OSTROWSKI TYPE INEQUALITY FOR ABSOLUTELY CONTINUOUS FUNCTIONS ON SEGMENTS IN LINEAR SPACES

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Abstract. An Ostrowski type inequality is developed for estimating the deviation of the integral mean of an absolutely continuous function and the linear combination of its values at $k+1$ partition points on a segment in (real) linear spaces. Some particular cases are provided which recapture earlier results along with the results for trapezoidal type inequalities and the classical Ostrowski inequality. Inequalities are obtained by applying these results for semi-inner products and some of these inequalities are proven to be sharp.

1. Introduction

In 1938, A. Ostrowski (see [25, p. 226]) considered the problem of estimating the deviation of a function from its integral mean. For any continuous function f on $[a,b] \subset \mathbb{R}$ which is differentiable on (a,b) and $|f'(x)| \le M$ for all $x \in (a,b)$, the inequality

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(x)dx \right| \le \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right](b-a)M, \tag{1.1}$$

holds for every $x \in [a,b]$ (see [25, p. 226–227] for the complete proof). This is then known as the Ostrowski inequality (see [24, p. 468]). The first factor on the right hand side of (1.1) reaches the value of $\frac{1}{4}$ at the midpoint and monotonically increases to $\frac{1}{2}$ which is attained at both endpoints [25, p. 226]. It implies that the constant $\frac{1}{4}$ is best possible, that is, it cannot be replaced by a smaller quantity (see also [2, p. 3775–3776], for an alternative proof).

The Ostrowski inequality has been generalised for functions of bounded variations (see [15, p. 374] and [19, p. 3–4]). For this class of functions, the results have been developed to estimate the absolute difference between the linear combination of values of a function at $k+1$ partition points (of a closed interval) from its integral mean (see [10]). A similar result has been obtained for the class of absolutely continuous functions (see [11,12,15,18,19]). The classical Ostrowski inequality and the trapezoidal type inequality were obtained by considering some particular cases of the generalised Ostrowski type inequality (see [15, p. 378–381]).

Another possibility of generalising the Ostrowski inequality is to consider the case of convex functions. Since any convex function is locally Lipschitzian (hence, it is locally absolutely continuous), thus it can be connected to the previous mentioned cases (see [15,17]). For other possible directions, we refer to the results in [4–9].

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An extension of the Ostrowski inequality to functions with values in Banach spaces has been given in [3]. The result can be stated as follows:

Theorem 1. *Let $(X, \|\cdot\|)$ be a Banach space with the Radon-Nikodym property and $f : [a,b] \to X$ an absolutely continuous function on $[a,b]$. Then we have the inequalities,*

$$\left\| f(s) - \frac{1}{b-a}(B)\int_a^b f(t)dt \right\|$$
$$\leq \begin{cases} \left[\frac{1}{4} + \left(\frac{s-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)|||f'|||_{[a,b],\infty}, & \text{if } f' \in L_\infty([a,b],X); \\ \frac{1}{(q+1)^{\frac{1}{q}}}\left[\left(\frac{s-a}{b-a}\right)^{q+1} + \left(\frac{b-s}{b-a}\right)^{q+1}\right]^{\frac{1}{q}}(b-a)^{\frac{1}{q}}|||f'|||_{[a,b],p}, & \\ \qquad \text{if } f' \in L_p([a,b],X),\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; & \\ \left[\frac{1}{2} + \left|s - \frac{a+b}{2}\right|\right]|||f'|||_{[a,b],1}, & \end{cases}$$

for any $s \in [a,b]$, where

$$|||f'|||_{[a,b],\infty} := \operatorname{ess}\sup_{t\in[a,b]} \|f'(t)\|, \quad \text{and} \quad |||f'|||_{[a,b],p} := \left(\int_a^b \|f'(t)\|^p dt\right)^{\frac{1}{p}}, \ p \geq 1.$$

A similar result has been established for functions defined on segments in general linear space (see [18]). The result can be summarised as follows:

Theorem 2. *Let X be a linear space, $x, y \in X$, $x \neq y$ and $f : [x,y] \subset X \to \mathbb{R}$ be a function defined on the segment $[x,y]$ and such that the Gâteaux differential $\nabla f[(1-\cdot)x + \cdot y](y-x)$ exists a.e. on $[0,1]$ and is Lebesgue integrable on $[0,1]$. Then for any $s \in [0,1]$ we have*

$$\left| \int_0^1 f[(1-t)x+ty]dt - f[(1-s)x+sy] \right|$$
$$(1.2) \qquad \leq \begin{cases} \left[\frac{1}{4} + \left(s-\frac{1}{2}\right)^2\right]\|\nabla f[(1-\cdot)x+\cdot y](y-x)\|_\infty, \\ \qquad \text{if } \nabla f[(1-\cdot)x+\cdot y](y-x) \in L_\infty[0,1]; \\ \frac{1}{(q+1)^{\frac{1}{q}}}[s^{q+1}+(1-s)^{q+1}]^{\frac{1}{q}}\|\nabla f[(1-\cdot)x+\cdot y](y-x)\|_p, \\ \qquad \text{if } \nabla f[(1-\cdot)x+\cdot y](y-x) \in L_p[0,1],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \left[\frac{1}{2} + \left|s-\frac{1}{2}\right|\right]\|\nabla f[(1-\cdot)x+\cdot y](y-x)\|_1, \end{cases}$$

where $\|\cdot\|_r$ ($r \in [1,\infty]$) are the usual Lebesgue norms on $L_r[x,y]$.

An application of Theorem 2 for semi-inner products in any normed linear spaces was also provided in [18, p. 95–99]. However, the sharpness for the constants of these inequalities has not been considered.

In this paper, we develop an Ostrowski type inequality for estimating deviation of the integral mean of an absolutely continuous function and the linear combination of its values at $k+1$ partition points on a segment in (real) linear spaces. We also provide some particular cases which recapture the results in [18] along with the results for trapezoidal type inequalities and the classical Ostrowski inequality. In a normed linear spaces, we obtain inequalities for semi-inner products by applying the obtained results and these inequalities are more general than those in [18]. Some of these inequalities are proven to be sharp and the proof also covers the sharpness of those in [18].

2. Definitions

Let X be a linear space (in this paper, we assume that the linear space is over the field of real numbers) and $x, y \in X$. We consider the Gâteaux lateral derivatives for any $x, y \in X$ and any function f defined on X, as

$$(\nabla_{\pm} f(x))(y) \quad := \quad \lim_{t \to 0^{\pm}} \frac{f(x+ty) - f(x)}{t},$$

if the above limits exist.

Let $x, y \in X,\ x \neq y$ and define the segment $[x, y] := \{(1-t)x + ty,\ t \in [0,1]\}$. Let $f : [x, y] \to \mathbb{R}$ and the associated function $h = g(x, y) : [0,1] \to \mathbb{R},\ h(t) = g(x, y)(t) := f[(1-t)x + ty],\ t \in [0,1]$. It is well known that the function h is absolutely continuous on $[0,1]$ if and only if h is differentiable almost everywhere, the derivative h' is Lebesgue integrable and $h(t) = \int_0^t h'(s)ds + h(0)$ (see [1, p. 263] and [27, p. 106–107]).

Lemma 1. *With the above notation, h is absolutely continuous if and only if f satisfies the following properties*

(1) $\nabla f[(1-\cdot)x + \cdot y](y-x)$ *exists almost everywhere on* $[0,1]$*;*
(2) $\nabla f[(1-\cdot)x + \cdot y](y-x)$ *is Lebesgue integrable on* $[0,1]$*;*
(3) $f[(1-t)x + ty] = \displaystyle\int_0^t \nabla f[(1-s)x + sy](y-x)ds + f(x).$

Definition 1. Let f be a real-valued function defined on a segment $[x, y]$ of a linear space X. We say that f is *absolutely continuous on segment* $[x, y]$ if f satisfies the conditions (1)-(3) of Lemma 1.

By Definition 1 and Lemma 1, we conclude that f is absolutely continuous on segment $[x, y]$ if and only if h is absolutely continuous on $[0,1]$.

Assume that $(X, \|\cdot\|)$ is a normed linear space. The function $f_0(x) = \frac{1}{2}\|x\|^2$ $(x \in X)$ is convex and the following limits

$$\langle x, y\rangle_{s(i)} \quad := \quad (\nabla_{+(-)} f_0(y))(x) = \lim_{t \to 0^{+(-)}} \frac{\|y + tx\|^2 - \|y\|^2}{2t},$$

exist for any $x, y \in X$. They are called the superior (inferior) semi-inner products associated to the norm $\|\cdot\|$ (see [16, p. 27–39] for further properties).

