# ON THE KY FAN INEQUALITY

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Abstract. Some inequalities related to the Ky Fan and C.-L. Wang inequalities for weighted arithmetic and geometric means are given.

## 1. Introduction

In 1961, E.F. Beckenbach and R. Bellman published in their well known book "Inequalities" the following "unpublished result due to Ky Fan" [2, p. 5] (see also [1, p. 150]).

**Theorem 1.** *If* $0 < x_i \leq \frac{1}{2}$, $(i = 1, \ldots, n)$; *then:*

$$\left[ \prod_{i=1}^{n} x_i \Big/ \prod_{i=1}^{n} (1 - x_i) \right]^{\frac{1}{n}} \leq \sum_{i=1}^{n} x_i \Big/ \sum_{i=1}^{n} (1 - x_i) \tag{1.1}$$

*with equality only if* $x_1 = \cdots = x_n$.

A generalisation of Ky Fan's inequality for weighted means was proved by C.-L. Wang in 1980, [9].

**Theorem 2.** *If* $0 < x_i \leq \frac{1}{2}$, $(i = 1, \ldots, n)$, *then*

$$\frac{A_n(\bar{p}, \bar{x})}{A_n(\bar{p}, 1 - \bar{x})} \geq \frac{G_n(\bar{p}, \bar{x})}{G_n(\bar{p}, 1 - \bar{x})}, \tag{1.2}$$

*where* $p_i > 0$ $(i = 1, \ldots, n)$ *with* $\sum_{i=1}^{n} p_i = 1$ *and* $A_n(\bar{p}, \bar{x}) := \sum_{i=1}^{n} p_i x_i$ *is the weighted arithmetic mean,* $G_n(\bar{p}, \bar{x}) := \prod_{i=1}^{n} x_i^{p_i}$ *is the weighted geometric mean. The equality holds in (1.2) iff* $x_1 = \cdots = x_n$.

For a survey on related results of Ky Fan's inequality, see [1] by H. Alzer.

For different refinements and generalisations, see [4] – [8].

## 2. The Results

The following result holds.

**Theorem 3.** *Assume that* $0 < m \leq x_i \leq M \leq \frac{1}{2}$, $(i = 1, \ldots, n)$, $p_i > 0$ $(i = 1, \ldots, n)$, *with* $\sum_{i=1}^{n} p_i = 1$, *then we have the inequalities:*

$$\frac{A_n(\bar{p}, \bar{x})}{G_n(\bar{p}, \bar{x})} \geq \left[ \frac{A_n(\bar{p}, \bar{x})}{G_n(\bar{p}, \bar{x})} \right]^{\frac{M^2}{(1-M)^2}} \geq \frac{A_n(\bar{p}, 1 - \bar{x})}{G_n(\bar{p}, 1 - \bar{x})} \geq \left[ \frac{A_n(\bar{p}, \bar{x})}{G_n(\bar{p}, \bar{x})} \right]^{\frac{m^2}{(1-m)^2}} \geq 1. \tag{2.1}$$

*The equality will hold in all inequalities iff* $x_1 = \cdots = x_n$.

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*Proof.* The first and the last inequality in (2.1) follow by the fact that $\frac{A_n(\bar{p},\bar{x})}{G_n(\bar{p},\bar{x})} \geq 1$ (by the weighted arithmetic mean - geometric mean inequality), $m \in \left(0, \frac{1}{2}\right]$ and $M \in \left(0, \frac{1}{2}\right]$.

We define the function $f : (0,1) \to \mathbb{R}$, $f(t) = \ln\left(\frac{1-t}{t}\right) + \alpha \ln t$ with $\alpha \in \mathbb{R}$. We have

$$f'(t) = -\frac{1}{t(1-t)} + \frac{\alpha}{t}, \quad t \in (0,1),$$

$$f''(t) = \frac{1-2t}{[t(1-t)]^2} - \frac{\alpha}{t^2} = \frac{1}{t^2}\left[\frac{1-2t}{(1-t)^2} - \alpha\right], \quad t \in (0,1).$$

If we consider the function $g : (0,1) \to \mathbb{R}$, $g(t) = \frac{1-2t}{(1-t)^2}$, then $g'(t) = \frac{2t(t-1)}{(t-1)^4}$, showing that the function $g$ is monotonically strictly decreasing on $(0,1)$.

