Black holes with a twist

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Dedicated to

Mum and Dad

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Abstract

In this thesis we study string theory on orbifolds of AdS_3. Non-extremal BTZ black holes have been shown to offer a good opportunity to study closed string tachyon condensation, as there are tachyons in the winding sector even in superstring theory. We study extremal BTZ black holes, both $M = 0$ and $M = J$ from a world sheet perspective. The string spectrum is calculated within bosonic string theory and tachyons are identified within the spectrum. The flat space limit of the $M = 0$ black hole is considered and an extension to superstring theory is discussed. In the second half of the thesis we discuss the self dual orbifold. The self dual orbifold is a simple example of a geometry which contains an AdS_2 factor. AdS_2 factors also appear in the near horizon limit of extremal Kerr and Reissner-Nordström black holes. Using the AdS/CFT correspondence we conjecture that the self dual orbifold is dual to a CFT on two distinct boundary regions and find evidence to support this statement. We consider asymptotically self dual orbifold spacetimes, one of which is dual to a single copy of the groundstate of the CFT.

Declaration

The work in this thesis is based on research carried out at the Centre for Particle Theory, the Department of Mathematical Sciences, the The University of Durham, England. No part of this thesis has been submitted elsewhere for any other degree or qualification and it all my own work unless referenced to the contrary in the text.

Chapter 5 is based on work done in collaboration with Simon Ross published in [1]. Chapter 6 is based on work done in collaboration with Simon Ross and Vijay Balasubramanian [2].

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Chapter 1

Introduction

At the beginning of the twentieth century the two great pillars of modern theoretical physics were established, general relativity and quantum mechanics. General relativity governs the world of the large scale, stars collapsing and galaxies being born. Quantum mechanics, which has since been extended to quantum field theory and the standard model, describes the microscopic landscape of atoms and nuclei. These two theories, two of the most accurate and precise in human history, fundamentally disagree with each other. Although this discrepancy is serious, it only becomes significant in a few extreme situations where the two regimes overlap. The most obvious example is a black hole singularity. Here the distance scales we wish to consider are very small, but space cannot be approximated to be flat; neither general relativity or field theory can explain physics here satisfactorily.

There have been several attempts in the latter half of the 20th century and the beginning of the 21st, to establish a 'theory of everything', which describes the four fundamental forces of nature and incorporates quantum mechanics. The most enduring of these is string theory. Instead of point particles string theory uses small pieces of 'string' as the fundamental objects of physics. These strings can either be open or closed. Closed strings are loops with periodic boundary conditions. Open strings instead have two end points, which are restricted to be on objects called D-branes [3, 4]. String theory has advantages and disadvantages over standard quantum field theory. Possibly the largest advantage is that the graviton naturally appears in the closed string spectrum. This means that string theory offers a way to combine general relativity with quantum mechanics. The two major disadvantages of bosonic string theory are that there are tachyons in the bulk spectrum and that it predicts 26 spacetime dimensions rather than the 4 which are observed. Tachyons represent instabilities in a theory, fortunately these can removed by imposing supersymmetry. Superstring theory lives in 10 dimensional space, which whilst less than bosonic string theory, is 6 more than would be liked. To rectify this problem these extra dimensions are considered to be compactified on a scale smaller than we can probe with current technology. The physics which emerges from string theory is highly dependent on the way in which the compactification is achieved, each different compactification leads to different string vacua. As of yet no compactification has been found which has a low energy limit containing solely the standard model of particle physics.

Tachyon condensation is a useful tool in the quest to understand the connection between these different vacua of string theory. Currently tachyon condensation is better understood for open rather than closed strings. In open string theory tachyon condensation leads to the annihilation of unstable D-branes, as these constrain the end points of open strings. As closed strings contain the graviton in their spectrum and gravity defines the geometry of spacetime, it is conjectured that closed string tachyon condensation results in some change to the structure of spacetime. This may be a change in topology or even the wholesale destruction of spacetime.

Asymptotically locally euclidean (ALE) spaces (i.e. spaces whose geometry at large distances are of the form $\mathbb{R}^k/\Gamma$, where Γ is a subgroup of the rotation group) have been studied [5] in connection with closed string tachyon condensation. These spaces have a singular point at the origin about which the rotation subgroup acts. Winding tachyons are present in this background and are localised near to this singularity. This localisation allows control in the condensation process. One of the simplest examples of an ALE spacetime is the $\mathbb{C}/\mathbb{Z}_n$ orbifold. There are several possibilities which could result from tachyon condensation in this case; a hole could appear at the singularity and spread to consume the whole space, the orbifold could elongate at the singularity to produce an infinite throat or there could be a topological change in which the singularity 'smoothed out' [6]. There is strong evidence that the last option is in fact the case [7]. This would be a useful result as it would remove two difficulties from the theory at once, both the tachyon and the singularity.

In recent times much of the progress which has been made within string theory has been due to the AdS/CFT conjecture [8–10]. This conjecture relates string theory living on an AdS_n spacetime to a gauge theory which lives on the $(n-1)$ dimensional boundary of that spacetime. Although this conjecture has not been rigorously proven there is a large amount of evidence in support of it. The conjecture implies that strongly coupled string theories are dual to weakly coupled gauge theories and vice versa. This duality means that calculations that were difficult within strongly coupled gauge theories can be dealt with perturbatively within string theory. The conjecture was originally conceived for an AdS_5 spacetime with a four dimensional gauge theory living on the boundary, it has however been extended to a variety of different situations [8].

If the gauge theory on the boundary is to be at a finite temperature, then the string theory in the bulk must have a finite temperature and the two theories must have matching entropy. A thermal object must be introduced into the bulk. The simplest object which can fulfil this role is a black hole. Classically black holes are very simple due to the black hole uniqueness theorems - black holes have no hair [11–13]. They can be completely determined by three quantities (at least in four dimensions [14]), their mass, charge and angular momentum. Simple objects having large entropy seems like a contradiction but it can be explained by considering quantum mechanics on a curved background, in a semi classical setting. Via black hole thermodynamics, black holes have a temperature associated with their surface gravity and an entropy associated with the area of their event horizon [15]. As black holes are thermal it is only natural that they radiate and this can be shown to be the case, with Hawking radiation [16, 17]. The corresponding entropy in the gauge theory is a result of entanglement between states on causally disconnected regions of the boundary.

This thesis looks at string theory on a background of various AdS_3 orbifolds. AdS_3 is the unique local solution to Einstein's equations in three dimensions with a negative cosmological constant. Note however that it is only locally unique, global identifications can be made such that the manifold is still a solution to Einstein's equations. Manifolds constructed in this manner are known as orbifolds. One of the most studied categories of AdS_3 orbifold is the Bañados-Teitelboim-Zanelli (BTZ) black hole [18,19]. As an example massive non-rotating BTZ black holes are orbifolds of AdS_3 by a hyperbolic generator of $SL(2,\mathbb{R})$, the mass and radius of these black holes are determined by the period of the identification [20]. These have similar causal structure to higher dimensional black holes, with a singularity created in order to remove closed timelike curves from the spacetime. As superstring theory lives in ten dimensions the spacetimes are completed with seven more compatified dimensions, which usually take the form of $S^3 \times T^4$. Although $AdS_3 \times S^3 \times T^4$ and its orbifolds are not good models for the real world, as there are only three large dimensions rather than four, they are useful models for learning more about string theory.

One of the reasons it is interesting to study BTZ black holes is that they may help to shed light on the process of tachyon condensation. In a similar manner to the $\mathbb{C}/\mathbb{Z}_n$ orbifold, the closed string spectrum of BTZ black holes contains a winding sector, where the strings are wrapped around the black hole. In certain circumstances this winding sector can contain a tachyon even in superstring theory. (There are no tachyons in the non-winding sector in superstring theory). The fact that these tachyons are localised means that the condensation process should be easier to follow, as the tachyons are restricted to one particular region of space[1].

The near horizon limit of extremal rotating BTZ black holes is known as the self-dual orbifold [21], which can itself be constructed directly as an orbifold of AdS_3. The self-dual orbifold is an interesting object as it is a simple spacetime which can be written as a fibration over AdS_2. AdS_2 is a factor which appears in the near horizon limit of a number of extremal black holes. For example, it appears in the near horizon limit of both extremal Kerr and Reissner-Nordström black holes [22, 23].

[1] In fact the winding modes are only 'quasi localised' due to long string states in the spectrum. These long string states exist because the string tension is balanced out by the B-field, which is necessary in constructing BTZ spacetimes [20]. Quasi-localised tachyons are still however (in principle) easier to deal with than tachyons in the bulk.

Extremal black holes have zero temperature but finite entropy. It is hoped that by considering the AdS_2/CFT correspondence the entropy of extremal black holes can be explained. The self-dual orbifold offers a prime opportunity to study this correspondence.

Chapter 2 motivates and supplies the tools required for the study of tachyons on BTZ black holes. The first section is on worldsheet string theory. The Wess Zumino Witten (WZW) model [24] and techniques used for calculating the string spectrum and locating tachyons are introduced. The second section of this chapter looks into tachyon condensation where different techniques for following the process are considered. The $\mathbb{C}/\mathbb{Z}_n$ orbifold is given as a brief example where tachyons can be located in a winding sector and the possible resulting spacetimes are explored.

Chapter 3 looks into the geometry of AdS_3 and some of its orbifolds. AdS_2, AdS_3, BTZ black holes and the self-dual orbifold are all discussed in various different coordinate systems. These differing coordinate systems will become useful for calculations in later chapters. The final section looks at calculating the closed string spectrum on a massive non-rotating BTZ black hole [20, 25–29] in bosonic string theory. Massive BTZ black holes are orbifolds of AdS_3, created by a hyperbolic generator. Here the twisted sector states (closed string states with a non-zero winding number about the black hole) are calculated in terms of a parafermionic representation of the current algebra, constructed using a twist operator. This technique is based on work previously done on long string sectors in global AdS_3 [30] and closely related to the work on the elliptic orbifold in [31]. This is a useful example before considering extremal BTZ black holes and also supplies results which will be used directly in calculating the extremal rotating BTZ black hole spectrum.

Chapter 4 introduces black hole thermodynamics and the AdS/CFT correspondence. The first half of this chapter concentrates on black hole thermodynamics, showing some of the techniques that can be used to calculate the temperature of different backgrounds and analyse the spectrum of Hawking radiation in different coordinate frames. The second half of the chapter concentrates on the AdS/CFT correspondence, motivating it from both a holographic and D-brane perspective. The chapter ends by looking at thermal states such as black holes within AdS/CFT and raising some questions about AdS_2/CFT_1.

Chapter 5 looks into extremal BTZ black holes in both the massless and extremal rotating cases. The spectrum is calculated in bosonic string theory and tachyons are found in the winding sectors. Extremal BTZ black holes have been less studied than the non-extremal black holes mentioned in chapter 3. They are in a different class of orbifold as they are constructed using a parabolic, rather than hyperbolic orbifold generator and therefore need to be studied separately. Different worldsheet techniques need to be employed as the parafermionic representation used for the non-extremal black hole cannot be applied in this case. A different representation of the vertex operators, which diagonalises the angular momentum for the extremal BTZ black hole must be found. Previous work on the winding sectors of extremal BTZ black holes was done in [32], here the relevance of the Wakimoto representation of currents used in [26] was noted. This chapter introduces the $M = 0$ BTZ black hole. The vertex operators in the untwisted sector are calculated using the Wakimoto representation. The twisted sector is then constructed using a twist operator and the tachyon is located within it. The flat space limit is discussed and the analysis is extended to the extremal rotating BTZ black hole. Finally the extension to superstring theory is considered, however difficulties arise in finding an appropriate representation for the spin fields.

In chapter 6, we consider the self-dual orbifold. Like all geometries with an AdS_2 factor it has two asymptotic boundaries. From an orbifold point of view the boundary has two disconnected regions because fixed points must be removed when making the orbifold identification. This chapter looks into whether the self-dual orbifold geometry is dual to a single copy of a gauge theory or two copies living on the two different boundaries. In aid of this we use coordinate systems which cover differing parts of the spacetime and calculate the dual gauge theories in these different coordinate systems. We also look at asymptotically self-dual orbifold spacetimes and ask whether any of these are dual to the ground state of the gauge theory.

Chapter 2

Introduction to worldsheet string theory and tachyon condensation

This chapter sets out some of the motivations and tools required to study twisted sector states in extreme BTZ black holes. Section 2.1 shows how tachyons can be located in a spectrum using worldsheet string theory. Section 2.2 outlines some background on tachyon condensation and why BTZ black holes might provide a useful laboratory for future study. A simple example of this is given, the tachyon is found in the twisted sector of the $\mathbb{C}/\mathbb{Z}_n$ orbifold and the possible consequences are discussed.

2.1 World-sheet string theory

String theory can be considered from two distinct points of view; the worldsheet or the target spacetime. From the target space perspective the string propagates through spacetime, mapping out a world sheet as it goes. From the worldsheet point of view the spacetime dimensions are bosonic fields on the world sheet. For a lot of the work in this thesis, especially calculating the string spectrum, it is this worldsheet perspective which provides the tools which are required. This section introduces the Wess Zumino Witten (WZW) model, which is a model of worldsheet string theory on a group manifold and a useful way to consider string theory on both AdS_3 and S^3 backgrounds. Using this model the worldsheet spectrum is calculated.

Finally the extension to superstring theory is considered.

2.1.1 The WZW model

We will discuss worldsheet string theory on a group manifold, this is an effective method for studying worldsheet string theory with both S^3 and AdS_3 target spaces. The model used here is known as the WZW model [24]. It is a nonlinear sigma model, with the action,

$$S_{WZW} = \frac{k}{8\pi\alpha'}\int d^2\sigma Tr(g^{-1}\partial_a g g^{-1}\partial^a g) + \frac{ik}{12\pi}\int Tr(g^{-1}dg\wedge g^{-1}dg\wedge g^{-1}dg). \quad (2.1)$$

Here the string lives in a spacetime with symmetries $G\times G$. g is an element of the symmetry group G. k is the level of the current algebra. As we shall see in section 3.1, AdS_3 has the symmetry group $SL(2,\mathbb{R})_L \times SL(2,\mathbb{R})_R$. We are therefore most interested in the case in which g is a group element of $SL(2,\mathbb{R})$. The worldsheet is a two dimensional manifold, covered by the coordinates (τ,σ). It is however in practice often easier to use the complex coordinates $z=\sigma+i\tau$, $\bar{z}=\sigma-i\tau$.

Any theory with symmetries such that the action remains invariant under $\phi_\alpha(z,\bar{z}) \rightarrow \phi'_\alpha(z,\bar{z}) = \phi_\alpha(z,\bar{z}) + \delta\phi_\alpha(z,\bar{z})$, with $\delta\phi_\alpha(z,\bar{z})$ small, will contain conserved currents and charges by Noether's theorem. The conserved current can be calculated by considering the transformation $\tilde{\phi}_\alpha(z,\bar{z}) = \phi_\alpha(z,\bar{z}) + \rho(z,\bar{z})\delta\phi_\alpha(z,\bar{z})$ where $\rho(z,\bar{z})$ is an arbitrary function and requiring $S[\phi]=S[\tilde{\phi}]$. In the case of the WZW model the action is invariant under [20],

$$g(z,\bar{z}) \rightarrow \Omega(z)g(z,\bar{z})\bar{\Omega}(\bar{z})^{-1}, \quad (2.2)$$

with $\Omega\in SL(2,\mathbb{R})$. Take the transformation to be small such that, $g\rightarrow g+\omega g - g\bar{\omega}$ with $\omega,\bar{\omega}\in sl(2,\mathbb{R})$ and $\Omega=e^{\omega}$.

Using this transformation leads to the worldsheet conserved currents [25],

$$J(z) = -\frac{k}{2}\partial_z g g^{-1}, \qquad \bar{J}(\bar{z}) = -\frac{k}{2}g^{-1}\partial_{\bar{z}} g. \quad (2.3)$$

J is conserved with respect to $\bar{z}$ and $\bar{J}$ is conserved with respect to z as they are holomorphic and antiholomorphic respectively. J is dependent only on z and $\bar{J}$

is dependent only on $\bar{z}$. These currents can then be rewritten as a vector using $J = J^a \tau_a$,

$$J^a = kTr(\tau^a \partial_z g g^{-1}), \qquad \bar{J}^a = kTr(\tau^a \partial_{\bar{z}} g^{-1} g). \tag{2.4}$$

τ^a are the generators of the $SL(2,\mathbb{R})$ symmetry defined by,

$$\tau^1 = \frac{i}{2}\sigma^3, \qquad \tau^2 = \frac{i}{2}\sigma^2, \qquad \tau^3 = \frac{1}{2}\sigma^2, \tag{2.5}$$

where σ^a are the Pauli matrices. Sometimes it will be useful to expand these currents in terms of modes.

$$J^a(z) = \sum_{n\in\mathbb{Z}} z^{-n-1} J^a_n, \tag{2.6}$$

The Ward identity can be used to then calculate the operator product expansions (OPEs) for these conserved currents [33]. The Ward identity is,

$$i\delta\mathcal{A}(z_0,\bar{z_0}) = \oint_{z_0} \frac{dz}{2\pi i}\omega J(z)\mathcal{A}(z_0,\bar{z_0}) + \oint_{\bar{z}_0} \frac{d\bar{z}}{2\pi i}\bar{\omega}\bar{J}(\bar{z})\mathcal{A}(z_0,\bar{z_0}), \tag{2.7}$$

where $\mathcal{A}$ is an arbitrary field. Consider then the case where $\bar{\omega} = 0$ and $\mathcal{A} = J$. Using the transformation above (2.2) and equation (2.4), it can be calculated that

$$\delta J = [\omega, J] - k\partial_z \omega, \tag{2.8}$$

ω can also be split up into vector form, $\omega = \tau^a \omega_a$. (2.8) can then be reexpressed as,

$$\delta J^a = i f_{abc} \omega^b J^c - k\partial_z \omega^a \tag{2.9}$$

where f_{abc} is a structure constant, in this case where $G = SL(2,\mathbb{R})$ it is the totally antisymmetric tensor, due to the commutation relations of τ^a. Using (2.7) and the residue theorem it can be shown that the OPE for the currents is given by,

$$J^a(z)J^b(w) \sim \frac{k}{2}\frac{\eta^{ab}}{(z-w)^2} + \frac{i\epsilon^{ab}{}_c J^c(w)}{(z-w)}, \tag{2.10}$$

with the conventions $\epsilon^{123} = 1$ and $\eta_{ab} = diag(1,1,-1)$. There is a similar expression for $\bar{J}$. Considering the case where $\bar{\omega} = 0$ and $\mathcal{A} = \bar{J}$, it is straightforward to show that $J^a \bar{J}^b = 0$.

As with any conserved current, there will be associated conserved charges. In this case the conserved charges will be the energy and angular momentum operators.

These can be calculated by making infinitesimal changes to the group element with respect to the spacetime angular and time coordinates.

$$-i\delta g = \omega(z)g - g\bar{\omega}(\bar{z}). \tag{2.11}$$

Applying this to the Ward identity then leads to the conserved charges. In the case of global coordinates (see chapter 3) for example the conserved charges become;

$$E_{SL(2,\mathbb{R})} = J_0^3 + \bar{J}_0^3 = \oint \frac{dz}{2\pi i} J^3 - \oint \frac{d\bar{z}}{2\pi i} \bar{J}^3, \tag{2.12}$$

$$L_{SL(2,\mathbb{R})} = J_0^3 - \bar{J}_0^3 = \oint \frac{dz}{2\pi i} J^3 + \oint \frac{d\bar{z}}{2\pi i} \bar{J}^3. \tag{2.13}$$

2.1.2 The spectrum

From now on we will concentrate on the holomorphic sector, with α' set to 1. Primary fields are defined as fields which transform covariantly, the same way as $g(z,\bar{z})$ in equation (2.2). This is equivalent to imposing that,

$$: J_0^a \Phi_\lambda : \;=\; -\sum_{\lambda'} t^a_{\lambda'\lambda} \Phi_{\lambda'} \tag{2.14}$$

$$: J_n^a \Phi_\lambda : \;=\; 0 \qquad n > 0 \tag{2.15}$$

where Φ_λ is a primary field and $t^a_{\lambda'\lambda}$ is a matrix element. :: means normal ordered

The energy momentum tensor can now be defined in terms of the worldsheet currents,

$$T(z) = \frac{\eta_{ab}}{k-2} : J^a(z) J^b(z) : \tag{2.16}$$

Acting with the energy momentum tensor on a primary operator gives the conformal dimension, h_λ, of that operator.

$$T\Phi_\lambda = h_\lambda \Phi_\lambda. \tag{2.17}$$

Once the spectrum has been found in this manner, it can be seen if any tachyons are present. To be tachyonic a mode must satisfy three conditions. It must grow exponentially in time, be a physical state (i.e. $(L_0 - 1)|phys\rangle = (\bar{L}_0 - 1)|phys\rangle = 0$, where L_0 is a worldsheet Virasoro generator) and be normalisable. From a spacetime point of view these conditions lead to the Breitenlohner-Freedman (BF) bound in AdS spacetimes, a tachyonic mode must have sufficiently negative mass squared. In AdS_3 the BF bound requires that $m^2 \leq -\frac{1}{4}$.

2.1.3 Superstring theory

Until this point we have been considering bosonic string theory. Whilst being an interesting theory in its own right it is unphysical; it does not allow for fermions in the theory and there are tachyons in the bulk spectrum. We must therefore turn our attention to superstring theory [4]. Superstring theory introduces fermions and imposes supersymmetry. As we will see it also leads to the Gliozzi-Scherk-Olive (GSO) projection which will remove tachyons from the bulk. We will concentrate on type II superstring theory.

Once fermions, $\psi^\mu(z)$, are introduced into the action for a closed string, certain restrictions must be placed on them. These are required so that Lorentz invariance is respected and so that the closed string boundary conditions make sense, the action must be invariant under $z \rightarrow z + 2\pi$. There are two ways to satisfy these conditions, either the Ramond (R) or Neveu-Schwarz (NS) boundary conditions,

$$R : \psi^\mu(z) = \psi^\mu(z + 2\pi), \tag{2.18}$$

$$NS : \psi^\mu(z) = -\psi^\mu(z + 2\pi). \tag{2.19}$$

These conditions can be applied independently to left ($\psi(z)$) and right ($\tilde{\psi}(\bar{z})$) movers. In this setting the mass shell condition becomes $\frac{1}{4}m^2 = N - v = \tilde{N} - \tilde{v}$. Where $v, \tilde{v} = 0$ for the R sector and $\frac{1}{2}$ for the NS sector.

When constructing type II closed superstring theories the tachyon with $m^2 = -2$ is only found in the (NS -, NS -) sector. (The minus sign here denotes $e^{i\pi F} = -1$, where F is the fermion number. A factor of -1 is contributed from the ghost ground state).

The GSO projection is applied to enforce consistency in the spectrum. For example it means that all pairs of vertex operators are mutually local and that OPEs close. It works by picking out which sectors can be combined together to form a consistent theory. It turns out that there are two ways of doing this, the resultant theories are called type IIA and type IIB. Type IIA keeps the sectors which satisfy,

$$exp(\pi i F) = +1, \qquad exp(\pi i \tilde{F}) = (-1)^{\tilde{\alpha}}, \tag{2.20}$$

where $\tilde{\alpha}$ is 1 if the right movers are in the R sector and 0 if the right movers are in the NS sector. Similarly for type IIB,

$$exp(\pi i F) = +1, \qquad exp(\pi i \tilde{F}) = +1. \tag{2.21}$$

Neither of these theories contain the sector (NS -, NS -), therefore neither of them contain a tachyon in the bulk.

2.2 Tachyon condensation

This section largely follows arguments laid out in [6]. The study of tachyon condensation is in effect the study of the string theory configuration space. A tachyon represents an inherent instability of a theory, it appears in a theory as an unstable equilibrium. A simple toy model for tachyon condensation is a charged particle in an electromagnetic potential located at a local maximum (figure 2.1). The potential, around this maximum, looks like an inverted harmonic oscillator. Under a small excitation the particle will move away from the maximum eventually settling at a local minimum. The potential energy which is lost will be converted into electromagnetic radiation. It's often difficult to solve for the problem for the full time evolution, so other techniques are employed.

