ON WEIGHTED OSTROWSKI TYPE INEQUALITIES FOR OPERATORS AND VECTOR-VALUED FUNCTIONS

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Abstract. Some weighted Ostrowski type integral inequalities for operators and vector-valued functions in Banach spaces are given. Applications for linear operators in Banach spaces and differential equations are also provided.

1. Introduction

In [12], Pečarić and Savić obtained the following Ostrowski type inequality for weighted integrals (see also [7, Theorem 3]):

Theorem 1. *Let $w : [a,b] \to [0,\infty)$ be a weight function on $[a,b]$. Suppose that $f : [a,b] \to \mathbb{R}$ satisfies*

$$|f(t) - f(s)| \leq N |t-s|^{\alpha}, \text{ for all } t,s \in [a,b], \tag{1.1}$$

where $N > 0$ and $0 < \alpha \leq 1$ are some constants. Then for any $x \in [a,b]$

$$\left| f(x) - \frac{\int_a^b w(t) f(t)\, dt}{\int_a^b w(t)\, dt} \right| \leq N \cdot \frac{\int_a^b |t-x|^{\alpha} w(t)\, dt}{\int_a^b w(t)\, dt}. \tag{1.2}$$

Further, if for some constants c and λ

$$0 < c \leq w(t) \leq \lambda c, \text{ for all } t \in [a,b],$$

then for any $x \in [a,b]$, we have

$$\left| f(x) - \frac{\int_a^b w(t) f(t)\, dt}{\int_a^b w(t)\, dt} \right| \leq N \cdot \frac{\lambda L(x) J(x)}{L(x) - J(x) + \lambda J(x)}, \tag{1.3}$$

where

$$L(x) := \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right]^{\alpha}$$

and

$$J(x) := \frac{(x-a)^{1+\alpha} + (b-x)^{1+\alpha}}{(1+\alpha)(b-a)}.$$

The inequality (1.2) was rediscovered in [4] where further applications for different weights and in Numerical Analysis were given.

For other results in connection to weighted Ostrowski inequalities, see [3], [8] and [10].

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In the present paper we extend the weighted Ostrowski's inequality for vector-valued functions and Bochner integrals and apply the obtained results for operatorial inequalities and linear differential equations in Banach spaces. Some numerical experiments are also conducted.

2. Weighted Inequalities

Let X be a Banach space and $-\infty < a < b < \infty$. We denote by $\mathcal{L}(X)$ the Banach algebra of all bounded linear operators acting on X. The norms of vectors or operators acting on X will be denoted by $\|\cdot\|$.

A function $f : [a,b] \to X$ is called *measurable* if there exists a sequence of simple functions $f_n : [a,b] \to X$ which converges punctually almost everywhere on $[a,b]$ at f. We recall also that a measurable function $f : [a,b] \to X$ is *Bochner integrable* if and only if its norm function (i.e. the function $t \mapsto \|f(t)\| : [a,b] \to \mathbb{R}_+$) is Lebesgue integrable on $[a,b]$.

The following theorem holds.

Theorem 2. *Assume that $B : [a,b] \to \mathcal{L}(X)$ is Hölder continuous on $[a,b]$, i.e.,*

$$\|B(t) - B(s)\| \leq H|t-s|^\alpha \quad \text{for all } \ t,s \in [a,b], \tag{2.1}$$

where $H > 0$ and $\alpha \in (0,1]$.

If $f : [a,b] \to X$ is Bochner integrable on $[a,b]$, then we have the inequality:

$$\left\| B(t)\int_a^b f(s)\,ds - \int_a^b B(s) f(s)\,ds \right\| \tag{2.2}$$

$$\leq H\int_a^b |t-s|^\alpha \|f(s)\|\,ds$$

$$\leq H \times \begin{cases} \dfrac{(b-t)^{\alpha+1} + (t-a)^{\alpha+1}}{\alpha+1} |||f|||_{[a,b],\infty} & \text{if} \quad f \in L_\infty([a,b];X); \\[2ex] \left[\dfrac{(b-t)^{q\alpha+1} + (t-a)^{q\alpha+1}}{q\alpha+1}\right]^{\frac{1}{q}} |||f|||_{[a,b],p} & \text{if} \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1 \\ & \text{and} \quad f \in L_p([a,b];X); \\ \left[\dfrac{1}{2}(b-a) + \left|t - \dfrac{a+b}{2}\right|\right]^\alpha |||f|||_{[a,b],1} & \end{cases}$$

for any $t \in [a,b]$.

Proof. Firstly, we prove that the $X-$valued function $s \mapsto B(s) f(s)$ is Bochner integrable on $[a,b]$. Indeed, let (f_n) be a sequence of $X-$valued, simple functions which converge almost everywhere on $[a,b]$ at the function f. The maps $s \mapsto B(s) f_n(s)$ are measurable (because they are continuous with the exception of a finite number of points s in $[a,b]$). Then

$$\|B(s) f_n(s) - B(s) f(s)\| \leq \|B(s)\| \|f_n(s) - f(s)\| \to 0 \ \text{ a.e. on } \ [a,b]$$

when $n \to \infty$ so that the function $s \mapsto B(s) f(s) : [a,b] \to X$ is measurable. Now, using the estimate

$$\|B(s) f(s)\| \leq \sup_{\xi \in [a,b]} \|B(\xi)\| \cdot \|f(s)\|, \ \text{ for all } s \in [a,b],$$

it is easy to see that the function $s \mapsto B(s) f(s)$ is Bochner integrable on $[a,b]$.

We have successively

$$\begin{aligned}
& \left\| B(t) \int_a^b f(s)\, ds - \int_a^b B(s) f(s)\, ds \right\| \\
= & \left\| \int_a^b (B(t) - B(s)) f(s)\, ds \right\| \leq \int_a^b \|(B(t) - B(s)) f(s)\|\, ds \\
\leq & \int_a^b \|(B(t) - B(s))\| \, \|f(s)\|\, ds \leq H \int_a^b |t-s|^{\alpha} \|f(s)\|\, ds =: M(t)
\end{aligned}$$

for any $t \in [a,b]$, proving the first inequality in (2.2).

Now, observe that

$$\begin{aligned}
M(t) & \leq H \, |||f|||_{[a,b],\infty} \int_a^b |t-s|^{\alpha}\, ds \\
& = H \, |||f|||_{[a,b],\infty} \cdot \frac{(b-t)^{\alpha+1} + (t-a)^{\alpha+1}}{\alpha+1}
\end{aligned}$$

and the first part of the second inequality is proved.

Using Hölder's integral inequality, we may state that

$$\begin{aligned}
M(t) & \leq H \left(\int_a^b |t-s|^{q\alpha}\, ds \right)^{\frac{1}{q}} \left(\int_a^b \|f(s)\|^p\, ds \right)^{\frac{1}{p}} \\
& = H \left[\frac{(b-t)^{q\alpha+1} + (t-a)^{q\alpha+1}}{q\alpha+1} \right]^{\frac{1}{q}} |||f|||_{[a,b],p} \,,
\end{aligned}$$

proving the second part of the second inequality.

Finally, we observe that

$$\begin{aligned}
M(t) & \leq H \sup_{s \in [a,b]} |t-s|^{\alpha} \int_a^b \|f(s)\|\, ds \\
& = H \max\left\{ (b-t)^{\alpha}, (t-a)^{\alpha} \right\} |||f|||_{[a,b],1} \\
& = H \left[\frac{1}{2}(b-a) + \left| t - \frac{a+b}{2} \right| \right]^{\alpha} |||f|||_{[a,b],1}
\end{aligned}$$

and the theorem is proved. ∎

The following corollary holds.

