Set Point Response and Disturbance Rejection Tradeoff for Second-Order Plus Dead Time Processes

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Abstract

In this paper, we shall present simple and effective tuning formulas for IMC controllers when they are applied to second-order plus dead-time processes (SOPDT). We have discovered that for controllers designed by applying IMC method, the proportional gain, the integral gain, the derivative gain of the PID part of the controller and an associate filter should all be modified according to the given formulas for the purpose of achieving set-point response and disturbance rejection tradeoff. The study has also shown that the tradeoff between set-point response and disturbance rejection is limited by normalised dead time of the SOPDT processes for the simple pole cases.

1 Introduction

According to [10], a survey carried out in Japan in 1989 revealed that proportional plus integral plus derivative (PID) controllers were employed in more than 90% of the control loops. This is because PID controllers are low order, have simple structures that are intuitively appealing, and tuning methods are widely available [13].

For many industrial and chemical plants that do not have integral and resonant characteristics, the dominant process dynamics can be represented by a first-order plus dead-time (FOPDT) transfer function [5]; that is, in Figure 1,

\[ G(s) = \frac{K}{\tau s + 1} e^{-Ls}, \]  \hspace{1cm} (1)

where \( K \) is the static process gain, \( \tau > 0 \) is the dominant time-constant in seconds, and \( L > 0 \) is the apparent dead time in seconds.

Many tuning formulas for PID controllers have been obtained for FOPDT processes [5, 7] by optimising some time-domain performance criteria. It was shown in [1] that, for FOPDT processes with a normalised dead time (defined as \( L/\tau \)) between 0.1 and 1, many of the known tuning methods often do not produce robust closed-loop systems, with a phase margin falling short of 30° and a gain margin of less than 4dB. Since stability robustness and performance robustness are important requirements, extensive research efforts have been directed towards discovering robust tuning formulas for PID controllers. For example, by considering gain and phase margin requirements with the minimum integral of squared error criterion, Ho and his co-workers [2] have successfully obtained empirical tuning formulas through curve fitting for optimal disturbance rejection when the process input is subjected to a step disturbance. Alternatively, by applying least-squares reduction to controllers designed with the Internal Model Control (IMC) method [4], Wang and his co-workers [8] have obtained PID controllers with good phase margin and set-point response. However, [2] and [8] did not provide any guidelines on how set-point response and disturbance rejection tradeoff could be accomplished.

In [6], a first-order all-pass transfer function was employed to interpolate the values of \( e^{-Ls} \) at \( s = 0 \) and \( s = j\omega_g \), where \( \omega_g \) is the specified gain crossover frequency. The IMC method is then applied to the rational function model of the plant to obtain analytically a set of PID tuning formulas for the FOPDT process.

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As a result, the actual gain crossover frequency (which is exactly $\omega_c$) have been predicted accurately and explicit formulas for the phase margin (to be denoted by $\phi_m$), the ratio of closed-loop bandwidth (to be denoted by $\omega_c$) to gain-crossover frequency, and the controller parameters in terms of $\omega_c L$ have been obtained. Moreover, it was also shown that, over the range of frequencies where the new approximation remained valid (i.e. when $\omega_c L < \pi/3$), the closed-loop system will have a guaranteed phase margin of at least 60°, and $\omega_c$ is limited solely by $L$ (as opposed to the commonly quoted $L/\tau$ when proportional controllers were employed [12]).

A procedure for tuning the IMC-PID controllers such that tradeoff is achieved between set-point response and disturbance rejection for FOPDT processes was also reported.

In [3]; a generalised PID controller was presented. This controller not only allows set-point response and disturbance rejection tradeoff to be achieved, but also possesses a guaranteed closed-loop nominal stability property. It was also illustrated by examples how generalised PID controllers and their associated tuning procedure can be applied to control FOPDT processes.

In this paper, we shall extend some of the work reported in [6] and [3] to second-order plus dead time (SOPDT) processes. The set-point response and disturbance rejection tradeoff for SOPDT processes will be discussed. A tuning procedure for the IMC-PID controller will be given and simulation examples will be presented.