The easiest way to see what will happen under this 'condensation' process is through inspection. The particle will clearly move from the unstable situation at the local maximum to the local minimum and if enough energy can be radiated away it will form a static solution there. There are also other ways to follow this process, one is to introduce a friction term to the equation of motion, such as;

$$\frac{d^2r}{dt^2} = -V'(r) - k\frac{dr}{dt} \tag{2.22}$$

k is the coefficient of friction and is always positive. With this technique it is possible to follow the condensation process whilst ignoring the details of the type of radiation given out. Note that the long term evolution of this system is independent of k. This means that we can make the change of variable $t = k\tilde{t}$ then take $k \to \infty$. This gives rise to following equation,

$$\frac{dr}{d\tilde{t}} = -V'(r). \tag{2.23}$$

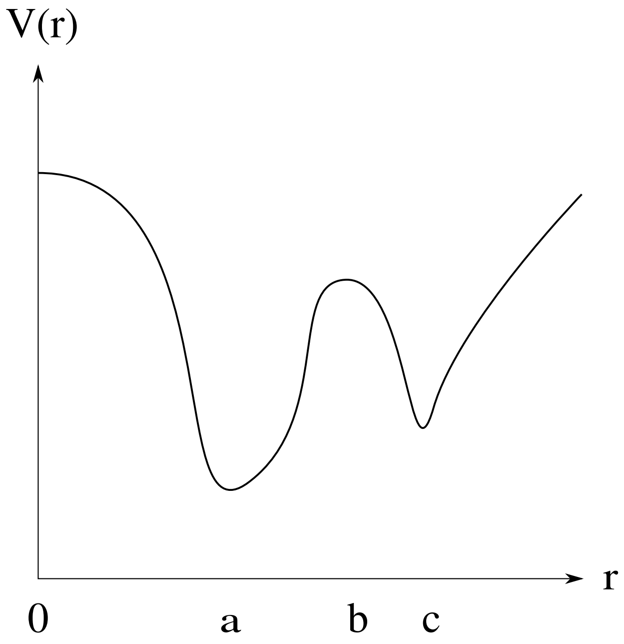

Figure 2.1: A possible potential for a charged particle. The particle stationary at any of 0,a,b,c represent static solutions, however of those solutions 0 and b are tachyonic as they would be unstable.

This process is analogous to the RG flow technique used in string theory to follow the results of tachyon condensation.

2.2.1 Open string tachyon condensation

For open string tachyon condensation Sen conjectured that (i) the endpoint of homogeneous tachyon condensation on the world-volume of the D-brane is the closed string vacuum, and (ii) the condensation of inhomogeneous modes of the tachyon field leads to lower dimensional D-branes. [6]

There are several lines of evidence which support these conjectures. The first mirrors the inspection method used for the toy model in the previous section. To use this technique we need a clear view of the potential energy landscape. This is not forthcoming from world sheet string theory as it deals with off shell quantities, such as the potential only indirectly.

A second technique, which this time can be conducted from the point of view of worldsheet string theory, is to follow the renormalisation group (RG) flow. This is can be seen as analogous to equation 2.23 for our toy model. RG flow is the process of integrating out high momentum modes from a theory, it ‘flows’ from the ultra violet (UV), high energy theory to the infra red (IR), low energy theory. This allows the same theory to be seen from different perspectives, so that factors that previously would only show up in ‘loop’ calculations now show up at tree level.

It is also possible to exactly solve the equations of classical time evolution for the decay of a D-brane via tachyon condensation. This solution is known as a rolling tachyon or S-brane [34]. Determining the long term behaviour of this system, however, is non trivial. This is because the solution is described in terms of an open string theory, Sen’s conjectures state that it should decay into a closed string vacuum state.

2.2.2 Closed string tachyon condensation

Closed string tachyon condensation is more complex than its open string counterpart. This is mainly because closed strings have the graviton in their spectrum. Rather than D-branes being modified or destroyed, in closed string tachyon condensation it is spacetime itself which is altered. If the asymptotics of spacetime are changed during this process (as has been suggested in certain cases) it is unclear if the energy can be compared between the two states even in principle. This makes some of the techniques used above for open strings inappropriate or very difficult to use.

As was discussed in section 2.1.3, bosonic string theory has a generic tachyon. This was removed by the GSO projection when considering type II superstring theory in flat space. The study of closed string tachyon condensation must therefore look into different situations where not all of the tachyons are removed by the GSO projection.

In general, research into closed string tachyon condensation has concentrated on situations where the tachyon has been localised. This has two advantages; firstly any effect which the process has on spacetime will happen locally first, making it easier to study. Secondly it allows any closed string radiation to dissipate into the bulk. If we considered non-local tachyons there would be nowhere for this radiation to go and it would not be clear that the system would end up in the ground state.

2.2.3 The $\mathbb{C}/\mathbb{Z}_n$ orbifold

This section offers a brief discussion about localised tachyons on a orbifold in superstring theory, based on the work in [5]. One of the most studied cases for closed string tachyon condensation is the $\mathbb{C}/\mathbb{Z}_n$ orbifold in a ten dimensional superstring theory. The $\mathbb{C}/\mathbb{Z}_n$ orbifold is a cone created by taking flat space and identifying the 8-9 plane under a rotation $2\pi/n$. The tip of the cone is a singular seven dimensional submanifold. There are two possible actions on the spinors,

$$R = exp(2\pi i J_{89}/n) \quad or \quad exp((n+1)2\pi i J_{89}/n), \tag{2.24}$$

where J_{89} is the rotation generator. For either choice R^n acts trivially on spacetime, which implies that it either equals 1 or $exp(2\pi i J_{89}) = (-1)^F$. If $R^n = (-1)^F$, then the orbifold group projects out fermions and introduces tachyons in the bulk. We wish to study an orbifold with localised tachyons and so choose $R^n = 1$. This means that only the second choice of R is valid and then only when n is odd.

In the sector twisted by R^k $(1 \le k \le n-1)$ the standard calculation for zero-point energy leads to,

$$\frac{\alpha'}{4}m^2 = -k/2n, \qquad \text{for k even} \tag{2.25}$$

$$= (k-n)/2n, \qquad \text{for k odd} \tag{2.26}$$

The lowest state is therefore tachyonic in every twisted sector. If n is odd spacetime supersymmetry is broken in such a way that tachyons in the twisted sector, winding around the cone, survive the GSO projection but tachyons in the bulk do not. These twisted sector tachyons cannot move away from the tip of the cone without stretching, which requires energy. They are therefore localised about the point.

Several suggestions were made for the fate of the $\mathbb{C}/\mathbb{Z}_n$ orbifold after tachyon condensation [5]. Firstly a hole may appear at the tip of the cone which then expands to engulf the whole of spacetime. Secondly the tip of the cone may expand to become an infinite throat. Thirdly the tip of the cone may smooth over, either to the string scale or continuing forever. It is the last of these suggestions which appears to be the most likely, based on current evidence. It has been suggested that, at least at late times, this smoothing out process travels outwards at the speed of light as a dilaton pulse. The region inside the pulse is flat. Evidence for this was obtained in [7] using RG flow on a gauged linear fixed sigma model.

Chapter 3

BTZ black holes

This chapter focuses on the geometry of AdS_3 and its various orbifolds, these include the BTZ black hole and the self-dual orbifold. The variety of coordinate systems introduced here will be used in the subsequent chapters to consider string theory and the AdS/CFT duality on these backgrounds. It is shown how AdS_2 arises as in near horizon limits of extremal black holes, including in the self-dual orbifold. The chapter concludes with the calculation of the bosonic string spectrum, including finding the winding tachyon, on a massive BTZ black hole.

3.1 AdS_3

AdS_3 is the unique local solution to Einstein's equations in three dimensions with a negative cosmological constant. It is a maximally symmetric space of constant negative curvature, with six Killing vectors. It is defined as a hyperboloid [35] [36],

$$-l^2 = -U^2 - V^2 + X^2 + Y^2, \tag{3.1.1}$$

embedded in a space $\mathbb{R}^{2,2}$ with a metric,

$$ds^2 = -dU^2 - dV^2 + dX^2 + dY^2. \tag{3.1.2}$$

The embedding space has six boost and rotation Killing vectors associated with the $SO(2,2)$ isometry group as well as four translation Killing vectors. The translation Killing vectors are not confined to the hyperboloid, but all of the boost and rotation Killing vectors are. AdS_3 therefore has the isometry group SO(2,2) with the associated Killing vectors,

$$J_{ab} = \eta_{ac}x^c\partial_b - \eta_{bc}x^c\partial_a, \tag{3.1.3}$$

with $x^a \equiv (U, V, X, Y)$ and η_{ab} is a diagonal matrix with entries (-1,-1,1,1).

The SO(2,2) isometry group can be decomposed into $SL(2,\mathbb{R})_L \otimes SL(2,\mathbb{R})_R$. The $SL(2,\mathbb{R})$ group element takes the form,

$$g = \begin{pmatrix} U+Y & V+X \\ X-V & U-Y \end{pmatrix}, \tag{3.1.4}$$

Taking $l = 1$. The associated Killing vectors are,

$$\zeta_1^{\pm} = \frac{1}{2}(J_{01} \pm J_{23}), \qquad \zeta_2^{\pm} = \frac{1}{2}(J_{02} \pm J_{13}), \qquad \zeta_3^{\pm} = \frac{1}{2}(J_{03} \mp J_{12}). \tag{3.1.5}$$

As with any manifold AdS_3 can be described in a variety of different coordinate systems. Two of the most important, global and Poincaré coordinates, are described below. Both of these systems will be important for work on both the BTZ black holes and the self-dual orbifold.

3.1.1 Global coordinates

Global coordinates for AdS_3 can be defined in terms of the embedding coordinates,

$$U = l\cosh\rho\sin\tau, \qquad V = l\cosh\rho\cos\tau, \tag{3.1.6}$$

$$X = l\sinh\rho\cos\theta, \qquad Y = l\sinh\rho\sin\theta. \tag{3.1.7}$$

It can clearly be seen that any point (ρ, τ, θ) will be on the hyperboloid defined by (3.1.1). The induced metric on AdS_3 in global coordinates is then,

$$ds^2 = l^2[-\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho d\theta^2]. \tag{3.1.8}$$

Here the coordinates have the ranges, $0 < \rho < \infty$, $0 < \theta < 2\pi$ and $0 < \tau < 2\pi$. This manifold as it stands is of limited interest as it contains closed timelike curves.

τ is written here as an angle so the curve $\rho = \rho_0$, $\theta = \theta_0$ is a closed timelike loop. This can be rectified by extending the manifold to its universal cover, $CAdS_3$, by unwrapping τ so that its range becomes $-\infty < \tau < \infty$. When discussing AdS_3 it will be the universal cover which we are interested in unless otherwise stated.

3.1.2 Poincaré coordinates

Poincaré coordinates can also be defined in terms of embedding coordinates,

$$U = \frac{1}{2r}(l^2 + r^2 + x^2 - t^2), \qquad V = l\frac{t}{r}, \tag{3.1.9}$$

$$X = \frac{-1}{2r}(-l^2 + r^2 + x^2 - t^2), \qquad Y = l\frac{x}{r}. \tag{3.1.10}$$

The metric in these coordinates is given by,

$$ds^2 = \frac{l^2}{r^2}(-dt^2 + dx^2 + dr^2). \tag{3.1.11}$$

The range of these coordinates are, $-\infty < x < \infty$, $-\infty < t < \infty$ and $0 < r < \infty$. This coordinate patch does not cover the whole of AdS_3, it only covers half of it, the other half can be covered using a similar coordinate patch as above, but allowing $-\infty < r < 0$. To cover the whole of $CAdS_3$ an infinite tower of Poincaré patches is required.

3.2 BTZ black holes

BTZ black holes first emerged in [18] and [19] and during this section we will follow the analysis in these two papers. Given the simplicity of gravity in (2+1) dimensions the variety of the BTZ black hole is remarkable. The geometries described in this section will allow the study of quantum mechanics and string theory on a relatively simple background, which still has enough variety to be of interest.

3.2.1 BTZ coordinates

As the name suggests BTZ coordinates will be useful in describing the BTZ black hole. Like the Poincaré coordinates it takes multiple coordinate patches to cover the whole of of AdS_3, in this case 12 patches will be required. Firstly the hyperboloid has to be split up into 3 different regions, with regard to the embedding coordinates,

$$Region1: \qquad -U^2 + X^2 \leq 0, \qquad -V^2 + Y^2 \geq 0, \tag{3.2.12}$$

$$Region2: \qquad -U^2 + X^2 \leq 0, \qquad -V^2 + Y^2 \leq 0, \tag{3.2.13}$$

$$Region3: \qquad -U^2 + X^2 \geq 0, \qquad -V^2 + Y^2 \leq 0. \tag{3.2.14}$$

Note that no part of the hyperboloid intersects the region $-U^2 + X^2 \geq 0$ and $-V^2 + Y^2 \geq 0$.

Each region can then be covered by a set of four coordinate patches, Region 1

$$U = \pm\hat{r}\cosh\hat{\phi}, \qquad V = \sqrt{\hat{r}^2 - l^2}\sinh\hat{t}, \tag{3.2.15}$$

$$X = \hat{r}\sinh\hat{\phi}, \qquad Y = \pm\sqrt{\hat{r}^2 - l^2}\cosh\hat{t}, \tag{3.2.16}$$

$$ds^2 = -(\hat{r}^2 - l^2)d\hat{t}^2 + l^2(\hat{r}^2 - l^2)^{-1}d\hat{r}^2 + \hat{r}^2 d\hat{\phi}^2,$$

$$l < \hat{r} < \infty, -\infty < \hat{t}, \hat{\phi} < \infty.$$

Region 2

$$U = \pm\hat{r}\cosh\hat{\phi}, \qquad V = \pm\sqrt{l^2 - \hat{r}^2}\cosh\hat{t}, \tag{3.2.17}$$

$$X = \hat{r}\sinh\hat{\phi}, \qquad Y = \sqrt{l^2 - \hat{r}^2}\sinh\hat{t}, \tag{3.2.18}$$

$$ds^2 = -(\hat{r}^2 - l^2)d\hat{t}^2 + l^2(\hat{r}^2 - l^2)^{-1}d\hat{r}^2 + \hat{r}^2 d\hat{\phi}^2,$$

$$0 < \hat{r} < l, -\infty < \hat{t}, \hat{\phi} < \infty.$$

Region 3

$$U = \hat{r}\sinh\hat{\phi}, \qquad V = \pm\sqrt{l^2 + \hat{r}^2}\cosh\hat{t}, \tag{3.2.19}$$

$$X = \pm\hat{r}\cosh\hat{\phi}, \qquad Y = \sqrt{l^2 + \hat{r}^2}\sinh\hat{t}, \tag{3.2.20}$$

$$ds^2 = (\hat{r}^2 + l^2)d\hat{t}^2 + l^2(\hat{r}^2 + l^2)^{-1}d\hat{r}^2 - \hat{r}^2 d\hat{\phi}^2,$$

$$0 < \hat{r} < \infty, -\infty < \hat{t}, \hat{\phi} < \infty.$$

For simplicity in the following sections we shall consider l set to equal 1.

3.2.2 Massive BTZ black holes

BTZ black holes are orbifolds of AdS_3 space, creating one involves making a periodic identification. In general, orbifolds can be created by mapping a point $P \rightarrow e^{2n\pi\zeta}P$, where ζ is a Killing vector and $n \in \mathbb{Z}$. Since the transformations are isometries, the quotient space inherits a well defined metric and so is a solution to Einstein's equations. All that remains to worry about is the causal structure of the space. These quotient spaces are black hole solutions if the Killing vector is of the form $\zeta = r_+ J_{12} - r_- J_{03} - J_{13} + J_{23} = \partial_{\hat{\phi}}$, where J_{ab} is as described in equation (3.1.3) and r_+, r_- are the positions of the outer and inner horizons of the BTZ black holes respectively. Out of convention we choose $r_- \leq r_+$. In cases where $r_+ \neq r_-$ this may be simplified by an SO(2,2) transformation to $\zeta' = r_+ J_{12} - r_- J_{13}$.

BTZ Black holes are characterised by three constants, their mass M as well as their angular momentum J and charge Q. This thesis will concentrate on uncharged black holes. The mass and angular momentum can be characterised by r_+ and r_- from the above transformations,

$$M = r_+^2 + r_-^2, \qquad |J| = 2r_+ r_-. \tag{3.2.21}$$

The other condition for these orbifolds to be valid spacetimes, besides being solutions to Einstein's equations, is that they must have admissible causal structure. They must not contain closed timelike loops. The quotient transformations mean that any curve joining two points on the same orbit will form a closed loop. A necessary (and sufficient for the type of Killing vectors used to make BTZ black holes) condition to avoid closed time like loops is that the Killing vector is spacelike, $\zeta.\zeta > 0$.

For the identifications being considered the Killing vectors are not spacelike everywhere. This is remedied by removing these areas with timelike Killing vectors from the spacetime. Once the quotient is made the surface with null Killing vectors $\zeta.\zeta = 0$ will be the singularity in the spacetime. This makes sense as to continue beyond the singularity would introduce closed timelike curves. Now the only incomplete geodesics on the quotient space are those which end on the singularity. The orbifold can be split up into three distinct regions; region I with $\zeta.\zeta > r_+$, region II with $r_- < \zeta.\zeta < r_+$ and region III with $\zeta.\zeta < r-$. The boundaries between these regions are lightlike and form the horizons of the black hole. The structure of the Penrose diagram is the same as for Kerr black hole (Figure 3.1).

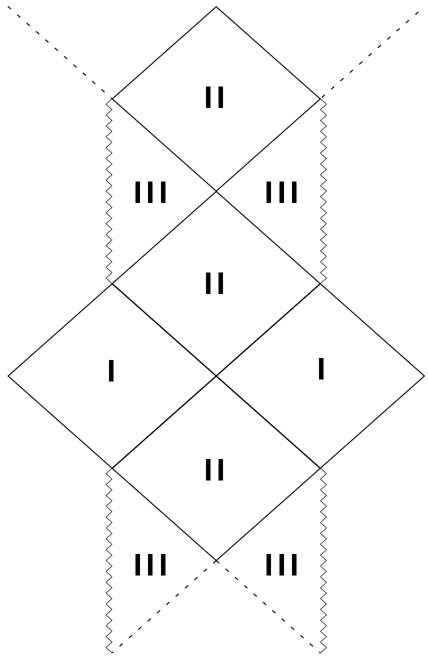

Figure 3.1: The Penrose diagram for a BTZ black hole $r_+ \neq r_- \neq 0$. The zigzagged line is the singularity at $r = 0$. The inner horizon is located between regions II and III at $r = r_-$. The outer horizon is located between regions I and II at $r = r_+$.

Solutions without naked singularities are limited to $-M \leq J \leq M$ with $M \geq 0$. This can be seen by solving for r_+, r_- to find the horizons. The exception to this rule is the case $M = -1$, $J = 0$ which although it has no horizon it also has no singularity, it is simply AdS_3 [19], see figure 3.2. To see this clearly it is best to use a coordinate transformation. Starting with the metric given for region 1 above let $\tilde{r} = r_+\hat{r}$, $\tilde{t} = t/r_+$ and $\tilde{\phi} = \phi/r_+$. Note that $\tilde{\phi}$ has the standard periodicity $\tilde{\phi} \to \tilde{\phi} + 2\pi$. The metric now becomes,

$$ds^2 = -(\tilde{r}^2 - M)d\tilde{t}^2 + (\tilde{r}^2 - M)^{-1}d\tilde{r}^2 + \tilde{r}^2 d\tilde{\phi}^2 \tag{3.2.22}$$

This clearly has a similar form to the metric of the Schwarzschild black hole. In the case $M = -1$ this metric can be seen to be identical to the metric for AdS_3 in global coordinates (3.1.8) with the coordinate transformation $\tilde{r} = \sinh \mu$.

In the case of a massive non-rotating black hole the quotient transformation is equivalent to the identification $\hat{\phi} \to \hat{\phi} + 2\pi r_+$ in the BTZ coordinate system. By

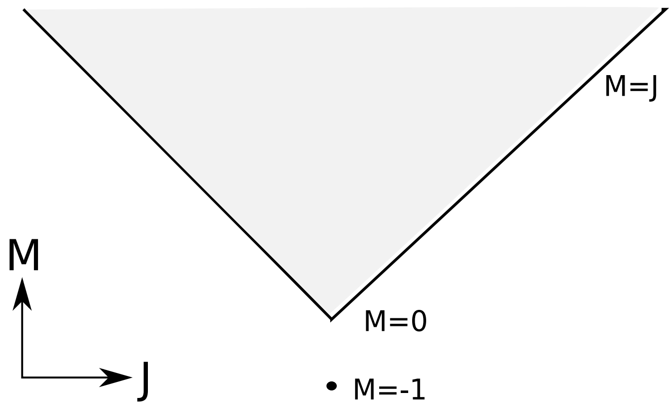

Figure 3.2: BTZ black holes in terms of mass and angular momentum. Pure AdS_3 is M=-1, J=0.

equation (3.2.21) $r_+^2 = M_{BTZ}$ and $J = r_- = 0$. It is interesting to compare the coordinate patch regions 1, 2 and 3 (3.2.15, 3.2.17 and 3.2.19) to the regions as mapped out in the Penrose diagram I, II and III, see figure 3.1. In the non rotating case however there is no region III, as $r_- = 0$. The Penrose diagram will instead look more like that of a Schwarzschild black hole. In the coordinate patch 'region 3' the Killing vector is timelike, these are therefore removed from the spacetime. In 'region 2', $0 \leq \zeta.\zeta \leq r_+^2$, this therefore maps to region II inside the black hole. In 'region 1', $r_+^2 \leq \zeta.\zeta$, so it maps to region I outside the event horizon. In general when thinking of the massive BTZ black hole we consider the 'region 1' coordinate patch and its metric, as this is the area between the event horizon and spatial infinity. [18].

There is a simple coordinate transformation to get from the non-rotating $(\hat{r}, \hat{t}, \hat{\phi})$ to rotating BTZ black hole (r, t, ϕ). Allow M,J to be as defined in equation (3.2.21), $0 < r_- < r_+$ and use the region 1 metric (3.2.15).

$$\hat{r}^2 = \frac{r_+^2(r^2 - r_-^2)}{r_+^2 - r_-^2}, \qquad \begin{pmatrix} \hat{t} \\ \hat{\phi} \end{pmatrix} = \begin{pmatrix} r_+ & -r_- \\ -r_- & r_+ \end{pmatrix} \begin{pmatrix} t \\ \phi \end{pmatrix} \tag{3.2.23}$$

The metric under this coordinate transformation becomes,

$$ds^2 = -(r^2 - M)dt^2 + \left(r^2 - M + \frac{J^2}{2r^2}\right)^{-1} dr^2 - Jdtd\phi + r^2 d\phi^2 \tag{3.2.24}$$

The identification to create the rotating BTZ black hole is $\phi = \phi + 2\pi$. Note that this is a different identification to that of the non rotating black hole as ϕ is not the same as $\hat{\phi}$. [25]

3.3 The self-dual orbifold

This section will focus on the geometry of the self-dual orbifold. The self-dual orbifold is another example of an orbifold of AdS_3, it will be studied in depth with regard to the AdS/CFT correspondence in chapter 6. The self-dual orbifold arises in a number of situations, directly as an orbifold of AdS_3, as a near horizon limit of an extremal BTZ black hole and as a near horizon, near extremal limit of a BTZ black hole. In this section we will discuss appropriate coordinate systems for each of these situations.