The function $f_r(x) = \|x\|^r$ $(x \in X$ and $1 \leq r < \infty)$ is also convex. Therefore, the following limits, which are related to superior (inferior) semi-inner products,

$$(\nabla_{+(-)} f_r(y))(x) \quad = \quad \lim_{t \to 0^{+(-)}} \frac{\|y + tx\|^r - \|y\|^r}{t} = r\|y\|^{r-2}\langle x, y\rangle_{s(i)}, \tag{2.1}$$

exist for all $x, y \in X$ whenever $r \geq 2$; otherwise, they exist for any $x \in X$ and nonzero $y \in X$. In particular, if $r = 1$, then the following limits

$$(\nabla_{+(-)} f_1(y))(x) \quad = \quad \lim_{t \to 0^{+(-)}} \frac{\|y + tx\| - \|y\|}{t} = \left\langle x, \frac{y}{\|y\|}\right\rangle_{s(i)},$$

exist for $x, y \in X$ and $y \neq 0$.

3. The Results

The following result is an Ostrowski type inequality for absolutely continuous functions defined on a segment in a linear space.

Theorem 3. *Let X be a linear space, $I_k : 0 = s_0 < s_1 < \cdots < s_{k-1} < s_k = 1$ be a division of the interval $[0,1]$ and α_i $(i = 0, \ldots, k+1)$ be $k+2$ points such that $\alpha_0 = 0$, $\alpha_i \in [s_{i-1}, s_i]$ $(i = 1, \ldots, k)$ and $\alpha_{k+1} = 1$. If $f : [x,y] \subset X \to \mathbb{R}$ is absolutely continuous on segment $[x,y]$, then*

$$\left| \int_0^1 f[(1-t)x+ty]dt - \sum_{i=0}^{k} (\alpha_{i+1} - \alpha_i) f[(1-s_i)x + s_i y] \right| \tag{3.1}$$

$$\leq \begin{cases} \left[\frac{1}{4} \sum_{i=0}^{k-1} h_i^2 + \sum_{i=0}^{k-1} \left(\alpha_{i+1} - \frac{s_i + s_{i+1}}{2} \right)^2 \right] \| \nabla f[(1-\cdot)x + \cdot y](y-x) \|_\infty, \\ \qquad \text{if } \nabla f[(1-\cdot)x + \cdot y](y-x) \in L_\infty[0,1]; \\ \frac{1}{(q+1)^{\frac{1}{q}}} \left[\sum_{i=0}^{k-1} \left[(\alpha_{i+1} - s_i)^{q+1} + (s_{i+1} - \alpha_{i+1})^{q+1} \right] \right]^{\frac{1}{q}} \| \nabla f[(1-\cdot)x + \cdot y](y-x) \|_p, \\ \qquad \text{if } \nabla f[(1-\cdot)x + \cdot y](y-x) \in L_p[0,1],\ p>1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ \left[\frac{1}{2} \nu(h) + \max_{i \in \{0,\ldots,k-1\}} \left| \alpha_{i+1} - \frac{s_i + s_{i+1}}{2} \right| \right] \| \nabla f[(1-\cdot)x + \cdot y](y-x) \|_1, \end{cases}$$

where $\nu(h) := \max\{h_i | i = 0, \ldots, k-1\}$, $h_i := s_{i+1} - s_i$ $(i = 0, \ldots, k-1)$ and $\| \cdot \|_p$ $(p \in [1, \infty])$ are the Lebesgue norms.
The constants $\frac{1}{4}$, $\frac{1}{(q+1)^{\frac{1}{q}}}$ and $\frac{1}{2}$ are sharp.

Proof. Under the assumptions, we have the Ostrowski type inequality for absolutely continuous function $h(\cdot)$ that has been established in [10–12, 15]

$$\left| \int_a^b h(t)dt - \sum_{i=0}^{k} (\alpha_{i+1} - \alpha_i) h(s_i) \right|$$

$$\leq \begin{cases} \left[\frac{1}{4} \sum_{i=0}^{k-1} h_i^2 + \sum_{i=0}^{k-1} \left(\alpha_{i+1} - \frac{s_i + s_{i+1}}{2} \right)^2 \right] \|h'\|_\infty, \\ \qquad \text{if } h' \in L_\infty[a,b]; \\ \frac{1}{(q+1)^{\frac{1}{q}}} \left[\sum_{i=0}^{k-1} \left[(\alpha_{i+1} - s_i)^{q+1} + (s_{i+1} - \alpha_{i+1})^{q+1} \right] \right]^{\frac{1}{q}} \|h'\|_p, \\ \qquad \text{if } h' \in L_p[a,b],\ p>1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ \left[\frac{1}{2} \nu(h) + \max_{i \in \{0,\ldots,k-1\}} \left| \alpha_{i+1} - \frac{s_i + s_{i+1}}{2} \right| \right] \|h'\|_1. \end{cases}$$

Consider the auxiliary function $h(t) = g(x,y)(t) = f[(1-t)x+ty]$ defined on $[0,1]$. Since f is absolutely continuous on the segment $[x,y]$, it follows that $h = g(x,y)$ is an absolutely continuous function and we may apply the above inequality. We obtain the desired result by writing the above inequality for $h(t) = g(x,y)(t)$. The sharpness of the constants follows by the particular cases which are given in Corollary 1 and Corollary 2. □

Corollary 1. *Let X be a linear space, $x, y \in X$, $x \neq y$ and $f : [x, y] \subset X \to \mathbb{R}$ be an absolutely continuous function on segment $[x, y]$. Then for any $s \in [0, 1]$ we have the inequalities*

$$\begin{aligned}
&\left|\int_0^1 f[(1-t)x+ty]dt - sf(x) - (1-s)f(y)\right| \\
(3.2)\qquad &\leq \begin{cases} \left[\frac{1}{4} + \left(s-\frac{1}{2}\right)^2\right] \|\nabla f[(1-\cdot)x+\cdot y](y-x)\|_\infty, \\ \qquad\qquad \text{if } \nabla f[(1-\cdot)x+\cdot y](y-x) \in L_\infty[0,1]; \\ \frac{1}{(q+1)^{\frac{1}{q}}}\left[s^{q+1}+(1-s)^{q+1}\right]^{\frac{1}{q}} \|\nabla f[(1-\cdot)x+\cdot y](y-x)\|_p, \\ \qquad\qquad \text{if } \nabla f[(1-\cdot)x+\cdot y](y-x) \in L_p[0,1], p>1, \frac{1}{p}+\frac{1}{q}=1; \\ \left[\frac{1}{2}+\left|s-\frac{1}{2}\right|\right] \|\nabla f[(1-\cdot)x+\cdot y](y-x)\|_1, \end{cases}
\end{aligned}$$

where $\|\cdot\|_p$ ($p \in [1,\infty]$) are the usual Lebesgue norms on $L_p[0, 1]$. Particularly, we have

$$\begin{aligned}
&\left|\frac{f(x)+f(y)}{2} - \int_0^1 f[(1-t)x+ty]dt\right| \\
(3.3)\qquad &\leq \begin{cases} \frac{1}{4}\|\nabla f[(1-\cdot)x+\cdot y](y-x)\|_\infty, \\ \frac{1}{2(q+1)^{\frac{1}{q}}}\|\nabla f[(1-\cdot)x+\cdot y](y-x)\|_p, \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{2}\|\nabla f[(1-\cdot)x+\cdot y](y-x)\|_1. \end{cases}
\end{aligned}$$

The constants in (3.2) and (3.3) are sharp.

Proof. Choose $s_0 = 0$, $s_1 = 1$ and $0 = \alpha_0 < \alpha_1 = s < \alpha_2 = 1$ in Theorem 3 to obtain (3.2). The sharpness of the constants in (3.2) follows by the particular case (that is, (3.3)). By choosing $s = \frac{1}{2}$ in (3.2), we obtain (3.3). Now, we will prove the sharpness of the constants in (3.3). Let α and β be real positive constants such that

$$\left|\frac{f(x)+f(y)}{2} - \int_0^1 f[(1-t)x+ty]dt\right| \leq \begin{cases} \alpha\|\nabla f[(1-\cdot)x+\cdot y](y-x)\|_\infty, \\ \beta\left(\frac{1}{(q+1)^{\frac{1}{q}}}\right)\|\nabla f[(1-\cdot)x+\cdot y](y-x)\|_p, \\ \qquad\qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \end{cases}$$

Take $X = \mathbb{R}$, $[x, y] = [a, b] \subset \mathbb{R}$ ($a \neq b$) and $f(x) = \left|x - \frac{b+a}{2}\right|$. Note that f is a convex function on the closed interval $[a, b]$, thus, it is an absolutely continuous function (see [27, Proposition 5.16]). Therefore,

$$\frac{1}{4}(b-a) \leq \begin{cases} \alpha(b-a), \\ \beta\left(\frac{1}{(q+1)^{\frac{1}{q}}}\right)(b-a)^{\frac{1}{q}},\ q>1. \end{cases}$$

From the first case, we obtain $\alpha \geq \frac{1}{4}$ since $b - a \neq 0$, which proves the sharpness of $\frac{1}{4}$ in the first case of (3.3). Now, let $q \to 1$ in the second case, we obtain $\frac{1}{4}(b-a) \leq \frac{1}{2}\beta(b-a)$, that is, $\beta \geq \frac{1}{2}$, since $b - a \neq 0$, which shows that $\frac{1}{2}$ is sharp in the second case of (3.3).