2 IMC Controllers for SOPDT Processes

2.1 IMC-PID Controller for Second-Order Plant without Dead Time

In order to understand the constant disturbance rejection property of an IMC controller for the system shown in Figure 1, we first consider a second-order plant without dead time:

$$G(s) = \frac{K}{(\tau_1 s + 1)(\tau_2 s + 1)}. \quad (2)$$

By applying the IMC method [4] with a second-order IMC filter

$$F(s) = \frac{\omega_c^2}{(s + \omega_c)^2}, \quad \omega_c > 0, \quad (3)$$

we obtain a controller in the form of a PID controller; that is,

$$K(s) = \left( K_p + \frac{K_i}{s} + \frac{K_d s}{1 + T_d s} \right), \quad (4)$$

where

$$K_i = \frac{\omega_c}{2K}, \quad K_p = K_i \left( \tau_1 + \tau_2 - \frac{1}{2\omega_c} \right), \quad (5)$$

$$T_d = \frac{1}{2\omega_c} \quad \text{and} \quad K_d = K_i \tau_1 \tau_2 - K_p T_d. \quad (6)$$

It is well known from the theory of IMC design that $\omega_c$ will be the -6dB designed closed-loop bandwidth.

2.2 Set-point Response and Disturbance Rejection Tradeoff for Second-Order Plant without Dead Time

From the designed sensitivity function relating the disturbance at the plant input to the system output (as shown in Figure 1),

$$S(s) = \frac{1}{K(s)} T(s),$$

where

$$T(s) = \frac{G(s)K(s)}{1 + G(s)K(s)} = \frac{\omega_c^2}{(s + \omega_c)^2}$$

is the designed closed-loop transfer function, it can be seen easily that, generally, all the poles of $G(s)$ that are cancelled by the zeros of $K(s)$ will become the poles of $S(s)$. As a result the disturbance rejection response is slow if $\tau_1$ or $\tau_2$ is large.

By writing the IMC-PID controller in the pole-zero form,

$$K(s) = \frac{\omega_c^2}{K} \left( s + \frac{1}{\tau_1} \right) \left( s + \frac{1}{\tau_2} \right), \quad (7)$$

we can see that the IMC-PID controller achieved good nominal set-point response by cancelling the poles of $G(s)$ at $-1/\tau_1$ and $-1/\tau_2$ by the corresponding zeros in $K(s)$. Therefore, it is clear that the IMC-PID controller will produce slow settling disturbance rejection if the disturbance enters the system via the plant input and if $\tau_1$ or $\tau_2$ is not small. This also implies that, in order to have a fast settling disturbance rejection, we should not cancel the slow plant poles at $-1/\tau_1$ and $-1/\tau_2$ by the corresponding controller zeros. Hence, instead of $K(s)$, we should employ a modified IMC-PID controller $K'(s)$ to prevent the problematic pole-zero cancellations:

$$K'(s) = \frac{\omega_c^2}{K} \left( s + z_1 \right) \left( s + z_2 \right), \quad (8)$$

or

$$K'(s) = K_p \frac{K'_i}{s} + \frac{K'_d s}{1 + T'_d s}, \quad (9)$$
with
\[ K'_i = \frac{\omega_c}{2K} \left( \frac{1}{z_1} + \frac{1}{z_2} - \frac{1}{2\omega_c} \right), \]
(10)

where
\[ \gamma_1 = \tau_1 z_1, \gamma_2 = \tau_2 z_2. \]
(11)

and
\[ K'_p = K'_i \left( \frac{1}{z_1} + \frac{1}{z_2} - \frac{1}{2\omega_c} \right), \]
(12)
\[ T'_d = \frac{1}{2\omega_c} = T_d, \]
(13)
\[ K'_d = K'_i \frac{1}{z_1 z_2} - K'_p T'_d. \]
(14)

(Note that \( K(s) \) can be recovered from \( K'(s) \) by setting \( z_1 = \frac{1}{\gamma_1} \) and \( z_2 = \frac{1}{\gamma_2} \) or \( \gamma_1 = \gamma_2 = 1 \).)

To prevent \(-z_1\) and \(-z_2\) from becoming the dominant poles of \( S(s) \), we would like to set \( z_1 > \frac{1}{\gamma_1} \) and \( z_2 > \frac{1}{\gamma_2} \) in equation (10) (or \( \gamma_1 > 1 \) and \( \gamma_2 > 1 \) in equation (10)) for fast disturbance rejection.