Corollary 1. *Assume that $B : [a,b] \to \mathcal{L}(X)$ is Lipschitzian with the constant $L > 0$. Then we have the inequality*

$$\begin{aligned}
& \left\| B(t) \int_a^b f(s)\, ds - \int_a^b B(s) f(s)\, ds \right\| \qquad (2.3) \\
\leq & \; L \int_a^b |t-s| \, \|f(s)\|\, ds
\end{aligned}$$

$$\leq L\times\begin{cases}\left[\frac{1}{4}\left(b-a\right)^{2}+\left(t-\frac{a+b}{2}\right)^{2}\right]\left\|\left|f\right|\right\|_{[a,b],\infty} & \text{if} \quad f\in L_{\infty}\left(\left[a,b\right];X\right);\\ \left[\frac{\left(b-t\right)^{q+1}+\left(t-a\right)^{q+1}}{q+1}\right]^{\frac{1}{q}}\left\|\left|f\right|\right\|_{[a,b],p} & \text{if} \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1\\ & \text{and} \quad f\in L_{p}\left(\left[a,b\right];X\right);\\ \left[\frac{1}{2}\left(b-a\right)+\left|t-\frac{a+b}{2}\right|\right]\left\|\left|f\right|\right\|_{[a,b],1} & \end{cases}$$

for any $t\in[a,b]$.

Remark 1. *If we choose* $t=\frac{a+b}{2}$ *in (2.2) and (2.3), then we get the following midpoint inequalities:*

$$\left\|B\left(\frac{a+b}{2}\right)\int_{a}^{b}f\left(s\right)ds-\int_{a}^{b}B\left(s\right)f\left(s\right)ds\right\| \tag{2.4}$$

$$\leq H\int_{a}^{b}\left|s-\frac{a+b}{2}\right|^{\alpha}\left\|f\left(s\right)\right\|ds$$

$$\leq H\times\begin{cases}\frac{1}{2^{\alpha}\left(\alpha+1\right)}\left(b-a\right)^{\alpha+1}\left\|\left|f\right|\right\|_{[a,b],\infty} & \text{if} \quad f\in L_{\infty}\left(\left[a,b\right];X\right);\\ \frac{1}{2^{\alpha}\left(q\alpha+1\right)^{\frac{1}{q}}}\left(b-a\right)^{\alpha+\frac{1}{q}}\left\|\left|f\right|\right\|_{[a,b],p} & \text{if} \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1\\ & \text{and} \quad f\in L_{p}\left(\left[a,b\right];X\right);\\ \frac{1}{2^{\alpha}}\left(b-a\right)^{\alpha}\left\|\left|f\right|\right\|_{[a,b],1} & \end{cases}$$

and

$$\left\|B\left(\frac{a+b}{2}\right)\int_{a}^{b}f\left(s\right)ds-\int_{a}^{b}B\left(s\right)f\left(s\right)ds\right\| \tag{2.5}$$

$$\leq L\int_{a}^{b}\left|s-\frac{a+b}{2}\right|\left\|f\left(s\right)\right\|ds$$

$$\leq L\times\begin{cases}\frac{1}{4}\left(b-a\right)^{2}\left\|\left|f\right|\right\|_{[a,b],\infty} & \text{if} \quad f\in L_{\infty}\left(\left[a,b\right];X\right);\\ \frac{1}{2\left(q+1\right)^{\frac{1}{q}}}\left(b-a\right)^{1+\frac{1}{q}}\left\|\left|f\right|\right\|_{[a,b],p} & \text{if} \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1\\ & \text{and} \quad f\in L_{p}\left(\left[a,b\right];X\right);\\ \frac{1}{2}\left(b-a\right)\left\|\left|f\right|\right\|_{[a,b],1} & \end{cases}$$

respectively.

Remark 2. *Consider the function* $\Psi_\alpha : [a,b] \to \mathbb{R}$, $\Psi_\alpha(t) := \int_a^b |t-s|^\alpha \|f(s)\| \, ds$, $\alpha \in (0,1)$. *If* f *is continuous on* $[a,b]$, *then* Ψ_α *is differentiable and*

$$\begin{aligned} \frac{d\Psi_\alpha(t)}{dt} &= \frac{d}{dt}\left[\int_a^t (t-s)^\alpha \|f(s)\| \, ds + \int_t^b (s-t)^\alpha \|f(s)\| \, ds\right] \\ &= \alpha \left[\int_a^t \frac{\|f(s)\|}{(t-s)^{1-\alpha}} ds - \int_t^b \frac{\|f(s)\|}{(s-t)^{1-\alpha}} ds\right]. \end{aligned}$$

If $t_0 \in (a,b)$ *is such that*

$$\int_a^{t_0} \frac{\|f(s)\|}{(t_0-s)^{1-\alpha}} ds = \int_{t_0}^b \frac{\|f(s)\|}{(s-t_0)^{1-\alpha}} ds$$

and $\Psi_s'(\cdot)$ *is negative on* (a,t_0) *and positive on* (t_0,b), *then the best inequality we can get in the first part of (2.2) is the following one*

$$\left\| B(t_0) \int_a^b f(s) \, ds - \int_a^b B(s) f(s) \, ds \right\| \le H \int_a^b |t_0 - s|^\alpha \|f(s)\| \, ds. \tag{2.6}$$

If $\alpha = 1$, *then, for*

$$\Psi(t) := \int_a^b |t-s| \|f(s)\| \, ds,$$

we have

$$\begin{aligned} \frac{d\Psi(t)}{dt} &= \int_a^t \|f(s)\| \, ds - \int_t^b \|f(s)\| \, ds, \quad t \in (a,b), \\ \frac{d^2\Psi(t)}{dt^2} &= 2\|f(t)\| \ge 0, \quad t \in (a,b), \end{aligned}$$

which shows that Ψ *is convex on* (a,b).

If $t_m \in (a,b)$ *is such that*

$$\int_a^{t_m} \|f(s)\| \, ds = \int_{t_m}^b \|f(s)\| \, ds,$$

then the best inequality we can get from the first part of (2.3) is

$$\left\| B(t_m) \int_a^b f(s) \, ds - \int_a^b B(s) f(s) \, ds \right\| \le L \int_a^b sgn(s - t_m) \, s \, \|f(s)\| \, ds. \tag{2.7}$$

Indeed, as

$$\begin{aligned} &\inf_{t\in[a,b]} \int_a^b |t-s| \|f(s)\| \, ds \\ &= \int_a^b |t_m - s| \|f(s)\| \, ds = \int_a^{t_m} (t_m - s) \|f(s)\| \, ds + \int_{t_m}^b (s - t_m) \|f(s)\| \, ds \\ &= t_m \left(\int_a^{t_m} \|f(s)\| \, ds - \int_{t_m}^b \|f(s)\| \, ds \right) + \int_{t_m}^b s \|f(s)\| \, ds - \int_a^{t_m} s \|f(s)\| \, ds \\ &= \int_{t_m}^b s \|f(s)\| \, ds - \int_a^{t_m} s \|f(s)\| \, ds = \int_a^b sgn(s - t_m) \, s \, \|f(s)\| \, ds, \end{aligned}$$

then the best inequality we can get from the first part of (2.3) is obtained for $t = t_m \in (a,b)$.

We recall that a function $F:[a,b]\to\mathcal{L}(X)$ is said to be *strongly continuous* if for all $x\in X$, the maps $s\mapsto F(s)x:[a,b]\to X$ are continuous on $[a,b]$. In this case the function $s\mapsto\|B(s)\|:[a,b]\to\mathbb{R}_+$ is (Lebesgue) measurable and bounded ([6]). The linear operator $L=\int_a^b F(s)\,ds$ (defined by $Lx:=\int_a^b F(s)\,xds$ for all $x\in X$) is bounded, because

$$\|Lx\|\le\left(\int_a^b\|F(s)\|\,ds\right)\cdot\|x\|\quad\text{for all }\ x\in X.$$

In a similar manner to Theorem 2, we may prove the following result as well.

