operating on real variable $x$ and on real constants.

We note first the following theorem of F. Cass and W. Kratz\cite{8}:

**Theorem 3.1.** Let $1 < p < \infty$ be fixed. Let $A$ be a weighted mean matrix given by (1.2). Suppose $\lim_{n \to \infty} \Lambda_n/n\lambda_n = L < p$, then $\frac{p}{(p-L)} \leq ||A||_{p,p}$.

It is easy to see $\lim_{n \to \infty} n^{\alpha-1}/(n^\alpha - (n-1)^\alpha) = 1/\alpha$ and the simplest Euler-Maclaurin formulae gives:

$$\sum_{i=1}^{n} f(i) = \int_{1}^{n} f(x)dx + f(1) + \int_{1}^{n} (x-[x])f'(x)dx,$$

for $f$ having continuous derivative $f'$, where $[x]$ denote the largest integer not exceeding the real number $x$. It then follows

$$\sum_{i=1}^{n} i^{\alpha-1} = n^{\alpha}/\alpha + o(n^{\alpha}).$$

Thus thanks to Theorem 3.1 we know if (1.4)-(1.5) hold for some $\alpha > 0, p > 1, \alpha p > 1$ then the constants $(\alpha p/ (\alpha p - 1))^p$ are best possible.

Now we apply Cartlidge’s Theorem to get

**Corollary 3.1.** Inequality (1.4) holds for $p > 1, \alpha \geq 2, \alpha p > 1$ and the constant there is best possible.

**Proof.** Apply Theorem 1.3 with $\lambda_i = i^\alpha - (i-1)^\alpha$. We define $f(x) = x^\alpha/(x^\alpha - (x-1)^\alpha), x \geq 1$ so that $\Lambda_{i+1}/\Lambda_i - \Lambda_i/\Lambda_i = f(i+1) - f(i) = f'(\xi), 1 \leq i < \xi < i+1$, with

$$0 < f'(\xi) = \frac{\alpha\xi^{\alpha-1}(\xi-1)^{\alpha-1}}{(\xi^\alpha - (\xi-1)^\alpha)^2} \leq \frac{1}{\alpha},$$

where the last inequality follows from Lemma 2.3 and the arithmetic-geometric inequality, since for $\alpha \geq 2,$

$$\xi^\alpha - (\xi-1)^\alpha \geq \alpha(\xi + (\xi-1)/2)^{\alpha-1} \geq \alpha(\xi(\xi-1))^{(\alpha-1)/2}.$$

This completes the proof. □

We note the corollary implies (1.7) for $\alpha \geq 2$. Now if we apply Theorem 1.3 to (1.5), we need to show

$$\sum_{i=1}^{n+1} i^{\alpha-1}/(n+1)^{\alpha-1} - \sum_{i=1}^{n} i^{\alpha-1}/n^{\alpha-1} = 1 + \left(\frac{1}{(n+1)^{\alpha-1}} - \frac{1}{n^{\alpha-1}}\right) \sum_{i=1}^{n} i^{\alpha-1} \leq 1/\alpha.$$

The second inequality above follows from Lemma 2.4 and we get

**Corollary 3.2.** Inequality (1.5) holds for $p > 1, \alpha \geq 2, \alpha p > 1$ and the constant there is best possible.
Theorem 4.1. Assume the same conditions in Lemma [2.7] and let 0 < p < 1 be fixed. Suppose there exists a positive constant $c$ such that $c^{-1} + 1 \leq c^{-1/p}$ and

\begin{equation}
(4.1) \quad c \leq 1 - p + (1 - p)(\lambda_i^{-q} - \lambda_{i-1}^{-q})\Lambda_{i-1}\Lambda_i^{q/p}, \ i \geq 2.
\end{equation}

Then for 0 < p < 1,

\begin{equation}
(4.2) \quad \sum_{i=1}^{\infty} (S_i/\Lambda_i)^{1/p} \leq c^{-1/p} \sum_{i=1}^{\infty} a_i^{1/p}.
\end{equation}

Proof. It suffices to prove the theorem for any integer $n \geq 1$. We note first the condition \ref{4.1} is equivalent to

\begin{equation}
(4.3) \quad q^{-1}(1 - c^{-1} + c^{-1}\Lambda_{i-1}\Lambda_i^{q/p}(\lambda_i^{-q} - \lambda_{i-1}^{-q})) \geq 1, \ i \geq 2.
\end{equation}

By setting $\eta_i = \lambda_i^{-1/p}, \mu_i^q = \lambda_i^{-q/p} + \Lambda_{i-1}/c\lambda_{i-1}^q$ in \ref{2.1}, we can rewrite the left-hand side of \ref{2.1} as

\begin{equation}
(1 - c^{-1/q})a_1^{1/p} + \sum_{i=2}^{n-1} (\lambda_i^{-q/p} + \Lambda_{i-1}/c\lambda_{i-1}^q)^{1/q} - (\Lambda_i/c) \sum_{i=1}^{n} a_i^{1/q} \sum_{i=1}^{n} a_i.
\end{equation}

By the mean value theorem,

\begin{align*}
(\lambda_i^{-q/p} + \Lambda_{i-1}/c\lambda_{i-1}^q)^{1/q} - (\Lambda_i/c) &= q^{-1}(\lambda_i^{-q/p} + \Lambda_{i-1}/c\lambda_{i-1}^q - \Lambda_i/c) \leq q^{-1}(1 - c^{-1} + c^{-1}\Lambda_{i-1}\Lambda_i^{q/p}(\lambda_i^{-q} - \lambda_{i-1}^{-q})(\Lambda_i/c))^{-1/p} \\
& \geq (\Lambda_i/c) \sum_{i=1}^{\infty} a_i.
\end{align*}