Also note that the integral gain \( K'_i \) should be increased by a factor \( \gamma_1 \gamma_2 \) and the proportional gain \( K'_p \) should be adjusted according to equations (12) and (14) to achieve set-point response and disturbance rejection tradeoff. This can be seen in the following simulation example.

Consider a second-order plant (without dead time) with \( K = 1 \), \( \gamma_1 = 1 \) and \( \gamma_2 = 10 \). A modified IMC-PID controller was employed where \( \gamma_1 \) and \( \gamma_2 \) were set to different values to obtain better disturbance rejection results. In Figure 2, subplot (a) refers to unit-step set-point responses, subplot (b) refers to control signals corresponding to a unit-step set-point responses, subplot (c) refers to unit-step disturbance responses, and subplot (d) refers to control signals corresponding to a unit-step disturbance responses. The solid curves in Figure 2 are the results with the original IMC-PID controller (i.e., \( \gamma_1 = \gamma_2 = 1 \)), the dashed curves are for \( \gamma_1 = 1 \), \( \gamma_2 = 2 \), the dotted curves are for \( \gamma_1 = 1 \), \( \gamma_2 = 3 \) and the dash-dotted curves are for \( \gamma_1 = 3 \), \( \gamma_2 = 1 \).

It can be observed that by setting the product \( \gamma_1 \gamma_2 \) to be greater than 1, we sacrifice the set-point performance to secure a faster settling in disturbance rejection. The tradeoff could be achieved over a wide range of \( \gamma_1 \gamma_2 \). Note that it is more effective to adjust the \( \gamma \) value that is related to the slow time constant (i.e., \( \gamma_2 \) in the above example) to prevent the slow plant pole to be cancelled by the corresponding controller zero. As can be seen in Figure 2, the unit-step set-point response and its corresponding control signal may have been sacrificed too much for achieving a faster disturbance rejection response with \( \gamma_1 = 3 \), \( \gamma_2 = 1 \).

2.3 IMC Controller for SOPDT Processes

The SOPDT transfer function can be expressed as:
\[ G(s) = \frac{K}{(\tau_1 s + 1)(\tau_2 s + 1)} e^{-Ls}. \]
(15)

Now consider a model
\[ \tilde{G}(s) = G_m(s) \tilde{G}_a(s), \]
where
\[ \tilde{G}_a(s) = \frac{1 - \alpha Ls}{1 + \alpha Ls} \]
with \( \alpha = 0.5 \) is the first-order Padé approximation \(^1\) of \( e^{-Ls} \) in equation (15), and
\[ G_m(s) = \frac{K}{(\tau_1 s + 1)(\tau_2 s + 1)} \]
is the minimum phase part of \( G(s) \).

By applying the IMC method \([4]\) with a second-order IMC filter defined previously in equation (3), the controller can be derived as:
\[ K_1(s) = K(s) H_1(s) \]
(16)
with
\[ H_1(s) = \frac{(s + \frac{\tau_1}{2})(s + 2\omega_c)}{(s + \frac{\tau_2}{2})(s + 2\omega_c) + 2\omega_c^2}, \]
and \( K(s) \) is the PID part of the IMC controller as defined previously via equations (4), (5), and (6).

2.4 Set-point Response and Disturbance Rejection Tradeoff for SOPDT processes

Following the same procedure described in Section 2.2, a modified IMC controller for set-point response and

\(^1\) It was described in [6] that a \( \tilde{G}_a(s) \) with \( \alpha = 0.5 \) will provide a good approximation to \( e^{-Ls} \) over a sufficiently wide control design regime.
disturbance rejection tradeoff for SOPDT processes is found to be:

\[ K'_1(s) = K'(s)H'_1(s) \]  

(17)

with

\[ H'_1(s) = \frac{(s + \frac{\gamma_1}{2})(s + 2\omega_c)(s + \frac{1}{\gamma_1})(s + \frac{1}{\gamma_2})}{D(s)} \]

and \( D(s) = (s + \frac{\gamma_1}{2})(s + 2\omega_c)(s + \frac{1}{\gamma_1})(s + \frac{1}{\gamma_2}) + 2\omega_c^2(s + z_1)(s + z_2) \). \( K'(s) \) is the modified IMC-PID controller given previously via equation (8) with \( K'_1, K'_d, T'_d, \) and \( K'_d \) defined in equations (10), (12), (13), and (14) respectively.