Theorem 3. *Assume that $f:[a,b]\to X$ is Hölder continuous, i.e.,*

$$\|f(t)-f(s)\|\le K|t-s|^{\beta}\quad\text{for all}\ \ t,s\in[a,b], \tag{2.8}$$

where $K>0$ and $\beta\in(0,1]$.

If $B:[a,b]\to\mathcal{L}(X)$ is strongly continuous on $[a,b]$, then we have the inequality:

$$\begin{aligned}
&\left\|\left(\int_a^b B(s)\,ds\right)f(t)-\int_a^b B(s)\,f(s)\,ds\right\| \qquad (2.9)\\
\le\ & K\int_a^b|t-s|^{\beta}\|B(s)\|\,ds\\
\le\ & K\times\begin{cases}\frac{(b-t)^{\beta+1}+(t-a)^{\beta+1}}{\beta+1}\,|||B|||_{[a,b],\infty} & \text{if}\quad \|B(\cdot)\|\in L_{\infty}([a,b];\mathbb{R}_+);\\[2ex]
\left[\frac{(b-t)^{q\beta+1}+(t-a)^{q\beta+1}}{q\beta+1}\right]^{\frac{1}{q}}|||B|||_{[a,b],p} & \text{if}\quad p>1,\ \frac{1}{p}+\frac{1}{q}=1\\
& \text{and}\quad \|B(\cdot)\|\in L_p([a,b];\mathbb{R}_+);\\[2ex]
\left[\frac{1}{2}(b-a)+\left|t-\frac{a+b}{2}\right|\right]^{\beta}|||B|||_{[a,b],1}\end{cases}
\end{aligned}$$

for any $t\in[a,b]$.

The following corollary holds.

Corollary 2. *Assume that f and B are as in Theorem 3. If, in addition, $\int_a^b B(s)\,ds$ is invertible in $\mathcal{L}(X)$, then we have the inequality:*

$$\begin{aligned}
&\left\|f(t)-\left(\int_a^b B(s)\,ds\right)^{-1}\int_a^b B(s)\,f(s)\,ds\right\| \qquad (2.10)\\
\le\ & K\left\|\left(\int_a^b B(s)\,ds\right)^{-1}\right\|\int_a^b|t-s|^{\beta}\|B(s)\|\,ds
\end{aligned}$$

for any $t\in[a,b]$.

Remark 3. *It is obvious that the inequality (2.10) contains as a particular case what is the so called Ostrowski's inequality for weighted integrals (see (1.2)).*

3. Inequalities for Linear Operators

Let $0\le a<b<\infty$ and $A\in\mathcal{L}(X)$. We recall that the operatorial norm of A is given by

$$\|A\|=\sup\{\|Ax\|:\|x\|\le 1\}.$$

The *resolvent set* of A (denoted by $\rho(A)$) is the set of all complex scalars λ for which $\lambda I - A$ is an invertible operator. Here I is the identity operator in $\mathcal{L}(X)$. The complementary set of $\rho(A)$ in the complex plane, denoted by $\sigma(A)$, is the *spectrum* of A. It is known that $\sigma(A)$ is a compact set in $\mathbb{C}$. The series $\left(\sum_{n\geq 0} \frac{(tA)^n}{n!}\right)$ converges absolutely and locally uniformly for $t \in \mathbb{R}$. If we denote by e^{tA} its sum, then

$$\left\|e^{tA}\right\| \leq e^{|t|\|A\|}, \quad t \in \mathbb{R}.$$

Proposition 1. *Let X be a real or complex Banach space, $A \in \mathcal{L}(X)$ and β be a non-null real number such that $-\beta \in \rho(A)$. Then for all $0 \leq a < b < \infty$ and each $s \in [a,b]$, we have*

$$\left\| \frac{e^{\beta b} - e^{\beta a}}{\beta} \cdot e^{sA} - (\beta I + A)^{-1} \left[e^{b(\beta I + A)} - e^{a(\beta I + A)} \right] \right\| \tag{3.1}$$
$$\leq \|A\| e^{b\|A\|} \cdot \left[\frac{1}{4}(b-a)^2 + \left(s - \frac{a+b}{2}\right)^2 \right] \cdot \max\left\{e^{\beta b}, e^{\beta a}\right\}.$$

Proof. We apply the second inequality from Corollary 1 in the following particular case.

$$B(\tau) := e^{\tau A}, \quad f(\tau) = e^{\beta\tau} x, \quad \tau \in [a,b], \ x \in X.$$

For all $\xi, \eta \in [a,b]$ there exists an α between ξ and η such that

$$\begin{aligned} \|B(\xi) - B(\eta)\| &= \left\| \sum_{n=1}^{\infty} \frac{(\xi^n - \eta^n)}{n!} A^n \right\| = \left\| (\xi - \eta) A \sum_{n=0}^{\infty} \frac{(\alpha A)^n}{n!} \right\| \\ &\leq \|A\| \left\|e^{\alpha A}\right\| \cdot |\xi - \eta| \leq \|A\| e^{b\|A\|} \cdot |\xi - \eta|. \end{aligned}$$

The function $\tau \mapsto e^{\tau A}$ is thus Lipschitzian on $[a,b]$ with the constant $L := \|A\| e^{b \cdot \|A\|}$. On the other hand we have

$$\begin{aligned} \int_a^b e^{\tau A} \left(e^{\beta\tau} x\right) d\tau &= \int_a^b e^{\tau A} \left(e^{\beta\tau} I x\right) d\tau = \int_a^b e^{\tau(A + \beta I)} x d\tau \\ &= (A + \beta I)^{-1} \left[e^{b(A+\beta I)} - e^{a(A+\beta I)} \right] x, \end{aligned}$$

and

$$|||f|||_{[a,b],\infty} = \sup_{\tau \in [a,b]} \left\|e^{\tau\beta} x\right\| = \max\left\{e^{\beta b}, e^{\beta a}\right\} \cdot \|x\|.$$

Placing all the above results in the second inequality from (2.3) and taking the supremum for all $x \in X$, we will obtain the desired inequality (3.1). ∎

Remark 4. *Let $A \in \mathcal{L}(X)$ such that $0 \in \rho(A)$. Taking the limit as $\beta \to 0$ in (3.1), we get the inequality*

$$\begin{aligned} &\left\|(b-a) e^{sA} - A^{-1}\left[e^{bA} - e^{aA}\right]\right\| \\ \leq\ &\|A\| e^{b\|A\|} \cdot \left[\frac{1}{4}(b-a)^2 + \left(s - \frac{a+b}{2}\right)^2 \right], \end{aligned}$$

where a, b and s are as in Proposition 1.

Proposition 2. *Let $A \in \mathcal{L}(X)$ be an invertible operator, $t \geq 0$ and $0 \leq s \leq t$. Then the following inequality holds:*

$$\left\| \frac{t^2}{2} \sin(sA) - A^{-2}[\sin(tA) - tA\cos(tA)] \right\| \leq \frac{2s^3 + 2t^3 - 3st^2}{6} \|A\|. \tag{3.2}$$

In particular, if $X = \mathbb{R}$, $A = 1$ and $s = 0$ it follows the scalar inequality

$$|\sin t - t\cos t| \leq \frac{t^3}{3}, \quad \textit{for all } t \geq 0.$$

Proof. We apply the inequality from (2.3) in the following particular case:

$$B(\tau) = \sin(\tau A) := \sum_{n=0}^{\infty} (-1)^n \frac{(\tau A)^{2n+1}}{(2n+1)!}, \quad \tau \geq 0,$$

and

$$f(\tau) = \tau \cdot x, \quad \text{for fixed } x \in X. \tag{3.3}$$

For each $\xi, \eta \in [0, t]$, we have

$$\begin{aligned} \|B(\xi) - B(\eta)\| &= \left\| A \left(\sum_{n=0}^{\infty} (-1)^n \frac{(\xi - \eta)\alpha^{2n}}{(2n)!} A^{2n} \right) \right\| \\ &\leq \|A\| |\xi - \eta| \cdot \|\cos(\alpha A)\| \leq \|A\| |\xi - \eta|, \end{aligned}$$

where α is a real number between ξ and η, i.e., the function $\tau \longmapsto B(\tau) : \mathbb{R}_+ \to \mathcal{L}(X)$ is $\|A\|$ −Lipschitzian.

