SOME REMARKS ON THE TRAPEZOID RULE IN NUMERICAL INTEGRATION

S.S. DRAGOMIR, P. CERONE, AND A. SOFO

ABSTRACT. In this paper, by the use of some classical results from the Theory of Inequalities, we point out quasi-trapezoid quadrature formulae for which the error of approximation is smaller than in the classical case. Examples are given to demonstrate that the bounds obtained within this paper may be tighter than the classical ones. Some applications for special means are also given.

1. INTRODUCTION

The following inequality is well known in the literature as the *trapezoid inequality*:

$$\left|\int_a^b f(x)\,dx - \frac{f(a)+f(b)}{2}(b-a)\right| \leq \frac{\|f''\|_\infty}{12}(b-a)^3, \tag{1.1}$$

where the mapping $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$ is supposed to be twice differentiable on the interval (a,b), with the second derivative bounded on (a,b), that is, $\|f''\|_\infty := \sup_{x\in(a,b)}|f''(x)| < \infty$.

Now if we assume that $I_h : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ is a partition of the interval $[a,b]$ and f is as above, then we can approximate the integral $\int_a^b f(x)\,dx$ by the *trapezoid quadrature formula* $A_T(f,I_h)$, having an error given by $R_T(f,I_h)$, where

$$A_T(f,I_h) := \frac{1}{2}\sum_{i=0}^{n-1}\left[f(x_i)+f(x_{i+1})\right]h_i,$$

and the remainder satisfies the estimation

$$|R_T(f,I_h)| \leq \frac{1}{12}\|f''\|_\infty\sum_{i=0}^{n-1}h_i^3,$$

where $h_i := x_{i+1}-x_i$ for $i=0,...,n-1$.

In this paper, via the use of some classical results from the Theory of Inequalities (Hölder's inequality, Grüss' inequality and the Hermite-Hadamard inequality), we produce some *quasi-trapezoid quadrature formulae* for which the remainder term is smaller than the classical one given above.

Some applications to special means: *arithmetic means, geometric means, identric means, logarithmic means,* etc., are also given.

For other results in connection with the trapezoid inequalities, see Chapter XV of the recent book by Mitrinović et al. [2].

Date: May, 1999.

1991 *Mathematics Subject Classification.* Primary 26D15, 26D20; Secondary 41A55.

Key words and phrases. Trapezoidal Rule, Analytic Inequalities.

2. Some Integral Inequalities

We shall start with the following lemma which is also interesting in its own right.

Lemma 1. *Let $f:[a,b]\to\mathbb{R}$ be a twice differentiable mapping on (a,b) and suppose that*

$$\|f''\|_\infty := \sup_{x\in(a,b)} |f''(x)| < \infty.$$

Then we have the estimation

$$\left|\int_a^b f(x)\,dx - \frac{f(a)+f(b)}{2}(b-a)\right| \tag{2.1}$$

$$\leq \begin{cases} \frac{\|f''\|_\infty}{12}(b-a)^3 \\ \\ \frac{1}{2}\|f''\|_q [B(p,p)]^{\frac{1}{p}} (b-a)^{2+\frac{1}{p}}, \ \frac{1}{p}+\frac{1}{q}=1, p>1 \\ \\ \frac{\|f''\|_1}{8}(b-a)^2 \end{cases},$$

where

$$\|f''\|_1 := \int_a^b |f''(t)|\,dt,$$

$$\|f''\|_q := \left(\int_a^b |f''(t)|^q\,dt\right)^{\frac{1}{q}},\ q>1$$

and B is the Beta function of Euler, that is,
$B(l,s) := \int_0^1 t^{l-1}(1-t)^{s-1}\,dt,\ l,s>0.$

Proof. Integrating by parts we can state that:

$$\begin{aligned} & \int_a^b (x-a)(b-x) f''(x)\,dx \\ = & \left[(x-a)(b-x)f'(x)\right]_a^b - \int_a^b [(a+b)-2x] f'(x)\,dx \\ = & \int_a^b [2x-(a+b)] f'(x)\,dx \\ = & f(x)[2x-(a+b)]_a^b - 2\int_a^b f(x)\,dx \\ = & (b-a)(f(a)+f(b)) - 2\int_a^b f(x)\,dx, \end{aligned}$$

from which we get the inequality

$$\int_a^b f(x)\,dx = \frac{f(a)+f(b)}{2}(b-a) - \frac{1}{2}\int_a^b (x-a)(b-x) f''(x)\,dx. \tag{2.2}$$

Thus,

$$\left|\int_a^b f(x)\,dx - \frac{f(a)+f(b)}{2}(b-a)\right| \leq \frac{1}{2}\int_a^b (x-a)(b-x)|f''(x)|\,dx. \tag{2.3}$$

