REFINEMENTS OF THE HERMITE-HADAMARD INTEGRAL INEQUALITY FOR LOG-CONVEX FUNCTIONS

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Abstract. Two refinements of the classical Hermite-Hadamard integral inequality for log-convex functions and applications for special means are given.

1. Introduction

Let I be an interval of real numbers.

The function $f : I \to \mathbb{R}$ is said to be *convex* on I if for all $x, y \in I$ and $t \in [0, 1]$, one has the inequality:

$$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y). \tag{1.1}$$

A function $f : I \to (0, \infty)$ is said to be *log-convex* or *multiplicatively convex* if $\log(f)$ is convex, or, equivalently, if for all $x, y \in I$ and $t \in [0, 1]$ one has the inequality:

$$f(tx + (1 - t)y) \leq [f(x)]^t [f(y)]^{1-t}. \tag{1.2}$$

We note that if f and g are convex functions and g is monotonic nondecreasing, then $g \circ f$ is convex. Moreover, since $f = \exp(\log f)$, it follows that a log-convex function is convex, but the converse is not true [2, p. 7]. This fact is obvious from (1.2) as by the arithmetic-geometric mean inequality, we have

$$[f(x)]^t [f(y)]^{1-t} \leq tf(x) + (1 - t)f(y) \tag{1.3}$$

for all $x, y \in I$ and $t \in [0, 1]$.

The next inequality (see for example [2, p. 137]) is well known in the literature as the Hermite-Hadamard inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a) + f(b)}{2}, \tag{1.4}$$

where $f : I \to \mathbb{R}$ is a convex function on the interval I and $a, b \in I$ with $a < b$.

For some recent results related to this classic result, see the papers [4] – [13] and the books [1], [2] and [3] where further references are given.

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In [13], S.S. Dragomir and B. Mond proved that the following inequalities of Hermite-Hadamard type hold for log-convex functions:

$$f\left(\frac{a+b}{2}\right) \leq \exp\left[\frac{1}{b-a}\int_a^b \ln\left[f(x)\right]dx\right] \tag{1.5}$$
$$\leq \frac{1}{b-a}\int_a^b G\left(f(x), f(a+b-x)\right)dx \leq \frac{1}{b-a}\int_a^b f(x)\,dx$$
$$\leq L\left(f(a), f(b)\right) \leq \frac{f(x)+f(b)}{2},$$

where $G(p,q) := \sqrt{pq}$ is the geometric mean and $L(p,q) := \frac{p-q}{\ln p - \ln q}$ $(p \neq q)$ is the logarithmic mean of the positive real numbers p, q (for $p = q$, we put $L(p,p) = p$).

In this paper we prove another refinement of the Hermite-Hadamard Inequality for differentiable log-convex functions. Some applications for special means are also given.

2. The Results

We shall start with the following refinement of the Hermite-Hadamard inequality for log-convex functions.

Theorem 1. *Let $f : I \to (0,\infty)$ be a differentiable log-convex function on the interval of real numbers $\mathring{I}$ (the interior of I) and $a, b \in \mathring{I}$ with $a < b$. Then the following inequalities hold:*

$$\frac{\frac{1}{b-a}\int_a^b f(x)\,dx}{f\left(\frac{a+b}{2}\right)} \tag{2.1}$$
$$\geq L\left(\exp\left[\frac{f'\left(\frac{a+b}{2}\right)}{f\left(\frac{a+b}{2}\right)}\left(\frac{b-a}{2}\right)\right], \exp\left[-\frac{f'\left(\frac{a+b}{2}\right)}{f\left(\frac{a+b}{2}\right)}\left(\frac{b-a}{2}\right)\right]\right) \geq 1.$$

Proof. Since f is differentiable and log-convex on $\mathring{\mathrm{I}}$, we have that

$$\log f(x) - \log f(y) \geq \frac{d}{dt}(\log f)(y)(x-y)$$

for all $x, y \in \mathring{\mathrm{I}}$, which gives that

$$\log\left[\frac{f(x)}{f(y)}\right] \geq \frac{f'(y)}{f(y)}(x-y)$$

for all $x, y \in \mathring{\mathrm{I}}$. That is,

$$f(x) \geq f(y)\exp\left[\frac{f'(y)}{f(y)}(x-y)\right] \quad \text{for all } x, y \in \mathring{\mathrm{I}}. \tag{2.2}$$

Now, if we choose $y = \frac{a+b}{2}$, we obtain:

$$\frac{f(x)}{f\left(\frac{a+b}{2}\right)} \geq \exp\left[\frac{f'\left(\frac{a+b}{2}\right)}{f\left(\frac{a+b}{2}\right)}\left(x - \frac{a+b}{2}\right)\right],\ x \in [a,b]. \tag{2.3}$$