The self-dual orbifold spacetime was introduced in [21] as a quotient of AdS_3 and its interpretation in the AdS/CFT correspondence was discussed by [35]. The spacetime is a quotient along the Killing vector $\xi = U\partial_X + X\partial_U + V\partial_Y + Y\partial_V$, where U, V, X, Y are the embedding coordinates (3.1.1). This Killing vector has a unit norm $||\xi||^2 = 1$, so the quotient has no fixed points in the bulk. The quotient preserves an $SL(2,\mathbb{R}) \times U(1)$ subgroup of the $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ symmetry of AdS_3, where the $U(1)$ factor is generated by ξ. The global self-dual orbifold coordinate system (t, ϕ, z) which covers the whole spacetime, relates to the embedding coordinates by

$$U + X = \frac{1}{\sqrt{2}} e^{\phi}(e^{z} \cos t - e^{-z} \sin t), \tag{3.3.25}$$

$$U - X = \frac{1}{\sqrt{2}} e^{-\phi}(e^{-z} \cos t - e^{z} \sin t), \tag{3.3.26}$$

$$V + Y = \frac{1}{\sqrt{2}} e^{\phi}(e^{-z} \cos t + e^{z} \sin t), \tag{3.3.27}$$

$$V - Y = \frac{1}{\sqrt{2}} e^{-\phi}(e^{z} \cos t + e^{-z} \sin t). \tag{3.3.28}$$

These coordinates are related to the usual global AdS_3 coordinates (ρ, τ, θ) (3.1.1) by

$$\cosh^2\rho = \cosh^2 z\cosh^2\phi + \sinh^2 z\sinh^2\phi, \tag{3.3.29}$$
$$\tan(\tau+\theta) = -\frac{\tanh 2z}{\sinh 2\phi},$$
$$\tan(\tau-\theta) = \frac{\tanh 2\phi\cos 2t + \sinh 2z\sin 2t}{-\tanh 2\phi\sin 2t + \sinh 2z\cos 2t}.$$

in terms of which the metric of AdS_3 is

$$ds^2 = -dt^2 + d\phi^2 + 2\sinh 2z dt d\phi + dz^2 = -\cosh^2 2z dt^2 + (d\phi + \sinh 2z dt)^2 + dz^2\,. \tag{3.3.30}$$

The self-dual orbifold is obtained by taking the quotient $\phi \sim \phi + 2\pi r_+$ for some r_+. The spacetime can be seen to be a $U(1)$ fibration over AdS_2 (this will be discussed further in section 3.4) and the Killing symmetries are

$$\xi = \frac{1}{2}\partial_\phi, \tag{3.3.31}$$
$$\chi_1 = \frac{1}{2}\partial_t, \tag{3.3.32}$$
$$\chi_2 = \frac{1}{2}\tanh 2z\cos 2t\partial_t + \frac{\cos 2t}{2\cosh 2z}\partial_\phi + \frac{1}{2}\sin 2t\partial_z, \tag{3.3.33}$$
$$\chi_3 = -\frac{1}{2}\tanh 2z\sin 2t\partial_t - \frac{\sin 2t}{2\cosh 2z}\partial_\phi + \frac{1}{2}\cos 2t\partial_z. \tag{3.3.34}$$

The factors of 2 in the χ_i are required to make them a representation of $SL(2,\mathbb{R})$; the one in ξ is simply conventional.

The spacetime has two boundaries, at $z \to \pm\infty$. Taking (3.3.30) as a coordinate system on all of AdS_3 (without a quotient) we would also have reached the boundary when $\phi \to \pm\infty$, which are fixed points of the quotient $\phi \sim \phi + 2\pi r_+$. From (3.3.29), when $\phi \to \pm\infty$,

$$\tan(\tau+\theta) = 0, \tag{3.3.35}$$

so this corresponds to $\tau + \theta = 0$ or π. That is, the quotient has fixed points on the null lines $\tau+\theta = 0, \pi$ in the conformal boundary. These lines separate the conformal boundary into two strips. These two regions are the two boundaries of the self-dual orbifold, at $z \to \pm\infty$. When $z \to \pm\infty$, (3.3.29) simplifes to

$$\tan(\tau+\theta) = \mp\frac{1}{\sinh 2\phi}, \quad \tan(\tau-\theta) = \tan 2t. \tag{3.3.36}$$

So on the boundary t is mapped to the null coordinate running up along the strips at $z \to \pm\infty$, while ϕ is the null coordinate running across the strips.

Consider a surface of constant t, say $t = 0$. In the strips at $z = \pm\infty$, this will map to $\tau - \theta = 0, \pi$. Let's choose $\tau - \theta = 0$ at $z = \infty$. At $t = 0$,

$$\tan\theta = -\frac{\tanh\phi - \tanh z}{\tanh\phi + \tanh z}. \tag{3.3.37}$$

At $z = \infty$, as ϕ ranges from $-\infty$ to ∞, θ ranges over $(\pi/2, 0)$. At $\phi = \infty$, as z ranges from ∞ to $-\infty$, θ ranges over $(0, -\pi/2)$. At $z = -\infty$, as ϕ ranges from ∞ to $-\infty$, θ ranges over $(-\pi/2, \pi)$. Finally, at $\phi = -\infty$, as z ranges from $-\infty$ to ∞, θ ranges from $(\pi, \pi/2)$. As a result, the surface $t = 0$ maps to a sawtoothed curve made from null segments:

$$t = 0 \leftrightarrow \begin{cases} \tau - \theta = 0, & \theta \in (\pi/2, 0), \\ \tau + \theta = 0, & \theta \in (0, -\pi/2), \\ \tau - \theta = \pi, & \theta \in (-\pi/2, -\pi), \\ \tau + \theta = \pi, & \theta \in (\pi, \pi/2). \end{cases} \tag{3.3.38}$$

This is shown in figure 3.3.

In addition to arising as a quotient of AdS_3, the self-dual orbifold can be obtained as a near-horizon limit of the extremal BTZ black hole, a point of view which was stressed in [37]. If we start with the BTZ black hole in a stationary coordinate system, we obtain the self-dual orbifold in a coordinate system which only covers a portion of the geometry. A convenient coordinate system is the near horizon limit (u, v, r) coordinates introduced in [37], which are related to the embedding coordinates by

$$U + X = e^{r+u}, \tag{3.3.39}$$

$$U - X = \frac{1}{2}(e^{-r-u} - 2ve^{r-u}), \tag{3.3.40}$$

$$V + Y = \frac{1}{2}(e^{-r+u} + 2ve^{r+u}), \tag{3.3.41}$$

$$V - Y = e^{r-u}. \tag{3.3.42}$$

The transformation between the global self-dual orbifold coordinates (t, ϕ, z) and

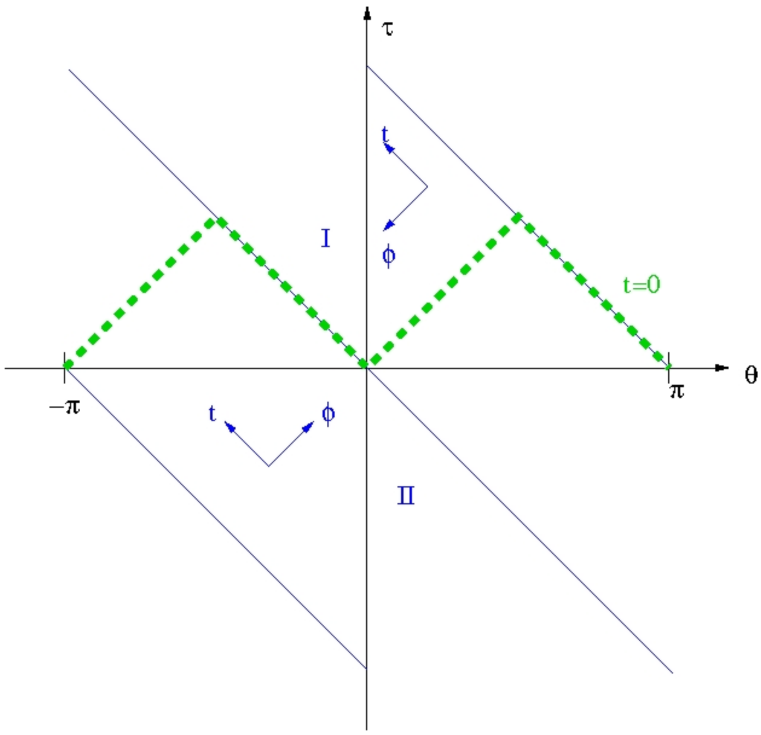

Figure 3.3: The relation between the global AdS_3 coordinates τ, θ and the global self-dual orbifold coordinates t, ϕ on the boundary. (The lines $\theta = \pm\pi$ are identified.) Region I is $z = \infty$, region II is $z = -\infty$. The direction of increasing t, ϕ in each region is indicated. The heavy dashed line is the surface $t = 0$ given in (3.3.39).

near horizon limit (u, v, r) coordinates is then

$$2e^{2r} = \cosh 2z \cos 2t + \sinh 2z, \tag{3.3.43}$$

$$v = \frac{\cosh 2z \sin 2t}{\cosh 2z \cos 2t + \sinh 2z}, \tag{3.3.44}$$

$$e^{2u} = e^{2\phi} \frac{(e^z \cos t - e^{-z} \sin t)}{(e^z \cos t + e^{-z} \sin t)}. \tag{3.3.45}$$

The near horizon limit coordinates are also simply related to the Poincaré coordinates (3.1.2) ($x^+ = \frac{(t+x)}{\sqrt{2}}, x^- = \frac{(x-t)}{\sqrt{2}}, Z = r$) on AdS_3,

$$Z = e^{-r+u}, \quad x^+ = 2e^{2u}, \quad x^- = v - \frac{1}{2}e^{-2r}. \tag{3.3.46}$$

The metric of AdS_3 in this coordinate system is

$$ds^2 = du^2 + 2e^{2r} du dv + dr^2 = -e^{4r} dv^2 + dr^2 + (du + e^{2r} dv)^2, \tag{3.3.47}$$

and the Killing vectors are

$$\xi = \frac{1}{2}\partial_u, \tag{3.3.48}$$

$$\chi_1 = -\left[1 + \frac{1}{4}(v^2 + e^{-4r})\right] \partial_v - \frac{1}{4}e^{-2r}\partial_u + \frac{1}{4}v\partial_r, \tag{3.3.49}$$

$$\chi_2 = -\left[1 - \frac{1}{4}(v^2 + e^{-4r})\right] \partial_v + \frac{1}{4}e^{-2r}\partial_u - \frac{1}{4}v\partial_r, \tag{3.3.50}$$

$$\chi_3 = -v\partial_v + \frac{1}{2}\partial_r. \tag{3.3.51}$$

Both the self-dual orbifold (t, ϕ, z) and AdS_3 (τ, θ, ρ) global coordinate system covers the whole of AdS_3, but from (3.3.46), we can see that near horizon limit coordinates (u, v, r) cover half a Poincaré patch of AdS_3, as they cover the region $x^+ > 0$.

From the AdS_2 point of view, the transformation (3.3.43, 3.3.44) is precisely the transformation from global coordinates to Poincaré coordinates on the AdS_2 factor, so these coordinates will cover the region of the self-dual orbifold corresponding to the Poincaré patch in AdS_2. It is interesting that half of the Poincaré patch in AdS_3 corresponds to the Poincaré patch in AdS_2. This patch includes a portion of the boundary at $z = \infty$ in the self-dual orbifold global coordinates (t, ϕ, z). The other half of the AdS_3 Poincaré patch corresponds to an AdS_2 Poincaré patch covering a portion of the other boundary at $z = -\infty$.

The coordinate transformation simplifies further on the boundary. At $z = \infty$,

$$u = \phi, \quad v = \tan t. \tag{3.3.52}$$

so the (u, v) coordinates cover the portion of the $z = \infty$ strip with $t \in (-\pi/2, \pi/2)$. The relation between Poincaré coordinates x^+, x^- and the u, v near horizon limit coordinates on the boundary is

$$x^+ = 2e^{2u}, \quad x^- = v. \tag{3.3.53}$$

The self-dual orbifold spacetime can be obtained as a near-horizon limit of an extreme BTZ black hole. In [37], it was observed that this near-horizon limit is very easy to describe in the (u, v, r) coordinates (3.3.47). The extreme BTZ black hole is given by this metric with the identifications

$$(\tilde{u}, \tilde{v}) \sim (\tilde{u} + 2\pi r_+, \tilde{v} + 2\pi r_+). \tag{3.3.54}$$

We call the extreme BTZ coordinates $(\tilde{u}, \tilde{v}, \tilde{r})$ to distinguish them from the near horizon limit coordinates, describing the self-dual orbifold, which we will shortly recover. To take the near-horizon limit, we want to write $\tilde{r} = r_0 + r$, and take $r_0 \to -\infty$. If we also write $\tilde{u} = u$, $\tilde{v} = e^{-2r_0} v$, then the metric in terms of (u, v, r) takes the same form (3.3.47) at finite r_0, but now with the identifications

$$(u, v) \sim (u + 2\pi r_+, v + 2\pi r_+ e^{2r_0}). \tag{3.3.55}$$

As we take the near-horizon limit $r_0 \to -\infty$ for fixed (u, v, r), this reduces to $u \sim u + 2\pi r_+$, giving us the self-dual orbifold.

3.3.1 Black hole coordinates

The self-dual orbifold spacetime has two boundaries, but when we obtain it as a near-horizon limit of the extreme BTZ black hole, we obtain it in a coordinate system which only covers one of the boundaries. To see that both boundaries play a role, it is useful to consider a different kind of near-horizon limit. We therefore consider the near-horizon, near-extremal limit of the non-extremal BTZ black hole.

We start with the BTZ black hole,

$$ds^2 = -\frac{(r^2-r_+^2)(r^2-r_-^2)}{r^2}dt^2 + \frac{r^2 dr^2}{(r^2-r_+^2)(r^2-r_-^2)} + r^2\left(d\phi - \frac{r_- r_+}{r^2}dt\right)^2, \quad (3.3.56)$$

which has two asymptotic regions in the full eternal black hole spacetime. The coordinate range $r \geq 0$ only covers one asymptotic boundary, but the metric can be extended in patches to cover both boundaries.

We first define a comoving coordinate system at the event horizon $r = r_+$ by setting $\phi' = \phi - \frac{r_-}{r_+}t$. Then define new coordinates $(\bar{t}, \bar{\phi}, \bar{r}^2)$ by

$$r^2 = r_+^2(1+\epsilon\bar{r}^2), \quad t = \frac{\bar{t}}{r_+\epsilon}, \quad \phi' = \frac{\bar{\phi}}{r_+}, \quad (3.3.57)$$

where $\epsilon = \frac{r_+^2 - r_-^2}{r_+^2}$. The metric in these coordinates is

$$ds^2 = -\frac{\bar{r}^2(\bar{r}^2+1)}{(1+\epsilon\bar{r}^2)}d\bar{t}^2 + \frac{(1+\epsilon\bar{r}^2)}{(\bar{r}^2+1)}d\bar{r}^2 + (1+\epsilon\bar{r}^2)\left(d\bar{\phi} + \frac{\sqrt{1-\epsilon^2}}{(1+\epsilon\bar{r}^2)}\bar{r}^2 d\bar{t}\right)^2. \quad (3.3.58)$$

We can then take a near-horizon, near-extremal limit by taking $\epsilon \to 0$ for finite values of $(\bar{t}, \bar{\phi}, \bar{r}^2)$. The resulting metric is

$$ds^2 = -\bar{r}^2(\bar{r}^2+1)d\bar{t}^2 + \frac{d\bar{r}^2}{(\bar{r}^2+1)} + \left(d\bar{\phi} + \bar{r}^2 d\bar{t}\right)^2 = -\bar{r}^2 d\bar{t}^2 + d\bar{\phi}^2 + 2\bar{r}^2 d\bar{t}d\bar{\phi} + \frac{d\bar{r}^2}{(\bar{r}^2+1)}. \quad (3.3.59)$$

This is the self-dual orbifold, in a "black-hole like" coordinate system. The AdS_2 part of the metric is written in the black hole coordinates of equation (2.12) of [38]. It is also worth noting that the inner horizon at $\bar{r}^2 = -1$ remains at a finite distance from the outer horizon at $\bar{r}^2 = 0$ as we take this limit.

To relate these coordinates to embedding coordinates, take the embedding of BTZ,

$$U = \sqrt{\frac{r^2-r_+^2}{r_+^2-r_-^2}}\sinh(r_+t - r_-\phi) = \sqrt{\frac{r^2-r_+^2}{r_+^2-r_-^2}}\sinh\left(\frac{r_+^2-r_-^2}{r_+}t - r_-\phi'\right), \quad (3.3.60)$$

$$V = \sqrt{\frac{r^2-r_-^2}{r_+^2-r_-^2}}\cosh(r_+\phi - r_-t) = \sqrt{\frac{r^2-r_+^2}{r_+^2-r_-^2}}\cosh(r_+\phi'),$$

$$X = -\sqrt{\frac{r^2-r_+^2}{r_+^2-r_-^2}}\cosh(r_+t - r_-\phi) = -\sqrt{\frac{r^2-r_+^2}{r_+^2-r_-^2}}\cosh\left(\frac{r_+^2-r_-^2}{r_+}t - r_-\phi'\right),$$

$$Y = \sqrt{\frac{r^2-r_-^2}{r_+^2-r_-^2}}\sinh(r_+\phi - r_-t) = \sqrt{\frac{r^2-r_+^2}{r_+^2-r_-^2}}\sinh(r_+\phi'),$$

and apply the same $\epsilon \to 0$ limit. This gives

$$\begin{aligned} U + X &= -\bar{r}e^{-\bar{t}+\bar{\phi}}, \\ U - X &= \bar{r}e^{\bar{t}-\bar{\phi}}, \\ V + Y &= \sqrt{\bar{r}^2+1}e^{\bar{\phi}}, \\ V - Y &= \sqrt{\bar{r}^2+1}e^{-\bar{\phi}}. \end{aligned} \tag{3.3.61}$$

These coordinates are related to the global self-dual orbifold coordinates (t, ϕ, z) by

$$\begin{aligned} \bar{r}^2 &= \sinh^2 z \cos^2 t - \cosh^2 z \sin^2 t, \\ e^{2\bar{t}} &= \frac{\tanh 2z + \sin 2t}{\tanh 2z - \sin 2t}, \\ e^{2\bar{\phi}} &= e^{2\phi}\frac{(\cosh z \cos t + \sinh z \sin t)}{(\cosh z \cos t - \sinh z \sin t)}. \end{aligned} \tag{3.3.62}$$

Thus, the $(\bar{t}, \bar{\phi}, \bar{r})$ coordinates for $\bar{r}^2 \geq 0$ cover a region $z \geq 0$, $\tan^2 t \leq \tanh^2 z$ in global self-dual orbifold coordinates (t, ϕ, z). Just as the non-extremal BTZ black hole has two asymptotic regions, which can be displayed by taking two patches with metrics (3.3.56), we can think of the black hole coordinates for the self-dual orbifold as covering two regions outside the “horizon” at $\bar{r}^2 = 0$, thus including patches of the two boundaries at $z \to \pm\infty$.

Note that in these black hole coordinates, $\bar{t}, \bar{\phi}$ are not null coordinates on the boundary of the self-dual orbifold. This can be corrected by defining $\tilde{\phi} = \bar{\phi} - \frac{\bar{t}}{2}$; it's also useful to set $\tilde{t} = \frac{\bar{t}}{2}$. Then the metric in these coordinates is

$$ds^2 = -4\bar{r}^2(\bar{r}^2+1)d\tilde{t}^2 + \frac{d\bar{r}^2}{(\bar{r}^2+1)} + \left(d\tilde{\phi} + (2\bar{r}^2+1)d\tilde{t}\right)^2, \tag{3.3.63}$$

and the relation to global self-dual orbifold coordinates (t, ϕ, z) is

$$\begin{aligned} \bar{r}^2 &= \sinh^2 z \cos^2 t - \cosh^2 z \sin^2 t, \\ e^{4\tilde{t}} &= \frac{\tanh 2z + \sin 2t}{\tanh 2z - \sin 2t}, \\ e^{4\tilde{\phi}} &= e^{4\phi}\frac{\sinh 2z - \tan 2t}{\sinh 2z + \tan 2t}. \end{aligned} \tag{3.3.64}$$

The Killing vectors in these coordinates are

$$\xi =\frac{1}{2}\partial_{\tilde{\phi}}, \tag{3.3.65}$$

$$\chi_1 =-\frac{1}{4}\left(\frac{\sqrt{\bar{r}^2+1}}{\bar{r}}+\frac{\bar{r}}{\sqrt{\bar{r}^2+1}}\right)\cosh 2\tilde{t}\partial_{\tilde{t}}+\frac{1}{4}\left(\frac{\sqrt{\bar{r}^2+1}}{\bar{r}}-\frac{\bar{r}}{\sqrt{\bar{r}^2+1}}\right)\cosh 2\tilde{t}\partial_{\tilde{\phi}} + \sqrt{\bar{r}^2+1}\sinh 2\tilde{t}\partial\bar{r}, \tag{3.3.66}$$

$$\chi_2 =\frac{1}{4}\left(\frac{\sqrt{\bar{r}^2+1}}{\bar{r}}+\frac{\bar{r}}{\sqrt{\bar{r}^2+1}}\right)\sinh 2\tilde{t}\partial_{\tilde{t}}-\frac{1}{4}\left(\frac{\sqrt{\bar{r}^2+1}}{\bar{r}}-\frac{\bar{r}}{\sqrt{\bar{r}^2+1}}\right)\sinh 2\tilde{t}\partial_{\tilde{\phi}} - \sqrt{\bar{r}^2+1}\cosh 2\tilde{t}\partial\bar{r}, \tag{3.3.67}$$

$$\chi_3 =-\frac{1}{2}\partial_{\tilde{t}}. \tag{3.3.68}$$

On the boundary at $z=\infty$,

$$e^{2\tilde{\phi}}=e^{2\phi}, \quad e^{2\tilde{t}}=\frac{\cos t+\sin t}{\cos t-\sin t}; \tag{3.3.69}$$

note that the rescaling of $\tilde{t}$ was chosen so that for t near 0, $\tilde{t}\approx t$. This coordinate system covers the portion of the $z=\infty$ strip with $t\in(-\pi/4,\pi/4)$; half as much as the u,v coordinates. In terms of the u,v coordinates,

$$e^{2\tilde{\phi}}=e^{2\phi}, \quad e^{2\tilde{t}}=\frac{1+v}{1-v}, \tag{3.3.70}$$

so it maps to the region $v\in(-1,1)$.

3.4 AdS$_2$

As we saw in the previous section the self-dual orbifold can be described as a U(1) fibration over AdS$_2$. The self-dual orbifold can also be described as a near horizon limit of an extremal BTZ black hole. As we shall see in this section AdS$_2$ also appears in the near horizon limit of other extremal black holes.

AdS$_2$ is the solution to Einstein's equations in (1+1) dimensions with a negative cosmological constant. It can be defined using embedding coordinates, as a hyperboloid on $\mathbb{R}^{1,2}$. The metric and restriction on the embedding space are given as,

$$ds^2=-dU^2-dX^2+dX^2 \qquad -l^2=-U^2-V^2+X^2 \tag{3.4.71}$$

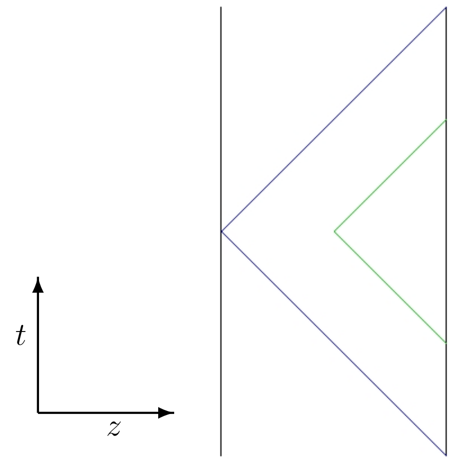

Figure 3.4: The regions in AdS$_2$ covered by the different coordinate systems. The global self-dual orbifold (t, z) coordinates cover the whole of AdS$_2$, the near horizon limit (v, r) coordinates cover the Poincaré wedge, and the $(\bar{t}, \bar{r})$ coordinates cover the smaller wedge.