First of all let us observe that

$$\begin{aligned}\int_a^b (x-a)(b-x)\left|f''(x)\right| dx &\leq \left\|f''\right\|_\infty \int_a^b (x-a)(b-x)\, dx \\ &= \frac{\left\|f''\right\|_\infty}{6}(b-a)^3,\end{aligned}$$

as

$$\int_a^b (x-a)(b-x)\, dx = \frac{(b-a)^3}{6}.$$

Thus, by (2.3), we get the first inequality in (2.1). Further, by Hölder's integral inequality we obtain:

$$\int_a^b (x-a)(b-x)\left|f''(x)\right| dx \leq \left(\int_a^b (x-a)^p (b-x)^p\, dx\right)^{\frac{1}{p}} \left\|f''\right\|_q,$$

where $\frac{1}{p}+\frac{1}{q}=1,\ p>1$ and $\left\|f''\right\|_q$ is as given above.
Now, using the transformation $x=(1-t)a+tb,\ t\in[0,1]$, we get

$$\begin{aligned}(x-a)^p (b-x)^p &= (b-a)^{2p} t^p (1-t)^p, \\ dx &= (b-a)\, dt\end{aligned}$$

and thus

$$\begin{aligned}\int_a^b (x-a)^p (b-x)^p\, dx &= (b-a)^{2p+1}\int_0^1 t^p (1-t)^p\, dt \\ &= (b-a)^{2p+1} B(p+1,p+1),\end{aligned}$$

where B is the Beta function of Euler; and the second inequality in (2.1) is proved.
Finally, we have that

$$\int_a^b (x-a)(b-x)\left|f''(x)\right| dx \leq \max_{x\in[a,b]}\left[(x-a)(b-x)\right]\left\|f''\right\|_1.$$

Also, since

$$\max_{x\in[a,b]}\left[(x-a)(b-x)\right] = \frac{(b-a)^2}{4},$$

we deduce the last part of (2.1). ∎

Some examples will now be presented to illustrate that the different norms in equation (2.1) provide better bounds on the error depending on the behaviour of the integrand.

Without loss of generality and for simplicity we take $a=0$ and $\beta=\frac{b-a}{2}$ in (2.1) to give as the right hand side of (2.1),

$$\begin{aligned}T_1 &= \frac{2}{3}\beta^3 \sup_{t\in(0,2\beta)}\left|f''(t)\right| \\ T_2 &= \frac{1}{2}\beta^2\left(\frac{4\sqrt{\pi}\beta}{\Gamma\left(p+\frac{1}{2}\right)}\right)^{\frac{1}{p}}\left(\int_0^{2\beta}\left|f''(t)\right|^q dt\right)^{\frac{1}{q}},\ \frac{1}{p}+\frac{1}{q}=1,\ p>1,\end{aligned}$$

Figure 1

FIGURE 1: Contours of $\frac{T_1}{T_2} = 1$ and $\frac{T_3}{T_2} = 1$ on the $\beta p-$plane. The regions from the bottom satisfy: $T_2 < T_1 < T_3$, $T_1 < T_2 < T_3$, and $T_1 < T_3 < T_2$ respectively.

and

$$T_3 = \beta^2 \int_0^{2\beta} |f''(t)|\, dt$$

respectively, where the Beta function has been expressed in terms of the gamma function and the duplication formula has been used.

Consider the example $f''(t) = 1 + \cos t$. It may be demonstrated through the use of some package such as Mathematica that T_1 is always smaller than T_3 since $\frac{T_1}{T_3}$ is always less than 1. Figure 1 shows two contours, the top one being $\frac{T_1}{T_2} = 1$ and the lower one being $\frac{T_3}{T_2} = 1$, relating to β and p. The regions from bottom to top satisfy: $T_2 < T_1 < T_3$, $T_1 < T_2 < T_3$, and $T_1 < T_3 < T_2$ respectively. This example demonstrates than one bound is not universally the best.

Another interesting and simple example is $f''(t) = e^t$. Figure 2 shows that $\frac{T_1}{T_3} = 1$ when $\beta = \beta^* = 1.41072$, and thus the traditional bound (T_1) is best for $\beta < \beta^*$ and so T_3 gives a tighter bound for larger integration intervals. The contours of $\frac{T_1}{T_2} = 1$ and $\frac{T_3}{T_2} = 1$ intersect with the line $\beta = \beta^*$, thus breaking the βp-plane into six regions. the regions represent $A : T_2 < T_1 < T_3$, $B : T_1 < T_2 < T_3$, $C : T_1 < T_3 < T_2$, $D : T_2 < T_3 < T_1$, $E : T_3 < T_2 < T_1$ and $F : T_3 < T_1 < T_2$. This further demonstrates that each of the bounds may be best under different circumstances. T_1 is best in regions B and C, T_2 in regions A and D, and T_3 in E and F. The regions D, E, and F relate to larger intervals of integration.

FIGURE 2

FIGURE 2: The diagram shows regions A,...,F of the $\beta p-$plane which are separated by the contours $\frac{T_1}{T_2} = 1$, $\frac{T_3}{T_2} = 1$ and the line $\beta = \beta^*$ (corresponding to $\frac{T_1}{T_3} = 1$).