Integrating this inequality over x on $[a,b]$ and using Jensen's integral inequality, we deduce that:

$$\frac{\frac{1}{b-a}\int_a^b f(x)\,dx}{f\left(\frac{a+b}{2}\right)} \geq \frac{1}{b-a}\int_a^b \exp\left[\frac{f'\left(\frac{a+b}{2}\right)}{f\left(\frac{a+b}{2}\right)}\left(x-\frac{a+b}{2}\right)\right]dx \tag{2.4}$$

$$\geq \exp\left[\frac{1}{b-a}\int_a^b \left[\frac{f'\left(\frac{a+b}{2}\right)}{f\left(\frac{a+b}{2}\right)}\left(x-\frac{a+b}{2}\right)\right]dx\right] = 1.$$

Now, as for $\alpha \neq 0$ we have that

$$\frac{1}{b-a}\int_a^b \exp(\alpha x)\,dx = \frac{\exp(\alpha b)-\exp(\alpha a)}{\alpha(b-a)} = L[\exp(\alpha b),\exp(\alpha a)],$$

where $L(\cdot,\cdot)$ is the usual logarithmic mean, then

$$\frac{1}{b-a}\int_a^b \exp\left[\alpha\left(x-\frac{a+b}{2}\right)\right]dx$$

$$= \frac{\exp\left[\alpha\left(\frac{b-a}{2}\right)\right]-\exp\left[-\alpha\left(\frac{b-a}{2}\right)\right]}{\alpha\left[\left(\frac{b-a}{2}\right)-\left(-\left(\frac{b-a}{2}\right)\right)\right]}$$

$$= L\left(\exp\left[\alpha\left(\frac{b-a}{2}\right)\right],\exp\left[-\alpha\left(\frac{b-a}{2}\right)\right]\right).$$

Using the above equality for $\alpha = \frac{f'\left(\frac{a+b}{2}\right)}{f\left(\frac{a+b}{2}\right)}$ the inequality (2.4) gives the desired result (2.1). ∎

The following corollary holds.

Corollary 1. *Let $g : I \to \mathbb{R}$ be a differentiable convex function on $\mathring{I}$ and $a, b \in \mathring{I}$ with $a < b$. Then we have the inequality:*

$$\frac{\frac{1}{b-a}\int_a^b \exp(g(x))\,dx}{\exp g\left(\frac{a+b}{2}\right)} \tag{2.5}$$

$$\geq L\left(\exp\left[g'\left(\frac{a+b}{2}\right)\left(\frac{b-a}{2}\right)\right],\exp\left[-g'\left(\frac{a+b}{2}\right)\left(\frac{b-a}{2}\right)\right]\right) \geq 1.$$

The following theorem also holds.

Theorem 2. *Let $f : I \to \mathbb{R}$ be as in Theorem 1. Then we have the inequality:*

$$\frac{\frac{f(a)+f(b)}{2}}{\frac{1}{b-a}\int_a^b f(x)\,dx} \geq 1+\log\left[\frac{\int_a^b f(x)\,dx}{\int_a^b f(x)\exp\left[\frac{f'(x)}{f(x)}\left(\frac{a+b}{2}-x\right)\right]dx}\right] \tag{2.6}$$

$$\geq 1+\log\left[\frac{\frac{1}{b-a}\int_a^b f(x)\,dx}{f\left(\frac{a+b}{2}\right)}\right] \geq 1.$$

Proof. From the inequality (2.2) we have

$$f\left(\frac{a+b}{2}\right)\geq f\left(y\right)\exp\left[\frac{f'\left(y\right)}{f\left(y\right)}\left(\frac{a+b}{2}-y\right)\right],$$

for all $y\in[a,b]$.

Integrating over y and using Jensen's integral inequality for $\exp\left(\cdot\right)$ functions, we have

$$\begin{aligned}\left(b-a\right)f\left(\frac{a+b}{2}\right) &\geq \int_a^b f\left(y\right)\exp\left[\frac{f'\left(y\right)}{f\left(y\right)}\left(\frac{a+b}{2}-y\right)\right]dy\\ &\geq \int_a^b f\left(y\right)dy\cdot\exp\left(\frac{\int_a^b f\left(y\right)\left[\frac{f'(y)}{f(y)}\left(\frac{a+b}{2}-y\right)\right]dy}{\int_a^b f\left(y\right)dy}\right)\\ &= \int_a^b f\left(y\right)dy\cdot\exp\left(\frac{\int_a^b f'\left(y\right)\left(\frac{a+b}{2}-y\right)dy}{\int_a^b f\left(y\right)dy}\right).\end{aligned}$$