Moreover, it is easy to see that

$$\int_0^t B(\tau) f(\tau) d\tau = A^{-2}[\sin(tA) - tA\cos(tA)] x$$

and

$$\int_0^t |s - \tau| |f(\tau)| d\tau = \frac{2s^3 + 2t^3 - 3st^2}{6} \|x\|. \tag{3.4}$$

Applying the first inequality from (2.3) and taking the supremum for $x \in X$ with $\|x\| \leq 1$, we get (3.2). ∎

4. Quadrature Formulae

Consider the division of the interval $[a, b]$ given by

$$I_n : a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b \tag{4.1}$$

and $h_i := t_{i+1} - t_i$, $\nu(h) := \max_{i=\overline{0,n-1}} h_i$. For the intermediate points $\boldsymbol{\xi} := (\xi_0, \ldots, \xi_{n-1})$ with $\xi_i \in [t_i, t_{i+1}]$, $i = \overline{0, n-1}$, define the sum

$$S_n^{(1)}(B, f; I_n, \boldsymbol{\xi}) := \sum_{i=0}^{n-1} B(\xi_i) \int_{t_i}^{t_{i+1}} f(s) ds. \tag{4.2}$$

Then we may state the following result in approximating the integral

$$\int_a^b B(s) f(s) ds,$$

based on Theorem 2.

Theorem 4. *Assume that $B:[a,b]\to\mathcal{L}(X)$ is Hölder continuous on $[a,b]$, i.e., it satisfies the condition (2.1) and $f:[a,b]\to X$ is Bochner integrable on $[a,b]$. Then we have the representation*

$$\int_a^b B(s)\,f(s)\,ds = S_n^{(1)}(B,f;I_n,\boldsymbol{\xi}) + R_n^{(1)}(B,f;I_n,\boldsymbol{\xi}), \tag{4.3}$$

where $S_n^{(1)}(B,f;I_n,\boldsymbol{\xi})$ is as given by (4.2) and the remainder $R_n^{(1)}(B,f;I_n,\boldsymbol{\xi})$ satisfies the estimate

$$\begin{aligned}
&\left\|R_n^{(1)}(B,f;I_n,\boldsymbol{\xi})\right\| \\
&\leq H\times\begin{cases}
\dfrac{1}{\alpha+1}\,|||f|||_{[a,b],\infty}\displaystyle\sum_{i=0}^{n-1}\left[(t_{i+1}-\xi_i)^{\alpha+1}+(\xi_i-t_i)^{\alpha+1}\right] \\[2ex]
\dfrac{1}{(q\alpha+1)^{\frac{1}{q}}}\,|||f|||_{[a,b],p}\left\{\displaystyle\sum_{i=0}^{n-1}\left[(t_{i+1}-\xi_i)^{q\alpha+1}+(\xi_i-t_i)^{q\alpha+1}\right]\right\}^{\frac{1}{q}}, \\
\qquad\qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1 \\[2ex]
\left[\dfrac{1}{2}\nu(h)+\displaystyle\max_{i=\overline{0,n-1}}\left|\xi_i-\dfrac{t_{i+1}+t_i}{2}\right|\right]^{\alpha}|||f|||_{[a,b],1}
\end{cases} \\
&\leq H\times\begin{cases}
\dfrac{1}{\alpha+1}\,|||f|||_{[a,b],\infty}\displaystyle\sum_{i=0}^{n-1}h_i^{\alpha+1} \\[2ex]
\dfrac{1}{(q\alpha+1)^{\frac{1}{q}}}\,|||f|||_{[a,b],p}\left(\displaystyle\sum_{i=0}^{n-1}h_i^{q\alpha+1}\right)^{\frac{1}{q}}, \qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1 \\[2ex]
\left[\dfrac{1}{2}\nu(h)+\displaystyle\max_{i=\overline{0,n-1}}\left|\xi_i-\dfrac{t_{i+1}+t_i}{2}\right|\right]^{\alpha}|||f|||_{[a,b],1}
\end{cases} \\
&\leq H\times\begin{cases}
\dfrac{1}{\alpha+1}\,|||f|||_{[a,b],\infty}[\nu(h)]^{\alpha} \\[2ex]
\dfrac{(b-a)^{\frac{1}{q}}}{(q\alpha+1)^{\frac{1}{q}}}\,|||f|||_{[a,b],p}[\nu(h)]^{\alpha} \\[2ex]
|||f|||_{[a,b],1}[\nu(h)]^{\alpha}.
\end{cases}
\end{aligned}$$

Proof. Applying Theorem 4 on $[x_i, x_{i+1}]$ $\left(i = \overline{0, n-1}\right)$, we may write that

$$\left\| \int_{t_i}^{t_{i+1}} B(s) f(s)\, ds - B(\xi_i) \int_{t_i}^{t_{i+1}} f(s)\, ds \right\|$$
$$\leq H \times \begin{cases} \left[\dfrac{(t_{i+1}-\xi_i)^{\alpha+1} + (\xi_i - t_i)^{\alpha+1}}{\alpha+1} \right] |||f|||_{[t_i,t_{i+1}],\infty} \\[2ex] \left[\dfrac{(t_{i+1}-\xi_i)^{q\alpha+1} + (\xi_i - t_i)^{q\alpha+1}}{q\alpha+1} \right]^{\frac{1}{q}} |||f|||_{[t_i,t_{i+1}],p} \\[2ex] \left[\dfrac{1}{2}(t_{i+1}-t_i) + \left| \xi_i - \dfrac{t_{i+1}+t_i}{2} \right| \right]^{\alpha} |||f|||_{[t_i,t_{i+1}],1} \end{cases}.$$

Summing over i from 0 to $n-1$ and using the generalised triangle inequality we get

$$\left\| R_n^{(1)}(B, f; I_n, \boldsymbol{\xi}) \right\|$$
$$\leq \sum_{i=0}^{n-1} \left\| \int_{t_i}^{t_{i+1}} B(s) f(s)\, ds - B(\xi_i) \int_{t_i}^{t_{i+1}} f(s)\, ds \right\|$$
$$\leq H \times \begin{cases} \dfrac{1}{\alpha+1} \sum_{i=0}^{n-1} \left[(t_{i+1}-\xi_i)^{\alpha+1} + (\xi_i - t_i)^{\alpha+1} \right] |||f|||_{[t_i,t_{i+1}],\infty} \\[2ex] \dfrac{1}{(q\alpha+1)^{\frac{1}{q}}} \left[\sum_{i=0}^{n-1} (t_{i+1}-\xi_i)^{q\alpha+1} + (\xi_i - t_i)^{q\alpha+1} \right]^{\frac{1}{q}} |||f|||_{[t_i,t_{i+1}],p} \\[2ex] \sum_{i=0}^{n-1} \left[\dfrac{1}{2} h_i + \left| \xi_i - \dfrac{t_{i+1}+t_i}{2} \right| \right]^{\alpha} |||f|||_{[t_i,t_{i+1}],1} \end{cases}.$$

Now, observe that

$$\sum_{i=0}^{n-1} \left[(t_{i+1}-\xi_i)^{\alpha+1} + (\xi_i - t_i)^{\alpha+1} \right] |||f|||_{[t_i,t_{i+1}],\infty}$$
$$\leq |||f|||_{[a,b],\infty} \sum_{i=0}^{n-1} \left[(t_{i+1}-\xi_i)^{\alpha+1} + (\xi_i - t_i)^{\alpha+1} \right]$$
$$\leq |||f|||_{[a,b],\infty} \sum_{i=0}^{n-1} h_i^{\alpha+1} \leq |||f|||_{[a,b],\infty} (b-a) [\nu(h)]^{\alpha}.$$