The above two examples demonstrate that the proposed bounds are not universally best. Work on determining *a priori* the best bound is the subject of future investigation.

The following lemma is of interest since it provides another integral inequality in connections with the trapezoid formula.

Lemma 2. *Let $f : [a, b] \to \mathbb{R}$ be a twice differentiable mapping on (a, b) and assume that*

$$m := \inf_{x \in (a,b)} f''(x) > -\infty \text{ and } M := \sup_{x \in (a,b)} f''(x) < \infty. \tag{2.4}$$

Then, we have the estimation

$$\left| \int_a^b f(x)\,dx - \frac{f(a) + f(b)}{2}(b-a) + \frac{(b-a)^2}{12}(f'(b) - f'(a)) \right| \leq \frac{(b-a)^3 (M-m)}{32}. \tag{2.5}$$

Proof. We shall apply the celebrated Grüss' inequality (see for example [2]) which says that:

$$\left|\frac{1}{b-a}\int_a^b h(x)\,g(x)\,dx-\frac{1}{b-a}\int_a^b h(x)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx\right|\leq\frac{(\Phi-\phi)(\Gamma-\gamma)}{4}, \tag{2.6}$$

where h,g are integrable mappings satisfying the conditions $\phi\leq h(x)\leq\Phi$ and $\gamma\leq g(x)\leq\Gamma$ for all $x\in[a,b]$.
Now, if we choose in (2.6), $h(x)=(x-a)(b-x)$, $g(x)=f''(x)$, $x\in[a,b]$, we get:

$$\phi=0,\ \Phi=\frac{(b-a)^2}{4},\ \gamma=m \text{ and } \Gamma=M,$$

and we can state that

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b (x-a)(b-x)f''(x)\,dx\right.\\&\left.-\frac{1}{b-a}\int_a^b (x-a)(b-x)\,dx\cdot\frac{1}{b-a}\int_a^b f''(x)\,dx\right|\\\leq\ &\frac{(b-a)^2(M-m)}{16}.\end{aligned} \tag{2.7}$$

A simple calculation gives us that

$$\int_a^b (x-a)(b-x)\,dx=\frac{(b-a)^3}{6} \text{ and } \int_a^b f''(x)\,dx=f'(b)-f'(a),$$

then, from (2.7),

$$\begin{aligned}&\left|\int_a^b (x-a)(b-x)f''(x)\,dx-\frac{(b-a)^2}{6}(f'(b)-f'(a))\right|\\\leq\ &\frac{(b-a)^3(M-m)}{16}.\end{aligned}$$

Finally, using the identity (2.2) gives

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}(b-a)-\int_a^b f(x)\,dx-\frac{(b-a)^2}{12}(f'(b)-f'(a))\right|\\\leq\ &\frac{(b-a)^3(M-m)}{32}\end{aligned}$$

and the lemma is proved. ∎

Finally, using a classical result on convex functions due to Hermite and Hadamard we have the following lemma concerning a double integral inequality.

Lemma 3. *Let $f:[a,b]\to\mathbb{R}$ be a twice differentiable mapping on (a,b) and suppose that $-\infty<m\leq f''(x)\leq M<\infty$ for all $x\in(a,b)$.*
Then we have the double inequality

$$\frac{m}{12}(b-a)^2\leq\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\leq\frac{M}{12}(b-a)^2, \tag{2.8}$$

and the estimation

(2.9)

$$\left|\frac{f(a)+f(b)}{2}(b-a)-\frac{M+m}{24}(b-a)^2-\int_a^b f(x)\,dx\right|\le\frac{(M-m)(b-a)^3}{24}.$$

Proof. We shall use the following inequality for convex mappings $g:[a,b]\to\mathbb{R}$:

$$\frac{1}{b-a}\int_a^b g(x)\,dx\le\frac{g(a)+g(b)}{2}, \tag{2.10}$$

which is well known in the literature as the Hermite-Hadamard inequality.
Let us choose firstly $g:[a,b]\to\mathbb{R}, g(x)=f(x)-\frac{m}{2}x^2$. Then g is twice differentiable on $[a,b]$ and

$$g'(x)=f'(x)-mx,\ g''(x)=f''(x)-m\ge 0 \text{ on } (a,b),$$

hence, g is convex on $[a,b]$. Thus, we can apply (2.10) for g to get

$$\frac{1}{b-a}\int_a^b\left(f(x)-\frac{m}{2}x^2\right)dx\le\frac{f(a)+f(b)}{2}-\frac{m}{4}\left(a^2+b^2\right),$$

giving on rearrangement

$$\frac{1}{b-a}\int_a^b f(x)\,dx\le\frac{f(a)+f(b)}{2}-\frac{m}{12}(b-a)^2,$$

which is ostensibly identical to the first inequality in (2.8).
The second part in (2.8) follows by (2.10) applied for the convex (and twice differentiable mapping) $h:[a,b]\to\mathbb{R},\ h(x)=\frac{M}{2}x^2-f(x)$.
Now, it is straightforward to see that, for $\alpha\le t\le\beta$ and thus $\left|t-\frac{\alpha+\beta}{2}\right|\le\frac{\beta-\alpha}{2}$, on taking $\alpha=\frac{m}{12}(b-a)^2$ and $\beta=\frac{M}{12}(b-a)^2$ we get the desired estimation (2.9). ∎