Now, suppose that

$$\left|\frac{f(x)+f(y)}{2} - \int_0^1 f[(1-t)x+ty]dt\right| \leq \gamma\|\nabla f[(1-\cdot)x+\cdot y](y-x)\|_1,$$

for a real constant $\gamma > 0$. By choosing $X = \mathbb{R}$ and the absolutely continuous function $f(x) = \frac{C}{C^2+x^2} - \tan^{-1}\left(\frac{1}{C}\right)$ $(C > 0)$ on the interval $[0, 1]$ (the proof of this part is due to Peachey, McAndrew and Dragomir [26, p. 99–100]), we obtain

$$\frac{1}{2C} - \tan^{-1}\left(\frac{1}{C}\right) + \frac{C}{2(C^2+1)} \leq \gamma\left[\frac{1}{C(C^2+1)}\right].$$

Thus,

$$\gamma \geq (C^2+1)\left[\frac{1}{2} - C\tan^{-1}\left(\frac{1}{C}\right) + \frac{C^2}{2(C^2+1)}\right],$$

and by taking $C \to 0^+$, we obtain $\gamma \geq \frac{1}{2}$ and the proof for the sharpness of the constants in (3.3) is complete. This implies that all constants in (3.1) and(3.2) are sharp. □

Remark 1. If we assume that the function f in (3.3) is convex, then the quantity $\left(\frac{f(x)+f(y)}{2} - \int_0^1 f[(1-t)x+ty]dt\right)$ is positive by the Hermite-Hadamard integral inequality (see [14, p. 2]).

Corollary 2. *Let X be a linear space, $x, y \in X$, $x \neq y$ and $f : [x, y] \subset X \to \mathbb{R}$ be an absolutely continuous function on segment $[x, y]$. Then for any $s \in [0, 1]$ we have the inequalities*

$$\left|\int_0^1 f[(1-t)x+ty]dt - f[(1-s)x+sy]\right|$$

$$(3.4)\qquad \leq \begin{cases} \left[\frac{1}{4} + \left(s-\frac{1}{2}\right)^2\right]\|\nabla f[(1-\cdot)x+\cdot y](y-x)\|_\infty, \\ \qquad\qquad \textit{if } \nabla f[(1-\cdot)x+\cdot y](y-x) \in L_\infty[0,1]; \\ \frac{1}{(q+1)^{\frac{1}{q}}}[s^{q+1}+(1-s)^{q+1}]^{\frac{1}{q}}\|\nabla f[(1-\cdot)x+\cdot y](y-x)\|_p, \\ \qquad \textit{if } \nabla f[(1-\cdot)x+\cdot y](y-x) \in L_p[0,1], p>1, \frac{1}{p}+\frac{1}{q}=1; \\ \left[\frac{1}{2} + \left|s-\frac{1}{2}\right|\right]\|\nabla f[(1-\cdot)x+\cdot y](y-x)\|_1, \end{cases}$$

where $\|\cdot\|_p$ $(p \in [1,\infty])$ are the usual Lebesgue norms on $L_p[0,1]$. Particularly, we have

$$\left|\int_0^1 f[(1-t)x+ty]dt - f\left(\frac{x+y}{2}\right)\right|$$

$$(3.5)\qquad \leq \begin{cases} \frac{1}{4}\|\nabla f[(1-\cdot)x+\cdot y](y-x)\|_\infty, \\ \frac{1}{2(q+1)^{\frac{1}{q}}}\|\nabla f[(1-\cdot)x+\cdot y](y-x)\|_p, \\ \qquad\qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{2}\|\nabla f[(1-\cdot)x+\cdot y](y-x)\|_1. \end{cases}$$

The constants in (3.4) and (3.5) are sharp.

Proof. Choose $s_0 = 0 \leq s_1 = s \leq 1 = s_2$ and $0 \leq \alpha_1 \leq s \leq \alpha_2 \leq 1$ in Theorem 3, then let $\alpha_1 = 0$ and $\alpha_2 = 1$, to obtain the (3.4). The sharpness of the constants in (3.4) would follow by the particular case (that is, (3.5)). By choosing $s = \frac{1}{2}$ in (3.4), we obtain (3.5). Now, we will prove the sharpness of the constants in (3.5). Let ζ and η be real positive constants such that

$$\left|\int_0^1 f[(1-t)x+ty]dt - f\left(\frac{x+y}{2}\right)\right| \leq \begin{cases} \zeta\|\nabla f[(1-\cdot)x+\cdot y](y-x)\|_\infty, \\ \eta\left(\frac{1}{(q+1)^{\frac{1}{q}}}\right)\|\nabla f[(1-\cdot)x+\cdot y](y-x)\|_p, \\ \qquad\qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1. \end{cases}$$

If we choose $X=\mathbb{R}$, $[x,y]=[a,b]$ $(a\neq b)$ and $f(x)=\left|x-\frac{b+a}{2}\right|$, then we obtain

$$\frac{1}{4}(b-a)\leq\begin{cases}\zeta(b-a),\\ \eta\left(\frac{1}{(q+1)^{\frac{1}{q}}}\right)(b-a)^{\frac{1}{q}},\ q>1.\end{cases}$$

From the first case, we obtain $\zeta\geq\frac{1}{4}$ since $b-a\neq0$, which proves the sharpness of $\frac{1}{4}$ in the first case of (3.5). Now, let $q\to1$ in the second case, we obtain $\frac{1}{4}(b-a)\leq\frac{1}{2}\eta(b-a)$, that is, $\eta\geq\frac{1}{2}$, since $b-a\neq0$, which shows that $\frac{1}{2}$ is sharp in the second case of (3.5).

Now, suppose that

$$\left|\frac{f(x)+f(y)}{2}-\int_0^1 f[(1-t)x+ty]dt\right|\leq\theta\|\nabla f[(1-\cdot)x+\cdot y](y-x)\|_1,$$

for a real constant $\theta>0$. By choosing $X=\mathbb{R}$ and the absolutely continuous function $f(x)=\frac{C}{C^2+x^2}-\tan^{-1}\left(\frac{1}{C}\right)$ $(C>0)$ on the interval $[-1,1]$ (the proof of this part is due to Peachey, McAndrew and Dragomir [26, p. 99–100]), we obtain

$$\left|-\frac{1}{C}+\tan^{-1}\left(\frac{1}{C}\right)\right|\leq\theta\left[\frac{2}{C(C^2+1)}\right].$$

Thus,

$$\theta\geq\frac{C^2+1}{2}\left|-1+C\tan^{-1}\left(\frac{1}{C}\right)\right|,$$

and by taking $C\to0^+$, we obtain $\theta\geq\frac{1}{2}$ and the proof for the sharpness of the constants in (3.5) is complete. This implies that all constants in (3.1) and (3.4) are sharp. □

Remark 2. If we assume that the function f in (3.5) is convex, then the quantity $\left(\int_0^1 f[(1-t)x+ty]dt-f\left(\frac{x+y}{2}\right)\right)$ is positive by the Hermite-Hadamard integral inequality (see [13, p. 2]).

Remark 3. The inequality (3.5) has been obtained in [18, Corollary 1]. We also note that the bounds in (3.4) and (3.5) are the same as the ones in (3.2) and (3.3), respectively. Cerone in [5, Remark 1] stated that there is a strong relationship between the Ostrowski and trapezoidal functionals which is highlighted by the symmetric transformations amongst their kernels. Particularly, the bounds in the Ostrowski and trapezoidal type inequalities are the same [5, p. 317].

Example 1. Let $(X,\|\cdot\|)$ be a normed linear space and consider the absolutely continuous function $f(x)=\ln(\|x\|)$, $x\in X\setminus\{0\}$. Applying this to (3.3) and (3.5), we obtain

$$\left|\ln(\sqrt{\|x\|\|y\|})-\int_0^1\ln(\|(1-t)x+ty\|)dt\right|$$
$$\leq\begin{cases}\frac{1}{4}\sup\limits_{u\in[0,1]}\left|\frac{\langle y-x,(1-u)x+uy\rangle_{s(i)}}{\|(1-u)x+uy\|^2}\right|,\\ \frac{1}{2(q+1)^{\frac{1}{q}}}\left(\int_0^1\left|\frac{\langle y-x,(1-u)x+uy\rangle_{s(i)}}{\|(1-u)x+uy\|^2}\right|^p du\right)^{\frac{1}{p}},\ p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \frac{1}{2}\int_0^1\left|\frac{\langle y-x,(1-u)x+uy\rangle_{s(i)}}{\|(1-u)x+uy\|^2}\right|du,\end{cases}$$