Using the discrete Hölder inequality, we may write that

$$\left[\sum_{i=0}^{n-1} (t_{i+1}-\xi_i)^{q\alpha+1} + (\xi_i - t_i)^{q\alpha+1} \right]^{\frac{1}{q}} |||f|||_{[t_i,t_{i+1}],p}$$
$$\leq \left[\sum_{i=0}^{n-1} \left(\left[(t_{i+1}-\xi_i)^{q\alpha+1} + (\xi_i - t_i)^{q\alpha+1} \right]^{\frac{1}{q}} \right)^q \right]^{\frac{1}{q}} \times \left[\sum_{i=0}^{n-1} |||f|||_{[t_i,t_{i+1}],p}^p \right]^{\frac{1}{p}}$$

$$= \left\{\sum_{i=0}^{n-1}\left[(t_{i+1}-\xi_i)^{q\alpha+1}+(\xi_i-t_i)^{q\alpha+1}\right]\right\}^{\frac{1}{q}}\left(\int_a^b \|f(t)\|^p\,ds\right)^{\frac{1}{p}}$$

$$\leq \left(\sum_{i=0}^{n-1}h_i^{q\alpha+1}\right)^{\frac{1}{q}}|||f|||_{[a,b],p}^{p}\leq (b-a)^{\frac{1}{q}}|||f|||_{[a,b],p}\,[\nu(h)]^{\alpha}.$$

Finally, we have

$$\sum_{i=0}^{n-1}\left[\frac{1}{2}h_i+\left|\xi_i-\frac{t_{i+1}+t_i}{2}\right|\right]^{\alpha}|||f|||_{[t_i,t_{i+1}],1}$$
$$\leq \left[\frac{1}{2}\max_{\overline{i=0,n-1}}h_i+\max_{\overline{i=0,n-1}}\left|\xi_i-\frac{t_{i+1}+t_i}{2}\right|\right]^{\alpha}|||f|||_{[a,b],1}$$
$$\leq [\nu(h)]^{\alpha}|||f|||_{[a,b],1}$$

and the theorem is proved. ∎

The following corollary holds.

Corollary 3. *If B is Lipschitzian with the constant L, then we have the representation (4.3) and the remainder $R_n^{(1)}(B,f;I_n,\boldsymbol{\xi})$ satisfies the estimates:*

$$\left\|R_n^{(1)}(B,f;I_n,\boldsymbol{\xi})\right\| \tag{4.4}$$

$$\leq L\times\begin{cases} |||f|||_{[a,b],\infty}\left[\frac{1}{4}\sum\limits_{i=0}^{n-1}h_i^2+\sum\limits_{i=0}^{n-1}\left(\xi_i-\frac{t_{i+1}+t_i}{2}\right)^2\right] \\ \frac{1}{(q+1)^{\frac{1}{q}}}|||f|||_{[a,b],p}\left\{\sum\limits_{i=0}^{n-1}\left[(t_{i+1}-\xi_i)^{q+1}+(\xi_i-t_i)^{q+1}\right]\right\}^{\frac{1}{q}}, \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1 \\ \left[\frac{1}{2}\nu(h)+\max\limits_{\overline{i=0,n-1}}\left|\xi_i-\frac{t_{i+1}+t_i}{2}\right|\right]|||f|||_{[a,b],1} \end{cases}$$

$$\leq L\times\begin{cases} \frac{1}{2}|||f|||_{[a,b],\infty}\sum\limits_{i=0}^{n-1}h_i^2 \\ \frac{1}{(q+1)^{\frac{1}{q}}}|||f|||_{[a,b],p}\left(\sum\limits_{i=0}^{n-1}h_i^{q+1}\right)^{\frac{1}{q}} \\ \left[\frac{1}{2}\nu(h)+\max\limits_{\overline{i=0,n-1}}\left|\xi_i-\frac{t_{i+1}+t_i}{2}\right|\right]|||f|||_{[a,b],1} \end{cases}$$

$$\leq L\times\begin{cases} \frac{1}{2}|||f|||_{[a,b],\infty}(b-a)\,\nu(h) \\ \frac{(b-a)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}|||f|||_{[a,b],p}\,\nu(h) \\ |||f|||_{[a,b],1}\,\nu(h) \end{cases}.$$

The second possibility we have for approximating the integral $\int_a^b B(s) f(s) ds$ is embodied in the following theorem based on Theorem 3.

Theorem 5. *Assume that $f : [a,b] \to X$ is Hölder continuous, i.e., the condition (2.8) holds. If $B : [a,b] \to \mathcal{L}(X)$ is strongly continuous on $[a,b]$, then we have the representation:*

$$\int_a^b B(s) f(s) ds = S_n^{(2)}(B, f; I_n, \boldsymbol{\xi}) + R_n^{(2)}(B, f; I_n, \boldsymbol{\xi}), \tag{4.5}$$

where

$$S_n^{(2)}(B, f; I_n, \boldsymbol{\xi}) := \sum_{i=0}^{n-1} \left(\int_{t_i}^{t_{i+1}} B(s) ds \right) f(\xi_i) \tag{4.6}$$

and the remainder $R_n^{(2)}(B, f; I_n, \boldsymbol{\xi})$ satisfies the estimate:

$$\begin{aligned}
&\left\| R_n^{(2)}(B, f; I_n, \boldsymbol{\xi}) \right\| \qquad (4.7)\\
&\leq K \times \begin{cases} \dfrac{1}{\beta+1} |||B|||_{[a,b],\infty} \sum_{i=0}^{n-1} \left[(t_{i+1} - \xi_i)^{\beta+1} + (\xi_i - t_i)^{\beta+1} \right] \\[2ex] \dfrac{1}{(q\beta+1)^{\frac{1}{q}}} |||B|||_{[a,b],p} \left\{ \sum_{i=0}^{n-1} \left[(t_{i+1} - \xi_i)^{q\beta+1} + (\xi_i - t_i)^{q\beta+1} \right] \right\}^{\frac{1}{q}}, \\ \qquad\qquad p > 1,\ \frac{1}{p} + \frac{1}{q} = 1 \\[2ex] \left[\dfrac{1}{2} \nu(h) + \max_{i=\overline{0,n-1}} \left| \xi_i - \dfrac{t_{i+1} + t_i}{2} \right| \right]^{\beta} |||B|||_{[a,b],1} \end{cases} \\
&\leq K \times \begin{cases} \dfrac{1}{\beta+1} |||B|||_{[a,b],\infty} \sum_{i=0}^{n-1} h_i^{\beta+1} \\[2ex] \dfrac{1}{(q\beta+1)^{\frac{1}{q}}} |||B|||_{[a,b],p} \left\{ \sum_{i=0}^{n-1} h_i^{q\beta+1} \right\}^{\frac{1}{q}}, \quad p > 1,\ \frac{1}{p} + \frac{1}{q} = 1 \\[2ex] \left[\dfrac{1}{2} \nu(h) + \max_{i=\overline{0,n-1}} \left| \xi_i - \dfrac{t_{i+1} + t_i}{2} \right| \right]^{\beta} |||B|||_{[a,b],1} \end{cases} \\
&\leq K \times \begin{cases} \dfrac{1}{\beta+1} |||B|||_{[a,b],\infty} (b-a) [\nu(h)]^{\beta} \\[2ex] \dfrac{(b-a)^{\frac{1}{q}}}{(q\beta+1)^{\frac{1}{q}}} |||B|||_{[a,b],p} [\nu(h)]^{\beta}, \quad p > 1,\ \frac{1}{p} + \frac{1}{q} = 1 \\[2ex] |||B|||_{[a,b],1} [\nu(h)]^{\beta}\ . \end{cases}
\end{aligned}$$