3. Some Trapezoid Quadrature Rules

We now consider applications of the integral inequalities developed in Section 2, to obtain, what is claimed to be, some new trapezoid or quasi-trapezoid rules.

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be as in Lemma 1.*
If $I_h: a=x_0<x_1<...<x_{n-1}<x_n=b$ is a partition of the interval $[a,b]$, then we have:

$$\int_a^b f(x)\,dx=A_T(f,I_h)+R_T(f,I_h), \tag{3.1}$$

where

$$A_T(f,I_h):=\frac{1}{2}\sum_{i=0}^{n-1}\left[f(x_i)+f(x_{i+1})\right]h_i$$

is the trapezoid quadrature rule and the remainder $R_T(f, I_h)$ *satisfies the relation:*

(3.2)

$$|R_T(f, I_h)| \leq \begin{cases} \frac{1}{2}\|f''\|_\infty \sum_{i=0}^{n-1} h_i^3 \\ \frac{1}{2}\|f''\|_q [B(p+1, p+1)]^{\frac{1}{p}} \left(\sum_{i=0}^{n-1} h_i^{2p+1}\right)^{\frac{1}{p}} & , \frac{1}{p}+\frac{1}{q}=1, \ p>1, \\ \frac{1}{8}\|f''\|_1 v^2(I_h) \end{cases}$$

where $h_i := x_{i+1} - x_i, \ i = 0, ..., n-1$ *and* $v(I_h) = \max_{i=\overline{0,n-1}} h_i$.

Proof. Applying the first inequality, (2.1), we get

$$\left|\int_{x_i}^{x_{i+1}} f(x)\,dx - \frac{f(x_i) + f(x_{i+1})}{2}(x_{i+1} - x_i)\right| \leq \frac{\|f''\|_\infty}{12} h_i^3$$

for all $i \in \{0, ..., n-1\}$.
Summing over i from 0 to $n-1$ we get the first part of (3.2).
The second inequality in (2.1) gives us:

$$\begin{aligned} & \left|\int_{x_i}^{x_{i+1}} f(x)\,dx - \frac{f(x_i) + f(x_{i+1})}{2}(x_{i+1} - x_i)\right| \\ \leq \ & \frac{1}{2} h_i^{2+\frac{1}{p}} [B(p+1, p+1)]^{\frac{1}{p}} \left(\int_{x_i}^{x_{i+1}} |f''(t)|^q\,dt\right)^{\frac{1}{q}}, \end{aligned}$$

for all $i = 0, ..., n-1$.
Summing and using Hölder's discrete inequality, we get:

$$\begin{aligned} & \left|\int_a^b f(x)\,dx - A_T(f, I_h)\right| \\ \leq \ & \frac{1}{2}[B(p+1, p+1)]^{\frac{1}{p}} \sum_{i=0}^{n-1} h_i^{\frac{2p+1}{p}} \left(\int_{x_i}^{x_{i+1}} |f''(t)|^q\,dt\right)^{\frac{1}{q}} \\ \leq \ & \frac{1}{2}[B(p+1, p+1)]^{\frac{1}{p}} \left[\sum_{i=0}^{n-1} \left(h_i^{\frac{2p+1}{p}}\right)^p\right]^{\frac{1}{p}} \left[\sum_{i=0}^{n-1} \left[\left(\int_{x_i}^{x_{i+1}} |f''(t)|^q\,dt\right)^{\frac{1}{q}}\right]^q\right]^{\frac{1}{q}} \\ = \ & \frac{1}{2}[B(p+1, p+1)]^{\frac{1}{p}} \left(\sum_{i=0}^{n-1} h^{2p+1}\right)^{\frac{1}{p}} \sum_{i=0}^{n-1} \left(\int_{x_i}^{x_{i+1}} |f''(t)|^q\,dt\right)^{\frac{1}{q}} \\ = \ & \frac{1}{2}[B(p+1, p+1)]^{\frac{1}{p}} \|f''\|_q \left(\sum_{i=0}^{n-1} h^{2p+1}\right)^{\frac{1}{p}}, \end{aligned}$$

and the second inequality in (3.2) is proved.
In the last part, we have by (2.1), that:

$$\begin{aligned}\left|R_T(f,I_h)\right| &\leq \frac{1}{8}\sum_{i=0}^{n-1}\left(\int_{x_i}^{x_{i+1}}\left|f''(t)\right|dt\right)h_i^2 \\ &\leq \frac{1}{8}\max_{i=0,n-1}h_i^2\sum_{i=0}^{n-1}\left(\int_{x_i}^{x_{i+1}}\left|f''(t)\right|dt\right) \\ &= \frac{1}{8}\upsilon^2(I_h)\left\|f''\right\|_1,\end{aligned}$$

and the theorem is proved. ∎

Remark 1. *We would like to note that in every book on numerical integration, encountered by the authors, only the first estimation in* (3.2) *is used. Sometimes, when* $\left\|f''\right\|_q$ $(q>1)$ *or* $\left\|f''\right\|_1$ *are easier to compute, it would perhaps be more appropriate to use the second or the third estimation.*

