EXPONENTIAL STABILITY AND BOUNDED CONVOLUTIONS

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Abstract

We consider a mild solution u_f of a well-posed inhomogeneous, Cauchy problem

$$\dot{u}(t) = A(t)u(t) + f(t), \quad u(0) = 0$$

on a Banach space X, where $A(\cdot)$ is periodic. We prove that if for every almost periodic X-valued functions f, with $f(0) = 0$, the solution u_f is almost periodic, then the solution of the well-posed Cauchy problem

$$\dot{u}(t) = A(t)u(t), \quad u(0) = x \in X,$$

is uniformly exponentially stable.

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1. Introduction

Let X be a complex Banach space and $\mathcal{L}(X)$ the space of all bounded and linear operators on X. We denote by $||\cdot||$ the norms of vectors and operators on X. Let $BUC(\mathbf{R}_+, X)$ the Banach space of all X-valued bounded and uniformly continuous functions on $\mathbf{R}_+$ endowed with sup-norm and $AP(\mathbf{R}_+, X)$ the space of almost periodic function in the sense of Bohr, i.e. the linear closed hull in $BUC(\mathbf{R}_+, X)$ of the set of all functions

$$\{e^{i\mu\cdot}x : \ \mu \in \mathbf{R}, \ x \in X\}.$$

Let $AP_0(\mathbf{R}_+, X)$ the set of all functions $f \in AP(\mathbf{R}_+, X)$ such that $f(0) = 0$. It is clear that $AP_0(\mathbf{R}_+, X)$ is a closed subspace of $AP(\mathbf{R}_+, X)$ or of $BUC(\mathbf{R}_+, X)$. We recall that a strongly continuous semigroup on X is a family $\mathbf{T} = \{T(t)\}_{t\geq 0}$ of bounded linear operators acting on the Banach space X which satisfies the following conditions:

- **(i)** $T(t+s) = T(t)T(s)$ for all $t, s \in \mathbf{R}_+ := [0, \infty)$;
- **(ii)** $T(0) = Id$, Id is the identity operator on $\mathcal{L}(X)$;
- **(iii)** the function $t \mapsto T(t)x : \mathbf{R}_+ \to X$ is continuous on $\mathbf{R}_+$ for all $x \in X$ (or, equivalently this function is continuous in $t = 0$).

Let $\mathbf{T}$ be a strongly continuous semigroup on X and A it's infinitesimal generator. It is well known that in this case the Cauchy problem

$$\dot{x}(t) = Ax(t), \quad x(0) = x \in X \tag{1.1}$$

is well-posed and the mild solution of (1.1) is defined by

$$x(t) = T(t)x \quad (t \geq 0).$$

For a locally integrable function $f : \mathbf{R}_+ \to X$, a mild solution of the inhomogeneous Cauchy problem

$$\dot{u}(t) = Au(t) + f(t) \quad (t \geq 0), \quad u(0) = 0$$

is defined by

$$u_f(t) = \int_0^t T(t-\xi)f(\xi)d\xi, \quad t \geq 0.$$

For a well-posed Cauchy problem

$$\dot{x}(t) = A(t)x(t) \quad (t \geq 0), \quad x(0) = x \in X \tag{1.2}$$

with (unbounded) linear operators $A(t)$ the solution lead to an evolution family $\mathcal{U} = \{U(t,s) : \ t \geq s \geq 0\}$ in the space $\mathcal{L}(X)$, that is

- $(\mathbf{e_1})$ $\quad U(t,t) = Id, U(t,\tau)U(\tau,s) = U(t,s)$ for $t \geq \tau \geq s \geq 0$;
- $(\mathbf{e_2})$ the map $(t,s) \mapsto U(t,s)x$ is continuous for every $x \in X$.

When the Cauchy problem (1.2) is periodic, i.e. there exists $q > 0$ such that $A(t+q) = A(t)$ for all $t \in \mathbf{R}_+$, the corresponding evolution family $\mathcal{U}$ on X has exponential growth, i.e. there exist $\omega \in \mathbf{R}$ and $M > 0$ such that

$$||U(t,s)|| \leq Me^{\omega(t-s)} \quad \forall t \geq s \geq 0, \tag{1.3}$$

see [BP, Lemma 4.1] or [DK, Theorem 6.6]. We recall that an evolution family $\mathcal{U}$, as above, is called uniformly exponentially stable if there are $\omega < 0$ and $M > 0$ such that (1.3) holds. For a locally integrable function $f : \mathbf{R}_+ \to X$ a mild solution of the well-posed inhomogeneous Cauchy problem

$$\dot{u}(t) = A(t)u(t) + f(t), \quad (t \geq 0), \quad u(0) = 0$$

is defined by

$$u_f(t) = \int_0^t U(t,\tau)f(\tau)d\tau \quad (t \geq 0).$$

We shall prove the following two theorems.