and

$$\left|\int_0^1 \ln(\|(1-t)x+ty\|)dt - \ln\left(\left\|\frac{x+y}{2}\right\|\right)\right|$$
$$\leq \begin{cases} \frac{1}{4}\sup\limits_{u\in[0,1]}\left|\frac{\langle y-x,(1-u)x+uy\rangle_{s(i)}}{\|(1-u)x+uy\|^2}\right|, \\ \frac{1}{2(q+1)^{\frac{1}{q}}}\left(\int_0^1 \left|\frac{\langle y-x,(1-u)x+uy\rangle_{s(i)}}{\|(1-u)x+uy\|^2}\right|^p du\right)^{\frac{1}{p}}, \ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{2}\int_0^1 \left|\frac{\langle y-x,(1-u)x+uy\rangle_{s(i)}}{\|(1-u)x+uy\|^2}\right| du, \end{cases}$$

for any linearly independent $x, y \in X$. Using the Cauchy-Schwarz inequality for superior (inferior) semi-inner products (see [16, p. 29]), we obtain

$$\left|\ln(\sqrt{\|x\|\|y\|}) - \int_0^1 \ln(\|(1-t)x+ty\|)dt\right|$$
$$\leq \|y-x\| \begin{cases} \frac{1}{4}\sup\limits_{u\in[0,1]}\|(1-u)x+uy\|^{-1}, \\ \frac{1}{2(q+1)^{\frac{1}{q}}}\left(\int_0^1 \|(1-u)x+uy\|^{-p}du\right)^{\frac{1}{p}}, \ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{2}\int_0^1 \|(1-u)x+uy\|^{-1}du, \end{cases}$$

and

$$\left|\int_0^1 \ln(\|(1-t)x+ty\|)dt - \ln\left(\left\|\frac{x+y}{2}\right\|\right)\right|$$
$$\leq \|y-x\| \begin{cases} \frac{1}{4}\sup\limits_{u\in[0,1]}\|(1-u)x+uy\|^{-1}, \\ \frac{1}{2(q+1)^{\frac{1}{q}}}\left(\int_0^1 \|(1-u)x+uy\|^{-p}du\right)^{\frac{1}{p}}, \ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{2}\int_0^1 \|(1-u)x+uy\|^{-1}du, \end{cases}$$

for any linearly independent $x, y \in X$.

4. Application for Semi-Inner Products

The following result holds in any normed linear space with the semi-inner products $\langle\cdot,\cdot\rangle_{s(i)}$.

Proposition 1. *Let $(X, \|\cdot\|)$ be a normed linear space, $I_k : 0 = s_0 < s_1 < \cdots < s_{k-1} < s_k = 1$ be a division of the interval $[0,1]$ and α_i $(i = 0, \ldots, k+1)$ be $k+2$ points such that $\alpha_0 = 0$, $\alpha_i \in [s_{i-1}, s_i]$ $(i = 1, \ldots, k)$ and $\alpha_{k+1} = 1$. If $1 \leq r < \infty$*

then

$$\left|\int_0^1 \|(1-t)x+ty\|^r dt - \sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)\|(1-s_i)x+s_iy\|^r\right|$$

$$(4.1) \leq \begin{cases} \left[\frac{1}{4}\sum_{i=0}^{k-1} h_i^2 + \sum_{i=0}^{k-1}\left(\alpha_{i+1}-\frac{s_i+s_{i+1}}{2}\right)^2\right] \\ \qquad \times \sup_{u\in[0,1]}\left[r\|(1-u)x+uy\|^{r-2}|\langle y-x,(1-u)x+uy\rangle_{s(i)}|\right], \\ \frac{1}{(q+1)^{\frac{1}{q}}}\left[\sum_{i=0}^{k-1}\left[(\alpha_{i+1}-s_i)^{q+1}+(s_{i+1}-\alpha_{i+1})^{q+1}\right]\right]^{\frac{1}{q}} \\ \qquad \times \left[\int_0^1 |r\|(1-u)x+uy\|^{r-2}\langle y-x,(1-u)x+uy\rangle_{s(i)}|^p du\right]^{\frac{1}{p}}, \\ \qquad\qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \left[\frac{1}{2}\nu(h)+\max_{i\in\{0,\ldots,k-1\}}\left|\alpha_{i+1}-\frac{s_i+s_{i+1}}{2}\right|\right] \\ \qquad \times\int_0^1 |r\|(1-u)x+uy\|^{r-2}\langle y-x,(1-u)x+uy\rangle_{s(i)}|du, \end{cases}$$

holds for any $x, y \in X$, whenever $r \geq 2$, otherwise it holds for any linearly independent $x, y \in X$. Here, $\nu(h) := \max\{h_i | i = 0,\ldots,k-1\}$, $h_i := s_{i+1} - s_i$ ($i = 0,\ldots,k-1$) and $\|\cdot\|_p$ ($p \in [1,\infty]$) are the Lebesgue norms.

Proof. Let $f(x) = \|x\|^r$, where $x \in X$, and $1 \leq r < \infty$. Since f is convex on X then $g(x,y)(\cdot) = f((1-\cdot)x + \cdot y)$ is convex on $[0,1]$ for any $1 \leq r < \infty$ and $x, y \in X$. It follows that $g(x,y)(\cdot) = \|(1-\cdot)x + \cdot y\|^r$ is an absolutely continuous function. Therefore, we may apply Theorem 3 for f (see (2.1)) and obtained the desired result. □

Remark 4. The result we obtain in Proposition 1 is "complicated" in the sense that the upper bound is not practical to apply. Here, we suggest a simpler, although coarser, upper bound (see [18, p. 97–98]) using the Cauchy-Schwarz inequality for semi-inner products. Under the assumptions of Proposition 1, and by Cauchy-Schwarz type inequality for superior (inferior) semi-inner products (see [16, p. 29]), we obtain

$$\begin{aligned} &\sup_{u\in[0,1]}\left[r\|(1-u)x+uy\|^{r-2}|\langle y-x,(1-u)x+uy\rangle_{s(i)}|\right. \\ \leq\ & r\|y-x\|\sup_{u\in[0,1]}\|(1-u)x+uy\|^{r-1} = r\|y-x\|\max\{\|x\|^{r-1},\|y\|^{r-1}\}, \end{aligned}$$

for all $x, y \in X$.

Note: Consider the function $f(t) = \|(1-t)x+ty\|^{r-1}$ on $[0,1]$. Since it is continuous and convex on $[0,1]$, then the supremum of f on $[0,1]$ is exactly its maximum, and it is attained at one of the endpoints. In other words,

$$\sup_{u\in[0,1]}\|(1-u)x+uy\|^{r-1} = \max\{\|x\|^{r-1},\|y\|^{r-1}\}.$$

We also have the following for any $x, y \in X$

$$
\begin{aligned}
& r\left(\int_0^1 |\|(1-u)x+uy\|^{r-2}\langle y-x,(1-u)x+uy\rangle_{s(i)}|^p du\right)^{\frac{1}{p}} \\
\leq\ & r\|y-x\|\left(\int_0^1 \|(1-u)x+uy\|^{p(r-1)}du\right)^{\frac{1}{p}} \leq r\|y-x\|\left(\frac{\|x\|^{p(r-1)}+\|y\|^{p(r-1)}}{2}\right)^{\frac{1}{p}},
\end{aligned}
$$

by the Hermite-Hadamard inequality for the norm (see [21, p. 3] and [28, p. 106]), and

$$
\begin{aligned}
& r\left(\int_0^1 |\|(1-u)x+uy\|^{r-2}\langle y-x,(1-u)x+uy\rangle_{s(i)}|du\right) \\
\leq\ & r\|y-x\|\left(\int_0^1 \|(1-u)x+uy\|^{r-1}du\right) \leq \frac{1}{2}r\|y-x\|(\|x\|^{r-1}+\|y\|^{r-1}),
\end{aligned}
$$

by the refined triangle inequality for the norm (see [21, p. 4] and [28, p. 106]). Therefore, we have the following inequalities

$$
\begin{aligned}
& \left|\int_0^1 \|(1-t)x+ty\|^r dt - \sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)\|(1-s_i)x+s_iy\|^r\right| \\
& \leq r\|y-x\| \begin{cases}
\left[\frac{1}{4}\sum_{i=0}^{k-1} h_i^2 + \sum_{i=0}^{k-1}\left(\alpha_{i+1}-\frac{s_i+s_{i+1}}{2}\right)^2\right] \sup_{u\in[0,1]} \|(1-u)x+uy\|^{r-1}, \\
\frac{1}{(q+1)^{\frac{1}{q}}}\left[\sum_{i=0}^{k-1}\left[(\alpha_{i+1}-s_i)^{q+1}+(s_{i+1}-\alpha_{i+1})^{q+1}\right]\right]^{\frac{1}{q}} \\
\qquad\qquad \times\left[\int_0^1 \|(1-u)x+uy\|^{p(r-1)}du\right]^{\frac{1}{p}}, \\
\qquad\qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\
\left[\frac{1}{2}\nu(h)+\max_{i\in\{0,\dots,k-1\}}\left|\alpha_{i+1}-\frac{s_i+s_{i+1}}{2}\right|\right]\int_0^1 \|(1-u)x+uy\|^{r-1}du,
\end{cases}
\end{aligned} \tag{4.2}
$$

$$
\leq r\|y-x\| \begin{cases}
\left[\frac{1}{4}\sum_{i=0}^{k-1} h_i^2 + \sum_{i=0}^{k-1}\left(\alpha_{i+1}-\frac{s_i+s_{i+1}}{2}\right)^2\right] \max\{\|x\|^{r-1},\|y\|^{r-1}\}, \\
\frac{1}{(q+1)^{\frac{1}{q}}}\left[\sum_{i=0}^{k-1}\left[(\alpha_{i+1}-s_i)^{q+1}+(s_{i+1}-\alpha_{i+1})^{q+1}\right]\right]^{\frac{1}{q}} \\
\qquad\qquad \times\left[\frac{\|x\|^{p(r-1)}+\|y\|^{p(r-1)}}{2}\right]^{\frac{1}{p}}, \\
\qquad\qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\
\frac{1}{2}\left[\frac{1}{2}\nu(h)+\max_{i\in\{0,\dots,k-1\}}\left|\alpha_{i+1}-\frac{s_i+s_{i+1}}{2}\right|\right](\|x\|^{r-1}+\|y\|^{r-1}),
\end{cases} \tag{4.3}
$$

which holds for any $x, y \in X$. The constants in the first and second cases of (4.2) and (4.3) are sharp. The proof follows by its particular cases which are mentioned in Remark 5 and Remark 6.