We shall now investigate the case where we have an equidistant partitioning of $[a,b]$ given by:

$$I_h : x_i = a + \frac{b-a}{n}\cdot i,\ i = 0,1,...,.n.$$

The following result is a consequence of Theorem 1.

Corollary 1. *Let* $f:[a,b]\to\mathbb{R}$ *be a twice differentiable mapping and* $\left\|f''\right\|_\infty<\infty$. *Then we have*

$$\int_a^b f(x)\,dx = A_{T,n}(f) + R_{T,n}(f),$$

where

$$A_{T,n}(f) = \frac{b-a}{2n}\sum_{i=0}^{n-1}\left[f\left(a+\frac{b-a}{n}i\right)+f\left(a+\frac{b-a}{n}(i+1)\right)\right]$$

and the remainder $R_{T,n}(f)$ *satisfies the estimation*

$$\left|R_{T,n}(f)\right| \leq \begin{cases} \frac{(b-a)^3\left\|f''\right\|_\infty}{12n^2} \\ \frac{(b-a)^2[B(p+1,p+1)]^{2+\frac{1}{p}}\left\|f''\right\|_q}{2n^2} & , \ \frac{1}{p}+\frac{1}{q}=1,\ p>1, \\ \frac{(b-a)^2\left\|f''\right\|_1}{8n^2} \end{cases}$$

for all $n\geq 1$.

The following theorem gives a quasi-trapezoid formula which is sometimes more appropriate.

Theorem 2. *Let* $f:[a,b]\to\mathbb{R}$ *be as in Lemma 2 and* I_h *is an arbitrary partition of the interval* $[a,b]$. *Then we have:*

$$\int_a^b f(x)\,dx = A_T(f,f',I_h) + \tilde{R}_T(f,I_h), \tag{3.3}$$

where

$$A_T(f,f',I_h) := \frac{1}{2}\sum_{i=0}^{n-1}[f(x_i)+f(x_{i+1})]h_i + \frac{1}{12}\sum_{i=0}^{n-1}(f'(x_{i+1})-f'(x_i))h_i^2$$

is a perturbed trapezoidal rule and the remainder term $\tilde{R}_T(f,I_k)$ *satisfies the estimation:*

$$\left|\tilde{R}_T(f,I_k)\right| \leq \frac{M-m}{32}\sum_{i=0}^{n-1}h_i^3, \tag{3.4}$$

where the h_i *are as above.*

Proof. Writing the inequality (2.5) on the intervals $[x_i, x_{i+1}]$ $(i-0,...,n-1)$ we get:

$$\left|\int_{x_i}^{x_{i+1}} f(x)\,dx - \frac{f(x_i)+f(x_{i+1})}{2}\cdot h_i + \frac{1}{12}(f'(x_{i+1})-f'(x_i))h_i^2\right| \leq \frac{M-m}{32}\cdot h_i^3$$

for all $i = 0,...,n-1$.
Summing over i from 0 to $n-1$, we deduce easily the desired estimation (3.4). ∎

Remark 2. *As*

$$0 \leq M-m \leq 2\|f''\|_\infty,$$

then

$$\frac{M-m}{32} \leq \frac{\|f''\|_\infty}{16} < \frac{\|f''\|_\infty}{12},$$

and so the approximation of the integral $\int_a^b f(x)\,dx$ *by the use of* $A_T(f,f',I_h)$ *is better than that provided by the classical trapezoidal formulae* $A_T(f,I_h)$ *for every partition* I_h *of the interval* $[a,b]$. *Atkinson* [1] *calls this the corrected trapezoidal rule. However, only the classical* $\|f''\|_\infty$ *norm is used as the bound on the error. Atkinson* [1] *uses the idea of an asymptotic error estimate rather than the inequality by Grüss.*

The following corollary of Theorem 2 holds:

Corollary 2. *Let* $f:[a,b]\to\mathbb{R}$ *be as in Lemma 2. Thus we have:*

$$\int_a^b f(x)\,dx = A_{T,n}(f,f') + \tilde{R}_{T,n}(f),$$

where

$$\begin{aligned} A_{T,n}(f,f') &= \frac{b-a}{2n}\sum_{i=0}^{n-1}\left[f\left(a+\frac{b-a}{n}\cdot i\right)+f\left(a+\frac{b-a}{n}\cdot(i+1)\right)\right] \\ &\quad + \frac{(b-a)^2}{12n^2}(f'(b)-f'(a)), \end{aligned}$$

and the remainder $\tilde{R}_T(f)$ *satisfies the estimation:*

$$\left|\tilde{R}_{T,n}(f)\right| \leq \frac{(M-m)(b-a)^3}{32n^2},$$

for all $n \geq 1$.