Here the last inequality follows from \ref{4.3}. Thus \ref{2.1} becomes

\begin{equation}
\sum_{i=1}^{\infty} (S_i/\Lambda_i)^{1/p} \leq (c^{-1} + 1) \sum_{i=1}^{\infty} a_i \leq c^{-1/p} \sum_{i=1}^{\infty} a_i.
\end{equation}

This completes the proof. \hfill \Box

We note here if 0 < $\lambda_1 \leq \lambda_2 \leq \cdots$, we can take $c = 1 - p$ in \ref{4.1} and one checks easily for 0 < $p < 1$, (1 − $p$)$^{-1} + 1 < (1 - p)^{-1/p}$. Theorem \ref{4.1} then implies Theorem \ref{1.2}.

We also note the constant given by the above theorem may be less satisfactory. For example the case $\alpha = 2, p = 2$ in \ref{4.4} corresponds to the case $\lambda_i = 2i - 1, p = 1/2, c = 3/4$ in \ref{4.2}. However, direct calculation shows \ref{4.1} is not satisfied in this case. Of course one may try to prove directly

\begin{equation}
(\lambda_i^{-q/p} + \Lambda_{i-1}/c\lambda_{i-1}^q)^{1/q} - (\Lambda_i/c) \geq (\Lambda_i/c)^{-1/p}.
\end{equation}

But one checks this fails for $i = 2$.

Similarly, the case $\alpha = 2, p = 2$ in \ref{4.5} corresponds to the case $\lambda_i = i, p = 1/2, c = 3/4$ in \ref{4.2}. One checks in this case \ref{4.1} holds for $i \geq 2$. However, $c^{-1} + 1 = 7/3 > 16/9 = c^{-2}$, so the coefficient of $a_1$ is slightly larger.

Now we focus our attention to Carleman-type inequalities.

Theorem 4.2. Assume the same conditions in Lemma \ref{2.2} and let $f(x)$ be a real valued function defined for $x \geq 2$ such that $f(n) = \Lambda_n/\lambda_n$ for $n \geq 2$ and 0 ≤ $f(x+1) - f(x) \leq 1/\alpha$ for some $\alpha > 0$. If \((1 + \Lambda_1/\lambda_2) \leq e^{1/\alpha}\) for the same $\alpha$ then

\begin{equation}
(4.4) \quad \sum_{n=1}^{\infty} (\prod_{i=1}^{n} \lambda_i)^{1/\Lambda_n} \leq (1 + \Lambda_1/\lambda_2) a_1 + \sum_{i=2}^{n} a_i (1 + (f(i+1) - f(i)) f(i)) \leq e^{1/\alpha} \sum_{n=1}^{\infty} a_n.
\end{equation}
Proof. It suffices to prove the theorem for any integer $n \geq 2$. Set $\mu_i = f(i+1)/f(i)$ in Lemma 2.2 we get
\[
\sum_{i=1}^{n} G_i \leq \sum_{i=1}^{n-1} G_i + f(n+1)G_N \leq (1 + \frac{A_1}{A_2})a_1 + \sum_{i=2}^{n} a_i (1 + \frac{f(i+1) - f(i)}{f(i)})f(i) \leq e^{1/\alpha} \sum_{n=1}^{\infty} a_n,
\]
by the conditions of the theorem and this completes the proof. \square

Apply Theorem 4.2 to $\lambda_1 = 1$, $\lambda_i = \alpha^{i-1} - \alpha^{i-2}$, $i \geq 2$ for some $\alpha > 1$, then $f(x) = \alpha/(\alpha - 1)$ and we get

Corollary 4.1. For $\alpha > 1$,
\[
\sum_{n=1}^{\infty} \left( a_1 \prod_{k=2}^{n} a_k^{\alpha k-1 - \alpha^{k-2}} \right)^{1/\alpha(n-1)} \leq (1 + \frac{1}{\alpha - 1})a_1 + \sum_{n=2}^{\infty} a_n.
\]

Apply Theorem 4.2 to $\lambda_i = \alpha^i$, $i \geq 1$ for some $\alpha > 0$, then $f(i+1) - f(i) = \alpha^{-i}$ and we get

Corollary 4.2. For $\alpha > 0$,
\[
\sum_{n=1}^{\infty} \left( \prod_{k=1}^{n} a_k^{\alpha k-1} \right)^{(\alpha n)/(\alpha - 1)} \leq (1 + \frac{1}{\alpha})a_1 + \sum_{n=2}^{\infty} e^{1/\alpha n} a_n \leq \sum_{n=1}^{\infty} e^{1/\alpha n} a_n.
\]

We end the paper by noting that if we take $\lambda_i = (i(i+1))^{-1}$ in Theorem 4.2 then $f(x) = x^2$ and we get back a result of Redheffer (see [14] page 693):

Corollary 4.3.
\[
\sum_{n=1}^{\infty} \left( \prod_{k=1}^{n} a_k^{1/k(k+1)} \right)^{(n+1)/n} \leq \sum_{n=1}^{\infty} e^{2n} a_n.
\]

REFERENCES


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