Corollary 3. *Let X be a normed linear space, $s \in [0,1]$ and $1 \leq r < \infty$. Then, we have the inequality*

$$\left|\int_0^1 \|(1-t)x+ty\|^r dt - s\|x\|^r - (1-s)\|y\|^r\right| \tag{4.4}$$
$$\leq r\|y-x\| \begin{cases} \left[\frac{1}{4}+\left(s-\frac{1}{2}\right)^2\right] \sup\limits_{u\in[0,1]} \|(1-u)x+uy\|^{r-1}, \\ \frac{1}{(q+1)^{\frac{1}{q}}}\left[s^{q+1}+(1-s)^{q+1}\right]^{\frac{1}{q}} \left(\int_0^1 \|(1-u)x+uy\|^{p(r-1)}du\right)^{\frac{1}{p}}, \\ \qquad\qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \left[\frac{1}{2}+\left|s-\frac{1}{2}\right|\right] \int_0^1 \|(1-u)x+uy\|^{r-1}du, \end{cases}$$

$$\leq r\|y-x\| \begin{cases} \left[\frac{1}{4}+\left(s-\frac{1}{2}\right)^2\right] \max\{\|x\|^{r-1}, \|y\|^{r-1}\}, \\ \frac{1}{(q+1)^{\frac{1}{q}}}\left[s^{q+1}+(1-s)^{q+1}\right]^{\frac{1}{q}} \left[\frac{\|x\|^{p(r-1)}+\|y\|^{p(r-1)}}{2}\right]^{\frac{1}{p}}, \\ \qquad\qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{2}\left[\frac{1}{2}+\left|s-\frac{1}{2}\right|\right] (\|x\|^{r-1}+\|y\|^{r-1}), \end{cases} \tag{4.5}$$

which holds for any $x, y \in X$. The constants in the first and second cases of (4.4) and (4.5) are sharp.

Proof. Choose $s_0 = 0$, $s_1 = 1$ and $0 = \alpha_0 < \alpha_1 = s < \alpha_2 = 1$ in (4.2) and (4.3). The sharpness of the constants follows by the particular case which is pointed out in Remark 5. □

Remark 5. Particularly,

$$\left|\int_0^1 \|(1-t)x+ty\|^2 dt - s\|x\|^2 - (1-s)\|y\|^2\right| \tag{4.6}$$
$$\leq 2\|y-x\| \begin{cases} \left[\frac{1}{4}+\left(s-\frac{1}{2}\right)^2\right] \sup\limits_{u\in[0,1]} \|(1-u)x+uy\|, \\ \frac{1}{(q+1)^{\frac{1}{q}}}\left[s^{q+1}+(1-s)^{q+1}\right]^{\frac{1}{q}} \left(\int_0^1 \|(1-u)x+uy\|^{p}du\right)^{\frac{1}{p}}, \\ \qquad\qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \left[\frac{1}{2}+\left|s-\frac{1}{2}\right|\right] \int_0^1 \|(1-u)x+uy\|du, \end{cases}$$

$$\leq 2\|y-x\| \begin{cases} \left[\frac{1}{4}+\left(s-\frac{1}{2}\right)^2\right] \max\{\|x\|, \|y\|\}, \\ \frac{1}{(q+1)^{\frac{1}{q}}}\left[s^{q+1}+(1-s)^{q+1}\right]^{\frac{1}{q}} \left(\frac{\|x\|^{p}+\|y\|^{p}}{2}\right)^{\frac{1}{p}}, \\ \qquad\qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{2}\left[\frac{1}{2}+\left|s-\frac{1}{2}\right|\right] (\|x\|+\|y\|), \end{cases} \tag{4.7}$$

holds for any $x, y \in X$. The constants in the first and second cases of (4.6) and (4.7) are sharp. This implies that the constants in the first and second cases of (4.2), (4.3), (4.4) and (4.5) are also sharp (the proof follows by considering a particular case which is presented in Remark 8).

We also have

$$\left|\int_0^1 \|(1-t)x+ty\|dt - s\|x\| - (1-s)\|y\|\right| \leq \|y-x\| \begin{cases} \frac{1}{4}+\left(s-\frac{1}{2}\right)^2, \\ \frac{1}{(q+1)^{\frac{1}{q}}}\left[s^{q+1}+(1-s)^{q+1}\right]^{\frac{1}{q}}, q>1; \\ \frac{1}{2}+\left|s-\frac{1}{2}\right|, \end{cases} \tag{4.8}$$

for any $x, y \in X$. Note that for all $1 < q < \infty$ and $s \in [0,1]$,

(1) $\frac{1}{4}+\left(s-\frac{1}{2}\right)^2 = \frac{s^2+(1-s)^2}{2} = \int_0^1 |t-s|dt$,

(2) $\frac{1}{(q+1)^{\frac{1}{q}}}\left[s^{q+1}+(1-s)^{q+1}\right]^{\frac{1}{q}} = \left(\int_0^1 |t-s|^q dt\right)^{\frac{1}{q}}$,

(3) $\frac{1}{2}+\left|s-\frac{1}{2}\right| = \max\{s, 1-s\} = \sup_{t\in[0,1]} |t-s|$,

and $\int_0^1 |t-s|dt \leq \left(\int_0^1 |t-s|^q dt\right)^{\frac{1}{q}} \leq \sup_{t\in[0,1]} |t-s|$ by the Hölder inequality. Thus,

$$\frac{1}{4}+\left(s-\frac{1}{2}\right)^2 \leq \frac{1}{(q+1)^{\frac{1}{q}}}[s^{q+1}+(1-s)^{q+1}]^{\frac{1}{q}} \leq \frac{1}{2}+\left|s-\frac{1}{2}\right|. \tag{4.9}$$

We conclude that the constant $\frac{1}{4}$ is best possible among the constants in all cases of (4.8) and rewrite (4.8) as

$$\left|\int_0^1 \|(1-t)x+ty\|dt - s\|x\| - (1-s)\|y\|\right| \leq \left[\frac{1}{4}+\left(s-\frac{1}{2}\right)^2\right]\|y-x\|. \tag{4.10}$$

The constant $\frac{1}{4}$ in (4.10) is sharp (the proof follows by considering a particular case which is given in Remark 8).

Corollary 4. *Let X be a normed linear space, $s \in [0,1]$ and $1 \leq r < \infty$. Then, we have the inequality*

$$\left|\|(1-s)x+sy\|^r - \int_0^1 \|(1-t)x+ty\|^r dt\right| \leq r\|y-x\| \begin{cases} \left[\frac{1}{4}+(s-\frac{1}{2})^2\right] \sup_{u\in[0,1]} \|(1-u)x+uy\|^{r-1}\}, \\ \frac{1}{(q+1)^{\frac{1}{q}}}[s^{q+1}+(1-s)^{q+1}]^{\frac{1}{q}}\left(\int_0^1 \|(1-u)x+uy\|^{p(r-1)}du\right)^{\frac{1}{p}}, \\ \qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \left[\frac{1}{2}+\left|s-\frac{1}{2}\right|\right]\left(\int_0^1 \|(1-u)x+uy\|^{r-1}du\right). \end{cases} \tag{4.11}$$

$$
(4.12) \quad \le \; r\|y-x\| \begin{cases} \left[\frac{1}{4}+(s-\frac{1}{2})^2\right]\max\{\|x\|^{r-1},\|y\|^{r-1}\}, \\ \frac{1}{(q+1)^{\frac{1}{q}}}\left[s^{q+1}+(1-s)^{q+1}\right]^{\frac{1}{q}}\left(\frac{\|x\|^{p(r-1)}+\|y\|^{p(r-1)}}{2}\right)^{\frac{1}{p}}, \\ \qquad\qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{2}\left[\frac{1}{2}+\left|s-\frac{1}{2}\right|\right]\left(\|x\|^{r-1}+\|y\|^{r-1}\right). \end{cases}
$$

for any $x, y \in X$. The constants in the first and second cases of (4.11) and (4.12) are sharp.