Now, of we apply Lemma 3, we can state the following quadrature formulae which is a quasi-trapezoid formula.

Theorem 3. *Let f be a in Lemma 3. If I_h is a partition of the interval $[a,b]$, then we have:*

$$\int_a^b f(x)\,dx = A_{T,m,M}(f,I_h) + R_{T,m,M}(f,I_h), \tag{3.5}$$

where

$$A_{T,m,M}(f,I_h) = \sum_{i=0}^{n-1} \frac{f(x_i)+f(x_{i+1})}{2}\cdot h_i - \frac{M+m}{24}\sum_{i=0}^{n-1} h_i^3$$

and

$$|R_{T,m,M}(f,I_h)| \le \frac{M+m}{24}\sum_{i=0}^{n-1} h_i^3. \tag{3.6}$$

Proof. Applying the inequality (2.9) on $[x_i, x_{i+1}]$, we get:

$$\left|\int_{x_i}^{x_{i+1}} f(x)\,dx - \frac{f(x_i)+f(x_{i+1})}{2}\cdot h_i + \frac{M+m}{24}\cdot h_i^2\right| \le \frac{M+m}{24}\cdot h_i^3$$

for all $i \in \{0,...,n-1\}$.
Summing over i from 0 to $n-1$ we get the representation (3.5) over the estimation (3.6). ∎

Corollary 3. *Let f be as above. Then we have:*

$$\int_a^b f(x)\,dx = A_{T,m,M,n}(f) + R_{T,m,M,n}(f),$$

where

$$\begin{aligned} A_{T,m,M,n}(f) &= \frac{1}{2}\sum_{i=0}^{n-1}\left[f\left(a+i\cdot\frac{b-a}{n}\right) - f\left(a+(i+1)\cdot\frac{b-a}{n}\right)\right] \\ &\quad + \frac{M+m}{12}\cdot\frac{(b-a)^2}{n} \end{aligned}$$

and the remainder term $R_{T,m,M,n}(f)$ satisfies the estimation:

$$|R_{T,m,M,n}(f)| \le \frac{(M-m)(b-a)^3}{24n^2}.$$

Remark 3. *As $0 \le M-m \le 2\|f''\|_\infty$, then the approximation given by $A_{T,m,M,n}(f)$ to the integral $\int_a^b f(x)\,dx$ is better than the classical trapezoidal rule.*

4. Applications for Some Special Means

Let us recall the following means:

(a) The *arithmetic mean:* $A = A(a,b) := \frac{a+b}{2},\ a,b \ge 0.$
(b) The *geometric mean:* $G(a,b) := \sqrt{ab},\ a,b \ge 0.$
(c) The *harmonic mean:* $H = H(a,b) := \frac{2}{\frac{1}{a}+\frac{1}{b}},\ a,b \ge 0.$
(d) The *logarithmic mean:* $L(a,b) := \begin{cases} a & \text{if } a=b \\ \frac{b-a}{\ln b - \ln a} & \text{if } a \ne b \end{cases};\ a,b \ge 0.$

(e) The *identric mean:*

$$I(a,b) := \begin{cases} a \text{ if } a = b \\ \frac{1}{e}\left(\frac{a^a}{b}\right)^{\frac{1}{b-a}} \end{cases} .$$

(f) *The p-logarithmic mean:*

$$L_p = L_p(a,b) := \begin{cases} a \text{ if } a = b \\ \left[\frac{b-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}} \text{ if } a \neq b \end{cases},$$

$p \in \mathbb{R}\backslash\{-1,0\}.$

The following inequality is well known in the literature:

$$H \leq G \leq L \leq I \leq A.$$

It is also well known that L_p is monotonically increasing, That is, $p \in \mathbb{R}$ (assuming that $L_0 := I$ and $L_{-1} := L$).

4.1. **Special Means: Results for the Traditional Trapezoidal Rule.** The inequality (2.1) is equivalent to:

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \begin{cases} \frac{\|f''\|_\infty}{12}(b-a)^2 \\ \frac{1}{2}\|f''\|_q [B(p+1,p+1)]^{\frac{1}{p}}(b-a)^{1+\frac{1}{p}} \\ \frac{\|f''\|_1}{8}(b-a). \end{cases} \tag{4.1}$$

We can now apply (4.1) to deduce some inequalities for the special means given above by the use of some particular mappings as follows.