In Poincaré coordinates this becomes,

$$U = \frac{1}{2y}(l^2 + y^2 - t^2), \quad V = \frac{lt}{y}, \quad X = \frac{-1}{2y}(-l^2 + y^2 - t^2) \tag{3.4.72}$$

$$ds^2 = \frac{-l^2(dt^2 + dy^2)}{y^2} \tag{3.4.73}$$

It can also be expressed in black hole like coordinates,

$$U = \sqrt{r^2 - l^2}\sinh(t), \quad V = r, \quad X = \sqrt{r^2 - l^2}\cosh(t) \tag{3.4.74}$$

$$ds^2 = -(r^2 - l^2)dt^2 + l^2(r^2 - l^2)^{-1}dr^2 \tag{3.4.75}$$

AdS$_2$ appears as a factor in near horizon extremal Kerr (NHEK) geometry. We will show this by following the calculation in [22]. As the name suggests this geometry is obtained by taking a near horizon limit on an extremal Kerr black hole in four dimensions. In Boyer-Lindquist coordinates the metric for a Kerr black hole is given by,

$$ds^2 = \frac{\Delta}{\rho^2}\left(d\hat{t} - a\sin^2(\theta)d\hat{\phi}\right)^2 + \frac{\sin^2(\theta)}{\rho^2}\left((\hat{r}^2 + a^2)d\hat{\phi} - ad\hat{t}\right)^2 + \frac{\rho^2}{\Delta}d\hat{r}^2 + \rho^2 d\theta^2, \tag{3.4.76}$$

$$\Delta \equiv \hat{r}^2 - 2Mr + a^2, \quad \rho^2 \equiv \hat{r}^2 + a^2\cos^2(\theta), \quad a \equiv \frac{J}{M}.$$

The angular momentum is restricted to avoid naked singularities, such that $-M^2 \leq J \leq M^2$. As we are interested in the extremal case we take $J = M^2$. NHEK geometry can be recovered using a change of coordinates,

$$t = \frac{\lambda \hat{t}}{2M}, \quad y = \frac{\lambda M}{\hat{r} - M}, \quad \phi = \hat{\phi} - \frac{\hat{t}}{2M}, \tag{3.4.77}$$

before taking the limit $\lambda \to 0$ whilst holding t, y, ϕ and θ constant. This results in the NHEK metric in Poincaré style coordinates,

$$ds^2 = 2M^2 \frac{1+\cos^2(\theta)}{2} \left[\frac{-dt^2 + dy^2}{y^2} + d\theta^2 + \left(\frac{2\sin(\theta)}{1+\cos^2(\theta)} \right)^2 \left(d\phi + \frac{dt}{y} \right)^2 \right], \tag{3.4.78}$$

with $\phi \sim \phi + 2\pi$ and $0 \leq \theta \leq \pi$. Similarly to Poincaré coordinates this patch only covers part of the NHEK space. A set of global coordinates may be obtained by making a further set of transformations,

$$y = \left(\cos(\tau)\sqrt{r^2+1} + r \right)^{-1}, \quad t = y \sin(\tau)\sqrt{r^2+1}, \quad \phi = \varphi + \ln\left(\frac{\cos(\tau) + r\sin(\tau)}{1 + \sin(\tau)\sqrt{r^2+1}} \right). \tag{3.4.79}$$

This results in the metric,

$$ds^2 = 2M^2 \frac{1+\cos^2(\theta)}{2} \left[-(1+r^2)d\tau^2 + \frac{dr^2}{1+r^2} + d\theta^2 + \left(\frac{2\sin(\theta)}{1+\cos^2(\theta)} \right)^2 (d\varphi + r d\tau)^2 \right]. \tag{3.4.80}$$

In this setting the r, τ plane is AdS_2 and the φ circle is a bundle over AdS_2.

A similar result can be obtained for an extremal charged black hole in an AdS background. The near horizon limit for certain extremal charged black holes in an AdS_{d+1} spacetime is $AdS_2 \times \mathbb{R}^{d-1}$ [23].

3.5 Winding tachyons in non-extremal BTZ black holes

In this section the knowledge acquired in this and the previous chapter will be put to use in calculating the string spectrum on a non-rotating BTZ black hole in both the winding and non-winding sectors. Using this spectrum the presence of tachyons can then be determined in the winding sector. This calculation will prove useful before looking at extremal black holes in chapter 5. Many of the results in this section will also be used directly when calculating the spectrum for the extremal rotating BTZ black hole. The spectrum for non-rotating BTZ black holes was studied in [20] and this section will follow their calculations.

For the non-rotating BTZ black hole it is convenient to work in BTZ coordinates (3.2.15), which corresponds to choosing a hyperbolic basis for the current algebra in which J^2 is diagonalised. In this coordinate system the conserved charges (2.12) are;

$$Q_t = J_0^2 - \bar{J}_0^2, \quad Q_\phi = J_0^2 + \bar{J}_0^2. \tag{3.5.81}$$

The periodic identification, $\phi \sim \phi + 2\pi r_+$, used to construct the BTZ black hole restricts the twisted sector states to quantised values of Q_ϕ;

$$r_+(J_0^2 + \bar{J}_0^2) \in \mathbb{Z}. \tag{3.5.82}$$

We will work with the currents J^2, $J^\pm = J^1 \pm J^3$, with the OPEs,

$$J^+(z)J^-(w) \sim \frac{k}{(z-w)^2} + \frac{2iJ^2}{(z-w)}, \tag{3.5.83}$$

$$J^2(z)J^2(w) \sim \frac{k/2}{(z-w)^2}, \tag{3.5.84}$$

$$J^2(z)J^\pm(w) \sim \pm\frac{iJ^\pm}{(z-w)}. \tag{3.5.85}$$

We will construct the untwisted sector states using a parafermionic representation and then impose a twist operator to construct the twisted sector states. Firstly the current J^2 is bosonised with a free field X

$$J^2 = -i\sqrt{\frac{k}{2}}\partial X, \tag{3.5.86}$$

where $X(z)X(w) \sim \ln(z-w)$. Parafermions $\xi^\pm$ are introduced to complete the current algebra,

$$J^\pm = \xi^\pm e^{\pm\sqrt{2/k}X}, \tag{3.5.87}$$

with,

$$\xi^+(z)\xi^-(w) \sim \frac{k}{(z-w)^{2+2/k}}, \quad \xi^\pm(z)\xi^\pm(w) \sim (z-w)^{2/k}. \tag{3.5.88}$$

In this paraferminonic representation the chiral primary operators of the current algebra are given as,

$$\Phi_{j\lambda}(w) = \Psi_{j\lambda}(w)e^{-i\lambda\sqrt{2/k}X}, \qquad (3.5.89)$$

where λ is the J^2 eigenvalue, which determines the spacetime energy. In the hyperbolic basis λ and j are unrelated. The conformal dimension of the primary operators is given by,

$$h(\Phi_{j\lambda}) = -\frac{j(j-1)}{k-2}, \qquad (3.5.90)$$

where $c_2 = -j(j-1)$ is the Casimir of the global SL(2,$\mathbb{R}$) symmetry generated by the zero modes of the currents. Using (2.17), (3.5.89), (3.5.90) it follows that,

$$h(\Psi_{j\lambda}) = -\frac{j(j-1)}{k-2} - \frac{\lambda^2}{k}. \qquad (3.5.91)$$

At this stage we must reintroduce the antiholomorphic sector. In the untwisted sector it was identical to the holomorphic sector. The condition (3.5.82) can be imposed by requiring that the physical states are mutually local with the twist operator t_n.

$$t_n = e^{ir_+\sqrt{k/2}n(X-\bar{X})}. \qquad (3.5.92)$$

It can then be seen that the n^{th} twisted sector is then given by the composite operator of the untwisted chiral primary with the twist operator.

$$\Phi^n_{j\lambda\bar{\lambda}} = \Psi_{j\lambda}\bar{\Psi}_{j\lambda}e^{-i\sqrt{2/k}[(\lambda+\frac{k}{2}nr_+)X+(\bar{\lambda}-\frac{k}{2}nr_+)\bar{X}]}. \qquad (3.5.93)$$

The twisted sector chiral primary operators then have conformal dimensions,

$$h(\Phi^n_{j\lambda\bar{\lambda}}) = -\frac{j(j-1)}{k-2} + \lambda r_+ n + \frac{kn^2r_+^2}{4}, \qquad (3.5.94)$$

$$h(\bar{\Phi}^n_{j\lambda\bar{\lambda}}) = -\frac{j(j-1)}{k-2} - \bar{\lambda} r_+ n + \frac{kn^2r_+^2}{4}. \qquad (3.5.95)$$

We now wish to look for tachyons in this spectrum. By continuity we will argue that if there are zero energy modes which satisfy the physical state condition, $(L_0 - 1)|phys\rangle = (\bar{L}_0 - 1)|phys\rangle = 0$, then there must also be tachyons which are physical in the spectrum. We set the energy and momentum to zero, $\lambda = \bar{\lambda} = 0$. After taking into account internal degrees of freedom, it can be shown that there are tachyons in the twisted sector (for large k) when $\sqrt{k}r_+ < 2$. (For a more detailed discussion refer back to the original paper [20]).

Similar to the $\mathbb{C}/\mathbb{Z}_n$ orbifold in section 2.2.3, tachyons in the ground state of the NS-NS sector can survive the GSO projection if we choose an antiperiodic spin structure for the fermions on spacetime. The tachyons which survive this GSO projection are in the odd twisted sectors.

Chapter 4

Introduction to the AdS/CFT correspondence and Hawking radiation

4.1 Black hole thermodynamics

Black holes offer one of the best opportunities to study quantum gravity, through the field of black hole thermodynamics. Hawking radiation gives a key insight into the connection between quantum mechanics and general relativity. This section will follow the arguments laid out in [15].

Black hole thermodynamics relates the gravitational quantities of surface gravity, κ and surface area, A to the thermodynamic quantities, temperature, T and entropy, S, by the relations,

$$T = \frac{\hbar\kappa}{2\pi}, \qquad S = \frac{A}{4G\hbar} \tag{4.1.1}$$

For stationary black holes, where the event horizon is also a Killing horizon the classical laws of black hole thermodynamics are as follows;

4.1.2 Zeroth Law. *The surface gravity κ is constant over the event horizon of a stationary black hole.*

4.1.3 First Law. *A change in surface area for a black hole is related to a change* *in mass, charge and spin of the black hole in the following manner,*

$$dM = \frac{\kappa}{8\pi} dA + \Omega dJ + \Phi dQ \tag{4.1.3}$$

Ω *is the angular velocity and* Φ *is the electric potential.*

The second law of black hole thermodynamics links the area of a black hole event horizon to the entropy of a black hole. This important relation will lead on to holography in the next section. The second law of thermodynamics states that entropy is non-decreasing, $\frac{dS}{dt} \geq 0$. It was noticed that classically this is a property which is shared with the black hole event horizon. As matter falls into a black hole the horizon will increase in size, but as nothing can escape a black hole the horizon area never decreases.

4.1.4 Second Law. *The surface area of a black hole event horizon is non-decreasing,*

$$\frac{dA}{dt} \geq 0 \tag{4.1.4}$$

This lead to the relation (4.1.1). As we shall see (4.1.4) only holds classically and can be violated by quantum effects. Black hole event horizons can decrease in size through Hawking radiation, as is discussed in section 4.1.2. In this case the Hawking radiation carries away the relevant entropy, so that overall entropy is non-decreasing.

4.1.5 Third Law. *The strong form of this law is that as temperature tends to zero, entropy tends to a universal constant. The weaker form is that it is impossible to create an extremal black hole by incremental changes to a non-extremal black hole.*

In practice there have been problems with an exact formulation of the third law [39].

4.1.1 Unruh radiation

Before looking at Hawking radiation it is useful to consider Unruh radiation. Unruh radiation is associated with an acceleration horizon. As general relativity is an observer dependent theory, any theory of quantum gravity must also be observer dependent. As we will see this means that even the vacuum state of a theory is not uniquely determined.

The Rindler horizon is the simplest example of a bifurcate Killing horizon. It occurs when Minkowski space is viewed from the point of view of an accelerating observer. Start with the usual Cartesian coordinate system (T, X) and apply a boost transformation,

$$X = x \cosh(\kappa t), \qquad T = x \sinh(\kappa t). \tag{4.1.6}$$

The metric in the Rindler coordinates becomes,

$$ds^2 = -\kappa^2 x^2 dt^2 + dx^2 + ds^2_{\mathbb{R}^{(d-2)}}. \tag{4.1.7}$$

The Rindler coordinates only cover part of flat space, called the Rindler wedge $X^2 - T^2 > 0$, $X > 0$ (figure 4.1). The surface located at the point $x = 0$ is a Killing horizon for the Killing vector $\zeta = \partial_t$, this is known as the Rindler horizon. Stationary black holes are typified by this type of bifurcate Killing horizon, so results here will have a useful application in a black hole setting later.

Rindler observers at constant x are accelerating with $a = 1/x$, the closer they are to the horizon the faster they are accelerating. The important thing to note here is that observers at constant x in the right wedge are not in causal contact with the left Rindler wedge and vice versa. They cannot be influenced by any events or fields which occur inside of the other region.

It can be shown using two point functions that any space, where the Euclideanised time is periodic with $\tau \sim \tau + \beta$, is thermal with a temperature $T = 1/\beta$. If we Euclideanise flat space in Rindler coordinates the metric becomes,

$$ds^2 = \kappa^2 x^2 d\tau^2 + dx^2 + ds^2_{\mathcal{R}^{(d-2)}}. \tag{4.1.8}$$

We wish for this to be a manifold and so be smooth at the horizon. As the metric looks very similar to polar coordinates it is easy to see that $\tau \sim \tau + \frac{2\pi}{\kappa}$. This implies that the temperature is $T = \frac{\kappa}{2\pi}$.

This form of thermal behaviour comes from the fact that the accelerated observer cannot see the whole of the spacetime, they are limited to the Rindler wedge. To describe the state seen by the accelerated observer a trace should be taken over the

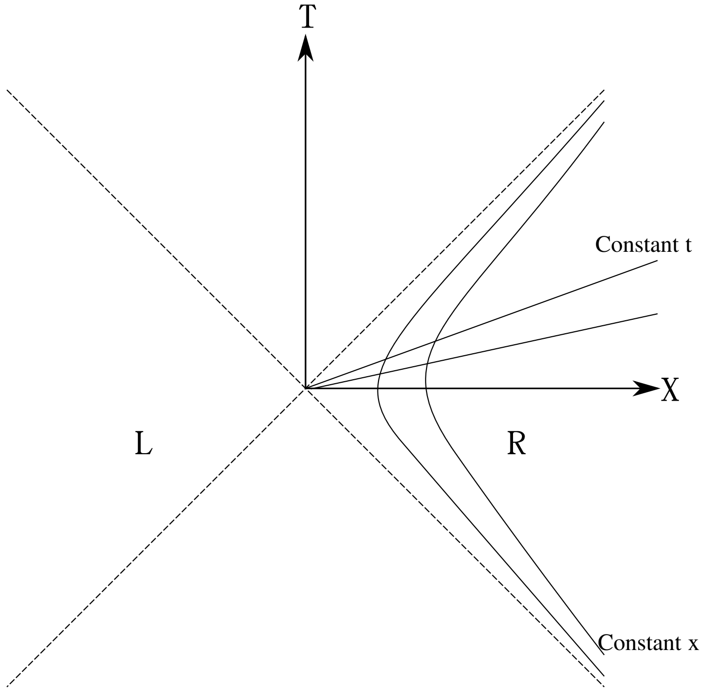

Figure 4.1: The Minkowski coordinate system (T, X) with lines of constant Rindler coordinates (x, t). The lines of constant x are the world lines for a constantly accelerating particles with acceleration $1/x$. Here the lines $x = 0$ which is $X = T$ and $X = -T$ define the Rindler horizons. These split the spacetime up into four separate regions, of which L and R can be covered by Rindler coordinates. These regions are spatially separated, meaning that any uniformly accelerating observer in R is causally disconnected from L and vice versa.

modes in the unseen region of space. Entropy will come from entanglement between modes in the two halves of spacetime.

Constructing a quantum theory on curved spacetimes presents many problems. One of the most pressing is that the vacuum is not unique, as we shall see. Not being able to define a unique vacuum means that there is no natural way to define positive frequency modes. To see this an inner product can be generalized from the Klein-Gordon inner product for flat space, for fields f and g,

$$\begin{aligned} (f,g) &= \int_\Sigma d\Sigma_\mu j^\mu_{f,g}, \\ j^\mu_{f,g} &= -i\sqrt{-g}g^{\mu\nu}(\bar{f}\partial_\nu g - \partial_\nu \bar{f} g). \end{aligned} \tag{4.1.9}$$

As this inner product is a scalar, it at most depends on the spacelike slice Σ which the current is integrated over, not on the coordinate system used to do the integration. However the equations of motion imply that the current is conserved, $\partial_\mu j^\mu = 0$. This means that the inner product is independent of the spacelike slice chosen and therefore the choice of time coordinate.

The inner product is not positive definite for the set of all modes. We wish to find a subset of modes for which the inner product is positive definite. We define the set of positive frequency modes, S_p to be the maximal set such that,

$$(f,f) > 0 \quad \forall f \in S_p \qquad (f,\bar{g}) = 0 \quad \forall f,g \in S_p. \tag{4.1.10}$$

The space of solutions of the field equations can now be decomposed into the positive norm modes and their conjugate,

$$S = S_p \oplus \bar{S}_p \tag{4.1.11}$$

It turns out however that this decomposition is not unique. The set of positive frequency modes can be chosen in multiple different ways. These different decompositions are not coordinate dependent (the inner product does not depend on the coordinate system used), some choices however will seem more 'natural' for a particular choice of coordinate system. Given this decomposition, annihilation and creation operators can be made in the usual manner. The vacuum can then be identified using the annihilation operators,

$$a(f)|0\rangle = 0 \quad \forall f \in S_p. \tag{4.1.12}$$

A Fock space can then be made using the creation operators acting on the vacuum. The problem arises that there is no natural specification for the positive norm subspace, S_p. If a different positive norm subspace is used then it results in a different vacuum state and a different Fock space. Modes of a different positive frequency subspace can always be expressed as a linear combination of the original positive frequency modes and their conjugates, via the Bogliubov transformation,

$$f'_n = \sum_m \alpha_{nm} f_m + \beta_{nm} \bar{f}_m, \tag{4.1.13}$$

for $f \in S_p$ and $f' \in S_{p'}$.

We will now look at plane wave modes for massless free scalar theory in both Minkowski and Rindler coordinates. They form two separate positive norm subspaces in massless free scalar theory. In Minkowski coordinates the plane wave solutions take the form,

$$u_k = e^{-i\omega_k V}, \tag{4.1.14}$$

where $V = T - X$ and ω_k is a positive frequency. These are obviously a convenient set of modes to use for Minkowski space. The Rindler modes must be defined separately for the left and right coordinate patches (see 4.1).

$$R_k = e^{-i\omega_k(v)} \quad \text{in } R \qquad = 0 \quad \text{in } L, \tag{4.1.15}$$

$$L_k = e^{i\omega_k(u)} \quad \text{in } L \qquad = 0 \quad \text{in } R, \tag{4.1.16}$$

with $v = t - x$ and $u = t + x$. We shall now look at the right wedge. Again these are a useful set of modes to consider the spectrum from the point of view of an accelerated observer. By itself R_k is not a positive frequency mode with respect to the Minkowski time T. A new mode, $\tilde{u}_k$, can however be constructed so that it is identical to R_k in R but positive frequency with respect to Minkowski time.

$$\begin{aligned} \tilde{u}_k &= e^{-i\omega_k \ln(-V)/\kappa} \quad \text{for } V < 0, \\ &= e^{-\pi\omega_k/\kappa} e^{i\omega_k \ln(V)/\kappa} \quad \text{for } V > 0. \end{aligned} \tag{4.1.17}$$

This can be shown to be a purely positive frequency mode with respect to Minkowski time. If T is allowed to be complex then $\tilde{u}_k$ is an analytic function on the lower half plane. This can be rewritten as,

$$\tilde{u}_k = R_k + e^{-\pi\omega_k/\kappa} L_k^*, \tag{4.1.18}$$

where * indicates complex conjugate. Since $\tilde{u}_k$ is positive frequency it can be used to construct annihilation operators, which destroy the Minkowski vacuum state, $a(\tilde{u}_k)|0\rangle_{Minkowski} = 0$. We can use the identity $a(p) = -a^\dagger(p^*)$. This gives us that,

$$a(R_k)|0\rangle_{Minkowski} = e^{-\pi\omega_k/\kappa} a^\dagger(L_k)|0\rangle_{Minkowski} \tag{4.1.19}$$

A similar process can be done concentrating on the left sector. This results in the relation,

$$a(L_k)|0\rangle_{Minkowski} = e^{-\pi\omega_k/\kappa} a^\dagger(R_k)|0\rangle_{Minkowski} \tag{4.1.20}$$

These two relations (4.1.19,4.1.20) can be formally solved by,

$$|0\rangle_{Minkowski} = \prod_k \exp\left[e^{-\pi\omega_k/\kappa} a^\dagger(L_k) a^\dagger(R_k)\right]|0\rangle_L \times |0\rangle_R \tag{4.1.21}$$

Where $|0\rangle_{L,R}$ are the Rindler vacuum states in the left and right sectors. This is only a formal relation as the Minkowski vacuum is not in the Hilbert space of the Rindler vacuum.

Using this relation we can see that the Minkowski vacuum is thermal from the point of view of accelerating observers . If there is an operator, $\mathcal{O}$, which acts on only one of the two sectors, then in calculating the operator expectation value a trace is taken over the other sector. Formally;

$$\langle 0|_{Mink} \mathcal{O}_R |0\rangle_{Mink} = \sum_k e^{-2\pi\omega_k/\kappa} \langle m|_R \mathcal{O}_R |m\rangle_R \tag{4.1.22}$$

Where $|m\rangle_R$ is the m particle state in the Fock space built on $|0\rangle_R$.

4.1.2 Hawking radiation

Hawking radiation has many properties in common with Unruh radiation. Again it is thermal radiation associated with a bifurcate Killing horizon. The difference is that this time the horizon in question is the event horizon for a black hole, rather than an acceleration horizon. In this case the observers which are static at a finite distance from the event horizon are equivalent to the Rindler observers in the Unruh case. It is these observers which will see the thermal Hawking radiation.

The temperature of the black hole can be calculated by Euclideanising the black hole metric and forcing it to be a regular manifold at the horizon, if the 'time' coordinate is taken to be periodic. This period is related to the surface gravity of the black hole, which in turn is related to its temperature. For example with the Schwarzchild black hole when the metric is Euclideanised under $t \to i\tau$ it becomes,

$$ds^2 = \left(1 - \frac{2M}{r}\right) d\tau^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \qquad (4.1.23)$$

Making the coordinate transformations $\tilde{r} = 4M\left(1 - \frac{2M}{r}\right)^{1/2}$ and $\tilde{\tau} = \frac{\tau}{4M}$, the event horizon is now located at $\tilde{r} = 0$. Taking the near horizon limit the metric becomes,

$$ds^2 = \tilde{r}^2 d\tilde{\tau}^2 + d\tilde{r}^2 \qquad (4.1.24)$$

For this to be a manifold at the event horizon $\tilde{\tau}$ must have a periodicity of 2π. This then implies that $\tau \to \tau + 2\pi/\kappa$ where $\kappa = 4M$ is the surface gravity. Using the KMS condition this leads to a temperature $T = \kappa/2\pi = 2M/\pi$. This shows that black holes are thermal objects.

4.2 Holography and the AdS/CFT correspondence

In the previous section we saw that a black hole has an associated entropy (4.1.1). Looking at this from a naive point of view there are two big surprises. The first is that the entropy is associated with the area of the event horizon rather than the volume or mass. The second is that black holes have a macroscopic entropy at all. Classically black holes are some of the simplest objects described in physics. The no hair theorems guarantee that black holes have a very limited number of parameters; mass, spin and charge. How then can this be reconciled with their large entropies?

The first of these problems is concerned with the holographic principle [40, 41].

4.2.25 The holographic principle. *The entropy of a black hole is proportional to its area in Planck units and this is the largest possible entropy for a system with given surface area.*

This is in essence a restating of the second law of black hole thermodynamics. It suggests that in quantum gravity the degrees of freedom for a given volume live on its surface, one per Planck area.

The AdS/CFT correspondence is a duality between a string theory which lives in anti de Sitter space and a gauge conformal field theory which lives on the boundary of that space. The simplest example of this correspondence is between type IIB string theory on $AdS_5 \times S^5$ and four dimensional $SU(N)$ Yang-Mills with $\mathcal{N} = 4$ supersymmetry [42], although it can be extended to a much larger class of duality. The correlation amounts to a one to one mapping of the spectrum, at any value of energy and other quantum numbers. There is also an equality of observables (i.e. correlation functions of operators) with an appropriate dictionary.

One way in which the AdS/CFT correspondence and holographic principle can be motivated is to consider the difference between gravitational and gauge theories [43]. For a given number of dimensions gravitational theories have fewer observables than non gravitational theories, this is because there is no invariant local way to specify position in general relativity. This leads to issues when trying to incorporate gravity into a gauge theory. The graviton is a massless spin-2 particle, but any entry in the energy momentum tensor which is associated with massless spin-2 particles leads to nonsensical results.