Consequently for $t \in (m, M)$, we have

$$\frac{1-2M}{(1-M)^2} = g(M) \leq g(t) \leq g(m) = \frac{1-2m}{(1-m)^2}. \tag{2.2}$$

Using (2.2) we observe that the function $f$ is strictly convex on $(m, M)$ if $\alpha \leq \frac{1-2M}{(1-M)^2}$.

Applying Jensen's discrete inequality for the function $f : (m, M) \to \mathbb{R}$, $f(t) = \ln\left(\frac{1-t}{t}\right) + \alpha \ln t$, with $\alpha \leq \frac{1-2M}{(1-M)^2}$, we deduce

$$\begin{aligned}\sum_{i=1}^{n} p_i \left[\ln\left(\frac{1-x_i}{x_i}\right) + \alpha \ln x_i\right] &= \sum_{i=1}^{n} p_i f(x_i) \geq f\left(\sum_{i=1}^{n} p_i x_i\right) \\ &= \ln\left(\frac{1-\sum_{i=1}^{n} p_i x_i}{\sum_{i=1}^{n} p_i x_i}\right) + \alpha \ln\left(\sum_{i=1}^{n} p_i x_i\right),\end{aligned}$$

which is equivalent to

$$\ln\left[\frac{G_n(\bar{p}, 1-\bar{x})}{G_n(\bar{p},\bar{x})}\right] + \alpha \ln G_n(\bar{p},\bar{x}) \geq \ln\left[\frac{A_n(\bar{p}, 1-\bar{x})}{A_n(\bar{p},\bar{x})}\right] + \alpha \ln A_n(\bar{p},\bar{x})$$

or, moreover, to

$$\ln\left[\frac{G_n(\bar{p},\bar{x})}{A_n(\bar{p},\bar{x})}\right]^{\alpha} \geq \ln\left[\frac{A_n(\bar{p}, 1-\bar{x})}{A_n(\bar{p},\bar{x})} \Big/ \frac{G_n(\bar{p}, 1-\bar{x})}{G_n(\bar{p},\bar{x})}\right],$$

that is,

$$\left[\frac{G_n(\bar{p},\bar{x})}{A_n(\bar{p},\bar{x})}\right]^{\alpha-1} \geq \frac{A_n(\bar{p}, 1-\bar{x})}{G_n(\bar{p}, 1-\bar{x})}. \tag{2.3}$$

Now, we observe that the inequality (2.3) is the best possible if $\alpha$ is maximal, i.e., $\alpha = \frac{1-2M}{(1-M)^2}$, getting

$$\left[\frac{G_n(\bar{p},\bar{x})}{A_n(\bar{p},\bar{x})}\right]^{\frac{1-2M}{(1-M)^2}-1} \geq \frac{A_n(\bar{p}, 1-\bar{x})}{G_n(\bar{p}, 1-\bar{x})},$$

which is clearly equivalent to the second inequality in (2.1).

The third inequality is produced in a similar fashion, using the function $h(t) = \beta \ln t - \ln\left(\frac{1-t}{t}\right)$ which is strictly convex on $(m, M)$ if $\beta \geq \frac{1-2m}{(1-m)^2}$.

The case of equality follows by the fact that in Jensen's inequality for strictly convex functions, the equality holds iff $x_1 = \cdots = x_n$.

We omit the details. ∎

**Remark 1.** *Since Wang's inequality (1.2) is equivalent to:*

$$\frac{A_n(\bar{p},\bar{x})}{G_n(\bar{p},\bar{x})} \geq \frac{A_n(\bar{p},1-\bar{x})}{G_n(\bar{p},1-\bar{x})}, \tag{2.4}$$

*then the first part of (2.1) may be seen as a refinement of Wang's result while the second part*

$$\frac{A_n(\bar{p},1-\bar{x})}{G_n(\bar{p},1-\bar{x})} \geq \left[\frac{A_n(\bar{p},\bar{x})}{G_n(\bar{p},\bar{x})}\right]^{\frac{m^2}{(1-m)^2}} \tag{2.5}$$

*can be considered a counterpart of (1.2).*

Now, let us recall the Lah-Ribarić inequality for convex functions (see for example [3, p. 140]).