Observe that the modified IMC controller for achieving set-point response and disturbance rejection tradeoff for SOPDT processes consists of the modified IMC-PID controller for second-order plant without dead time (i.e. \( K'(s) \)) cascaded with a fourth order filter \( H'_1(s) \).

3 Tuning Procedure and Simulations

Before describing a tuning procedure of the IMC controller for SOPDT processes, we would make the following important observations. Recall that the modified IMC-PID controller for achieving set-point response and disturbance rejection tradeoff for second-order plant without dead time is defined by three tuning parameters (namely, \( \omega_c, \gamma_1 \) and \( \gamma_2 \)). By setting \( \gamma_1 = \gamma_2 = 1 \) (corresponding to setting \( z_1 = 1/\gamma_1 \) and \( z_2 = 1/\gamma_2 \) ), we recover the original IMC-PID controller \( K(s) \) shown in equation (4) from the modified IMC-PID controller \( K'(s) \) shown in equation (9). As shown in equation (16), the original IMC controller for SOPDT process consists of the original IMC-PID controller for second-order process without dead time cascaded with a second order filter \( H_1(s) \) while, as shown in equation (17), the modified IMC controller for achieving set-point response and disturbance rejection tradeoff for SOPDT processes consists of the modified IMC-PID controller for second-order processes without dead time \( K'(s) \) cascaded with a fourth order filter \( H'_1(s) \). Once we have observed these relationships between \( K'(s) \) and \( K(s), K'_1(s) \) and \( K_1(s) \), the tuning procedure of the modified IMC-PID controllers for SOPDT processes can be described as follows:

1. Specify the desired closed-loop performance in terms of the designed closed-loop bandwidth \( \omega_c \) as if we are going to control the plant by the original IMC controller \( K_1(s) \).

2. Set \( \gamma_1 = \gamma_2 = 1 \) and apply the value of \( \omega_c \) obtained from the previous step to the modified IMC controller \( K'_1(s) \). That is, initialise the modified IMC controller \( K'_1(s) \) to give good set-point step response (and possibly slow settling disturbance rejection).

3. If the disturbance rejection is not sufficiently fast, increase the value of the appropriate \( \gamma \) from 1 to speed up the disturbance rejection. For processes with real and distinct poles, increase the value of \( \gamma \) related to the slower time constant. For processes with equal real poles or complex conjugate poles, increase the values of \( \gamma_1 \) and \( \gamma_2 \) equally.

4. Fine tune \( K'_1(s) \) by making incremental changes to the values of the appropriate \( \gamma \) (and \( \omega_c \) if necessary) until the desired results are obtained.

We shall now present some simulation examples. In each of the following figures, subplot (a) refers to unit-step set-point responses, subplot (b) refers to control signals corresponding to a unit-step set-point responses, subplot (c) refers to unit-step disturbance responses, and subplot (d) refers to control signals corresponding to a unit-step disturbance responses.

**Example 1**

In this example, a SOPDT plant with \( \gamma_1 = 1 \ sec, \gamma_2 = 10 \ sec, K = 1, \) and \( L = 1 \ sec \) is used. The dominant time constant is 10 sec and the normalised dead time is \( L/\tau = 0.1 \) in this case. We used \( \omega_c = 0.5 \ rad/sec \). The results are shown in Figure 3.

The solid curves in Figure 3 are the results with the original IMC controller (corresponding to \( \gamma_1 = \gamma_2 = 1 \)), the dashed curves are for \( \gamma_1 = 1, \gamma_2 = 2 \) and the dotted curves are for \( \gamma_1 = 1, \gamma_2 = 3 \). Observe how we sacrifice the set-point performance to secure a faster settling disturbance rejection.

**Example 2**

In this example, a SOPDT plant with \( \gamma_1 = 1 \ sec, \gamma_2 = 10 \ sec, K = 1, \) and \( L = 5 \ sec \) is used. The dominant
The dominant time constant is 10 sec and the normalised dead time is $L/\tau = 0.5$ in this case. We again used $\omega_c = 0.5 \text{ rad/s}$. The results are shown in Figure 4.

The solid curves in Figure 4 are the results with the original IMC controller, the dashed curves are for $\gamma_1 = 1, \gamma_2 = 2$ and the dotted curves are for $\gamma_1 = 1, \gamma_2 = 3$. Observe again that we sacrifice the set-point performance to secure a faster settling disturbance rejection.