If we consider the quadrature

$$M_n^{(1)}(B, f; I_n) := \sum_{i=0}^{n-1} B\left(\frac{t_i + t_{i+1}}{2} \right) \int_{t_i}^{t_{i+1}} f(s) ds, \tag{4.8}$$

then we have the representation

$$\int_a^b B(s) f(s)\, ds = M_n^{(1)}(B,f;I_n) + R_n^{(1)}(B,f;I_n), \tag{4.9}$$

and the remainder $R_n^{(1)}(B,f;I_n)$ satisfies the estimate:

$$\left\|R_n^{(1)}(B,f;I_n)\right\| \tag{4.10}$$

$$\leq H \times \begin{cases} \frac{1}{2^\alpha(\alpha+1)} \||f|\|_{[a,b],\infty} \sum_{i=0}^{n-1} h_i^{\alpha+1} \\ \frac{1}{2^\alpha(q\alpha+1)^{\frac{1}{q}}} \||f|\|_{[a,b],p} \left[\sum_{i=0}^{n-1} h_i^{q\alpha+1}\right]^{\frac{1}{q}}, \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{2^\alpha}[\nu(h)]^\alpha \||f|\|_{[a,b],1} \end{cases}$$

$$\leq H \times \begin{cases} \frac{1}{2^\alpha(\alpha+1)} (b-a) \||f|\|_{[a,b],\infty} [\nu(h)]^\alpha \\ \frac{(b-a)^{\frac{1}{q}}}{2^\alpha(q\alpha+1)^{\frac{1}{q}}} \||f|\|_{[a,b],p} [\nu(h)]^\alpha, \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{2^\alpha} \||f|\|_{[a,b],1} [\nu(h)]^\alpha \end{cases},$$

provided that B and f are as in Theorem 4.

Now, if we consider the quadrature

$$M_n^{(2)}(B,f;I_n) := \sum_{i=0}^{n-1}\left(\int_{t_i}^{t_{i+1}} B(s)\, ds\right) f\left(\frac{t_i+t_{i+1}}{2}\right), \tag{4.11}$$

then we also have

$$\int_a^b B(s) f(s)\, ds = M_n^{(2)}(B,f;I_n) + R_n^{(2)}(B,f;I_n), \tag{4.12}$$

and in this case the remainder satisfies the bound

$$\left\|R_n^{(2)}(B,f;I_n)\right\| \tag{4.13}$$

$$\leq K \times \begin{cases} \frac{1}{2^\beta(\beta+1)} \||B|\|_{[a,b],\infty} \sum_{i=0}^{n-1} h_i^{\beta+1} \\ \frac{1}{2^\beta(q\beta+1)^{\frac{1}{q}}} \||B|\|_{[a,b],p} \left(\sum_{i=0}^{n-1} h_i^{q\beta+1}\right)^{\frac{1}{q}}, \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{2^\beta}[\nu(h)]^\beta \||B|\|_{[a,b],1} \end{cases}$$

$$\leq K\times\begin{cases}\dfrac{1}{2^{\beta}(\beta+1)}(b-a)\,|||B|||_{[a,b],\infty}\,[\nu(h)]^{\beta}\\[2ex]\dfrac{(b-a)^{\frac{1}{q}}}{2^{\beta}(q\beta+1)^{\frac{1}{q}}}\,|||B|||_{[a,b],p}\,[\nu(h)]^{\beta}, \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2ex]\dfrac{1}{2^{\beta}}\,|||B|||_{[a,b],1}\,[\nu(h)]^{\beta}\end{cases}, \tag{4.14}$$

provided B and f satisfy the hypothesis of Theorem 5.

Now, if we consider the equidistant partitioning of $[a,b]$,

$$E_n : t_i := a+\left(\frac{b-a}{n}\right)\cdot i, \quad i=\overline{0,n},$$

then $M_n^{(1)}(B,f;E_n)$ becomes

$$M_n^{(1)}(B,f) := \sum_{i=0}^{n-1} B\left(a+\left(i+\frac{1}{2}\right)\cdot\frac{b-a}{n}\right)\int_{a+\frac{b-a}{n}\cdot i}^{a+\frac{b-a}{n}\cdot(i+1)} f(s)\,ds \tag{4.15}$$

and then

$$\int_a^b B(s)f(s)\,ds = M_n^{(1)}(B,f)+R_n^{(1)}(B,f), \tag{4.16}$$

where the remainder satisfies the bound

$$\left\|R_n^{(1)}(B,f)\right\| \leq H\times\begin{cases}\dfrac{(b-a)^{\alpha+1}}{2^{\alpha}(\alpha+1)n^{\alpha}}\,|||f|||_{[a,b],\infty}\\[2ex]\dfrac{(b-a)^{\alpha+\frac{1}{q}}}{2^{\alpha}(\alpha+1)n^{\alpha}}\,|||f|||_{[a,b],p}, \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2ex]\dfrac{(b-a)^{\alpha}}{2^{\alpha}n^{\alpha}}\,|||f|||_{[a,b],1}\end{cases}. \tag{4.17}$$

Also, we have

$$\int_a^b B(s)f(s)\,ds = M_n^{(2)}(B,f)+R_n^{(2)}(B,f), \tag{4.18}$$

where

$$M_n^{(2)}(B,f) := \sum_{i=0}^{n-1}\left(\int_{a+\frac{b-a}{n}\cdot i}^{a+\frac{b-a}{n}\cdot(i+1)} B(s)\,ds\right) f\left(a+\left(i+\frac{1}{2}\right)\cdot\frac{b-a}{n}\right),$$

and the remainder $R_n^{(2)}(B,f)$ satisfies the estimate

$$\left\|R_n^{(2)}(B,f)\right\| \leq K\times\begin{cases}\dfrac{(b-a)^{\beta+1}}{2^{\beta}(\beta+1)n^{\beta}}\,\|B\|_{[a,b],\infty}\\[2ex]\dfrac{(b-a)^{\beta+\frac{1}{q}}}{2^{\beta}(\beta+1)n^{\beta}}\,|||B|||_{[a,b],p}, \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2ex]\dfrac{(b-a)^{\beta}}{2^{\beta}n^{\beta}}\,|||B|||_{[a,b],1}\end{cases}. \tag{4.19}$$

5. Application for Differential Equations in Banach Spaces

We recall that a family of operators $\mathcal{U} = \{U(t,s) : t \geq s\} \subset \mathcal{L}(X)$ with $t,s \in \mathbb{R}$ or $t,s \in \mathbb{R}_+$ is called an *evolution family* if:

(i) $U(t,t) = I$ and $U(t,s)U(s,\tau) = U(t,\tau)$ for all $t \geq s \geq \tau$; and
(ii) for each $x \in X$, the function $(t,s) \longmapsto U(t,s)x$ is continuous for $t \geq s$.

Here I is the identity operator in $\mathcal{L}(X)$.

An evolution family $\{U(t,s) : t \geq s\}$ is said to be *exponentially bounded* if, in addition,

(iii) there exist the constants $M \geq 1$ and $\omega > 0$ such that

$$\|U(t,s)\| \leq Me^{\omega(t-s)}, \quad t \geq s. \tag{5.1}$$

Evolution families appear as solutions for abstract Cauchy problems of the form

$$\dot{u}(t) = A(t)u(t), \ u(s) = x_s, \ x_s \in \mathcal{D}(A(s)), \ t \geq s, \ t,s \in \mathbb{R} \text{ (or } \mathbb{R}_+\text{)}, \tag{5.2}$$

where the domain $\mathcal{D}(A(s))$ of the linear operator $A(s)$ is assumed to be dense in X. An evolution family is said to solve the abstract Cauchy problem (5.2) if for each $s \in \mathbb{R}$ there exists a dense subset $Y_s \subseteq D(A(s))$ such that for each $x_s \in Y_s$ the function

$$t \longmapsto u(t) := U(t,s)x_s : [s,\infty) \to X,$$

is differentiable, $u(t) \in D(A(t))$ for all $t \geq s$ and

$$\frac{d}{dt}u(t) = A(t)u(t), \quad t \geq s.$$

This later definition can be found in [15]. In this definition the operators $A(t)$ can be unbounded. The Cauchy problem (5.2) is called *well-posed* if there exists an evolution family $\{U(t,s) : t \geq s\}$ which solves it.