If $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ is convex on $[a,b]$, $x_i \in [a,b]$, $p_i \geq 0$ $(i = 1, \ldots, n)$ and $\sum_{i=1}^n p_i = 1$, then

$$\sum_{i=1}^n p_i f(x_i) \leq \frac{b - \sum_{i=1}^n p_i x_i}{b-a} \cdot f(a) + \frac{\sum_{i=1}^n p_i x_i - a}{b-a} \cdot f(b). \tag{2.6}$$

Now, we can state and prove the following inequality related to the Ky Fan result.

**Theorem 4.** *Assume that* $0 < m \leq x_i \leq M \leq \frac{1}{2}$, $p_i > 0$ $\quad (i = 1, \ldots, n)$ *with* $\sum_{i=1}^n p_i = 1$, *then we have the inequalities:*

$$\begin{aligned} &\left(\frac{1-m}{m^{\left(\frac{m}{1-m}\right)^2}}\right)^{\frac{M-A_n(\bar{p},\bar{x})}{M-m}} \left(\frac{1-M}{M^{\left(\frac{m}{1-m}\right)^2}}\right)^{\frac{A_n(\bar{p},\bar{x})-m}{M-m}} \cdot G_n(\bar{p},\bar{x})^{\left(\frac{m}{1-m}\right)^2} \\ \leq\ & G_n(\bar{p},1-\bar{x}) \\ \leq\ & \left(\frac{1-m}{m^{\left(\frac{M}{1-M}\right)^2}}\right)^{\frac{M-A_n(\bar{p},\bar{x})}{M-m}} \left(\frac{1-M}{M^{\left(\frac{M}{1-M}\right)^2}}\right)^{\frac{A_n(\bar{p},\bar{x})-m}{M-m}} G_n(\bar{p},\bar{x})^{\left(\frac{M}{1-M}\right)^2}. \end{aligned} \tag{2.7}$$

*Proof.* From the proof of Theorem 3, we know that the function $f : (m,M) \subset \left(0, \frac{1}{2}\right] \to \mathbb{R}$, $f(t) = \ln\left(\frac{1-t}{t}\right) + \frac{1-2M}{(1-M)^2}\ln t$ is strictly convex on $(m,M)$. Now, if we apply the Lah-Ribarić inequality for $f$ as above, $a = m$ and $b = M$, we get:

$$\begin{aligned} &\sum_{i=1}^n p_i \left[\ln\left(\frac{1-x_i}{x_i}\right) + \frac{1-2M}{(1-M)^2}\ln x_i\right] \\ =\ & \sum_{i=1}^n p_i f(x_i) \leq \frac{M - \sum_{i=1}^n p_i x_i}{M-m} f(m) + \frac{\sum_{i=1}^n p_i x_i - m}{M-m} f(M) \\ =\ & \frac{M - \sum_{i=1}^n p_i x_i}{M-m}\left[\ln\left(\frac{1-m}{m}\right) + \frac{1-2M}{(1-M)^2}\ln m\right] \\ &+ \frac{\sum_{i=1}^n p_i x_i - m}{M-m}\left[\ln\left(\frac{1-M}{M}\right) + \frac{1-2M}{(1-M)^2}\ln M\right], \end{aligned}$$

which is equivalent to

$$\begin{aligned}
&\ln\left[\frac{G_n(\bar{p},1-\bar{x})}{G_n(\bar{p},\bar{x})}\right]+\frac{1-2M}{(1-M)^2}\ln G_n(\bar{p},\bar{x})\\
\leq\ &\frac{M-A_n(\bar{p},\bar{x})}{M-m}\left[\ln\left(\frac{1-m}{m}\right)+\ln(m)^{\frac{1-2M}{(1-M)^2}}\right]\\
&+\frac{A_n(\bar{p},\bar{x})-m}{M-m}\left[\ln\left(\frac{1-M}{M}\right)+\ln(M)^{\frac{1-2M}{(1-M)^2}}\right],
\end{aligned}$$

that is,

$$\begin{aligned}
&\frac{G_n(\bar{p},1-\bar{x})}{G_n(\bar{p},\bar{x})}\cdot[G_n(\bar{p},\bar{x})]^{\frac{1-2M}{(1-M)^2}}\\
\leq\ &\left((1-m)\,m^{\left\{\frac{1-2M}{(1-M)^2}-1\right\}}\right)^{\frac{M-A_n(\bar{p},\bar{x})}{M-m}}\cdot\left((1-M)\,M^{\left\{\frac{1-2M}{(1-M)^2}-1\right\}}\right)^{\frac{A_n(\bar{p},\bar{x})-m}{M-m}}
\end{aligned}$$

from which we obtain the second inequality in (2.7).