A simple integration by parts gives

$$\int_a^b f'\left(y\right)\left(\frac{a+b}{2}-y\right)dy=\int_a^b f\left(y\right)dy-\frac{f\left(a\right)+f\left(b\right)}{2}\left(b-a\right).$$

Then we have

$$\begin{aligned}\exp\left[1-\frac{\frac{f(a)+f(b)}{2}\left(b-a\right)}{\int_a^b f\left(x\right)dx}\right] &\leq \frac{\int_a^b f\left(y\right)\exp\left[\frac{f'(y)}{f(y)}\left(\frac{a+b}{2}-y\right)\right]dy}{\int_a^b f\left(y\right)dy}\\ &\leq \frac{\left(b-a\right)f\left(\frac{a+b}{2}\right)}{\int_a^b f\left(y\right)dy},\end{aligned}$$

which is equivalent to

$$\begin{aligned}1-\frac{\frac{f(a)+f(b)}{2}\left(b-a\right)}{\int_a^b f\left(x\right)dx} &\leq \log\left[\frac{\int_a^b f\left(y\right)\exp\left[\frac{f'(y)}{f(y)}\left(\frac{a+b}{2}-y\right)\right]dy}{\int_a^b f\left(y\right)dy}\right]\\ &\leq \log\left[\frac{f\left(\frac{a+b}{2}\right)}{\frac{1}{b-a}\int_a^b f\left(x\right)dx}\right]\end{aligned}$$

from where we get the desired inequality. ∎

The following corollary is a natural consequence of the above theorem.

Corollary 2. *Let $g:I\to\mathbb{R}$ be as in Corollary 1. Then we have the inequality:*

$$\begin{aligned}\frac{\frac{\exp g(a)+\exp g(b)}{2}}{\frac{1}{b-a}\int_a^b\exp g\left(x\right)dx} &\geq 1+\log\left[\frac{\int_a^b\exp g\left(x\right)dx}{\int_a^b\exp\left[g\left(x\right)-\left(x-\frac{a+b}{2}\right)g'\left(x\right)\right]dx}\right]\\ &\geq 1+\log\left[\frac{\frac{1}{b-a}\int_a^b\exp g\left(x\right)dx}{\exp g\left(\frac{a+b}{2}\right)}\right]\geq 1.\end{aligned}$$

3. Applications

The function $f(x)=\frac{1}{x}$, $x\in(0,\infty)$ is log-convex on $(0,\infty)$. Then we have

$$\begin{aligned}\frac{1}{b-a}\int_a^b\frac{dx}{x} &= L^{-1}(a,b),\\ f\left(\frac{a+b}{2}\right) &= A^{-1}(a,b),\\ \frac{f'\left(\frac{a+b}{2}\right)}{f\left(\frac{a+b}{2}\right)} &= -\frac{1}{A}.\end{aligned}$$

Now, applying the inequality (2.1) for the function $f(x)=\frac{1}{x}$, we get the inequality:

$$\frac{A(a,b)}{L(a,b)}\geq L\left(\exp\left(-\frac{b-a}{2A}\right),\exp\left(\frac{b-a}{2A}\right)\right)\geq 1, \tag{3.1}$$

which is a refinement of the well-known inequality

$$A(a,b)\geq L(a,b), \tag{3.2}$$

where $A(a,b)$ is the *arithmetic mean* and $L(a,b)$ is the *logarithmic mean* of a,b, that is, $A(a,b)=\frac{a+b}{2}$, and $L(a,b)=\frac{a-b}{\ln a-\ln b}$.

For $f(x)=\frac{1}{x}$, we also get

$$\frac{f(a)+f(b)}{2}=H^{-1}(a,b),$$

where $H(a,b):=\frac{1}{\frac{1}{a}+\frac{1}{b}}$ is the *harmonic mean* of a,b. Now, using the inequality (2.6) we obtain another interesting inequality:

$$\frac{L(a,b)}{H(a,b)}\geq 1+\log\left[\frac{A(a,b)}{L(a,b)}\right]\geq 1, \tag{3.3}$$

which is a refinement of the following well-known inequality

$$L(a,b)\geq H(a,b). \tag{3.4}$$

Similar inequalities may be stated for the log-convex functions $f(x)=x^x$, $x>0$ or $f(x)=e^x+1$, $x\in\mathbb{R}$, etc. We omit the details.

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