Consider instead a graviton as a bound state of two spin-1 gauge bosons. In this way the graviton gains an extra degree of freedom. As well as the spacetime position of its center of mass the graviton is also dependent on the separation of the two spin-1 gauge bosons, in effect the graviton lives in one extra dimension. We need to impose a couple more restrictions to see how this connects with AdS/CFT. Firstly quantum field theory is easiest to deal with when it is scale invariant. We therefore wish for the separation dimension z to scale in the same way as the centre of mass coordinates x^μ. The most general metric which respects this condition along with the usual symmetries of spacetime is;

$$ds^2 = L^2 \frac{dz^2 + \eta_{\mu\nu} dx^\mu dx^\nu}{z^2}. \tag{4.2.26}$$

This may be recognised as the Poincaré coordinate system for AdS space (3.1.11). Two other conditions must be met. For the gauge bosons to act as a graviton rather than a pair of gauge bosons we must take a strong coupling limit. The other requirement is that the AdS scale L is large in comparison to the Planck length. This allows room for a black hole with a large surface area and therefore large entropy, corresponding to a field theory with a large number of degrees of freedom.

So in a hand waving manner we have managed to find a duality between a five dimensional gravitational theory on AdS space and a four dimensional gauge theory on flat space. Another useful way to motivate the AdS/CFT correspondence is to consider D-branes. This introduces some of the more 'stringy' aspects of the correspondence. This was in fact the original motivation for the AdS/CFT correspondence [44].

Consider type IIB string theory in flat Minkowski space [42]. Now to this set up add a stack of N coincident D3 branes. In this situation type IIB string theory contains two types of excitation. The first is the closed string which is in the bulk and contains the graviton in its spectrum. The second is the open string which has its end points on the D3 branes and describe excitations of the D3 branes. We wish to consider the low energy limit of this theory where only massless string states are excited. The action can be split up into the bulk action, the action on the branes and the interaction between the two, the massive modes are integrated out. The low energy limit for the closed strings in type IIB string theory is 10 dimensional supergravity. The massless states for the open strings form a $\mathcal{N} = 4$ supermultiplet on the D3 branes and the low energy effective Lagrangian is that of $\mathcal{N} = 4$ $SU(N)$ super Yang-Mills theory. At low energies the open and closed string sectors decouple and the interaction terms vanish. This leaves behind free supergravity in the bulk and a gauge theory on the branes.

There is a second perspective from which this set up can be seen, that of an observer at spatial infinity. The D branes are massive charged objects which source supergravity fields and warp spacetime. This leads to a solution with the metric,

$$ds^2 = \left(1 + \frac{R^4}{r^4}\right)^{-\frac{1}{2}} \left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right) + \left(1 + \frac{R^4}{r^4}\right)^{\frac{1}{2}} \left(dr^2 + r^2 d\Sigma_5^2\right), \tag{4.2.27}$$

where R^4 is a constant proportional to the string coupling and N. As g_{tt} is non-constant energy is redshifted from the point of view of an observer at infinity.

$$E_\infty = \left(1 + \frac{R^4}{r^4}\right)^{-\frac{1}{4}} E_r \tag{4.2.28}$$

E_∞ is the energy as measured by an observer at infinity and E_r is the energy measured by an observer at constant radius r. This means that as an object of fixed energy moves towards $r = 0$ the observer at infinity sees its energy decrease towards zero. Again we are interested in the low energy limit of this system, this time in terms of E_∞. There are two types of excitation in the low energy limit, massless states with long wavelength in the bulk and any excitation which approaches $r = 0$. These two types of excitation decouple from each other in the low energy limit. This is because the bulk excitations have much longer wavelengths than the size of the D branes. They therefore do not interact with states localised near the D brane. Again we have two decoupled sectors bulk supergravity and the 'near horizon' excitations.

In the near horizon region $r << R$ and the metric becomes,

$$ds^2 = \frac{r^2}{R^2}\left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right) + \frac{R^2}{r^2}dr^2 + R^2 d\Sigma_5^2. \quad (4.2.29)$$

This the metric for $AdS_5 \times S^5$. The near horizon excitations are therefore type IIB superstring theory on a $AdS_5 \times S^5$ background.

So from the two different perspectives we have taken the low energy limit and obtained supergravity on flat space decoupled from a second sector. This second sector is either type IIB superstring theory on $AdS_5 \times S^5$ or $\mathcal{N} = 4$ $SU(N)$ super Yang-Mills theory. Since these two perspectives describe the same physics it is natural to identify the two descriptions. It is therefore conjectured that type IIB superstring theory on $AdS_5 \times S^5$ is 'dual' to $\mathcal{N} = 4$ $SU(N)$ super Yang-Mills on $3 + 1$ dimensions.

4.2.1 Evidence for the AdS/CFT correspondence

The AdS/CFT correspondence is a conjecture rather than a formally proved theory. There is however an array of evidence to support it. [42, 43]

- The symmetries of the two theories match. Considering the $AdS_5 \times S^5$ case. $\mathcal{N} = 4$ $SU(N)$ super Yang-Mills has a $SU(4) \sim SO(6)$ R symmetry which rotates its six scalar fields and four fermions. It also has the conformal symmetry in 4 dimensions, SO(2,4). These symmetries match to the space time isometries of $AdS_5 \times S^5$.

- The spectra of the supersymmetric states on the two sides match. This includes all modes of the graviton on $AdS_5 \times S^5$.

- All of the amplitudes which are protected by supersymmetry and thus can be easily compared are equal.

- When the duality is perturbed in a suitable manner to break the supersymmetry or conformal symmetry the $AdS_5 \times S^5$ geometry behaves in the expected way from the field theory, i.e. confinement.

- Higher symmetries exist on both sides which allow certain quantities to be calculated for all values of the coupling constant g, these have been matched with consistency.

- Long string states can be matched together from the two theories.

4.2.2 The dictionary

Whilst we have so far noted that there is a duality between the gauge and gravity theories, it is important to define exactly which quantities correspond for it to be of any use. In particular we wish to be able to map the fields on the string side to states in the conformal field theory. The duality may be expressed as a one to one mapping between particle species in AdS_{d+1} and the single trace chiral primary operators in the CFT. For simplicity we will consider scalar fields, ϕ, in Poincaré coordinates. If the bulk field goes as z^{Δ} as it approaches the boundary, then this mapping can be realised as a scaled boundary limit, leading to a CFT operator, $\mathcal{O}$, of dimension Δ.

$$\mathcal{O}(x) = C \lim_{z \to 0} z^{-\Delta} \phi(x, z) \tag{4.2.30}$$

C is a constant which is dependent on convention and $z = 0$ defines the boundary of the AdS space. This mapping can be checked under a scale transformation $\phi(x, z) \to \phi(\zeta x, \zeta z)$. This results in a correctly scale transformed boundary operator $\mathcal{O}(x) \to \zeta^{\Delta} \mathcal{O}(\zeta x)$.

Similar analysis can be carried out for tensor fields. For example whenever a CFT has a conserved current then there will be a corresponding gauge field in the bulk. Once the mapping between states has been defined (4.2.30), it can be used to calculate other quantities such as correlation functions.

4.2.3 AdS$_3$

In the previous sections we have discussed the AdS/CFT correspondence with regard to AdS_5, but this is not the only duality between gauge and gravity theories. This section will concentrate on AdS_3 space, as this is required to study BTZ black holes and the self-dual orbifold.

When we talk about the AdS_3/CFT_2 correspondence, we are in reality talking about the duality between type IIB string theory on $AdS_3 \times S^3 \times M^4$ and the conformal field theory living on the two dimensional boundary of AdS_3 space. M^4 is a four dimensional manifold, for convenience we will usually take it to be T^4. The AdS_3 spacetime in global coordinates can be viewed as a cylinder (see figure 4.2). This duality can be derived again using a stack of D-branes, this time however it requires two different types [42]. We start with a manifold which is $M^4 \times (6 \quad dimensional \quad flat \quad space)$, D1 and D5 branes are then introduced. N_1 D1 branes are introduced in a non-compact direction. N_5 D5 branes are also added, these wrap around the M^4 dimensions, their remaining dimension is coincident with the D1 brane. Again looking at different perspectives we can relate the near horizon gravity theory with the gauge field which lives on the branes. The near horizon geometry becomes $AdS_3 \times S^3$ where the radius of both the AdS space and the sphere are given by the number of D branes and the string coupling.

$$ds^2 = \alpha'[\frac{z^2}{R^2}(-dt^2 + dx^2) + \frac{R^2}{z^2}dz^2 + R^2 d\Omega_3^2] \tag{4.2.31}$$

$$R^2 = g_6\sqrt{N_1 N_5} \tag{4.2.32}$$

The resultant M^4 volume in the near horizon geometry is then proportional to N_1/N_5 and independent of the volume of the original M^4 factor.

The low energy dual field theory lives on the D1-D5 brane system. It is a 1+1 dimensional CFT with $\mathcal{N} = (4, 4)$ supersymmetry.

4.2.4 Thermal states and AdS_2

The AdS/CFT correspondence doesn't only link pure AdS space to the ground state of the gauge theory on the boundary, it also links bulk spacetimes which are asymptotically AdS to other (possibly excited) states on the boundary. Often the states that are considered are thermal or at least have finite entropy. In AdS/CFT the thermal objects that we consider are usually black holes. The eternal black hole has two boundaries, on which the dual CFTs live [15]. These CFTs are entangled together as information cannot pass from one to the other, they are causally disconnected. To look at the state of a CFT on one particular boundary a trace must be taken over the other boundary. The state will then be seen to be purely thermal, that is to say the minimum amount of information, just the temperature, is known about it. In AdS/CFT eternal black holes have two important features, horizons in the bulk and two disconnected boundaries. We can see that the AdS/CFT correspondence links thermal states in the bulk to thermal states on the boundary. The horizon gives entropy to the gravitational theory and entanglement between the boundaries give rise to entropy in the gauge theory. Extremal black holes have zero temperature but finite entropy (this can be seen from their surface gravity and horizon area). As the self dual orbifold is the near horizon limit of an extremal black hole it also falls into this category.

AdS_2 is an interesting case, in which the AdS/CFT correspondence is still poorly understood. Whereas higher dimensional AdS spaces have a single boundary and are dual to a pure CFT state, AdS_2 has two distinct boundaries. Although the boundaries are distinct, they are causally connected through the bulk and there is no horizon present (see figure 4.2).

AdS_2 geometry appears in the process of taking a near horizon limit for extremal black holes in three, four and five dimensions (see section 3.4). It is thought therefore that the AdS_2/CFT correspondence will have a direct bearing on microscopic explanations for the Bekenstein-Hawking entropy of extremal black holes [45].

This leads to a number of issues; from a gravitational perspective there are no horizons in the bulk. As entropy is proportional to horizon area it is unclear how AdS_2 can provide the entropy for extremal black holes. AdS_2 has two boundaries on which a gauge theory can live, if separate CFTs live on these boundaries then they can be entangled to provide the entropy required. The two boundaries however are in causal contact, this should mean that there is some relation between the Hamiltonians of the two CFTs.

Higher dimensional anti de Sitter spaces are dual to the ground state of the CFT living on their boundaries. As AdS_2 has finite entropy it cannot be dual to the groundstate of a theory. It would be interesting to see if there is a geometry which is dual to the ground state of a single copy of the CFT which lives on the boundary of AdS_2.

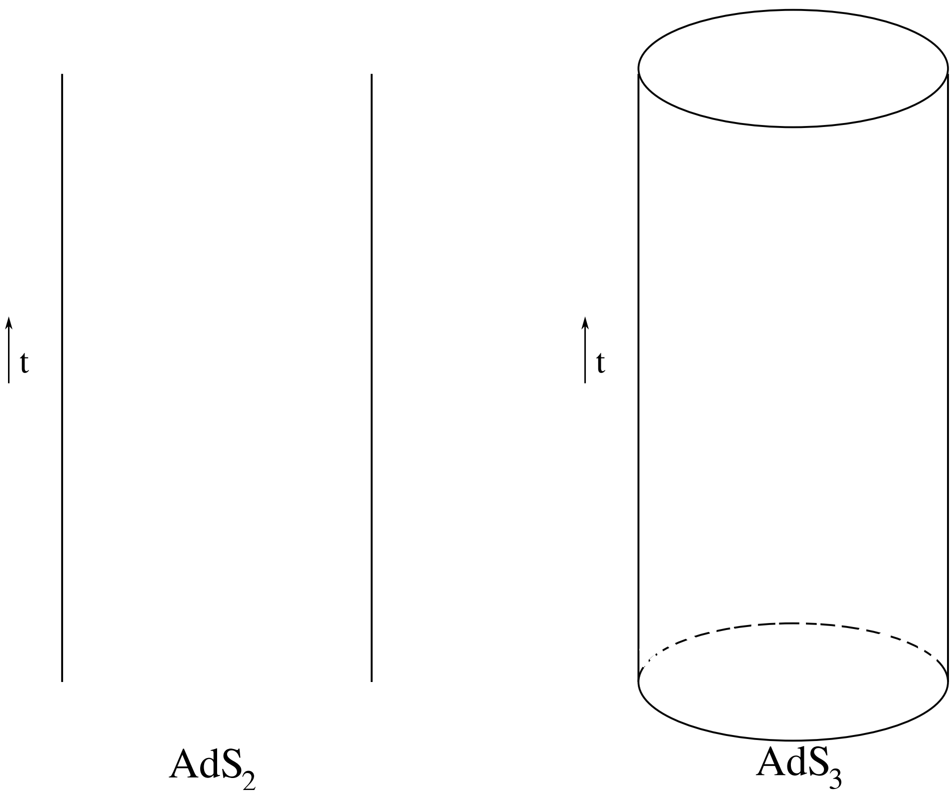

Figure 4.2: AdS_3 space is conformal to a cylinder with a single boundary, whereas AdS_2 is conformal to a strip which has two boundaries.

There have been a few attempts to construct a CFT which is dual to AdS_2. [46] deals with a super-conformal theory on a M2-brane moduli space, arising from a near horizon limit in a multi-black hole background which will share many properties with the CFT living on the boundary/boundaries of AdS_2 space. [47] conjectures a matrix model to be dual to type 0A string theory on AdS_2. [45] supports the idea that the entropy in extremal black holes comes from the entanglement between states on the two boundaries of the AdS_2 factor in their near horizon limit. Calculations can be done to show that the entanglement entropy between the two boundaries on the AdS_2 factor are equal to the black hole entropy. In many of these cases the entanglement entropy has to be calculated holographically as the exact dual CFT is unknown. We will return to these issues in chapter 6.

Chapter 5

Extremal BTZ black holes

5.1 Introduction

The Banados-Teitelboim-Zanelli (BTZ) black hole [18,19] is a very useful laboratory for exploring aspects of black holes and geometry in a simplified setting, as the geometry is simply an orbifold of AdS_3.

This chapter addresses the problem of finding an explicit set of vertex operators for the untwisted and twisted sectors for the bosonic string on the zero-mass and extremal rotating BTZ black holes. We consider an $\mathrm{AdS}_3 \times S^3$ geometry supported by an NS-NS flux, corresponding to an F1-NS5 system compactified on a Ricci-flat internal manifold. The world-sheet theory is a CFT with an $\widehat{SL(2,\mathbb{R})}_k \times \widehat{SU(2)}_k$ current algebra, with the level k being set by the NS-NS flux. We will discuss the bosonic string in detail; the problem of extending our analysis to the superstring will be discussed at the end of the chapter. We want to work in a parabolic basis for $SL(2,\mathbb{R})$, which diagonalises the combination of generators corresponding to the momentum along the compact circle in zero-mass BTZ. Here we show that the hyperbolic Wakimoto representation introduced in [26], provides an appropriate representation of the $\widehat{SL(2,\mathbb{R})}_k$ current algebra. This representation has the advantage that the expression for the vertex operators is more explicit than in the parafermionic representation used in the non-extremal case.

We apply this calculation of the spectrum to study the tachyons in this background. Tachyons in the non-extremal non-rotating BTZ black hole were studied in [20], as an explicit example of the kind of quasi-localised closed string tachyons discussed in [48].[1] The idea is that if we consider string theory compactified on a circle, when the size of the circle is smaller than the string length ℓ_s, there are tachyonic winding modes. If the size of this circle varies over some base space, one heuristically expects a tachyon which is confined to the region where the size of the circle $\leq \ell_s$. It was found in [20] that there is a tachyon in the twisted sector NS-NS ground state if the size of the circle at the black hole horizon is smaller than the string scale, $\sqrt{k}r_+ \leq \ell_s$. However, this tachyon is found not to be localised in the near-horizon region, due to the coupling to the NS-NS field. As the zero-mass BTZ black hole is the limit as $r_+ \to 0$, we would expect that in this case, the NS-NS ground state in twisted sectors should always be tachyonic, and using the explicit representation of the spectrum we obtain, that is indeed what we find. We also extend the analysis to the rotating case, showing that tachyons arise in the twisted sectors if $\sqrt{k}r_+ \leq 2\ell_s$, as in the previous discussion of the non-rotating non-extremal case. The study of tachyons in the $M = 0$ BTZ black hole has a couple of advantages over the previous non-extremal case. Firstly, the expressions for the vertex operators in the Wakimoto representation are more explicit than the parafermionic representation used in the previous case. Secondly, the geometry has a causal Killing vector everywhere, so issues of tachyon condensation could be addressed in the $M = 0$ BTZ black hole without having to deal with the complications of studying the behaviour on a time-dependent background geometry in the region behind the horizon. We will not address the question of the condensation of the tachyon, which remains a challenging direction for future work.

In section 5.2, we review aspects of the $M = 0$ BTZ black hole. In section 5.3, we introduce the hyperbolic Wakimoto representation of the current algebra, and use it to construct vertex operators for the untwisted sector states. We then introduce a twist operator enforcing the orbifold condition, and use it to obtain the twisted sector vertex operators. We discuss the condition for a tachyon to exist in the spectrum, and argue that the NS-NS ground states in the twisted sector are

[1] As BTZ arises as the near-horizon limit of a charged black string, it is directly related to the examples discussed in [49, 50]. Other examples with a quasi-localised tachyon include [51, 52].

tachyonic, as expected. In section 5.4, we discuss the flat space limit, taking $k \to \infty$ while focusing on the neighbourhood of the singularity. In this limit, the zero mass BTZ black hole reduces to the null orbifold of flat space [53, 54].

In section 5.5, we extend the analysis to the extremal rotating BTZ black hole. We show that the relevant orbifold action is chiral, with the action on left-movers the same as for the zero-mass black hole while the action on right-movers is the same as for the non-zero mass black hole. We can thus construct appropriate vertex operators by combining the previous results for these two cases. We show that the resulting set of vertex operators for twisted sectors is mutually local, and argue that a tachyon appears when $\sqrt{k}r_+ \leq 2\ell_s$, as expected.

In section 5.6, we discuss the extension of our results to the superstring. The main open problem is to find a representation of the spin fields which diagonalises the action of the spacetime angular momentum. Without such a representation, we cannot explicitly construct vertex operators corresponding to the modes which survive the orbifold projection in the NS-R and R-R sectors. The final section summarises the chapter and considers possible routes for further investigation.

5.2 AdS$_3$ worldsheet theory

Bosonic string theory on AdS_3 is described by the $SL(2,\mathbb{R})$ WZW model as given in section 2.1. We can find the OPEs for the currents. They are

$$\mathcal{J}^b \mathcal{J}^c = \frac{i\epsilon^{bc}{}_a \mathcal{J}^a}{z-w} + \frac{\frac{k}{2}\eta^{bc}}{(z-w)^2}, \qquad \bar{\mathcal{J}}^b \bar{\mathcal{J}}^c = -\frac{i\epsilon^{bc}{}_a \bar{\mathcal{J}}^a}{\bar{z}-\bar{w}} + \frac{\frac{k}{2}\eta^{bc}}{(\bar{z}-\bar{w})^2}, \tag{5.2.1}$$

where ϵ^{abc} is the totally antisymmetric tensor, with $\epsilon^{123} = 1$, and η^{ab} is the metric defined by $\eta^{ab} = diag(1, 1, -1)$. There is a relative minus sign between the OPEs for the left and the right moving currents, as noted in [31]. This minus sign can be fixed using a relabelling process, setting $J^a = \mathcal{J}^a$, $\bar{J}^1 = \bar{\mathcal{J}}^1$, $\bar{J}^3 = \bar{\mathcal{J}}^3$, $\bar{J}^2 = -\bar{\mathcal{J}}^2$. The OPEs for both the left and right moving sectors are then identical for the new currents,

$$J^b J^c = \frac{i\epsilon^{bc}{}_a J^a}{z-w} + \frac{\frac{k}{2}\eta^{bc}}{(z-w)^2}, \qquad \bar{J}^b \bar{J}^c = \frac{i\epsilon^{bc}{}_a \bar{J}^a}{\bar{z}-\bar{w}} + \frac{\frac{k}{2}\eta^{bc}}{(\bar{z}-\bar{w})^2}. \tag{5.2.2}$$

Assuming the currents have trivial monodromies, they will have a mode expansion

$$J^a = \sum_{n \epsilon \mathbb{Z}} z^{-n-1} J^a_n \qquad \bar{J}^a = \sum_{n \epsilon \mathbb{Z}} \bar{z}^{-n-1} \bar{J}^a_n. \tag{5.2.3}$$

The commutation relations for these modes are then

$$[J^a_n, J^b_m] = i\epsilon^{ab}{}_c J^c_{m+n} + \frac{k}{2} n\eta^{ab}\delta_{m+n,0}, \tag{5.2.4}$$

and similarly for the $\bar{J}^a$. In particular the zero modes form an $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ subalgebra, corresponding to the spacetime isometries.

5.2.1 Zero mass black hole

The $M = 0$ BTZ black hole corresponds to writing the space in Poincaré coordinates (3.1.2) and making an identification. In these coordinates the $SL(2,\mathbb{R})$ group element is

$$g = \begin{pmatrix} \frac{1}{z} & \frac{(t+x)}{z} \\ \frac{(x-t)}{z} & \frac{(x^2+z^2-t^2)}{z} \end{pmatrix}, \tag{5.2.5}$$

so the metric is

$$ds^2 = \frac{k}{z^2}(-dt^2 + dz^2 + dx^2) \tag{5.2.6}$$

and the NSNS 2-form field is

$$B = \frac{k}{z^2} dt \wedge dx. \tag{5.2.7}$$

In Poincaré coordinates, the currents are

$$\begin{aligned} J^1 &= -ik\left[(\partial x + \partial t)\frac{(x-t)}{z^2} + \frac{\partial z}{z}\right], \\ J^2 &= ik\left[-\frac{(x-t)}{z}\partial z + \frac{(\partial x - \partial t)}{2} + \frac{(\partial x + \partial t)}{2z^2}(2tx + 1 - x^2 - t^2)\right], \\ J^3 &= -ik\left[-\frac{(x-t)}{z}\partial z + \frac{(\partial x - \partial t)}{2} + \frac{(\partial x + \partial t)}{2z^2}(2tx - 1 - x^2 - t^2)\right], \end{aligned} \tag{5.2.8}$$

and

$$\begin{aligned} \bar{J}^1 &= -ik\left[(\bar{\partial} x - \bar{\partial} t)\frac{(x+t)}{z^2} + \frac{\bar{\partial} z}{z}\right], \\ \bar{J}^2 &= ik\left[-\frac{(x+t)}{z}\bar{\partial} z + \frac{(\bar{\partial} x + \bar{\partial} t)}{2} + \frac{(\bar{\partial} x - \bar{\partial} t)}{2z^2}(-2tx + 1 - x^2 - t^2)\right], \\ \bar{J}^3 &= -ik\left[-\frac{(x+t)}{z}\bar{\partial} z + \frac{(\bar{\partial} x + \bar{\partial} t)}{2} + \frac{(\bar{\partial} x - \bar{\partial} t)}{2z^2}(-2tx - 1 - x^2 - t^2)\right]. \end{aligned} \tag{5.2.9}$$

(In this notation $\partial = \partial_z$ and $\bar{\partial} = \partial_{\bar{z}}$, where z and $\bar{z}$ are the worldsheet coordinates).