(a) Consider the mapping $f(x) = x^r,\ f:(0,\infty) \to \mathbb{R}$, where $r \in \mathbb{R}\backslash\{0,1\}$. Then we have for $a < b$:

$$\begin{aligned} \frac{1}{b-a}\int_a^b f(x)\,dx &= L_r^r(a,b), \\ \frac{f(a)+f(b)}{2} &= A(a^r,b^r), \\ \|f''\|_\infty &= |r(r-1)| \times \begin{cases} b^{r-2} \text{ if } r \in [2,\infty] \\ a^{r-2} \text{ if } r \in (-\infty,2)\backslash\{-1,0\} \end{cases}, \\ \|f''\|_q &= |r(r-1)|(b-a)^{\frac{1}{q}} L_{q(r-1)}^{r-1}(a,b), \\ \|f''\|_1 &= |r(r-1)| L_{r-1}^{r-1}(a,b)(b-a). \end{aligned}$$

Thus, the inequality (4.1) gives us that:

$$(4.2)\qquad \left|A\left(a^r,b^r\right)-L_r^r\left(a,b\right)\right| \leq \begin{cases} \frac{\left|r(r-1)\right|\delta_r(a,b)}{12}\left(b-a\right)^2 \\ \\ \frac{1}{2}\left|r\left(r-1\right)\right|\left(b-a\right)^2 L_{q(r-1)}^{r-1}\left(a,b\right)\left[B\left(p+1,p+1\right)\right]^{\frac{1}{p}} \\ \\ \frac{r(r-1)L_{r-1}^{r-1}(a,b)(b-a)^2}{8}. \end{cases}$$

where

$$\delta_r\left(a,b\right) := \begin{cases} b^{r-2} \text{ if } r\in[2,\infty] \\ \\ a^{r-2} \text{ if } r\in(-\infty,2)\setminus\{-1,0\} \end{cases}$$

and $\frac{1}{p}+\frac{1}{q}=1,\ p>1.$

(b) Consider now the mapping $f\left(x\right)=\frac{1}{x},\ x\in[a,b]\subset(0,\infty)$. Then we have:

$$\begin{aligned} \frac{1}{b-a}\int_a^b f\left(x\right)dx &= L_{-1}^{-1}\left(a,b\right), \\ \frac{f\left(a\right)+f\left(b\right)}{2} &= \frac{A\left(a,b\right)}{G^2\left(a,b\right)}, \\ \left\|f''\right\|_\infty &= \frac{2}{a^3}, \\ \left\|f''\right\|_q &= 2\left(b-a\right)^{\frac{1}{q}}L_{-3q}^{-1}, \\ \left\|f''\right\|_1 &= 2\left(b-a\right)L_{-3}^{-3}\left(a,b\right). \end{aligned}$$

Then the inequality (4.1) gives us that:

$$\left|\frac{A}{G^2}-\frac{1}{L}\right| \leq \begin{cases} \frac{(b-a)^2}{6a^3} \\ \\ \left(b-a\right)^2\left[B\left(p+1,p+1\right)\right]^{\frac{1}{p}}L_{-3q}^{-3} & ,\frac{1}{p}+\frac{1}{q}=1,\ p>1. \\ \\ \frac{(b-a)^2}{4}L_{-3}^{-3} \end{cases}$$

which is equivalent to:

$$(4.3)\qquad 0\leq AL-G^2\leq \begin{cases} \frac{(b-a)^2}{6a^3}LG^2 \\ \\ \left(b-a\right)^2\left[B\left(p+1,p+1\right)\right]^{\frac{1}{p}}L_{-3q}^{-3}G^2L & ,\frac{1}{p}+\frac{1}{q}=1,\ p>1. \\ \\ \frac{(b-a)^2}{4}L_{-3}^{-3}G^2L. \end{cases}$$

(c) Let us consider the mapping $f(x)=\ln x,\ x\in[a,b]\subset(0,\infty)$.
Thus we have:

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)\,dx &= \ln I(a,b),\\ \frac{f(a)+f(b)}{2} &= \ln G,\\ \|f''\|_\infty &= \frac{1}{a^2},\\ \|f''\|_q &= (b-a)^{\frac{1}{q}} L_{-2q}^{-2},\\ \|f''\|_1 &= (b-a)\,L_{-2}^{-2}.\end{aligned}$$

Then the inequality (4.1) gives us that:

$$|\ln G-\ln I|\le\begin{cases}\frac{(b-a)^2}{12a^2}\\ \frac{1}{2}(b-a)^2 L_{-2q}^{-2}\left[B(p+1,p+1)\right]^{\frac{1}{p}}\\ \frac{(b-a)^2}{8}L_{-2}^{-2}\end{cases},$$

which is equivalent to:

$$1\le\frac{I}{G}\le\begin{cases}\exp\left[\frac{(b-a)^2}{12a^2}\right]\\ \exp\left[\frac{1}{2}(b-a)^2 L_{-2q}^{-2}\left[B(p+1,p+1)\right]^{\frac{1}{p}}\right] & ,\ \frac{1}{p}+\frac{1}{q}=1,\ p>1.\\ \exp\left[\frac{(b-a)^2}{8}L_{-2}^{-2}\right].\end{cases}\tag{4.4}$$

4.2. **Special Means: Results for the Quasi-trapezoidal Rule.** The inequality (2.5) is equivalent to:

$$\left|\frac{f(a)+f(b)}{2}-\frac{(b-a)^2}{12}\cdot\frac{(f'(b)-f'(a))}{b-a}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{(M-m)(b-a)^2}{32}.\tag{4.5}$$

We can now apply (4.5) to deduce some inequalities for the special means given above by the use of some particular mappings as follows.