To prove the first inequality, we apply the Lah-Ribarić inequality for the function $h:(m,M)\to\mathbb{R}$, $h(t)=\frac{1-2m}{(1-m)^2}\ln t-\ln\left(\frac{1-t}{t}\right)$ which is strictly convex on $(m,M)$.

We omit the details. ∎

Finally, let us recall Dragomir-Ionescu's inequality for differentiable convex functions (see [7])

$$\begin{aligned}
0\ &\leq\ \sum_{i=1}^{n}p_if(x_i)-f\left(\sum_{i=1}^{n}p_ix_i\right)\\
&\leq\ \sum_{i=1}^{n}p_ix_if'(x_i)-\sum_{i=1}^{n}p_ix_i\sum_{i=1}^{n}p_if'(x_i)
\end{aligned}\tag{2.8}$$

provided $f:(a,b)\subseteq\mathbb{R}\to\mathbb{R}$ is differentiable convex on $(a,b)$, $x_i\in(a,b)$ and $p_i>0$ $(i=1,\dots,n)$ with $\sum_{i=1}^{n}p_i=1$.

If $f$ is strictly convex on $(a,b)$, then the equality holds in (2.8) iff $x_1=\cdots=x_n$, we may state the following result.

**Theorem 5.** *With the assumptions of Theorem 4, we have*

$$\begin{aligned}
&\exp\left[A_n(\bar{p},\bar{x})A_n\left(\bar{p},\frac{1}{\bar{x}(1-\bar{x})}\right)-A_n\left(\bar{p},\frac{1}{1-\bar{x}}\right)\right]\\
&\times\left[\frac{1-2M}{(1-M)^2}\left\{1-A_n(\bar{p},\bar{x})A_n\left(\bar{p},\frac{1}{\bar{x}}\right)\right\}\right]\times\left[\frac{A_n(\bar{p},\bar{x})}{G_n(\bar{p},\bar{x})}\right]^{\frac{1-2M}{(1-M)^2}}\\
\geq\ &\left[\frac{G_n(\bar{p},1-\bar{x})}{G_n(\bar{p},\bar{x})}\right]\Big/\left[\frac{A_n(\bar{p},1-\bar{x})}{A_n(\bar{p},\bar{x})}\right]\\
\geq\ &\exp\left[A_n(\bar{p},\bar{x})A_n\left(\bar{p},\frac{1}{\bar{x}(1-\bar{x})}\right)-A_n\left(\bar{p},\frac{1}{1-\bar{x}}\right)\right]\\
&\times\left[\frac{1-2m}{(1-m)^2}\left\{1-A_n(\bar{p},\bar{x})A_n\left(\bar{p},\frac{1}{\bar{x}}\right)\right\}\right]\times\left[\frac{A_n(\bar{p},\bar{x})}{G_n(\bar{p},\bar{x})}\right]^{\frac{1-2m}{(1-m)^2}},
\end{aligned}\tag{2.9}$$

*where* $\frac{1}{\bar{x}}$ *denotes the vector* $\left(\frac{1}{x_1},\dots,\frac{1}{x_n}\right)$, $\bar{y}\cdot\bar{z}:=(y_1z_1,\dots,z_ny_n)$, *and* $\bar{x}\in\mathbb{R}^n$, $\bar{x}>\bar{0}$ *(i.e.,* $x_i>0$ *for any* $i\in\{1,\dots,n\}$*),* $\bar{y},\bar{z}\in\mathbb{R}^n$.