Note that the tradeoff achieved with the same values of $\gamma_1$ and $\gamma_2$ is more limited in this example than that of Example 1 due to the higher value of the normalised dead time $L/\tau$.

Example 3

In this example, a SOPDT plant with $\tau_1 = 1\text{ sec}, \tau_2 = 10\text{ sec}, K = 1$, and $L = 10\text{ sec}$ is used. The dominant time constant is 10 sec and the normalised dead time is $L/\tau = 1$ in this case. We have kept $\omega_c$ at 0.5 rad/s. The results are shown in Figure 5.

The solid curves in Figure 5 are the results with the original IMC controller, the dashed curves are for $\gamma_1 = 1, \gamma_2 = 2$ and the dotted curves are for $\gamma_1 = 1, \gamma_2 = 3$. Note that the tradeoff achieved with the same values of $\gamma_1$ and $\gamma_2$ is even more limited in this example than that of Example 2. This is due to the much higher value of the normalised dead time $L/\tau$.

Example 4

In this example, a SOPDT plant with equal time constants is used (i.e., $\tau_1 = 10\text{ sec}, \tau_2 = 10\text{ sec}, K = 1$, and $L = 1\text{ sec}$). The normalised dead time is $L/\tau = 0.1$ in this case. We used $\omega_c = 0.2 \text{ rad/s}$. The results are shown in Figure 6.

The solid curves in Figure 6 are the results with the original IMC controller, the dashed curves are for $\gamma_1 = \gamma_2 = \sqrt{2}$ and the dotted curves are for $\gamma_1 = \gamma_2 = \sqrt{3}$. Note that the tradeoff is achieved with the equal values of $\gamma_1$ and $\gamma_2$ (since $\tau_1 = \tau_2$).

The following examples deal with SOPDT plant with complex conjugate poles. The SOPDT plant transfer function is in the form of

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} e^{-Ls}$$

where $\omega_n$ is the undamped natural frequency and $\zeta$ is the damping factor of the SOPDT plant.

Example 5

In this example, a SOPDT plant with complex conjugate poles is used (i.e., $p_1 = -0.5 + 0.866j$ and $p_2 = -0.5 - 0.866j, K = 1$, and $L = 2\text{ sec}$). These poles are corresponding to $\omega_n = 1 \text{ rad/sec}$, and $\zeta = 0.5$. We used $\omega_c = 0.2 \text{ rad/s}$. The results are shown in Figure 7.

The solid curves in Figure 7 are the results with the original IMC controller, the dashed curves are for $\gamma_1 = \ldots$
The solid curves in Figure 9 are the results with the original IMC controller, the dashed curves are for $\gamma_1 = \gamma_2 = \sqrt{2}$ and the dotted curves are for $\gamma_1 = \gamma_2 = \sqrt{3}$. Note that the tradeoff is achieved with equal values of $\gamma_1$ and $\gamma_2$.

Note that in Examples 1, 2, 3 and 4 only real poles in the SOPDT plant have been considered. However, the tuning procedure can be easily extended to SOPDT plant with complex conjugate poles as illustrated in Examples 5, 6, and 7.

From the results of the Examples 1, 2, 3, and 4 we can make the following important observation. The achievable tradeoff between set-point response and disturbance rejection for SOPDT processes under IMC control is limited by $L/\tau$ of the processes for the simple pole cases, where $\tau$ is the dominant time constant of the SOPDT processes. For SOPDT plant with complex conjugate poles, the factor which limits the achievable tradeoff between set-point response and disturbance rejection under IMC control needs to be examined further in future work.

4 Conclusions

In this paper, we have presented some derivation of IMC controllers and tuning procedures when they are applied to SOPDT processes for achieving set-point response and disturbance rejection tradeoff. We have discovered that for controllers designed by following the IMC approach, the integral gain, the proportional gain, the derivative gain plus a fourth-order filter of the controller should all be adjusted according to the given formulas and tuning procedure presented for the purpose of achieving set-point response and disturbance rejection tradeoff. The study has also shown that the tradeoff between set-point response and disturbance rejection is again limited by the normalised dead time for the simple pole cases. For SOPDT plant with complex conjugate poles, the factor which limits the achievable tradeoff between set-point response and disturbance rejection under IMC control needs to be examined further in future work.

References