THEOREM 1. Let $\mathbf{T} = \{T(t)\}_{t\geq 0}$ be a strongly continuous semigroup on X. The following statements are equivalent:

- **(1)** $\mathbf{T}$ *is uniformly exponentially stable, i.e. it's growth bound*

$$\omega_0(\mathbf{T}) := \lim_{t\to\infty} \frac{\ln ||T(t)||}{t}$$

  is negative;

- **(2)** *the function* $t \mapsto \int_0^t T(\xi)f(t-\xi)d\xi : \mathbf{R}_+ \to X$ *belongs to* $AP_0(\mathbf{R}_+, X)$ *for all* $f \in AP_0(\mathbf{R}_+, X)$*;*

- **(3)** $\sup_{t\geq 0} || \int_0^t T(\xi)f(t-\xi)d\xi|| = M_f < \infty, \ \forall \ f \in AP_0(\mathbf{R}_+, X)$.

THEOREM 2. Let $\mathcal{U} = \{U(t,s) : \ t \geq s \geq 0\}$ be a q-periodic evolution family on X. The following statements are equivalent:

- **(i)** $\mathcal{U}$ *is uniformly exponentially stable;*

- **(ii)** *the function* $t \mapsto u_f(t) = \int_0^t U(t,\xi)f(\xi)d\xi : \mathbf{R}_+ \to X$ *belongs to* $AP_0(\mathbf{R}_+, X)$ *for all* $f \in AP_0(\mathbf{R}_+, X)$*;*

- **(iii)** $\sup_{t\geq 0} || \int_0^t U(t,\xi)f(\xi)d\xi|| = K_f < \infty$, *for all* $f \in AP_0(\mathbf{R}_+, X)$.

2. Proofs of the theorems

Proof of Theorem 1. The proof of implications (**1**) $\Rightarrow$ (**3**) and (**2**) $\Rightarrow$ (**3**) are obvious and we omit the details. The proof of (**3**) $\Rightarrow$ (**1**) is based on the following result which has been proved in [BDL, Proposition 4], see also [VS, Corollary 4.5 and its Reformulation] for a related result:

If $\sup_{t>0} || \int_0^t e^{-i\mu\xi} T(t-\xi) g(\xi) d\xi|| < \infty$ *for every* $g \in P_q^0(\mathbf{R}_+, X)$ *and some* $\mu \in R$ *then* $T(q)$ *is power bounded and* $e^{i\mu} \in \rho(T(q))$. Here $P_q^0(\mathcal{R}_+, X)$ is the set of all X-valued continuous functions such that $f(t+q) = f(t)$ for any $t \geq 0$ and $f(0) = 0$.

Now we prove that **(1)** implies **(2)**. Let $\mathcal{T} = \{\mathcal{T}^t\}_{t\geq 0}$ the evolution semigroup associated of $\mathbf{T}$ on the space $AP_0(\mathbf{R}_+, X)$, i.e.,

$$(\mathcal{T}^t f)(s) = \begin{cases} T(t)f(s-t), & s \geq t \\ 0, & 0 \leq s \leq t \end{cases}$$

for every $f \in AP_0(\mathbf{R}_+, X)$. It is easy to see that $\mathcal{T}^t$ acts on $AP_0(\mathbf{R}_+, X)$ for all $t \geq 0$ and, in addition, $\mathcal{T}$ is strongly continuous, see [NM, Lemma 2]. Let $(G, D(G))$ the infinitesimal generator of $\mathcal{T}$ and $u, f \in AP_0(\mathbf{R}_+, X)$. As in [MRS, Lemma 1.1] it is can be proves that $u \in D(G)$ and $Gu = -f$ if and only if $u = u_f$. Moreover, if $\mathbf{T}$ is uniformly exponentially stable then the growth bound of $\mathcal{T}$ is negative, hence G is invertible. It follows that $u_f \in D(G) \subset AP_0(\mathbf{R}_+, X)$ and the proof of Theorem 1 is finished.

Proof of Theorem 2. The proof of (**iii**) $\Rightarrow$ (**i**) follows from the fact that if $u_{e^{-i\mu\cdot}g(\cdot)}$ is bounded for every $g \in P_q^0(\mathbf{R}_+, X)$ and some $\mu \in \mathbf{R}$ then the monodromy operator $V := U(q, 0)$ is power bounded and $e^{i\mu} \in \rho(V)$, see [B, Proof of Theorem 4]. Here $\rho(V)$ is the resolvent set of V. The proofs of (**i**) $\Rightarrow$ (**iii**) and (**ii**) $\Rightarrow$ (**iii**) are obvious and the proof of (**i**) $\Rightarrow$ (**ii**) follows along the lines of (**1**) $\Rightarrow$ (**2**) from Theorem 1. Another proof for the implication (**ii**) $\Rightarrow$ (**i**) we will give here. This proof is based on a method indicated in [CLMR, Theorem 2.5]. Let $h : \mathbf{R} \to AP_0(\mathbf{R}_+, X)$, $(G, D(G))$ the infinitesimal generator of the evolutionary semigroup E associated to $\mathcal{U}$ on $AP_0(\mathbf{R}_+, X)$, and