To relate the spacetime energy and momentum to these currents, we consider infinitesimal time and space translations. For the time-translation, the infinitesimal transformation is

$$\delta_{(t)} g \equiv i\epsilon_{(t)}(z) g - i g \bar{\epsilon}_{(t)}(\bar{z}) = \frac{\partial g}{\partial t}\delta t, \tag{5.2.10}$$

where

$$i\epsilon_{(t)}(z) = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \delta t, \qquad i\bar{\epsilon}_{(t)}(\bar{z}) = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \delta t. \tag{5.2.11}$$

Thus, in terms of the $SL(2,\mathbb{R})$ generators

$$\epsilon_{(t)}(z) = (\tau^2 + \tau^3)\delta t \equiv \tau^+ \delta t, \qquad \bar{\epsilon}_{(t)}(\bar{z}) = (\tau^2 - \tau^3)\delta t \equiv -\tau^- \delta t. \tag{5.2.12}$$

A similar calculation can be performed for the infinitesimal transformations in the x-direction, $\delta_{(x)} g$. Using the Ward identity (2.7) and substituting in ϵ and $\bar{\epsilon}$, it can then be seen that

$$Q_t = (\mathcal{J}_0^+ + \bar{\mathcal{J}}_0^-), \qquad Q_x = (-\mathcal{J}_0^+ + \bar{\mathcal{J}}_0^-). \tag{5.2.13}$$

In terms of the modified currents, the charges are then

$$Q_t = (J_0^+ + \bar{J}_0^+), \qquad Q_x = -(J_0^+ - \bar{J}_0^+). \tag{5.2.14}$$

We obtain the $M = 0$ BTZ black hole by making periodic identifications along the ∂_x direction. The period of the identification can be changed by rescaling x, so it is not a physical parameter. For convenience, we choose $x \sim x + 2\pi$. Invariance under this orbifold restricts states to have a quantised value of Q_x,

$$(J_0^+ - \bar{J}_0^+) \in \mathbb{Z}. \tag{5.2.15}$$

We therefore need to work in a parabolic basis for $SL(2,\mathbb{R})$, which diagonalises J_0^+. In the next section, we will use a Wakimoto representation for these currents to implement this constraint.

5.3 Vertex operators on zero mass black hole

In this section, we construct vertex operators for the untwisted and twisted sector states of the bosonic string on the $M = 0$ BTZ black hole.

To implement the constraint (5.2.15), it is crucial to have a set of vertex operators which diagonalise the action of J_0^+. It is therefore useful to have a representation of the current algebra where J^+ is as simple as possible. In the case of non-extreme BTZ black hole in [20], we needed a representation which diagonalised J^2, and we could simply introduce a free boson representing the current J^2, writing the remainder of the vertex operator in terms of a parafermion. Since the current J^+ is null, a simple free boson representation will not be possible. However, it turns out that the hyperbolic Wakimoto representation of the $\widehat{SL(2,\mathbb{R})}$ current algebra introduced in [26] provides a simple representation for J^+ (the relevance of this representation for the $M = 0$ BTZ black hole was previously noted in [32]). The Wakimoto representation constructs the conserved currents in terms of a free boson ϕ and anticommuting $\beta - \gamma$ bosonic ghosts:

$$
\begin{aligned}
iJ^+(z) &= \beta(z), && (5.3.16)\\
iJ^-(z) &= \gamma^2(z)\beta(z) + \sqrt{2k'}\gamma(z)\partial\phi(z) + k\partial\gamma(z), && (5.3.17)\\
iJ^1(z) &= -\gamma(z)\beta(z) - \sqrt{\frac{k'}{2}}\partial\phi(z), && (5.3.18)
\end{aligned}
$$

where $k' \equiv k - 2$, and the OPEs for β,γ and ϕ are

$$
\begin{aligned}
\beta(z)\gamma(w) &= -\gamma(z)\beta(w) \sim \frac{1}{z-w}, && (5.3.19)\\
\phi(z)\phi(w) &\sim -\ln(z-w). && (5.3.20)
\end{aligned}
$$

This leads to the required OPEs for the conserved currents,

$$
\begin{aligned}
J^+(z)J^-(w) &\sim \frac{-k}{(z-w)^2} + \frac{2iJ^1(w)}{z-w} && (5.3.21)\\
J^1(z)J^\pm(w) &\sim \frac{\mp iJ^\pm(w)}{z-w} && (5.3.22)\\
J^1(z)J^1(w) &\sim \frac{\frac{k}{2}}{(z-w)^2} && (5.3.23)
\end{aligned}
$$

Unlike in the non-zero mass BTZ black hole case, this is an explicit representation of the full current algebra. We introduce an identical representation for the antiholomorphic currents $\bar{J}^a$, in terms of $\bar{\beta}(\bar{z})$, $\bar{\gamma}(\bar{z})$, $\bar{\phi}(\bar{z})$.

5.3.1 Untwisted sector vertex operators

We want to take a basis of vertex operators which diagonalise J_0^+. A vertex operator V has J_0^+ eigenvalue λ if $J^+V(z) = \frac{\lambda V(z)}{z-w}$. Using (5.3.19), this implies that

$$V(z) = e^{i\lambda\gamma} f(\beta, \phi). \tag{5.3.24}$$

For AdS$_3$, the $\widehat{SL(2,\mathbb{R})}$ current algebra is a spectrum generating algebra, and the spectrum contains short string states in highest weight representations of the current algebra: the continuous representations $\hat{\mathcal{C}}_j^\alpha \times \hat{\mathcal{C}}_j^\alpha$ for $j = \frac{1}{2} + is$ which correspond to spacetime tachyons, and the discrete representations $\hat{\mathcal{D}}_j^\pm \times \hat{\mathcal{D}}_j^\pm$ for $\frac{1}{2} < j < \frac{k-1}{2}$. The spectrum in global AdS$_3$ also contains long string states, but these do not survive the orbifold projection, being replaced instead by the twisted sector states. Diagonalising J_0^+ corresponds to considering the representations of $SL(2,\mathbb{R})$ in a parabolic basis. For both the continuous and discrete representations of the current algebra, in this parabolic representation, the eigenvalue λ can take all real values.

Since these are highest weight representations of the current algebra, we can focus on the chiral primary operators; other vertex operators will be obtained as descendents. Requiring that (5.3.24) be a chiral primary operator implies that $f(\beta, \phi)$ is independent of β, as V would otherwise have too singular an OPE with J^-. Including the anti-holomorphic sector, we can therefore take a basis of chiral primary vertex operators in the untwisted sector of the form

$$V_{j\lambda\bar{\lambda}}(z, \bar{z}) = e^{i\lambda\gamma - \sqrt{\frac{2}{k'}}j\phi} e^{i\bar{\lambda}\bar{\gamma} - \sqrt{\frac{2}{k'}}j\bar{\phi}}. \tag{5.3.25}$$

To implement the orbifold, we need to quantise the eigenvalue of $J_0^+ - \bar{J}_0^+$, that is, we need

$$\lambda - \bar{\lambda} \in \mathbb{Z}. \tag{5.3.26}$$

In the next subsection, we will see how this quantisation condition can be implemented using a twist operator. This will also allow us to construct vertex operators for the twisted sector modes.

The energy momentum tensor for the WZW model is

$$T = \frac{1}{k-2}\eta_{ab} : J^a J^b :, \tag{5.3.27}$$

which can be rewritten in terms of J^+ and J^- as

$$T = \frac{1}{(k-2)} : \left(J^1 J^1 - \frac{1}{2} J^+ J^- - \frac{1}{2} J^- J^+ \right) : \; . \tag{5.3.28}$$

Working in the Wakimoto representation,

$$T = \beta\partial\gamma - \frac{\partial^2\phi}{\sqrt{2k'}} - \frac{(\partial\phi)^2}{2}. \tag{5.3.29}$$

The conformal dimensions of the vertex operators (5.3.25) are then

$$h = \bar{h} = \frac{-j(j-1)}{(k-2)}. \tag{5.3.30}$$

Thus, the label j on the vertex operators corresponds to the label on representations of the current algebra.

5.3.2 Twist operator

So far, we have just described the vertex operators describing strings on AdS_3 in a basis which is adapted to working in Poincaré coordinates. To describe the $M = 0$ BTZ black hole, we would now like to impose the quantisation condition (5.2.15). Following the same route as in the analysis of the non-extremal BTZ black hole [20], we would like to impose this condition by requiring mutual locality of the untwisted sector vertex operators (5.3.25) with an appropriate twist operator. The twisted sector vertex operators will then be obtained by closure of the OPE including the twist operator.

To do this we have to change our representation again and bosonize the $\beta - \gamma$ system as in [26],

$$\beta = \partial\phi_+, \tag{5.3.31}$$

$$\gamma = \phi_-, \tag{5.3.32}$$

where

$$\phi_\pm = \frac{1}{\sqrt{2}}(\phi_0 \pm \phi_1), \tag{5.3.33}$$

$$\phi_i(z)\phi_j(w) \sim -\eta_{ij} \ln(z-w), \qquad \eta_{ij} = diag(-1,1). \tag{5.3.34}$$

We introduce a similar bosonization for $\bar{\beta}, \bar{\gamma}$. In this representation, the untwisted sector chiral primary fields become

$$V_{j\lambda\bar{\lambda}}(z,\bar{z}) = e^{i\lambda\phi_- - \sqrt{\frac{2}{k'}}j\phi} e^{i\bar{\lambda}\bar{\phi}_- - \sqrt{\frac{2}{k'}}j\bar{\phi}}. \tag{5.3.35}$$

This representation was used to discuss strings on AdS_3 in Poincaré coordinates in [55]. It was noted there that there are potential logarithmic branch cuts associated with the definition of the boson ϕ_+. We will now see (as also noted in [32]) that these branch cuts are correctly interpreted as reflecting winding around the x direction, describing the twisted sectors in the string on the $M = 0$ BTZ black hole.

The appropriate twist operators are

$$t_n = e^{in(\phi_+ + \bar{\phi}_+)}. \tag{5.3.36}$$

This will impose the correct quantisation condition, as

$$t_n(z)V_{j\lambda\bar{\lambda}}(w) \sim \frac{\exp(in\phi_+ + i\lambda\phi_- - \sqrt{\frac{2}{k'}}j\phi)}{(z-w)^{n\lambda}} \frac{\exp(in\bar{\phi}_+ + i\bar{\lambda}\bar{\phi}_- - \sqrt{\frac{2}{k'}}j\bar{\phi})}{(\bar{z}-\bar{w})^{n\bar{\lambda}}}, \tag{5.3.37}$$

so the OPE will only be mutually local for $\lambda - \bar{\lambda} \in \mathbb{Z}$.

We can also read off the twisted sector vertex operators from this OPE, obtaining

$$V_{jn\lambda\bar{\lambda}} = \exp(in\phi_+ + i\lambda\phi_- - \sqrt{\frac{2}{k'}}j\phi) \exp(in\bar{\phi}_+ + i\bar{\lambda}\bar{\phi}_- - \sqrt{\frac{2}{k'}}j\bar{\phi}). \tag{5.3.38}$$

Note that the current algebra generators J^-, J^2 will have non-trivial monodromies around a twisted sector vertex operator because of the dependence on ϕ_+, reflecting the twisting. The conformal dimensions for these twisted sector operators are

$$h = -\frac{j(j-1)}{k'} - n\lambda, \quad \bar{h} = -\frac{j(j-1)}{k'} - n\bar{\lambda}. \tag{5.3.39}$$

The level-matching condition $h - \bar{h} \in \mathbb{Z}$ is satisfied as a consequence of the quantisation of the charge Q_x in (5.3.26). Comparing this to the spectrum obtained for the twisted sectors of the non-extreme BTZ black hole in [20], we see that the spectrum there reduces to this one in the limit $r_+ \to 0$, as expected, with $r_+\lambda_{there} = -\lambda_{here}$, $r_+\bar{\lambda}_{there} = \bar{\lambda}_{here}$. The full vertex operators involve taking descendents of these chiral primary operators, and include a contribution from the internal CFT. The physical state conditions will then be

$$-\frac{j(j-1)}{k'} - n\lambda + h_{int} + N = -\frac{j(j-1)}{k'} - n\bar{\lambda} + \bar{h}_{int} + \bar{N} = 1, \tag{5.3.40}$$

where $h_{int}, \bar{h}_{int}$ are the dimensions of the operator from the internal CFT, and $N, \bar{N}$ are oscillator numbers for the current algebra. We assume that the internal CFT is unitary, so $h_{int}, \bar{h}_{int} \geq 0$.

This is one of the main results of the chapter: by adopting this bosonised version of the Wakimoto representation for the currents, we see that we can give completely explicit expressions for the vertex operators for the full spectrum of string states on the M=0 BTZ black hole, including twisted sector states. In [32], this representation of the currents was also used to construct the Virasoro generators associated with the asymptotic isometries of the spacetime in terms of the worldsheet currents. This makes it possible to study the relation of the worldsheet theory to the dual CFT in AdS/CFT, as was done for global AdS_3 in [56, 57].

5.3.3 Tachyons in twisted sectors

As we are working with the bosonic string, we know there is a tachyonic ground state in the untwisted sectors. We would like to see if there is a tachyon in the twisted sectors. In the extension to the supersymmetric case, the expectation is that there will be a choice of GSO projection which eliminates the ground state in the untwisted sectors but retains it in the odd twisted sectors (corresponding to choosing an antiperiodic spin structure on the orbifold circle in spacetime). Given the appearance of a tachyon in twisted sectors for the non-extreme BTZ black hole with $\sqrt{k} r_+ < 2 l_s$ [20], we would expect that there will be one here as well.

We need to consider carefully the definition of a tachyon. Classically, a spacetime field is tachyonic if it has normalisable solutions which grow exponentially in time. It is difficult to look directly for modes which grow exponentially in time from the worldsheet point of view, as this would require complex $\lambda, \bar{\lambda}$, which makes it difficult to see how we can satisfy the physical state condition $h = \bar{h} = 1$ in the twisted sectors. We will therefore look instead for modes with zero energy; if a spacetime field has a normalisable solution of zero energy, it should generically also have solutions which grow exponentially in time, by continuity.[2] Since the x direction

[2] In a black hole background, modes supported close to the horizon have low energy as a result

is spacelike everywhere in the $M = 0$ BTZ black hole, it is physically reasonable to further restrict to modes which also have zero momentum along x; these should be the most tachyonic modes for a field with a given mass-squared. Thus, it suffices for us to consider modes with $\lambda = \bar{\lambda} = 0$.

Thus, we are looking for normalisable modes with $\lambda = \bar{\lambda} = 0$, satisfying the physical state condition. In this case, the physical state condition for the twisted sectors is identical to that in the corresponding untwisted sector, and the twisted sector states will satisfy the physical state condition whenever the corresponding untwisted sector states do. In particular, there are physical states obtained by spectral flow from the tachyon in the untwisted sector, which have $j(j-1) < 0$.

The remaining condition is normalisability. In the untwisted sector, we know that we have the usual bosonic string tachyon, which has normalisable solutions with $\lambda = \bar{\lambda} = 0$. The twist operator is expressed in terms of the Wakimoto representation, and not in terms of the spacetime coordinates, so it is not possible to rigorously relate the normalisability of twisted sector modes to the corresponding untwisted sector ones, but we expect that at least for large k, the twisting will not significantly modify the dependence on the radial coordinate, so twisted sector modes will be normalisable if the corresponding untwisted sector mode is. Thus, we expect that the $\lambda = \bar{\lambda} = 0$ vertex operators in twisted sectors obtained from the tachyon in the untwisted sector will be normalisable, and hence give modes of a tachyon in the twisted sectors.

As in [20], these modes have roughly the same radial behaviour as for the untwisted sector tachyon, so they are not well-localised in the neighbourhood of the horizon. The twisted sector states are essentially long string states, which can propagate to the asymptotic boundary of the $M = 0$ BTZ black hole at low cost in energy because of the coupling the NSNS 2-form field.

The expectation is that while the untwisted sector tachyons are removed by the GSO projection in the supersymmetric theory, the twisted sector tachyons will

of the gravitational redshift, so a non-tachyonic field will have an energy spectrum starting from zero. However, it will not have a normalisable mode of strictly zero energy, so we believe this condition is still physically appropriate even in the presence of a black hole horizon.

remain. We will comment on this again when we discuss the extension of our work to the superstring later.

5.4 The Null orbifold limit

An important source of intuition and a useful check on the calculations in studying orbifolds of AdS_3 is to consider the limit $k \to \infty$, in which the space becomes flat. For the non-zero mass BTZ black hole studied in [20], there were two flat space limits of interest, the near horizon limit which focused on the region near the event horizon, and the Milne limit, which focused on the singularity. For the $M = 0$ BTZ black hole the event horizon and the singularity are at the same point in space, $z = \infty$ in the coordinates of (5.2.6). The two limits are therefore replaced by one, the null orbifold limit. In this section, we consider the behaviour of the untwisted and twisted sector states we constructed above as we take this limit. This limit is most closely analogous to the Milne limit in [20].

To show that the $M = 0$ BTZ black hole reduces to the null orbifold as we take $k \to \infty$ focusing on the region near the Poincaré horizon at $z = \infty$, we need to make a change of coordinates. If we define new coordinates

$$y^+ = \frac{\sqrt{k}}{z}, \quad y^- = \sqrt{k}(t+z), \quad y = x, \tag{5.4.41}$$

then the metric (5.2.6) becomes

$$ds^2 = -\frac{(y^+)^2(dy^-)^2}{k} - 2dy^+dy^- + (y^+)^2 dy^2. \tag{5.4.42}$$

Taking the limit $k \to \infty$ for fixed $y^\pm, y$, the first term vanishes, and we can see that the metric reduces to the null orbifold of [53, 54], which was analysed in string theory in [58, 59]. In these coordinates, the null orbifold is simply the identification $y \sim y + 2\pi$. The quantization condition associated with this orbifold identification remains simply $\lambda - \bar{\lambda} \in \mathbb{Z}$. We should also consider the limit for the NSNS 2-form field (5.2.7). To obtain a finite limit as $k \to \infty$, we first need to make a gauge transformation to write

$$B = \frac{k}{z^2}(dt + dz) \wedge dx = \frac{(y^+)^2}{\sqrt{k}} dy^- \wedge dy, \tag{5.4.43}$$

so in this gauge the 2-form vanishes in the limit as $k \to \infty$. The contribution from the 2-form is still important to see that the currents J^a, $\bar{J}^a$ are conserved to sub-leading order as we take the limit, as in [20].

It is also convenient to rewrite the null orbifold in Cartesian coordinates; the relation is

$$x^+ = y^+, \quad x^- = y^- + \frac{1}{2} y^+ y^2, \quad x = y^+ y. \tag{5.4.44}$$

In these coordinates, the null orbifold metric is simply flat, but the identification is more complicated:

$$(x^+, x^-, x) \sim (x^+, x^- + 2\pi x + 2\pi^2 x^+, x + 2\pi x^+). \tag{5.4.45}$$

Using (5.2.8) the currents can be calculated in terms of $y^\pm, y$:

$$\begin{aligned}
J^1 &= -i\sqrt{k}(y^+\partial y + y\partial y^+) - iy^+\partial y^- + iy^-\partial y^+ - i(y^+)^2 y\partial y + O(\frac{1}{\sqrt{k}}), \quad (5.4.46)\\
J^2 &= -i\sqrt{k}(y^+ y\partial y + \partial y^- + \frac{y^2\partial y^+}{2} - \frac{\partial y^+}{2})\\
&\quad +i(-\frac{(y^+)^2 y^2 \partial y}{2} + y^+ y^- \partial y + \frac{(y^+)^2\partial y}{2} - y^+ y\partial y^- + y^- y\partial y^+) + O(\frac{1}{\sqrt{k}}),\\
J^3 &= i\sqrt{k}(y^+ y\partial y + \partial y^- + \frac{y^2\partial y^+}{2} + \frac{\partial y^+}{2})\\
&\quad -i(-\frac{(y^+)^2 y^2 \partial y}{2} + y^+ y^- \partial y - \frac{(y^+)^2\partial y}{2} - y^+ y\partial y^- + y^- y\partial y^+) + O(\frac{1}{\sqrt{k}}).
\end{aligned}$$

This can be more simply re-expressed in terms of $x^\pm, x$:

$$\begin{aligned}
J^1 &= -i\sqrt{k}\partial x - ix^+\partial x^- + ix^-\partial x^+ + O(\frac{1}{\sqrt{k}}), \quad (5.4.47)\\
J^2 &= -i\sqrt{k}(\partial x^- - \frac{\partial x^+}{2}) + i(x^-\partial x - x\partial x^- + \frac{1}{2}(x^+\partial x - x\partial x^+)) + O(\frac{1}{\sqrt{k}}),\\
J^3 &= i\sqrt{k}(\partial x^- + \frac{\partial x^+}{2}) - i(x^-\partial x - x\partial x^- - \frac{1}{2}(x^+\partial x - x\partial x^+)) + O(\frac{1}{\sqrt{k}}).
\end{aligned}$$

We see that in the flat space limit $k \to \infty$, the leading order ($O(\sqrt{k})$) terms reproduce the Cartesian translation currents on flat space. As in [20], the subleading parts involve Lorentz transformations in the flat space limit, and are required to make the total current conserved to subleading order taking into account the effects of the 2-form field. The same can be done for the antiholomorphic sector.

We can use this expression for the currents to relate the Wakimoto variables to the coordinates in this limit,

$$\beta = -\sqrt{k}\partial x^+ - (x^+\partial x - x\partial x^+), \tag{5.4.48}$$

$$\gamma = \frac{2}{\sqrt{k}}x^- + \frac{2}{k}xx^-, \tag{5.4.49}$$

$$\phi = -\sqrt{\frac{2}{k'}}\left(\sqrt{k}x + x^+x^-\right). \tag{5.4.50}$$

This in turn implies that the bosonised versions of the Wakimoto variables are closely related to the $x^{\pm}, x$ coordinates in the flat space limit; to leading order,

$$\phi^+ = -\sqrt{k}x^+, \quad \phi^- = \frac{2}{\sqrt{k}}x^-, \quad \phi = -\sqrt{2}x. \tag{5.4.51}$$

By studying the flat space limit of the antiholomorphic currents $\bar{J}^a$, one can similarly learn that

$$\bar{\phi}^+ = -\sqrt{k}\bar{x}^+, \quad \bar{\phi}^- = -\frac{2}{\sqrt{k}}\bar{x}^-, \quad \bar{\phi} = \sqrt{2}\bar{x} \tag{5.4.52}$$

to leading order. Note that the factors of 2 in these expressions appear because in units with $\alpha' = 1$, the flat space coordinates have OPEs $x^\mu x^\nu \sim \frac{1}{2}\eta^{\mu\nu}\ln(z-w)$.

5.4.1 States and vertex operators

In the untwisted sector, the states which survive in this flat space limit are those with $\lambda - \bar{\lambda} \in \mathbb{Z} \sim \mathcal{O}(1)$ and $\lambda + \bar{\lambda} \sim \mathcal{O}(\sqrt{k})$ (since we hold $y^- = \sqrt{k}(t+r)$ fixed in the limit). To have $h \sim \mathcal{O}(1)$, we need $j \sim \mathcal{O}(\sqrt{k})$, and similarly for the barred quantities. In the twisted sectors, we still have $\lambda - \bar{\lambda} \in \mathbb{Z} \sim \mathcal{O}(1)$ and $\lambda + \bar{\lambda} \sim \mathcal{O}(\sqrt{k})$, since the Q_x and Q_t eigenvalues are unaffected by twisting. However, in twisted sectors $h = -\frac{j(j-1)}{k-2} - n\lambda$, so $h \sim \mathcal{O}(1)$ in twisted sectors requires $j \sim \mathcal{O}(k^{3/4})$ to cancel the $\sqrt{k}$ contribution from λ. This cancellation can be achieved for modes in the continuous representations of the current algebra if $\lambda > 0$, and for modes in the discrete representations of the current algebra if $\lambda < 0$. Thus, both tachyonic and non-tachyonic twisted sector modes survive in the flat space limit, but with this curious correlation with the sign of λ. We expect that the resulting spectrum in the flat space limit should agree with the one obtained in [58].

We note that as in the Milne limit in [20], in general the modes which survive in twisted sectors in this flat space limit are not the ones which are obtained by twisting from the untwisted sector states which survive in the limit. This can also be seen by observing that the twist operator (5.3.36) becomes, to leading order,

$$t_n = e^{-n\sqrt{k}(x_+ + \bar{x}_+)}, \tag{5.4.53}$$

and hence does not have a well-defined flat space limit. Thus, our twist operator construction does not have a counterpart in the null orbifold.