It is known that the well-posedness of (5.2) can be destroyed by a bounded and continuous perturbation [13]. Let $f : \mathbb{R} \to X$ be a locally integrable function. Consider the inhomogeneous Cauchy problem:

$$\dot{u}(t) = A(t)u(t) + f(t), \ u(s) = x_s \in X, \ t \geq s, \ t,s \in \mathbb{R} \text{ (or } \mathbb{R}_+\text{)}. \tag{5.3}$$

A continuous function $t \longmapsto u(t) : [s,\infty) \to X$ is said to a *mild solution* of the Cauchy problem (5.3) if $u(s) = x_s$ and there exists an evolution family $\{U(t,\tau) : t \geq \tau\}$ such that

$$u(t) = U(t,s)x_s + \int_s^t U(t,\tau)f(\tau)d\tau, \ t \geq s, \ x_s \in X, \ t,s \in \mathbb{R} \text{ (or } \mathbb{R}_+\text{)}. \tag{5.4}$$

The following theorem holds.

Theorem 6. *Let $\mathcal{U} = \{U(\nu,\eta) : \nu \geq \eta\} \subset \mathcal{L}(X)$ be an evolution family and $f : \mathbb{R} \to X$ be a locally Bochner integrable and locally bounded function. We assume that for all $\nu \in \mathbb{R}$ (or $\mathbb{R}_+$) the function $\eta \longmapsto U(\nu,\eta) : [\nu,\infty) \to \mathcal{L}(X)$ is locally Hölder continuous (i.e. for all $a,b \geq \nu$, $a < b$, there exist $\alpha \in (0,1]$ and $H > 0$ such that*

$$\|U(\nu,t) - U(\nu,s)\| \leq H|t-s|^{\alpha}, \quad \text{for all } t,s \in [a,b] \text{).}$$

We use the notations in Section 4 for $a = 0$ and $b = t > 0$. The map $u(\cdot)$ from (5.4) can be represented as

$$u(t) = U(t,0)x_0 + \sum_{i=0}^{n-1} U(t,\xi_i)\int_{t_i}^{t_{i+1}} f(s)ds + R_n^{(1)}(\mathcal{U},f,I_n,\boldsymbol{\xi}) \tag{5.5}$$

where the remainder $R_n^{(1)}(\mathcal{U}, f, I_n, \boldsymbol{\xi})$ *satisfies the estimate*

$$\left\|R_n^{(1)}(\mathcal{U}, f, I_n, \boldsymbol{\xi})\right\| \le \frac{H}{\alpha+1} |||f|||_{[0,t],\infty} \sum_{i=0}^{n-1} \left[(t_{i+1}-\xi_i)^{\alpha+1} + (\xi_i - t_i)^{\alpha+1}\right].$$

Proof. It follows by representation (4.3) and the first estimate after it. ∎

Moreover, if n is a natural number, $i \in \{0, \dots, n\}$, $t_i := \frac{t \cdot i}{n}$ and $\xi_i := \frac{(2i+1)t}{2n}$, then

$$u(t) = U(t,0)x_0 + \sum_{i=0}^{n-1} U\left(t, \frac{(2i+1)t}{2n}\right) \int_{\frac{t \cdot i}{n}}^{\frac{t\cdot(i+1)}{n}} f(s)\,ds + R_n^{(1)} \tag{5.6}$$

and the remainder $R_n^{(1)}$ satisfies the estimate

$$\left\|R_n^{(1)}\right\| \le \frac{H}{\alpha+1} \cdot \frac{t^{\alpha+1}}{2^\alpha \cdot n^\alpha} |||f|||_{[0,t],\infty}. \tag{5.7}$$

The following theorem also holds.

Theorem 7. *Let* $\mathcal{U} = \{U(\nu, \eta) : \nu \ge \eta\} \subset \mathcal{L}(X)$ *be an exponentially bounded evolution family of bounded linear operators acting on the Banach space* X *and* $f : \mathbb{R} \to X$ *be a locally Hölder continuous function, i.e., for all* $a, b \in \mathbb{R}$, $a < b$ *there exist* $\beta \in (0,1]$ *and* $K > 0$ *such that (2.8) holds. We use the notations of Section 4 for* $a = 0$ *and* $b = t > 0$*. The map* $u(\cdot)$ *from (5.4) can be represented as*

$$u(t) = U(t,0)x_0 + \sum_{i=0}^{n-1} \left(\int_{t_i}^{t_{i+1}} U(t,\tau)\,d\tau\right)(f(\xi_i)) + R_n^{(2)}(\mathcal{U}, f, I_n, \boldsymbol{\xi}) \tag{5.8}$$

where the remainder $R_n^{(2)}(\mathcal{U}, f, I_n, \boldsymbol{\xi})$ *satisfies the estimate*

$$\left\|R_n^{(2)}(\mathcal{U}, f, I_n, \boldsymbol{\xi})\right\| \le \frac{KM}{\beta+1} e^{\omega t} \sum_{i=0}^{n-1} \left[(t_{i+1}-\xi_i)^{\beta+1} + (\xi_i - t_i)^{\beta+1}\right].$$

Proof. It follows from the first estimate in (4.7) for $B(s) := U(t,s)$, using the fact that

$$|||B(\cdot)|||_{[0,t],\infty} = \sup_{\tau \in [0,t]} \|U(t,\tau)\| \le \sup_{\tau \in [0,t]} M e^{\omega(t-\tau)} \le M e^{\omega t}.$$

∎

Moreover, if n is a natural number, $i \in \{0, \dots, n\}$, $t_i := \frac{t \cdot i}{n}$ and $\xi_i := \frac{(2i+1)t}{2n}$ then

$$u(t) = U(t,0)x_0 + \sum_{i=0}^{n-1} \left(\int_{\frac{t \cdot i}{n}}^{\frac{t\cdot(i+1)}{n}} U(t,\tau)\,d\tau\right) f\left(\frac{(2i+1)t}{2n}\right) + R_n^{(2)} \tag{5.9}$$

and the remainder $R_n^{(2)}$ satisfies the estimate

$$\left\|R_n^{(2)}\right\| \le \frac{KM}{\beta+1} e^{\omega t} \cdot \frac{t^{\beta+1}}{2^\beta \cdot n^\beta}. \tag{5.10}$$

6. Some Numerical Examples

1. Let $X = \mathbb{R}^2$, $x = (\xi, \eta) \in \mathbb{R}^2$, $\|x\|_2 = \sqrt{\xi^r + \eta^2}$. We consider the linear 2-dimensional system

$$\left\{ \begin{array}{l} \dot{u}_1(t) = \left(-1 - \sin^2 t\right) u_1(t) + \left(-1 + \sin t \cos t\right) u_2(t) + e^{-t}; \\ \\ \dot{u}_2(t) = \left(1 + \sin t \cos t\right) u_1(t) + \left(-1 - \cos^2 t\right) u_2(t) + e^{-2t}; \\ \\ u_1(0) = u_2(0) = 0. \end{array} \right. \tag{6.1}$$