$$[(\tilde{G}h)(\theta)](t) := [Gh(\theta)](t) = \int_0^t U(t, t-\xi)(h(\theta))(t-\xi) d\xi, \quad \theta \in \mathbf{R}, t \geq 0.$$

It is easy to see that the function

$$\theta \mapsto \int_0^{\cdot} U(\cdot, \cdot - \xi) h(\theta)(\cdot - \xi) d\xi \text{ belongs to } AP(\mathbf{R}, AP_0(\mathbf{R}_+, X))$$

for all $h \in AP(\mathbf{R}, AP_0(\mathbf{R}_+, X))$, i.e. $\tilde{G}$ is a linear operator on $AP(\mathbf{R}, AP_0(\mathbf{R}_+, X))$. Moreover $\tilde{G}$ is bounded on $AP(\mathbf{R}, AP_0(\mathbf{R}_+, X))$, because

$$\begin{aligned} ||\tilde{G}h||_{AP(\mathbf{R}, AP_0(\mathbf{R}, X))} &= \sup_{\theta \in \mathbf{R}} ||Gh(\theta)||_{AP_0(\mathbf{R}_+, X)} \\ &\leq ||G||_{\mathcal{L}(AP_0(\mathbf{R}_+, X))} ||h||_{AP(\mathbf{R}, AP_0(\mathbf{R}_+, X))}. \end{aligned}$$

For the isometry J defined on the space $AP(\mathbf{R}, AP_0(\mathbf{R}_+, X))$ by

$$[(Jh)(\theta)](t) = (h(\theta + t))(t),$$

we have

$$[(J^{-1}\tilde{G}Jh)(\theta)](t) = \int_0^t U(t, t - \xi)(h(\theta - \xi))(t - \xi) d\xi$$

Let $E = \{E^t\}_{t \geq 0}$ be the evolution semigroup on $AP_0(\mathbf{R}_+, X)$, defined by

$$(E^t f)(\xi) = \begin{cases} U(t, t - \xi) f(t - \xi), & t \geq \xi \\ 0, & 0 \leq t \leq \xi \end{cases}$$

and

$$(G_* h)(\theta) := \int_0^\infty E^\tau h(\theta - \tau) d\tau \quad \theta \in \mathbf{R}, \quad h \in AP(\mathbf{R}, AP_0(\mathbf{R}_+, X)).$$

A simple calculus show that $G_* = J^{-1}\tilde{G}J$, therefore G_* is a bounded operator on $AP(\mathbf{R}, AP_0(\mathbf{R}_+, X))$. Each function $g_+ \in AP_0(\mathbf{R}_+, AP_0(\mathbf{R}_+, X))$ can be extented to a function $g \in AP(\mathbf{R}, AP_0(\mathbf{R}_+, X))$ by setting

$$g(\theta) = \begin{cases} g_+(\theta), & if\ \theta \geq 0 \\ 0, & if\ \theta < 0 \end{cases}$$

It is clear that $G_*g \in AP(\mathbf{R}, AP_0(\mathbf{R}_+, X))$. Consider the function $f_+ : \mathbf{R}_+ \to AP_0(\mathbf{R}_+, X)$, defined by

$$f_+(r) = \int_0^r E^\tau g_+(r-\tau)d\tau \quad (r \geq 0).$$

It is easy to see that for all $t \geq 0$, we have

$$[f_+(\theta)](t) = \int_0^{\min(\theta,t)} U(t,t-\tau)(g_+(\theta-\tau))(t-\tau)d\tau, \quad \theta \geq 0,$$

and

$$[(G_*g)(\theta)](t) = \begin{cases} (f_+(\theta))(t), & if\ \theta \geq 0 \\ 0, & if\ \theta < 0 \end{cases}$$

Then $G_*g|_{\mathbf{R}_+} = f_+$ belongs to $AP_0(\mathbf{R}_+, AP_0(\mathbf{R}_+, X))$. ¿From Theorem 1 ((**3**) $\Rightarrow$ (**1**)) with $\mathbf{T}$ replaced by E and X replaced by $AP_0(\mathbf{R}_+, X)$ it results that E is uniformly exponentially stable. Now is easy to see that $\mathcal{U}$ is uniformly exponentially stable, cf. [CLMR, Theorem 2.2].

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