Despite the failure of the twist operator to survive in the flat space limit, one might still hope that we could follow our vertex operators in this limit, since we have an explicit construction of the vertex operators in terms of the Wakimoto representation and we understand how these Wakimoto fields are related to flat space coordinates in the limit. Disappointingly, this does not work. If we consider the vertex operator (5.3.25) and substitute in the leading order relations between the Wakimoto fields and the coordinates (5.4.51), we obtain

$$V_{j\lambda\bar{\lambda}}(z) = e^{i\frac{2\lambda}{\sqrt{k}}x^- + j\frac{2\sqrt{k}}{k-2}x} e^{-i\frac{2\bar{\lambda}}{\sqrt{k}}\bar{x}^- - j\frac{2\sqrt{k}}{k-2}\bar{x}}. \tag{5.4.54}$$

In the limit, let us write $2\lambda = \sqrt{k}E + P_x$, $2\bar{\lambda} = \sqrt{k}E - P_x$, $j = \sqrt{k}m$. Then the vertex operator is to leading order

$$V_{j\lambda\bar{\lambda}}(z) = e^{iEx^- + 2mx} e^{-iE\bar{x}^- - 2m\bar{x}}. \tag{5.4.55}$$

We see that this expression has lost the dependence on P_x, so it degenerates in the flat space limit. This failure to obtain a good representation of the vertex operators in the flat space limit is in retrospect not unexpected: the $\widehat{SL(2,\mathbb{R})} \times \widehat{SL(2,\mathbb{R})}$ structure we used in constructing our vertex operators degenerates in this limit. Although the metric and spectrum smoothly go over to the flat space orbifold in this limit, a new representation of the vertex operators is necessary. Similarly, when we take the $r_+ \to 0$ limit of the non-zero mass BTZ black hole, the spectrum of [20] reduces to the one we have obtained here for the zero-mass black hole, but the vertex operators do not have a smooth limit. A new representation is needed in the limit, reflecting the fact that we are considering a different orbifold.

Thus, while the orbifold we are considering reduces to the null orbifold in the limit $k \to \infty$, the representation of the vertex operators in terms of Wakimoto fields degenerates in this limit, so we do not seem to be able to glean new insight into the null orbifold from our construction.

5.5 The extremal rotating black hole

We can easily extend this investigation of the zero mass BTZ black hole to study extremal rotating BTZ black holes, with $M = J$. Rotating BTZ black holes are orbifolds of AdS_3 with an asymmetric action on the worldsheet; as we will see below, the extremal rotating black holes correspond to an orbifold where the action on the left-movers is the same as for the zero mass BTZ black hole, while the action on the right movers is the same as for the non-zero mass black hole studied in [20]. Thus, by combining our previous results, it is easy to determine the spectrum in this case as well. The construction of the twisted sectors is based on an appropriate ansatz, for which we then explicitly check the mutual locality.

The extremal rotating BTZ black hole has the metric

$$ds^2 = k\left(-\frac{(r^2-r_+^2)^2}{r^2}d\tau^2 + \frac{r^2}{(r^2-r_+^2)^2}dr^2 + r^2(d\phi - \frac{r_+^2}{r^2}dt)^2\right), \qquad (5.5.56)$$

where ϕ is a periodic coordinate, $\phi \sim \phi + 2\pi$. The spacetime is locally AdS_3, with periodicity of ϕ corresponding to the action of an orbifold. If we write AdS_3 in the embedding coordinates x^0, x^1, x^2, x^3 in $\mathbb{R}^{2,2}$ as the hyperboloid $-x_0^2 - x_1^2 + x_2^2 + x_3^3 = -k^2$, the orbifold which gives us the above extremal rotating black hole is along the Killing vector [18]

$$\xi = r_+(J_{03} + J_{12}) + J_{01} - J_{02} - J_{13} + J_{23}, \qquad (5.5.57)$$

up to conjugation, where J_{ab} are the Lorentz transformations on $\mathbb{R}^{2,2}$, $J_{ab} = \eta_{bc}x^c\partial_a - \eta_{ac}x^c\partial_b$. In the coordinate system of (5.5.56), this Killing vector is $\xi = \partial_\phi$.

The extremal rotating BTZ metric is locally AdS_3, so it can be related to the Poincaré coordinate system (3.1.11) we used earlier. The coordinate transformation

$$z = R^{-1/2}e^{r_+(\phi-t)}, \quad t + x = e^{2r_+(\phi-t)}, \quad t - x = -(T+\phi), \qquad (5.5.58)$$

where

$$R = \frac{1}{2r_+}(r^2 - r_+^2), \quad T = t - \frac{r_+}{r^2 - r_+^2}, \qquad (5.5.59)$$

converts the metric (5.2.6) into the metric (5.5.56). This corresponds to writing the group element of $SL(2,\mathbb{R})$ as

$$g = \begin{pmatrix} R^{\frac{1}{2}}e^{-r_+(\phi-t)} & R^{\frac{1}{2}}e^{r_+(\phi-t)} \\ (T+\phi)R^{\frac{1}{2}}e^{-r_+(\phi-t)} & R^{\frac{1}{2}}e^{r_+(\phi-t)}\left(T+\phi+\frac{1}{R}\right) \end{pmatrix}. \qquad (5.5.60)$$

As in section 5.2.1, we can determine the conserved charge associated with ϕ translation. The action of ϕ translation on the group element is

$$\delta_{(\phi)} g = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} g + r_+ g \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{5.5.61}$$

so the conserved charge is

$$Q_\phi = -J_0^+ + r_+ \bar{J}_0^1. \tag{5.5.62}$$

Note that this naive expression will apply for the untwisted sectors; for the twisted sectors, there is the possibility of a total derivative term, which we need to determine.

Thus, we see that the action in terms of $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ is chiral, with the left moving part looking like that of a massless BTZ black hole we have studied above, while the right moving part looks like that of the massive BTZ black hole. It is therefore natural to choose the parafermionic representation for the right movers, and the Wakimoto representation introduced above for the left movers. The parafermionic representation for the right movers involves writing the currents as

$$\bar{J}^1 = -i\sqrt{\frac{k}{2}}\partial \bar{X}, \quad \bar{J}^\pm = \bar{\xi}^\pm e^{\pm\sqrt{\frac{2}{k}}\bar{X}}, \tag{5.5.63}$$

where $\bar{X}$ is a free boson, $\bar{X}(\bar{z})\bar{X}(\bar{w}) \sim -\ln(\bar{z}-\bar{w})$, and $\bar{\xi}^\pm$ are parafermions representing the remaining $\widehat{SL(2,\mathbb{R})}_k/\widehat{U(1)}$ algebra 3.2.2. Thus, the vertex operators in the untwisted sector are

$$V_{j\lambda\bar{\lambda}}(z) = e^{i\lambda\gamma - j\sqrt{\frac{2}{k'}}\phi}\bar{\Psi}_{j\bar{\lambda}} e^{-i\sqrt{\frac{2}{k}}\bar{\lambda}\bar{X}} \tag{5.5.64}$$

where $\bar{\Psi}_{j\bar{\lambda}}$ are parafermionic operators with conformal dimension $\bar{h}_{\bar{\Psi}} = -\frac{j(j-1)}{k'} - \frac{\bar{\lambda}^2}{k}$. We know that these untwisted sector vertex operators are mutually local for the operators corresponding to modes of fields on AdS. This gives us some information about the OPE of the parafermionic operators, as the OPE of two such vertex operators will only be mutually local if

$$m - \frac{2}{k'}jj' - \frac{2}{k}\bar{\lambda}\bar{\lambda}' \in \mathbb{Z} \tag{5.5.65}$$

where m characterises the leading singularity in the OPE of the parafermionic operators,

$$\bar{\Psi}_{j\bar{\lambda}}\bar{\Psi}'_{j'\bar{\lambda}'} \sim \frac{\mathcal{O}}{(z-w)^m}. \tag{5.5.66}$$

We construct an ansatz for the twisted sector states in this orbifold by combining the results for the twisted sectors from our earlier analysis of the massive and massless black holes: that is, we guess that the twisted sector vertex operators are simply the right moving part of the twisted sector state from the massive black hole, combined with the left moving part of the twisted sector state from the massless black hole. This gives

$$V_{j\lambda\bar{\lambda}n}(z) = \exp\left(in\phi_+ + i\lambda\phi_- - j\sqrt{\frac{2}{k'}}\phi\right)\bar{\Psi}_{j\bar{\lambda}}\exp\left(-i\sqrt{\frac{k}{2}}\left[\bar{\lambda} - \frac{k}{2}nr_+\right]\bar{X}\right) \tag{5.5.67}$$

The conformal dimensions for this operator are

$$h = -\frac{j(j-1)}{k'} - n\lambda, \quad \bar{h} = -\frac{j(j-1)}{k'} - \bar{\lambda}r_+n + \frac{kn^2r_+^2}{4}. \tag{5.5.68}$$

As in section 5.3.3, we should have a tachyon in the twisted sectors if we can satisfy the physical state condition for a mode with $j(j-1) < 0$ and $\lambda = \bar{\lambda} = 0$. This requires $\sqrt{k}r_+ < 2$, as in [20], so there is a tachyon if the size of the spatial circle at the black hole horizon is small enough. As in section 5.3.3, this tachyon will not be well localised in the region near the horizon.

Level matching will require that $h - \bar{h} \in \mathbb{Z}$, which implies

$$-n\lambda + r_+n\bar{\lambda} - \frac{kn^2r_+^2}{4} \in \mathbb{Z}. \tag{5.5.69}$$

We would like to see this arise as a consequence of the quantisation of angular momentum imposed by the orbifold. Naively, the generator of translation in ϕ is (5.5.62), which would imply a quantisation condition $-\lambda + r_+(\bar{\lambda} - knr_+/2) \in \mathbb{Z}$, which does not agree with (5.5.69). Hence, as in [27], we will need to include a total derivative term in the definition of Q_ϕ, so that in twisted sectors

$$Q_\phi = -J_0^+ + r_+\bar{J}_0^1 + \frac{knr_+^2}{4}, \tag{5.5.70}$$

implying a quantization condition

$$-\lambda + r_+\bar{\lambda} - \frac{knr_+^2}{4} \in \mathbb{Z}, \tag{5.5.71}$$

consistent with (5.5.69). Note that unlike in non-rotating cases, the quantization condition on $\lambda, \bar{\lambda}$ depends on the twist n, so the twisted sector states cannot be obtained by considering the OPE of untwisted sector states with an appropriate twist operator. Such a procedure would get the quantization condition wrong.

We then need to verify mutual locality of these twisted sector states. To determine mutual locality consider an OPE between $V_{j\lambda\bar{\lambda}n}(z)$ and $V_{j'\lambda'\bar{\lambda}'n'}(w)$. It turns out that the relevant condition for mutual locality is

$$m - \frac{2}{k'}jj' - n\lambda' - n\lambda - \frac{2}{k}\left(\bar{\lambda} - \frac{k}{2}nr_+\right)\left(\bar{\lambda}' - \frac{k}{2}n'r_+\right) \in \mathbb{Z}. \tag{5.5.72}$$

This is satisfied as a consequence of (5.5.69) and (5.5.65). Therefore the correct spectrum for the twisted sector is indeed given by (5.5.67); this completes the untwisted sector spectrum to obtain a mutually local set of operators with the appropriate set of twisted sectors, indexed by a single integer denoting the twist. It would be interesting to see if the modular invariance of the resulting partition function could be explicitly verified, but this will be complicated to check, as is generally the case for asymmetric orbifolds, so we will not attempt to do so explicitly here.

5.6 Supersymmetry

In this section, we will discuss the extension of our analysis of the spectrum to the superstring. This analysis is particularly interesting for the $M = J$ extremal rotating black holes considered here, as these are supersymmetric backgrounds for the superstring. Also, from the point of view of considering the tachyons, the tachyons in twisted sectors are most interesting in the superstring, where we expect the GSO projection to eliminate the tachyon in the untwisted sector. For the supersymmetric choice of spin structure on spacetime, spacetime supersymmetry implies that there are no tachyons, but if we make the opposite choice of spin structure, we expect the tachyons in odd twisted sectors to survive the GSO projection.

Unfortunately, we are not able to construct the spectrum for the superstring explicitly. We have not succeeded in extending the nice representation of $SL(2,\mathbb{R})$ in terms of Wakimoto fields to the superstring, so we have not succeeded even in constructing an explicit representation for the untwisted sector modes which survive the orbifold projection condition. In this section, we will describe the problem, focusing on the case of the $M = 0$ black hole for simplicity.

We consider Type II String theory on BTZ $\times\mathbf{S^3}\times\mathbf{T^4}$ as described by a $\widehat{SL(2,\mathbb{R})}$ super-WZW model at level k. The super-current algebra is

$$J^a = j^a - \frac{i}{k}\epsilon^a{}_{bc}\psi^b\psi^c \tag{5.6.73}$$

with ψ^a having the usual OPE structure for a fermion,

$$\psi^a(z)\psi^b(w) \sim \frac{k}{2}\frac{\eta^{ab}}{(z-w)} \tag{5.6.74}$$

$$j^a(z)\psi^b(w) \sim 0 \tag{5.6.75}$$

The OPEs for the bosonic currents j^a are almost identical to the previous section, while the OPEs for the supercurrent J^a are very similar,

$$j^a(z)j^b(w) \sim \frac{\tilde{k}}{2}\frac{\eta^{ab}}{(z-w)^2} + i\frac{\epsilon^{ab}{}_c j^c}{(z-w)} \tag{5.6.76}$$

$$J^a(z)J^b(w) \sim \frac{k}{2}\frac{\eta^{ab}}{(z-w)^2} + i\frac{\epsilon^{ab}{}_c j^c}{(z-w)} \tag{5.6.77}$$

where $\tilde{k} = k+2$. The world-sheet $\mathcal{N}=1$ super-current is

$$G(z) = \frac{2}{k}\left(g_{ab}\psi^a j^b - \frac{i}{3k}\epsilon_{abc}\psi^a\psi^b\psi^c\right). \tag{5.6.78}$$

For the superstring, the generator of x translations is again (5.2.13), but where J^a are now the total currents given in (5.6.73). We therefore want to find a representation of the total currents which simplifies the action of J^+, and use this representation to write the untwisted sector vertex operator in a form which diagonalises the action of J^+. We can use a Wakimoto representation as before for the bosonic currents j^a, but the fermionic contribution to J^a is more problematic.

In [20,31], the fermions were rewritten in terms of a set of bosons H_I, $I = 1,\ldots,5$, with OPEs

$$H_I(z)H_J(w) = -\delta_{IJ}\ln(z-w), \tag{5.6.79}$$

such that the spin fields

$$S_\alpha = e^{\frac{i}{2}\epsilon_I H_I} \tag{5.6.80}$$

with $\epsilon_I = \pm 1$ diagonalise the action of J^2 and hence Q_ϕ. In our case, it is J^+ which is relevant, and this involves a combination $\psi_1(\psi_3+\psi_2) \equiv \psi_1\psi_+$. We could formally define a field H_* by

$$\partial H_* = \psi_1\psi_+, \tag{5.6.81}$$

but this will not have the OPE of a free boson, so we cannot use it in constructing the spin fields in a way analogous to [20,31]. As a result, we have no simple route to constructing an appropriate basis of spin fields in this case which diagonalises the action of Q_x. We leave this as an open problem for future work. It is clearly very interesting to try to understand these simple examples of BPS black holes from the worldsheet perspective, so we hope further progress will be possible.

5.7 Discussion

The main result of this chapter is that we have obtained the full spectrum and discussed the tachyons appearing in this spectrum for the bosonic string on the $M = 0$ BTZ black hole and on the extremal rotating $M = J$ black hole. The spectrum on the zero mass black hole is just the limit of the spectrum obtained for the massive black hole in [20], as we would expect. However, because the zero mass black hole corresponds to a parabolic orbifold of AdS_3, the description of the states in this case is quite different from that of [20]. The use of the Wakimoto representation in the $M = 0$ BTZ black hole enables us to give a fully explicit description of the vertex operators for states in both untwisted and twisted sectors. This makes this example a particularly interesting laboratory for further explorations of worldsheet string theory on these black hole backgrounds; compared to the parafermionic representation of the vertex operators employed for the massive black hole in [20], this more explicit representation ought to give us greater control. Unfortunately, however, this description of the vertex operators appears to degenerate in the flat space limit, so our understanding of the limit where we zoom in on the region near the singularity is not significantly improved by the use of this representation.

For the zero mass black hole, we argued that the twisted sector modes obtained from the bulk tachyon in the untwisted sector are also tachyonic. The analysis of these tachyons will closely parallel the corresponding analysis in [20], so we have not given much detail in our discussion of the tachyons. These modes are not localised in the region near the horizon, as the coupling to the NSNS 2-form field makes a negative contribution to the energy of the string, allowing it to propagate to large distances. The study of the condensation of these tachyons would be an interesting direction for further work, but because the tachyon is not localised, it may be quite challenging.

We extended our work on the zero mass black hole by considering the extremal rotating black hole, which corresponds to an asymmetric orbifold, with the action on left movers the same as for the zero mass black hole and the action on right movers the same as for the massive non-rotating black hole. We were therefore able to construct a proposal for the spectrum of strings on this background by combining our work on the zero mass black hole with previous work on the massive black hole.

Finally, we considered the extension to the superstring. For the elliptic or hyperbolic orbifolds, it was possible to extend the orbifold construction to the superstring by choosing an appropriate set of spin fields which were eigenfunctions of the momentum along the compact direction, allowing us to construct superstring vertex operators which satisfy the appropriate quantisation condition. We were unable to find a corresponding basis for the parabolic orbifold which gives the $M = 0$ BTZ black hole; as a result, we cannot construct superstring vertex operators which are well-defined on the orbifold spacetime. This technical problem appears to be the most important direction for further work: obviously, the main motivation for interest in the $M = 0$ and $M = J$ black holes is that they are supersymmetric solutions in appropriate supergravity theories [60]. Also, the study of the tachyons in twisted sectors is mainly interesting in the context of the superstring, where we expect to be able to choose a GSO projection which will project out the untwisted sector tachyon but keep the tachyonic modes in odd twisted sectors. Any further progress on these directions will require an appropriate construction of vertex operators for the superstring.

Chapter 6

The self-dual orbifold

6.1 Introduction

There has recently been interest in the holographic relation between spacetime geometries with AdS_2 factors and their dual field theories. One example is the proposed Kerr/CFT correspondence [22], which attempts to extend holography to provide a description of the near-horizon region of uncharged extreme black holes. Another comes from studies of field theories at finite charge density in the AdS/CFT correspondence, which involve a particular Reissner-Nordström AdS black hole. This black hole has a near-horizon $AdS_2 \times \mathbb{R}^n$ geometry in the low-temperature limit, which controls the long-distance transport properties of the field theory [23]. String theory in AdS_2 arises in the near-horizon limits of a wide variety of four and five dimensional black holes in both asymptotically flat (e.g., [61]) and asymptotically AdS space (e.g., [62]).

The self-dual orbifold of AdS_3 [21] (see section 3.3) is a simple example of a geometry with an AdS_2 factor. This spacetime is a circle fibration over AdS_2, with an $SL(2,\mathbb{R}) \times U(1)$ isometry group, and can be viewed as arising either as a quotient of AdS_3, or as the near-horizon limit of a BTZ black hole [35, 37]. We can use these descriptions to understand the dual field theory description in detail. It has two asymptotic boundaries; from the quotient point of view, this is because the quotient has fixed points on the conformal boundary of AdS_3, and excising these fixed points divides the boundary into two disconnected regions. The conformal geometry on these boundaries is a null cylinder; that is, a flat two-dimensional spacetime with a null direction compactified. Working in coordinates which only covered one boundary, it was argued in [35,37] (see section 6.2) that the dual field theory is the Discrete Lightcone Quantization (DLCQ) of the original two-dimensional field theory dual to AdS_3, with the chiral sector which survives DLCQ in a thermal state (see also [61]). This raises a question: *Are there any other states of the DLCQ theory that have a dual description as classical spacetimes that are asymptotic to the self-dual orbifold?*

In [37], the geometry was considered in the analogue of Poincaré coordinates, which only see one boundary of the spacetime, but the global self-dual orbifold has two disjoint boundaries [35]. This raises a second long-standing question in the dual CFT description of asymptotically AdS_2 spacetimes: *Are they dual to a single CFT, or to two copies of the CFT living on the two boundaries?*

In Sec. 6.3 we will argue that the self-dual orbifold should be thought of as dual to two copies of the CFT. We will first give a general argument based on the bulk diffeomorphism symmetries. We will then consider the orbifold global coordinates as coordinates on AdS_3 (without considering the quotient); here it is clear that there are independent CFT degrees of freedom on the two boundaries. Finally, we will consider the self-dual orbifold in the "black hole" coordinate system which was obtained by considering a near-horizon, near-extremal limit of the non-extremal BTZ black hole in section 3.3.1. These coordinates cover regions of both boundaries. Since the non-extremal BTZ black hole is described by an entangled state in two copies of a CFT, in the near-horizon infrared limit we still have a pair of entangled CFTs (a similar argument was previously given in [45]). However, the infrared limit restricts both theories to a chiral sector, and these sectors are entangled. This is consistent with the picture of [37]; tracing over one boundary will give us a thermal state in a chiral CFT. We can also see the proposed entanglement by thinking of the orbifold global coordinates as coordinates on AdS_3: these cover the conformal boundary of AdS_3 in two patches, and rewriting the vacuum state of a CFT on the conformal boundary in these coordinates gives rise to entanglement of the right-moving degrees of freedom. This interpretation is also consistent with the general picture that the connectivity of regions of the boundary through the bulk is dual to entanglement between these regions in the CFT [63].

Extrapolating from this example suggests that the dual of any spacetime with an AdS_2 factor is a $1+1$ CFT with one chiral sector in its ground state, and the other in an entangled state with non-zero entropy. The entanglement plays a crucial role in explaining the spacetime structure. This is quite different from the dual description of higher-dimensional AdS spaces, which correspond to ground states of a dual CFT, with no entropy.

The global description of AdS_2 also leads to a puzzle – the two boundaries are *causally* connected, implying non-zero commutators for operators in the two copies of the CFT (Sec. 6.3.4)[1]. Entanglement cannot reproduce these commutators: if the two copies of the CFT are independent, the operators should commute. Within the patches of AdS_2 that are recovered by the near-horizon limit of higher dimensional black holes, the puzzle is avoided because there is no causal connection between the regions of the two boundaries covered by the "black hole" coordinate systems. The puzzle would be resolved if we were restricted to only consider correlation functions between spacelike separated operators on the two boundaries of AdS_2, perhaps in view of the AdS_2 instability of [38]. Understanding whether such restrictions can be implemented is an important goal for future work.

The other main aim of this chapter is to construct new examples of asymptotically self-dual orbifold spacetimes (Sec. 6.4) corresponding to different states of the dual field theory. One motivation is to identify geometries dual to particular pure states that contribute to the entropy of the spacetime as in the black hole microstates program (see [65] for a review). Another motivation is that in the Kerr-CFT correspondence, it has been shown that there are no non-trivial asymptotically near-horizon extremal Kerr geometries [66, 67]; it would be interesting to know if this is a general feature of spacetimes with AdS_2 factors.

We first show that there is a quotient of AdS_3 which is the natural dual to the ground state for a single copy of a CFT on a null cylinder, and which can be obtained

[1]It is interesting to note that this problem is special to the two-dimensional case; in higher dimensions, [64] shows that generically spacetimes cannot have disconnected boundaries that are causally connected through the bulk.

as a near-horizon limit of the $M = 0$ BTZ black hole.[2] This is a quotient along a null direction in the bulk, so the geometry has closed null curves, and the duality may only be a formal correspondence in this case. This is similar to the Schrödinger spacetime [69, 70], where this issue was pointed out in [71]. We then construct a rich class of examples of asymptotically self-dual orbifold spacetimes by applying a solution-generating transformation [72]. These geometries could be interpreted in the dual CFT as more restricted thermal ensembles where the local charge density is prescribed and not just the total charge. However, they are mildly singular in the bulk on the boundary of the region covered by the orbifold Poincaré coordinates. It is worth noting that this solution-generating approach relies on the existence of a globally null Killing vector, which appears in the self-dual orbifold but not in the higher-dimensional solutions with AdS_2 factors such as near-horizon extremal Kerr or the near-horizon limits of extreme Reissner-Nordström AdS black holes.