*Proof.* Since the function $f:(m,M)\subset\left(0,\frac{1}{2}\right]\to\mathbb{R},\ f(t)=\ln\left(\frac{1-t}{t}\right)+\frac{1-2M}{(1-M)^2}\ln t$ is strictly convex on $(m,M)$, by (2.8) we may state that

$$\begin{aligned}
&\sum_{i=1}^{n}p_i\left[\ln\left(\frac{1-x_i}{x_i}\right)+\frac{1-2M}{(1-M)^2}\ln x_i\right]-\ln\left(\frac{1-\sum_{i=1}^{n}p_ix_i}{\sum_{i=1}^{n}p_ix_i}\right)\\
&-\frac{1-2M}{(1-M)^2}\ln\left(\sum_{i=1}^{n}p_ix_i\right)\\
=\ &\sum_{i=1}^{n}p_if(x_i)-f\left(\sum_{i=1}^{n}p_ix_i\right)\le\sum_{i=1}^{n}p_ix_if'(x_i)-\sum_{i=1}^{n}p_ix_i\sum_{i=1}^{n}p_if'(x_i)\\
=\ &\sum_{i=1}^{n}p_ix_i\left[\frac{1-2M}{(1-M)^2}\cdot\frac{1}{x_i}-\frac{1}{x_i(1-x_i)}\right]\\
&-\sum_{i=1}^{n}p_ix_i\sum_{i=1}^{n}p_i\left[\frac{1-2M}{(1-M)^2}\cdot\frac{1}{x_i}-\frac{1}{x_i(1-x_i)}\right],
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&\ln\left[\frac{G_n(\bar{p},1-\bar{x})}{G_n(\bar{p},\bar{x})}\right]+\frac{1-2M}{(1-M)^2}\ln G_n(\bar{p},\bar{x})-\ln\left[\frac{A_n(\bar{p},1-\bar{x})}{A_n(\bar{p},\bar{x})}\right]\\
&-\frac{1-2M}{(1-M)^2}\ln A_n(\bar{p},\bar{x})\\
\le\ &\frac{1-2M}{(1-M)^2}-A_n\left(\bar{p},\frac{1}{1-\bar{x}}\right)\\
&-A_n(\bar{p},\bar{x})\times\left[\frac{1-2M}{(1-M)^2}A_n\left(\bar{p},\frac{1}{\bar{x}}\right)-A_n\left(\bar{p},\frac{1}{\bar{x}(1-\bar{x})}\right)\right],
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&\ln\left[\left[\frac{G_n(\bar{p},1-\bar{x})}{G_n(\bar{p},\bar{x})}\right]\Big/\left[\frac{A_n(\bar{p},1-\bar{x})}{A_n(\bar{p},\bar{x})}\right]\right]\\
\le\ &\ln\left[\frac{A_n(\bar{p},\bar{x})}{G_n(\bar{p},\bar{x})}\right]^{\frac{1-2M}{(1-M)^2}}+\frac{1-2M}{(1-M)^2}\left[1-A_n(\bar{p},\bar{x})A_n\left(\bar{p},\frac{1}{\bar{x}}\right)\right]\\
&+A_n(\bar{p},\bar{x})A_n\left(\bar{p},\frac{1}{\bar{x}(1-\bar{x})}\right)-A_n\left(\bar{p},\frac{1}{1-\bar{x}}\right)\\
=\ &\ln\left\{\left[\frac{A_n(\bar{p},\bar{x})}{G_n(\bar{p},\bar{x})}\right]^{\frac{1-2M}{(1-M)^2}}\cdot\exp\left[\frac{1-2M}{(1-M)^2}\left\{1-A_n(\bar{p},\bar{x})A_n\left(\bar{p},\frac{1}{\bar{x}}\right)\right\}\right]\right.\\
&\left.\times\exp\left[A_n(\bar{p},\bar{x})A_n\left(\bar{p},\frac{1}{\bar{x}(1-\bar{x})}\right)-A_n\left(\bar{p},\frac{1}{1-\bar{x}}\right)\right]\right\},
\end{aligned}$$

hence the first inequality in (2.9).

The second inequality follows by (2.8) applied for the strictly convex function $h(t)=\frac{1-2m}{(1-m)^2}\ln t-\ln\left(\frac{1-t}{t}\right),\ t\in(m,M)$.

We omit the details. ∎

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