If we denote

$$A(t) := \begin{pmatrix} -1 - \sin^2 t & -1 + \sin t \cos t \\ \\ 1 + \sin t \cos t & -1 - \cos^2 t \end{pmatrix}, \quad f(t) = \left(e^{-t}, e^{-2t}\right), \quad x = (0, 0)$$

and we identify (ξ, η) with $\begin{pmatrix} \xi \\ \eta \end{pmatrix}$, then the above system is a Cauchy problem. The evolution family associated with $A(t)$ is

$$U(t, s) = P(t) P^{-1}(s), \quad t \geq s, \ t, s \in \mathbb{R},$$

where

$$P(t) = \begin{pmatrix} e^{-t} \cos t & e^{-2t} \sin t \\ \\ -e^{-t} \sin t & e^{-2t} \cos t \end{pmatrix}, \quad t \in \mathbb{R}. \tag{6.2}$$

The exact solution of the system (6.1) is $u = (u_1, u_2)$, where

$$u_1(t) = \left(e^{-t} \cos t\right) E_1(t) + \left(e^{-2t} \sin t\right) E_2(t)$$

$$u_2(t) = -\left(e^{-t} \sin t\right) E_1(t) + \left(e^{-2t} \cos t\right) E_2(t), \quad t \in \mathbb{R},$$

and

$$\begin{aligned} E_1(t) &= \sin t + \frac{1}{2} e^{-t} (\cos t + \sin t) - \frac{1}{2}, \\ E_2(t) &= \sin t + \frac{1}{2} (\sin t - \cos t) \cdot e^t + \frac{1}{2}, \end{aligned}$$

see [2, Section 4] for details. The function $t \mapsto A(t)$ is bounded on $\mathbb{R}$ and therefore there exist $M \geq 1$ and $\omega > 0$

$$\|U(t, s)\| \leq M e^{\omega |t-s|}, \quad \text{for all } t, s \in \mathbb{R}.$$

Let $\xi \geq 0$ be fixed and $t, s \geq \xi$. Then there exists a real number μ between t and s such that

$$\|U(\xi, t) - U(\xi, s)\| = |t - s| \, \|U(\xi, \mu) A(\mu)\| \leq M e^{\omega \mu} \, \||A(\cdot)\||_\infty \cdot |t - s|,$$

that is, the function $\eta \mapsto U(\xi, \eta)$ is locally Lipschitz continuous on $[\xi, \infty)$.

Using (6.2), it follows

$$U(t, s) = \begin{pmatrix} a_{11}(t, s) & a_{12}(t, s) \\ \\ a_{21}(t, s) & a_{22}(t, s) \end{pmatrix},$$

where

$$\begin{aligned}
a_{11}(t,s) &= e^{(s-t)}\cos t\cos s + e^{2(s-t)}\sin t\sin s;\\
a_{12}(t,s) &= -e^{(s-t)}\cos t\sin s + \frac{1}{2}e^{2(s-t)}\sin t\cos s;\\
a_{21}(t,s) &= -e^{(s-t)}\sin t\cos s + e^{2(s-t)}\cos t\sin s;\\
a_{22}(t,s) &= e^{(s-t)}\sin t\sin s + \frac{1}{2}e^{2(s-t)}\cos t\cos s.
\end{aligned}$$

Then from (5.6) we obtain the following approximating formula for $u(\cdot)$:

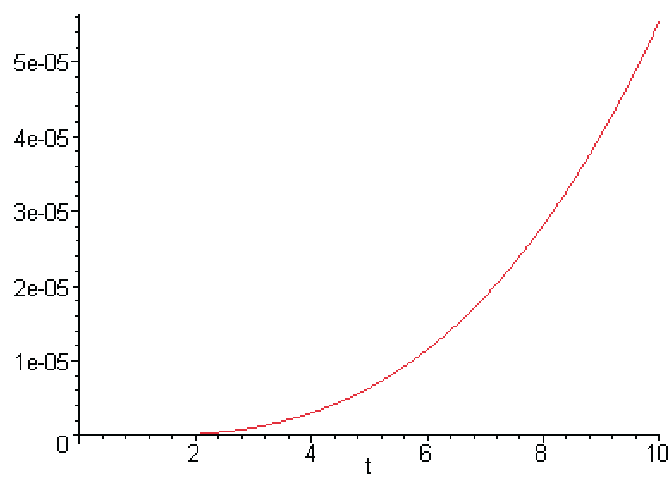
$$\begin{aligned}
u_1(t) &= -\sum_{i=0}^{n-1}\left[a_{11}\left(t,\frac{(2i+1)t}{2n}\right)\left(e^{-\frac{t(i+1)}{n}}-e^{-\frac{ti}{n}}\right)\right.\\
&\quad\left.+\frac{1}{2}a_{12}\left(t,\frac{(2i+1)t}{2n}\right)\left(e^{-\frac{2t(i+1)}{n}}-e^{-\frac{2ti}{n}}\right)\right]+R_{1,n}^{(1)}
\end{aligned}$$

and

$$\begin{aligned}
u_2(t) &= -\sum_{i=0}^{n-1}\left[a_{21}\left(t,\frac{(2i+1)t}{2n}\right)\left(e^{-\frac{t(i+1)}{n}}-e^{-\frac{ti}{n}}\right)\right.\\
&\quad\left.+\frac{1}{2}a_{22}\left(t,\frac{(2i+1)t}{2n}\right)\left(e^{-\frac{2t(i+1)}{n}}-e^{-\frac{2ti}{n}}\right)\right]+R_{2,n}^{(1)},
\end{aligned}$$

where the remainder $R_n^{(1)} = \left(R_{1,n}^{(1)}, R_{2,n}^{(1)}\right)$ satisfies the estimate (5.7) with $\alpha = 1$, $H = Me^{\omega t}\,|||A(\cdot)|||_{\infty}$ and $|||f|||_{[0,t],\infty} \le 2$.

The Figure 1 contains the behaviour of the error $\varepsilon_n(t) := \left\|\left(R_{1,n}^{(1)}, R_{2,n}^{(1)}\right)\right\|_2$ for $n = 200$.



2. Let $X = \mathbb{R}$ and $U(t,s) := \frac{t+1}{s+1}$, $t \ge s \ge 0$. It is clear that the family $\{U(t,s) : t \ge s \ge 0\} \subset \mathcal{L}(\mathbb{R})$ is an exponentially bounded evolution family which solves the Cauchy problem

$$\dot{u}(t) = \frac{1}{t+1}u(t), \quad u(s) = x_s \in \mathbb{R}, \quad t \ge s \ge 0.$$

Consider the inhomogeneous Cauchy problem

$$\left\{\begin{array}{l}\dot{u}(t)=\frac{1}{t+1}u(t)+\cos[\ln(t+1)], \quad t\geq 0\\ \\ u(0)=0.\end{array}\right. \tag{6.3}$$

The solution of (6.3) is given by

$$u(t)=\int_0^t\frac{t+1}{\tau+1}\cos(\ln(\tau+1))\,d\tau=(t+1)\sin[\ln(t+1)],\ t\geq 0.$$

From (5.9) we obtain the approximating formula for $u(\cdot)$ as,

$$u(t)=(t+1)\sum_{i=0}^{n-1}\ln\left[\frac{n+ti+t}{n+ti}\right]\cos\left\{\ln\left[1+\frac{(2i+1)t}{2n}\right]\right\}+R_n,$$

where R_n satisfies the estimate (5.10) with $K=M=\omega=\beta=1$. Indeed,

$$\frac{t+1}{s+1}\leq e^t,\ \text{ for all } t\geq s\geq 0$$

and

$$|\cos[\ln(t+1)]-\cos[\ln(s+1)]|=|t-s|\left|\frac{1}{c+1}\sin[\ln(c+1)]\right|\leq|t-s|$$

for all $t\geq s\geq 0$, where c is some real number between s and t.

The Figure 2 contains the behaviour of the error $\varepsilon_n(t):=|R_n|$ for $n=400$.

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