(a) Let us consider the mapping $f(x)=x^r,\ f:(0,\infty)\to\mathbb{R}$, where $r\in\mathbb{R}\setminus\{0,1\}$.
Thus we have

$$\frac{f'(b)-f'(a)}{b-a}=r(r-1)L_{r-2}^{r-2},$$

$$m=\inf_{x\in[a,b]}f''(x)=\begin{cases}r(r-1)a^{r-2} \text{ if } r\in(0,1)\cup(2,\infty)\\ r(r-1)b^{r-2} \text{ if } r\in(-\infty,0)\cup(1,2)\setminus\{-1\}\end{cases},$$

$$M=\sup_{x\in[a,b]}f''(x)=\begin{cases}r(r-1)b^{r-2} \text{ if } r\in(0,1)\cup(2,\infty)\\ r(r-1)a^{r-2} \text{ if } r\in(-\infty,0)\cup(1,2)\setminus\{-1\}\end{cases},$$

$$M - m = r\left(r-1\right)\left(b^{r-2} - a^{r-2}\right) = r\left(r-1\right)\left(r-2\right)\left(b-a\right)L_{r-3}^{r-3},$$

and thus the inequality (4.5) becomes:

$$\left|A\left(a^r, b^r\right) - \frac{\left(b-a\right)^2}{12} r\left(r-1\right)L_{r-2}^{r-2}\left(a,b\right) - L_r^r\left(a,b\right)\right| \leq \frac{r\left(r-1\right)\left(r-2\right)\left(b-a\right)^3}{32} L_{r-3}^{r-3} \tag{4.6}$$

(b) Let us consider the mapping $f\left(x\right) = \frac{1}{x},\ x \in [a,b] \subset (0,\infty)$.
Then,

$$\begin{aligned} \frac{f'\left(b\right) - f'\left(a\right)}{b-a} &= \frac{2A\left(a,b\right)}{G^4\left(a,b\right)}, \\ m &= \inf_{x\in[a,b]} f''\left(x\right) = \frac{2}{b^2}, \\ M &= \sup_{x\in[a,b]} f''\left(x\right) = \frac{2}{a^2}, \\ M - m &= \frac{4\left(b-a\right)A\left(a,b\right)}{G^4\left(a,b\right)}, \end{aligned}$$

and then the inequality (4.5) becomes:

$$\left|\frac{A}{G^2} - \frac{\left(b-a\right)^2 A}{6G^4} - \frac{1}{L}\right| \leq \frac{\left(b-a\right)^3 A}{8G^4}. \tag{4.7}$$

(c) Let us consider the mapping $f\left(x\right) = \ln x,\ x \in [a,b] \subset (0,\infty)$.
Then we have:

$$\begin{aligned} \frac{f'\left(b\right) - f'\left(a\right)}{b-a} &= -\frac{1}{G^2}, \\ m &= \inf_{x\in[a,b]} f''\left(x\right) = -\frac{1}{a^2}, \\ M &= \sup_{x\in[a,b]} f''\left(x\right) = -\frac{1}{b^2}, \\ M - m &= \frac{2\left(b-a\right)A}{G^4} \end{aligned}$$

and then the inequality (4.5) becomes:

$$\left|\ln G + \frac{\left(b-a\right)^2}{12G^2} - \ln I\right| \leq \frac{\left(b-a\right)^3 A}{16G^4}. \tag{4.8}$$

Remark 4. *If we use inequality* (2.8) *and* (2.9) *we can deduce similar results. We shall omit the details.*

References

[1] K.E. ATKINSON, *An Introduction to Numerical Analysis,* Wiley and Sons, Second Edition, 1989.

[2] D.S. MITRINOVIĆ, J.E.PEČARIĆ and A.M. FINK, *Inequalities Involving Functions and their Integrals and Derivatives,* Kluwer Academic Publishers, 1994.

(S.S. Dragomir, P. Cerone and A. Sofo), Department of Computer and Mathematical Sciences, Victoria University of Technology, PO Box 14428, MCMC, Melbourne, Victoria 8001, Australia

E-mail address: sever@matilda.vu.edu.au

E-mail address: pc@matilda.vu.edu.au

E-mail address: sofo@matilda.vu.du.au