Proof. Choose $s_0 = 0 \le s_1 = s \le 1 = s_2$ and $0 \le \alpha_1 \le s \le \alpha_2 \le 1$, then let $\alpha_1 = 0$ and $\alpha_2 = 1$ in (4.2) and (4.3). The sharpness of the constants follows by the particular case which is pointed out in Remark 6. □

Remark 6. Particularly, we have

$$
\left|\|(1-s)x+sy\|^2-\int_0^1\|(1-t)x+ty\|^2dt\right|
$$

$$
(4.13) \quad \le \; 2\|y-x\| \begin{cases} \left[\frac{1}{4}+(s-\frac{1}{2})^2\right]\max\{\|x\|,\|y\|\}, \\ \frac{1}{(q+1)^{\frac{1}{q}}}\left[s^{q+1}+(1-s)^{q+1}\right]^{\frac{1}{q}}\left(\int_0^1\|(1-u)x+uy\|^p du\right)^{\frac{1}{p}}, \\ \qquad\qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \left[\frac{1}{2}+\left|s-\frac{1}{2}\right|\right]\left(\int_0^1\|(1-u)x+uy\|du\right), \end{cases}
$$

$$
(4.14) \quad \le \; 2\|y-x\| \begin{cases} \left[\frac{1}{4}+\left(s-\frac{1}{2}\right)^2\right]\max\{\|x\|,\|y\|\}, \\ \frac{1}{(q+1)^{\frac{1}{q}}}\left[s^{q+1}+(1-s)^{q+1}\right]^{\frac{1}{q}}\left(\frac{\|x\|^p+\|y\|^p}{2}\right)^{\frac{1}{p}}, \\ \qquad\qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{2}\left[\frac{1}{2}+\left|s-\frac{1}{2}\right|\right]\left(\|x\|+\|y\|\right), \end{cases}
$$

and

$$
(4.15) \qquad \left|\|(1-s)x+sy\|-\int_0^1\|(1-t)x+ty\|dt\right| \le \left[\frac{1}{4}+\left(s-\frac{1}{2}\right)^2\right]\|y-x\|
$$

for any $x, y \in X$. The constants in the first and second cases of (4.13) and (4.14) are sharp as well as the one in (4.15) (the proof follows by considering a particular case which is pointed out in Remark 10). It implies the sharpness of the constants in the first and second cases of (4.2), (4.3), (4.11) and (4.12).

Remark 7. Again, we note that the bounds in (4.11), (4.12), (4.13), (4.14) and (4.15) are the same as the ones in (4.4), (4.5), (4.6), (4.7) and (4.10), respectively (see Remark 3).

5. Some Particular Cases of Interest

Proposition 2. *Let X be a normed linear space and $1 \le r < \infty$. Then*

$$
\begin{aligned}
0 &\le \frac{\|x\|^r + \|y\|^r}{2} - \int_0^1 \|(1-t)x + ty\|^r dt \\
&\le r\|y-x\| \begin{cases} \frac{1}{4} \sup\limits_{u\in[0,1]} \|(1-u)x+uy\|^{r-1}, \\ \frac{1}{2(q+1)^{\frac{1}{q}}} \left(\int_0^1 \|(1-u)x+uy\|^{p(r-1)} du \right)^{\frac{1}{p}}, \ p>1, \ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{2}\int_0^1 \|(1-u)x+uy\|^{r-1} du, \end{cases} \qquad (5.1) \\
&\le r\|y-x\| \begin{cases} \frac{1}{4} \max\{\|x\|^{r-1}, \|y\|^{r-1}\}, \\ \frac{1}{2(q+1)^{\frac{1}{q}}} \left(\frac{\|x\|^{p(r-1)} + \|y\|^{p(r-1)}}{2} \right)^{\frac{1}{p}}, \ p>1, \ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{4}(\|x\|^{r-1} + \|y\|^{r-1}), \end{cases} \qquad (5.2)
\end{aligned}
$$

holds for any $x, y \in X$. The constants in the first and second cases of (5.1) and (5.1) are sharp.

Proof. Choose $s = \frac{1}{2}$ in (4.4) and (4.5). The sharpness of the constants follows by the particular case which is presented in Remark 8. ☐

Remark 8. Particularly,

$$
\begin{aligned}
0 &\le \frac{\|x\|^2 + \|y\|^2}{2} - \int_0^1 \|(1-t)x + ty\|^2 dt \\
(5.3) \quad &\le \|y-x\| \begin{cases} \frac{1}{2} \sup\limits_{u\in[0,1]} \|(1-u)x+uy\|, \\ \frac{1}{(q+1)^{\frac{1}{q}}} \left(\int_0^1 \|(1-u)x+uy\|^{p} du \right)^{\frac{1}{p}}, \quad p>1, \ \frac{1}{p}+\frac{1}{q}=1; \\ \int_0^1 \|(1-u)x+uy\| du, \end{cases} \\
(5.4) \quad &\le \|y-x\| \begin{cases} \frac{1}{2} \max\{\|x\|, \|y\|\}, \\ \frac{1}{(q+1)^{\frac{1}{q}}} \left(\frac{\|x\|^{p} + \|y\|^{p}}{2} \right)^{\frac{1}{p}}, \quad p>1, \ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{2}(\|x\| + \|y\|), \end{cases}
\end{aligned}
$$

holds for any $x, y \in X$. The constant $\frac{1}{2}$ is sharp in the first case of (5.3) and (5.4).

The proof is as follows: suppose that the inequality holds for the constant $A > 0$ instead of $\frac{1}{2}$, that is,

$$\frac{\|x\|^2 + \|y\|^2}{2} - \int_0^1 \|(1-t)x + ty\|^2 dt \le A\|y-x\| \max\{\|x\|, \|y\|\}.$$

Note that it is sufficient for us to prove the sharpness of the constant in the first case of (5.4), since both quantities are equal.

Choose $(X, \|\cdot\|) = (\mathbb{R}, \|\cdot\|_1)$, $x = (\frac{1}{n}, n)$, and $y = (-\frac{1}{n}, n)$ for $n \in \mathbb{N}$, then we have

$$\frac{3n^2+2}{3n^2} \leq A\left(\frac{2n^2+2}{n^2}\right).$$

Taking $n \to \infty$, we obtain $1 \leq 2A$, that is, $A \geq \frac{1}{2}$. This implies that the constants in the first case of (4.6), (4.7), (5.1) and (5.2) are also sharp.

Note that the constants in the second case of (5.3) and (5.4) are also sharp. Suppose that the inequality holds for the constants $B, C > 0$ instead of the multiplicative constant 1, that is,

$$\begin{aligned} &\frac{\|x\|^2+\|y\|^2}{2} - \int_0^1 \|(1-t)x+ty\|^2 dt \\ \leq\ & B\frac{1}{(q+1)^{\frac{1}{q}}}\left(\int_0^1 \|(1-u)x+uy\|^p du\right)^{\frac{1}{p}} \leq C\frac{1}{(q+1)^{\frac{1}{q}}}\left(\frac{\|x\|^p+\|y\|^p}{2}\right)^{\frac{1}{p}}. \end{aligned}$$

Choose $(X, \|\cdot\|) = (\mathbb{R}, \|\cdot\|_1)$, $x = (\frac{1}{n}, n)$, and $y = (-\frac{1}{n}, n)$ for $n \in \mathbb{N}$, then we have

$$\frac{3n^2+2}{3n^2} \leq 2B\left(\frac{(n^2(n^2+1)^p+(n^2+1)^p-n^{2p+2})^{\frac{1}{p}}}{n^2(q+1)^{\frac{1}{q}}(p+1)^{\frac{1}{p}}}\right) \leq C\left(\frac{2(n^2+1)}{n^2(q+1)^{\frac{1}{q}}}\right)$$

Taking $q \to 1$ and $n \to \infty$, we obtain $B \geq 1$ and $C \geq 1$. Therefore, the constants in the second case of (4.6), (4.7), (5.1) and (5.2) are also sharp.

We also have

$$0 \leq \frac{\|x\|+\|y\|}{2} - \int_0^1 \|(1-t)x+ty\|dt \leq \begin{cases} \frac{1}{4}\|y-x\|, \\ \frac{1}{2(q+1)^{\frac{1}{q}}}\|y-x\|, \ q>1; \\ \frac{1}{2}\|y-x\|, \end{cases} \tag{5.5}$$

for any $x, y \in X$. Note that for any $1 < q < \infty$, we have $\frac{1}{4} \leq \frac{1}{2(q+1)^{\frac{1}{q}}} \leq \frac{1}{2}$ (the proof follows by choosing $s = \frac{1}{2}$ in (4.9)). Therefore, $\frac{1}{4}$ is the best possible among the constants of all cases in (5.5) and now we have

$$0 \leq \frac{\|x\|+\|y\|}{2} - \int_0^1 \|(1-t)x+ty\|dt \leq \frac{1}{4}\|y-x\|. \tag{5.6}$$

The constant $\frac{1}{4}$ in (5.6) is the best possible constant.

The proof is as follows: suppose that the inequality holds for any constant $D > 0$ instead of $\frac{1}{4}$, that is,

$$\frac{\|x\|+\|y\|}{2} - \int_0^1 \|(1-t)x+ty\|dt \leq D\|y-x\|.$$

Choose $(X, \|\cdot\|) = (\mathbb{R}^2, \|\cdot\|_1)$, $x = (2, 1)$, and $y = (2, -1)$ to obtain $\frac{1}{2} \leq 2D$, that is, $D \geq \frac{1}{4}$. Thus, the constant $\frac{1}{4}$ is sharp (this implies that the constant $\frac{1}{4}$ in (4.10) is sharp).

Remark 9 (The case of inner product space)**.** If X is an inner product space, the constant in the first case of (5.4) is not sharp, since

$$\frac{\|x\|+\|y\|}{2} - \int_0^1 \|(1-t)x+ty\|dt = \frac{1}{6}\|y-x\|^2,$$

and the fact that

$$\frac{1}{6}\|y-x\|^2 \le \frac{1}{6}\|y-x\|(\|x\|+\|y\|) = \frac{1}{3}\|y-x\|\max\{\|x\|,\|y\|\}.$$

The sharpness of the constant in the second case of (5.4) is not preserved in this case, since we have the fact that

$$\frac{1}{6}\|y-x\|^2 \le \frac{1}{6}\|y-x\|(\|x\|+\|y\|) \le \frac{1}{3}\|y-x\|(\|x\|^p+\|y\|^p)^{\frac{1}{p}},$$

and that $\frac{1}{3} \le \frac{1}{2^{\frac{1}{p}}(q+1)^{\frac{1}{q}}}$. The constant in the third case of (5.4) is not sharp, since

$$\frac{1}{6}\|y-x\|^2 \quad \le \quad \frac{1}{6}\|y-x\|(\|x\|+\|y\|).$$

The constant $\frac{1}{4}$ in (5.6) remains sharp in this case. The proof follows by choosing $(X,\|\cdot\|) = (\mathbb{R},|\cdot|)$, $x=1$, and $y=-1$.

Proposition 3. *Let X be a normed linear space and $1 \le r < \infty$. Then*

$$0 \le \int_0^1 \|(1-t)x+ty\|^r dt - \left\|\frac{x+y}{2}\right\|^r \tag{5.7}$$

$$\le r\|y-x\| \begin{cases} \frac{1}{4}\sup_{u\in[0,1]}\|(1-u)x+uy\|^{r-1}, \\ \frac{1}{2(q+1)^{\frac{1}{q}}}\left(\int_0^1 \|(1-u)x+uy\|^{p(r-1)}du\right)^{\frac{1}{p}}, p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{2}\int_0^1 \|(1-u)x+uy\|^{r-1}du, \end{cases} \tag{5.8}$$

$$\le r\|y-x\| \begin{cases} \frac{1}{4}\max\{\|x\|^{r-1},\|y\|^{r-1}\}, \\ \frac{1}{2(q+1)^{\frac{1}{q}}}\left(\frac{\|x\|^{p(r-1)}+\|y\|^{p(r-1)}}{2}\right)^{\frac{1}{p}}, p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{4}(\|x\|^{r-1}+\|y\|^{r-1}), \end{cases}$$

holds for any $x, y \in X$. The constants in the first and second cases of (5.7) and (5.8) are sharp.

Proof. Choose $s=\frac{1}{2}$ in (4.11) and (4.12). The sharpness of the constants follows by the particular case which is pointed out in Remark 10. □

Remark 10. Particularly,

$$
\begin{aligned}
0 &\leq \int_0^1 \|(1-t)x+ty\|^2 dt - \left\|\frac{x+y}{2}\right\|^2 \\
&\leq \|y-x\| \begin{cases} \frac{1}{2}\sup\limits_{u\in[0,1]} \|(1-u)x+uy\|, \\ \frac{1}{(q+1)^{\frac{1}{q}}}\left(\int_0^1 \|(1-u)x+uy\|^p du\right)^{\frac{1}{p}}, \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \int_0^1 \|(1-u)x+uy\| du, \end{cases} \qquad (5.9)\\
&\leq \|y-x\| \begin{cases} \frac{1}{2}\max\{\|x\|,\|y\|\}, \\ \frac{1}{(q+1)^{\frac{1}{q}}}\left(\frac{\|x\|^p+\|y\|^p}{2}\right)^{\frac{1}{p}}, \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{2}(\|x\|+\|y\|), \end{cases} \qquad (5.10)
\end{aligned}
$$

holds for any $x,y\in X$. The constant $\frac{1}{2}$ is sharp in the first case of (5.9) and (5.10).

The proof is as follows: suppose that the inequality holds for the constant $E>0$ instead of $\frac{1}{2}$, that is,

$$\int_0^1 \|(1-t)x+ty\|^2 dt - \left\|\frac{x+y}{2}\right\|^2 \leq E\|y-x\|\max\{\|x\|,\|y\|\}.$$

Again, it is sufficient for us to prove the sharpness of the constant in the first case of (5.10).

Choose $(X,\|\cdot\|)=(\mathbb{R},\|\cdot\|_1)$, $x=(\frac{1}{n},n)$, and $y=(-\frac{1}{n},n)$ for $n\in\mathbb{N}$, then we have

$$\frac{3n^2+1}{3n^2} \leq E\left(\frac{2n^2+2}{n^2}\right).$$

Taking $n\to\infty$, we obtain $1\leq 2E$, that is, $E\geq\frac{1}{2}$. This implies that the constants in the first case of (4.13), (4.14), (5.7) and (5.8) are also sharp.

Note that the constants in the second case of (5.9) and (5.10) are also sharp. Suppose that the inequality holds for the constants $F,G>0$ instead of the multiplicative constant 1, that is,

$$
\begin{aligned}
&\int_0^1 \|(1-t)x+ty\|^2 dt - \left\|\frac{x+y}{2}\right\|^2 \\
&\leq F\frac{1}{(q+1)^{\frac{1}{q}}}\left(\int_0^1 \|(1-u)x+uy\|^p du\right)^{\frac{1}{p}} \leq G\frac{1}{(q+1)^{\frac{1}{q}}}\left(\frac{\|x\|^p+\|y\|^p}{2}\right)^{\frac{1}{p}}.
\end{aligned}
$$

Choose $(X,\|\cdot\|)=(\mathbb{R},\|\cdot\|_1)$, $x=(\frac{1}{n},n)$, and $y=(-\frac{1}{n},n)$ for $n\in\mathbb{N}$, then we have

$$\frac{3n^2+1}{3n^2} \leq 2F\left(\frac{(n^2(n^2+1)^p+(n^2+1)^p-n^{2p+2})^{\frac{1}{p}}}{n^2(q+1)^{\frac{1}{q}}(p+1)^{\frac{1}{p}}}\right) \leq G\left(\frac{2(n^2+1)}{n^2(q+1)^{\frac{1}{q}}}\right)$$

Taking $q\to 1$ and $n\to\infty$, we obtain $F\geq 1$ and $G\geq 1$. Therefore, the constants in the second case of (4.13), (4.14), (5.7) and (5.8) are also sharp.

We also have

$$0 \leq \int_0^1 \|(1-t)x+ty\| dt - \left\|\frac{x+y}{2}\right\| \leq \frac{1}{4}\|y-x\| \qquad (5.11)$$

The constant $\frac{1}{4}$ in (5.11) is the best possible constant.

The proof is as follows: suppose that the inequality holds for any constant $H > 0$ instead of $\frac{1}{4}$, that is,

$$\int_0^1 \|(1-t)x + ty\|dt - \left\| \frac{x+y}{2} \right\| \leq H\|y-x\|.$$

Choose $(X, \|\cdot\|) = (\mathbb{R}^2, \|\cdot\|_1)$, $x = (2,1)$, and $y = (2,-1)$ to obtain $\frac{1}{2} \leq 2H$, that is, $H \geq \frac{1}{4}$. Thus, the constant $\frac{1}{4}$ is sharp (this implies that the constant $\frac{1}{4}$ in (4.15) is sharp).

Remark 11. Again, we note that the bounds in (5.7), (5.8), (5.9), (5.10) and (5.11) are the same as the ones in (5.1), (5.2), (5.3), (5.4) and (5.6), respectively (see Remark 3).

Remark 12 (The case of inner product space)**.** If X is an inner product space, the constant in the first case of (5.10) is not sharp, since

$$\int_0^1 \|(1-t)x + ty\|dt - \left\| \frac{x+y}{2} \right\| = \frac{1}{12}\|y-x\|^2,$$

and the fact that

$$\frac{1}{12}\|y-x\|^2 \leq \frac{1}{12}\|y-x\|(\|x\| + \|y\|) = \frac{1}{6}\|y-x\| \max\{\|x\|, \|y\|\}.$$

The sharpness of the constant in the second case of (5.10) is not preserved in this case, since we have the fact that

$$\frac{1}{12}\|y-x\|^2 \leq \frac{1}{12}\|y-x\|(\|x\| + \|y\|) \leq \frac{1}{6}\|y-x\|(\|x\|^p + \|y\|^p)^{\frac{1}{p}},$$

and that $\frac{1}{6} \leq \frac{1}{2^{\frac{1}{p}}(q+1)^{\frac{1}{q}}}$. The constant in the third case of (5.10) is not sharp, since

$$\frac{1}{12}\|y-x\|^2 \quad \leq \quad \frac{1}{12}\|y-x\|(\|x\| + \|y\|).$$

The constant $\frac{1}{4}$ in (5.11) remains sharp in this case. The proof follows by choosing $(X, \|\cdot\|) = (\mathbb{R}, |\cdot|)$, $x = 1$, and $y = -1$.

6. Comparison Analysis

In [21, p. 11, 15], we considered an Ostrowski type inequality for convex functions on linear spaces and obtained the following result in any normed linear space $(X, \|\cdot\|)$

$$\frac{\|x\|^r + \|y\|^r}{2} - \int_0^1 \|(1-t)x + ty\|^r dt \leq \frac{1}{8} r(\langle y-x, y\|y\|^{r-2}\rangle_i - \langle y-x, x\|x\|^{r-2}\rangle_s),$$

$$\int_0^1 \|(1-t)x + ty\|^r dt - \left\| \frac{x+y}{2} \right\|^r \leq \frac{1}{8} r(\langle y-x, y\|y\|^{r-2}\rangle_i - \langle y-x, x\|x\|^{r-2}\rangle_s),$$

for any $x, y \in X$ whenever $r \geq 2$; otherwise they hold for nonzero $x, y \in X$, and $\langle\cdot,\cdot\rangle_{s(i)}$ is the superior (inferior) semi-inner product with respect to the norm $\|\cdot\|$.

In this paper, we have considered the Ostrowski type inequality for absolutely continuous functions, which is more general than [21] and have obtained the following bounds for the left-hand side of the inequalities above (see (5.1), (5.2), (5.7) and (5.8)). The bound that we have obtained is,

$$\frac{1}{4} r\|y-x\| \sup_{u\in[0,1]} \|(1-u)x + uy\|^{r-1} = \frac{1}{4} r\|y-x\| \max\{\|x\|^{r-1}, \|y\|^{r-1}\}.$$

Note: Since the last two bounds in (5.1), (5.2), (5.7) and (5.8) are not significant for the case of $r = 1$ (see Remark 8 and Remark 10), we consider only the first bound for this section.

We want to compare the two bounds $\frac{1}{8}r(\langle y-x, y\|y\|^{r-2}\rangle_i - \langle y-x, x\|x\|^{r-2}\rangle_s)$ and $\frac{1}{4}r\|y-x\|\max\{\|x\|^{r-1}, \|y\|^{r-1}\}$. The bound that we obtained in this paper is simpler in the sense that it only involves the given norm, while the other one involves not only the given norm, but also the superior (inferior) semi-inner product associated to the norm. However, the bounds in [21] are proven better for the case of inner product spaces. The verification is as follows:

In an inner product space $(X, \langle\cdot,\cdot\rangle)$, consider the particular case of $r = 2$, we want to compare $\frac{1}{4}\langle y-x, y-x\rangle = \frac{1}{4}\|y-x\|^2$ and $\frac{1}{2}\|y-x\|\max\{\|x\|, \|y\|\}$. For all $x, y \in X$, we have

$$\frac{1}{4}\|y-x\|^2 \leq \frac{1}{4}\|y-x\|(\|x\|+\|y\|) \leq \frac{1}{2}\|y-x\|\max\{\|x\|, \|y\|\}.$$

We conclude that the bounds in [21] are better.

Next, we consider the particular case of $r = 1$, that is, we wish to compare $\frac{1}{8}\langle y-x, \frac{y}{\|y\|} - \frac{x}{\|x\|}\rangle$ and $\frac{1}{4}\|y-x\|$ (for nonzero $x, y \in X$). We recall the Dunkl-Williams inequality (see [20, p. 53], [22, p. 890] and [23, p. 448])

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \frac{2\|x-y\|}{\|x\|+\|y\|}, \tag{6.1}$$

which holds for nonzero x and y in an inner product space X. Now, for $x, y \in X$ where $x, y \neq 0$, we have

$$\begin{aligned} \frac{1}{8}\left\langle y-x, \frac{y}{\|y\|} - \frac{x}{\|x\|} \right\rangle &\leq \frac{1}{8}\|y-x\| \left\| \frac{y}{\|y\|} - \frac{x}{\|x\|} \right\| \\ &\leq \frac{1}{4}\|y-x\|\frac{\|x-y\|}{\|x\|+\|y\|} \leq \frac{1}{4}\|y-x\|\frac{\|x\|+\|y\|}{\|x\|+\|y\|} = \frac{1}{4}\|y-x\|. \end{aligned}$$

We conclude that the bounds in [21] are better.

For general $1 \leq r < \infty$, we conjecture that

Conjecture 1. In an inner product space $(X, \langle\cdot,\cdot\rangle)$, the following inequality

$$\frac{1}{8}r\langle y-x, y\|y\|^{r-2} - x\|x\|^{r-2}\rangle \leq \frac{1}{4}r\|y-x\|\max\{\|x\|^{r-1}, \|y\|^{r-1}\}$$

holds for any $x, y \in X$ whenever $r \geq 2$; otherwise it holds for any nonzero $x, y \in X$.

We observe that the above statement is true in some cases. Taking $X = \mathbb{R}$ and multiplication as its inner product and utilizing Maple for the following functions

$$\Phi(x,y) := \frac{1}{4}r|y-x|\max\{|x|^{r-1}, |y|^{r-1}\} - \frac{1}{8}r(y-x)(y|y|^{r-2} - x|x|^{r-2}),$$

for $x, y \in \mathbb{R}$, we observe that for several values of r, we have $\Phi(x,y) \geq 0$ for any $x, y \in X$ (see Figure 1 for the plot of Φ with the choice of $r = 3$). However, we have no analytical proof for this statement.

Conjecture 2. In a normed linear space $(X, \|\cdot\|)$, the following inequality

$$\frac{1}{8}r(\langle y-x, y\|y\|^{r-2}\rangle_i - \langle y-x, x\|x\|^{r-2}\rangle_s) \leq \frac{1}{4}r\|y-x\|\max\{\|x\|^{r-1}, \|y\|^{r-1}\}$$

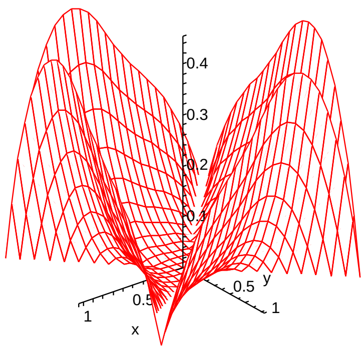

FIGURE 1. Plot of Φ for $r = 3$.

holds for any $x, y \in X$ whenever $r \geq 2$; otherwise it holds for any nonzero $x, y \in X$ (here, $\langle \cdot, \cdot \rangle_{s(i)}$ is the superior (inferior) semi-inner product respect to the norm $\|\cdot\|$).

We observe that the above statement is true in some cases. Taking $(X, \|\cdot\|) = (\mathbb{R}^2, \|\cdot\|_1)$ and consider the case of $r = 1$, we have the functions

$$\begin{aligned} f(x,y) &:= \frac{1}{8}\left[\sum_{y_i \neq 0} \frac{y_i}{|y_i|}(y_i - x_i) - \sum_{y_i = 0}|y_i - x_i| - \sum_{x_i \neq 0} \frac{x_i}{|x_i|}(y_i - x_i) - \sum_{x_i = 0}|y_i - x_i|\right], \\ g(x,y) &:= \frac{1}{4}\|y - x\|_1, \end{aligned}$$

for $x, y \in \mathbb{R}^2$. We observe that $f(x,y) \leq g(x,y)$ for some $x, y \in X$ (We choose $x = (1,0)$ and $y = (a,b)$ $(a, b \neq 0)$ and plot the non negative function $\Psi(a,b) := g(x,y) - f(x,y) = \frac{1}{4}(|a-1| + |b|) - \frac{1}{8}\left(\frac{a(a-1)}{|a|} + \frac{b^2}{|b|} - (a-1) - |b|\right)$ in Figure 2). However, we do not have an analytical proof for this statement.

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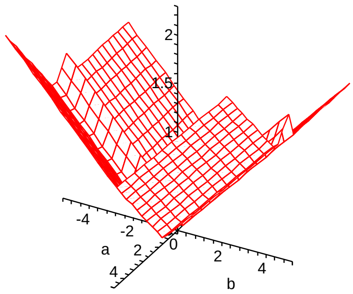

FIGURE 2. Plot of Ψ.

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