6.2 Chiral CFT

The geometry of the self-dual orbifold was discussed in various coordinate systems in section (3.3). Here we use the near horizon limit coordinates to show that the dual CFT is chiral.

The identifications (3.3.55) describe a circle of proper size $2\pi r_+ e^{r_0}$ viewed in a boosted frame. So taking this limit corresponds to a DLCQ limit in the CFT on the (u, v) cylinder, where we take the size of the circle to zero and the boost to infinity to recover a null identification, giving the CFT on a null cylinder. In general, this DLCQ limit will restrict us to the ground state for the right-moving excitations, as the energy of right-moving excitations is infinitely blueshifted by the boost: $E_v = E_{\tilde{v}} e^{-2r_0}$. We might think that we could take $E_{\tilde{v}} \to 0$ at the same time to recover a finite energy in the boosted frame, but since the theory lives in finite volume, there is a finite density of states, and the energy spectrum is quantised. The spacing is at least $\Delta E_{\tilde{v}} \sim e^{-S} \sim e^{-c}$, where c is the central charge. Although this discreteness in the spectrum cannot be seen from the classical spacetime point

[2]This material, which appears in Sec. 4.2, has overlap with [68].

of view, it will ensure that we are ultimately left with only the ground states for the right-movers, with $L_0 = \frac{c}{24}$. The dual CFT is thus a chiral theory, with only left-moving excitations.

The left-moving excitations are unaffected by this DLCQ procedure, so we would expect them to be in the same state as before we took the near-horizon limit. The dual of the extreme BTZ black hole is a thermal state for the left-movers, at temperature $T_L = r_+/2\pi$. Since we have included a factor of r_+ in our definition of the identifications, the temperature with respect to our coordinate u is in fact $T = 1/2\pi$.

Thus, the proposal of [37] was that the dual description was a chiral CFT, with the left-movers in a thermal state at temperature $T = 1/2\pi$. In section 6.3, we will refine this proposal by considering the description of the self-dual orbifold including both boundaries, and propose that the dual is an entangled state which reduces to this thermal state on tracing over the CFT degrees of freedom on one of the boundaries.

6.3 Holographic dual of the global spacetime

The self-dual orbifold has two boundaries. We would like to understand whether this should be interpreted as the dual of a pair of field theories on these two boundaries, or whether there should be a single CFT dual to the spacetime, as has been proposed for AdS_2 in [61]. We will argue from several points of view that the correct interpretation seems to be as an entangled state in two disjoint copies of the CFT, living on the two boundaries of the spacetime. We will see that a challenge to this interpretation arises because the two boundaries of the self-dual orbifold are causally connected, suggesting that there is an interaction between them.

6.3.1 Diffeomorphisms and Hamiltonians

There is a simple general argument using the bulk diffeomorphism symmetry which implies that the dual description should indeed involve two copies of the field theory. Consider a spacetime with two conformal boundaries. If we consider a bulk surface of constant t, the bulk diffeomorphism freedom allows us to shift the surface arbitrarily. Changes in the constant t surface which do not affect its intersection with the boundary are pure gauge, and are not seen from the boundary field theory point of view. But diffeomorphisms which shift the intersection with the boundary correspond to the action of a boundary Hamiltonian from the field theory point of view [73, 74]. Since the bulk diffeomorphism symmetry includes transformations which independently deform the intersection of the bulk constant t surface with the two boundaries, there are two such Hamiltonians, acting separately on the two boundaries. These generate a $\mathbb{R} \times \mathbb{R}$ symmetry of the theory, corresponding to arbitrary time translations in the two CFTs.[3] We take the presence of these two independent time-translation generators to imply that there are two separate physical systems on the two boundaries. If the bulk geometry corresponded in the dual theory to a state in a single field theory, we would expect to have just one Hamiltonian, not two. This argument is quite general and would apply whenever the spacetime has two boundaries.

In the self-dual orbifold, the bulk spacetime has a time-translation symmetry that corresponds to the diagonal $\mathbb{R}$ subgroup of the $\mathbb{R} \times \mathbb{R}$ symmetry of the theory. This implies that the dual CFT description should be in terms of a state in the two CFTs which preserves this diagonal $\mathbb{R}$ subgroup. This state should involve some non-trivial entanglement between the two theories, as the connectedness of the spacetime in the bulk implies that there will be non-zero correlation functions $\langle \mathcal{O}_1 \mathcal{O}_2 \rangle$ between operators on the two boundaries. This relation between entanglement and connections in the bulk spacetime was investigated in general in [63]. We will work out the form of the entanglement in the case of the self-dual orbifold a little further on.

[3] Actually, since the bulk surface meets the boundary in a one-dimensional surface in the self-dual orbifold, there is an infinite-dimensional group of translations of this surface in each boundary, but let us restrict for simplicity to the overall translational subgroup.

6.3.2 Quotient perspective

To understand the holographic description in more detail, we consider the spacetime from the quotient perspective. As a first step, we can consider the metric (3.3.30) with ϕ non-compact as simply a new choice of coordinates on AdS_3, where we understand holography well.

From this perspective, it is clear that there are independent CFT degrees of freedom in the two boundaries, as these correspond to different regions in the single boundary of AdS_3. The two Hamiltonians found above correspond to translations of the two segments of the $t = 0$ surface in figure 3.3.

If we take a linear quantum theory on the boundary as a toy model for the CFT on the cylinder, we can use the coordinate transformations given above to work out the description of its vacuum state in terms of the self-dual orbifold coordinates.[4] It is most straightforward to do this using the transformation (3.3.53) between the (x^+, x^-) Poincaré coordinates on AdS_3 and the (u, v) coordinates. There is a non-trivial coordinate transformation between x^+ and u, so the ground state for left-movers with respect to x^+ will map to an entangled state where the modes on one boundary are entangled with the corresponding modes on the other. Tracing over one boundary will then leave us with a thermal state for the left-movers on the other boundary (see section 4.1.1).

If we consider a massless scalar field on the boundary, the positive frequency left and right-moving mode solutions in (u, v) coordinates are, up to normalisation,

$$p^1_{\omega,l} = e^{-i\omega u}, \quad p^1_{\omega,r} = e^{-i\omega v}, \tag{6.3.1}$$

$$p^2_{\omega,l} = e^{i\omega u}, \quad p^2_{\omega,r} = e^{-i\omega v}, \tag{6.3.2}$$

where $1, 2$ denote the two (u, v) coordinate patches, each of which covers half of the Poincaré patch. In global coordinates, these two patches lie on the boundaries at $z = \pm\infty$ respectively. The subscripts l, r denote left and right-moving modes. We can construct modes which are purely positive frequency with respect to Poincaré

[4]This perspective was used to identify the state dual to the non-extremal BTZ black hole in [75]; this was extended to the extremal black hole in [76].

coordinates by rewriting these solutions in terms of x^+, x^- and analytically continuing in the lower-half complex $x^\pm$ plane, following [77]. This is a standard exercise; the only difference here is that the relation is only non-trivial for the left-movers. Using the coordinate transformation $x^+ = \frac{1}{2}e^{-2u}$, we see that solutions which are pure positive frequency with respect to x^+ are

$$W^1_{\omega,l} = p^1_{\omega,l} + e^{-\frac{\pi\omega}{2}} \bar{p}^2_{\omega,l}, \tag{6.3.3}$$

and similarly analytically continuing the solution $p^2_{\omega,l}$ will give

$$W^2_{\omega,l} = p^2_{\omega,l} + e^{-\frac{\pi\omega}{2}} \bar{p}^1_{\omega,l}. \tag{6.3.4}$$

These are purely positive frequency modes with respect to x^+, so the corresponding annihilation operators $a^1_{\omega,l}, a^2_{\omega,l}$ will annihilate the ground state. Using the above expressions, we can write these annihilation operators in terms of the annihilation operators $b^1_{\omega,l}, b^2_{\omega,l}$ for the modes $p^1_{\omega,l}$, $p^2_{\omega,l}$ as

$$a^1_{\omega,l} = b^1_{\omega,l} - e^{-\frac{\pi\omega}{2}} b^{2\dagger}_{\omega,l}, \tag{6.3.5}$$

$$a^2_{\omega,l} = b^2_{\omega,l} - e^{-\frac{\pi\omega}{2}} b^{1\dagger}_{\omega,l}. \tag{6.3.6}$$

This indicates that the vacuum $|0\rangle$ in Poincaré coordinates can be formally written as an entangled state in the Hilbert space built on the vacua $|0\rangle_1$, $|0\rangle_2$ annihilated by the $b^1_{\omega,l}, b^2_{\omega,l}$:

$$|0\rangle = e^{-i\int_0^\infty d\omega e^{-\pi\omega} b^{1\dagger}_{\omega,l} b^{2\dagger}_{\omega,l}} |0\rangle_1 \otimes |0\rangle_2. \tag{6.3.7}$$

This demonstrates that from the point of view of self-dual orbifold coordinates, the vacuum on the boundary of AdS_3 is an entangled state where the left-movers on the two boundaries of the self-dual orbifold are entangled with each other. If we trace over one boundary, we would have a thermal state for the left-movers on the other boundary, at a temperature $1/2\pi$. This is consistent with the analysis of [37] because we have hidden the scale in u; the quotient identification is $u \sim u + 2\pi r_+$.

We should next consider the effect of the quotient. The quotient will have three effects; first, the points at $\phi = \pm\infty$, corresponding to $\tau + \theta = 0, \pi$, are fixed points of the quotient. We must therefore excise them from the boundary manifold, turning it into two genuinely disconnected surfaces. Secondly, if we write the metric on one of these surfaces in terms of the (u, v) coordinates, the effect of the quotient is then to make u periodic. This will restrict the momentum in the u direction to discrete values, leaving us with the same entangled state, but with the integral in (6.3.7) replaced by a sum. This reduces the $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ symmetry of the state to $SL(2,\mathbb{R}) \times U(1)$.

Thirdly, the quotient will project out the right-movers. This cannot be seen directly by imposing the quotient on the boundary, as the left movers have no u dependence. However, we can think of this null identification as a limit of a spacelike identification, by thinking of the spacetime as cutoff at some finite r, and taking the limit $r \to \infty$ will give us an infinite boost. Thus, in this quotient perspective we can also see the DLCQ of the field theory that was seen in the near-horizon limit in [37]. The infinite boost as we take the limit $r \to \infty$ sets the right-movers to the ground state. For a simple scalar field model, this is easy to see: the metric on a surface at $r = r_0$ is given by

$$ds^2 = du^2 + 2e^{2r_0} du dv. \tag{6.3.8}$$

Thus the null coordinates at finite r_0 are $\tilde{u} = u$, $\tilde{v} = v - 2e^{-2r_0}u$. Right moving modes are $e^{i\omega_R \tilde{v}}$, whilst left moving modes are $e^{i\omega_L \tilde{u}}$. Looking at the right movers, the modes which survive the quotient are those with $\omega_R = ne^{2r_0}$ for $n \in \mathbb{Z}$. Therefore as the cutoff is removed, $r_0 \to \infty$ and the only right mover remaining is the ground state, $n = 0$. For the actual CFT on the boundary, the picture is a little more subtle; the theory lives in finite volume, so the energy spectrum is quantised, but as was observed in our review of the DLCQ argument of [37], the level spacing is $\Delta L_0 \sim e^{-c}$. This discreteness cannot be seen from the spacetime point of view. Nonetheless, as the boost is taken to infinity, we need to take $L_0 - \frac{c}{24} \to 0$, and the energy will ultimately become smaller than this gap, forcing us to set $L_0 = \frac{c}{24}$.

This discussion considered a simple toy model of a single scalar field, but the key point is just the non-trivial transformation between x^+ and u, so the $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ invariant vacuum state for the full boundary CFT will similarly be an entangled state for the left-movers on the two regions $z = \pm\infty$. The fact that the CFT is in a non-trivial excited state can be seen directly from the fact that the boundary stress tensor takes a non-vanishing value. This can be calculated from the bulk metric (3.3.47) using the usual holographic dictionary [78, 79]. The boundary stress tensor is

$$T_{\alpha\beta} = \frac{c}{6}[\pi_{\alpha\beta} + h_{\alpha\beta}] = \frac{c}{6}[K_{\alpha\beta} - (K-1)h_{\alpha\beta}], \tag{6.3.9}$$

where $K_{\alpha\beta}$ is the extrinsic curvature, $h_{\alpha\beta}$ is the induced metric on the boundary, and $c = 3/2G_3$ is the central charge of the boundary CFT. The metric (3.3.47) gives $K_{uv} = e^{2r}$, so $K = 2$, and the only non-vanishing component of $T_{\alpha\beta}$ is

$$T_{uu} = -\frac{c}{6}. \tag{6.3.10}$$

Note that this is a momentum density in coordinates where the u direction is periodic with period $2\pi r_+$; the total left-moving momentum on the boundary is hence $c\pi r_+/3$.

This stress tensor could also be obtained by considering the Schwarzian derivative associated with the coordinate transformation (3.3.53) between the Poincaré coordinates x^+, x^- and the u, v coordinates. In terms of holomorphic coordinates $w, \bar{w}$, the Schwarzian derivative gives in general

$$T(w') = T(w)(\partial_{w'}w)^2 + \frac{c}{12}\left[\frac{\partial^3_{w'}w}{\partial_{w'}w} - \frac{3}{2}\left(\frac{\partial^2_{w'}w}{\partial_{w'}w}\right)^2\right], \tag{6.3.11}$$

and similarly for $\bar{T}(\bar{w})$. Since the stress tensor in Poincaré coordinates vanishes in the global vacuum state on the boundary cylinder, and the coordinate transformation for the left-moving coordinates is trivial, this implies that $T_{vv} = 0$, and

$$T_{uu} = \frac{c}{12}\left[\frac{\partial^3_u x^+}{\partial_u x^+} - \frac{3}{2}\left(\frac{\partial^2_u x^+}{\partial_u x^+}\right)^2\right] = \frac{c}{12}[4 - 6] = -\frac{c}{6}, \tag{6.3.12}$$

reproducing the direct result. Thus, from the quotient point of view, the boundary stress tensor is accounted for by the non-trivial conformal transformation from the Poincaré coordinates to the self-dual orbifold coordinates (u, v), and is associated with the fact that the field theory is in a non-trivial entangled state on the two boundaries.

Thus, the self-dual orbifold is identified with a non-trivial entangled state of two copies of the CFT, living on the two boundaries of the spacetime. The fact that the dual description of a spacetime with an AdS_2 factor involves an excited state is quite different from the description of higher-dimensional AdS spacetimes, which are usually dual to the vacuum state in the dual CFT. This description of AdS_2 is more analogous to the description of black holes in higher-dimensional AdS spacetimes.

However, we would argue that the description of AdS_2 will always be qualitatively similar to this. There should be some entanglement to account for the connectivity between the two boundaries in the AdS_2 spacetime [63], and geometries with an AdS_2 factor are usually obtained as the near-horizon limit of black holes with a non-zero entropy, which is reproduced by the entropy of the mixed state obtained by tracing over one of the boundaries.

6.3.3 Near-horizon near-extremal limit

We can also argue for this description of the self-dual orbifold by taking the non-extremal BTZ black hole and considering the near-horizon, near-extremal limit introduced in section 3.3.1. The dual description of the non-extremal BTZ black hole is as a saddle-point corresponding to an entangled state of two copies of the CFT on the two boundaries of the maximal analytic extension of the black hole,[5] with the left- and right-movers at temperatures

$$T_L = \frac{(r_+ + r_-)}{4\pi}, \quad T_R = \frac{(r_+ - r_-)}{4\pi}. \tag{6.3.13}$$

Taking the near-horizon, near-extremal limit of the geometry is a DLCQ limit from the point of view of the field theory. This DLCQ limit does not affect the structure of the field theory, so it will give us two copies of the CFT in an entangled state on the regions of the boundary of the self-dual orbifold covered by the black hole coordinates (3.3.63). This confirms the entanglement description of the self-dual orbifold geometry.

However, there is a small subtlety in the nature of the entangled state in these coordinates. Naively one would say that as we take the extremal limit, $T_R \to 0$, but T_L remains finite, reproducing the entangled state we saw above in (u, v) coordinates. However, T_R is the temperature with respect to the null coordinate

[5]This description was obtained by a similar quotient argument in [75] and by considering the analytic continuation from the Euclidean black hole in [80].

x^- in the boundary of the black hole spacetime; this is related to the right-moving coordinate $\tilde{t}$ in the near-horizon region by an infinite boost. Taking this boost into account, the temperature with respect to $\tilde{t}$ is

$$T_{\tilde{t}} = \frac{T_R}{r_+ \epsilon} = \frac{1}{2\pi}, \tag{6.3.14}$$

while the temperature with respect to $\tilde{\phi}$ is $T_{\tilde{\phi}} = 1/2\pi$, as it was in the analysis above from the quotient point of view. Thus, when we consider the spacetime in black hole coordinates, it appears to be dual to an entangled state where both the left and right-movers are at finite temperature. This can also be seen by considering the boundary stress tensor in these coordinates, which is given by

$$T_{\tilde{\phi}\tilde{\phi}} = -\frac{c}{6}, \quad T_{\tilde{t}\tilde{t}} = -\frac{c}{6}. \tag{6.3.15}$$

This appears to be inconsistent with the statement that $L_0 = \frac{c}{24}$, which should follow from the DLCQ here as it did in [37]. However, the two statements are in fact perfectly consistent. In general, for the CFT on a spacelike circle, the translation generators are

$$L_0 - \frac{c}{24} = \oint T_{z\mu} n^\mu, \quad \bar{L}_0 - \frac{c}{24} = \oint T_{\bar{z}\mu} n^\mu, \tag{6.3.16}$$

where the integral is around the spacelike circle, and n^μ is the unit normal to this circle in the boundary metric. Since $T_{z\bar{z}} = 0$ for a spacelike circle, this reduces to

$$L_0 - \frac{c}{24} = \oint T_{zz} n^z, \quad \bar{L}_0 - \frac{c}{24} = \oint T_{\bar{z}\bar{z}} n^{\bar{z}}, \tag{6.3.17}$$

but in the limit as the circle becomes null, $n^z = 0$. Thus, for the CFT on the null circle, $L_0 - \frac{c}{24} = 0$, whether or not the right-moving component of the stress tensor vanishes. A finite right-moving energy density translates to a vanishing right-moving energy in the limit because the proper size of the compact direction is going to zero. So for the near-horizon, near-extremal limit of the non-extreme black hole, we get the CFT on a null cylinder in a state with entanglement for both the left and right movers, but this still has $L_0 = \frac{c}{24}$.

This entangled state is not a different candidate description of the self-dual orbifold; the state we have obtained in the near-horizon limit is in fact the same as the state we obtained above from the quotient perspective, just described in a different conformal frame on the boundary. The entanglement of the right-movers comes from the further coordinate transformation between $\tilde{t}$ and v coordinates (3.3.70). In particular, the non-zero stress tensor $T_{\tilde{t}\tilde{t}}$ in (6.3.15) can be seen to arise from applying the Schwarzian derivative formula (6.3.11) to the conformal transformation between $\tilde{t}$ and v coordinates. Thus, we obtain a consistent picture of the self-dual orbifold as dual to a particular entangled state in two copies of the CFT on the two boundaries.

6.3.4 Causal connection

We have obtained a description of the self-dual orbifold in terms of an entangled state in two copies of the CFT from two independent points of view. This description is also consistent with the description obtained in [37] by considering the near-horizon limit of the extremal BTZ black hole. However, there is a problem with this description, as it fails to account for the *causal* connections between the two boundaries, which would appear to imply direct interactions between the theories living on them.

To see that the boundaries at $z = \pm\infty$ are causally connected in the bulk, consider the metric (3.3.30). We see that the conserved quantity from ϕ-translation invariance is $L = \dot{\phi} + \sinh 2z\dot{t}$, and the minimum elapsed t is along curves of $L = 0$, for which along null geodesics

$$\Delta t = \int_{-\infty}^{\infty} \frac{dz}{\cosh 2z} = \frac{\pi}{2}. \tag{6.3.18}$$

If we think about (3.3.30) as a coordinate system on AdS_3, there is no mystery about this causal connection: it corresponds to causal connections in the boundary. The part of the strip at $z = -\infty$ with $t > \pi/2$ is in the causal future of the surface at $z = \infty$, $t = 0$ in the boundary geometry. The two parts of a surface of constant t were offset in $\tau - \theta$ by π, so when $\Delta t > \pi/2$, this offset is overcome and the surfaces at $z = \pm\infty$ are connected by causal curves in the boundary. From this AdS_3 point of view, this is not a surprise; it is well-known that in AdS_d, points in the boundary which are causally connected in the bulk are also causally connected in the boundary: the bulk and boundary light cones agree for pure AdS geometries.

However, when we take the quotient, we must first delete from the conformal boundary the points at $\tau+\theta = 0, \pi$, which are fixed points of the identification acting on the conformal boundary. This breaks the causal connection between the two strips on the boundary at $z = \pm\infty$. This breaking of the explicit causal connection does not immediately cause problems, as the connection could be retained by a boundary condition linking the behaviour of fields at $\tau + \theta = -\epsilon$ to the behaviour at $\tau + \theta = +\epsilon$. However, when we make the identification, we replace such a boundary condition with a periodic boundary condition in the ϕ direction, and there is no longer any connection between the behaviour of boundary fields on the two strips. So in the quotient space, causal connection in the bulk is not reproduced by causal connection in the boundary.

This is a problem because an AdS/CFT calculation with causal connection in the bulk would usually predict a non-zero value for the commutator of operators on the two boundaries. If $\mathcal{O}_1$ is a scalar operator on the boundary at $z = \infty$ and $\mathcal{O}_2$ is an insertion of the same scalar operator on the boundary at $z = -\infty$,

$$\langle[\mathcal{O}_1(0), \mathcal{O}_2(\Delta t)]\rangle = \Delta^{\phi}_{bulk}(\Delta t) \neq 0 \quad \text{for } \Delta t > \pi/2, \tag{6.3.19}$$

where Δ^{ϕ}_{bulk} is the half advanced minus half retarded propagator for the corresponding bulk field ϕ.

This non-trivial commutator between operators on the two boundaries cannot arise simply from entanglement between the quantum states of the theories on the two boundaries, as the expectation value of the commutator is independent of the state that the expectation value is evaluated in. Thus, this seems to require some explicit interaction between the two boundary theories. The bulk prediction (6.3.19) is not consistent with our proposed description of the self-dual orbifold in terms of two independent, but entangled, boundary theories.

Indeed, it is very hard to see how the CFT description could be modified to produce such interactions. From the quotient perspective, we would expect local operators on the two strips to be simply independent once we delete the fixed points. From the near-horizon point of view it is even harder to see how some interactions could arise from taking the DLCQ limit; the two CFTs should still simply be entangled. Specifically, in a black hole coordinate system, the portions of the boundaries that are captured are not in causal contact. So when we take this near-horizon limit, the commutator between fields on the two boundaries vanish in the region we are covering.

A possible resolution of our problem would be an obstruction to the extension of the CFT to the full boundary of the self-dual orbifold spacetime. However, there is a barrier to finding such an obstruction. The bulk spacetime has an $SL(2,\mathbb{R}) \times U(1)$ isometry which acts transitively, mapping any point in the spacetime to any other point. Any obstruction to extending the geometry from the region covered by the black hole coordinates to the full spacetime must break this symmetry. The entangled state that we constructed preserves the spacetime isometries, as we would expect. We therefore expect the CFT in this state to live naturally on the conformal boundary of the full global self-dual orbifold spacetime, by the analogue of the argument of [81] in the higher-dimensional case. Finite $SL(2,\mathbb{R}) \times U(1)$ actions can map a point on one boundary to any other point on that boundary in the full spacetime.

Another resolution would be a restriction on the types of correlation functions we can consider. The isometries can map a point on one boundary to any other point on that boundary, but will only map a pair of spacelike separated points to spacelike separated points. From the near-horizon point of view, we obtain correlation functions or commutators of operators on the two boundaries at spacelike separated points as a limit of the same observables in the theory on the boundary of the BTZ black hole. If we restrict to considering just such observables, there will be no conflict with our entanglement description even when we consider the full self-dual orbifold spacetime.

Such a restriction may be necessary because of the instability of AdS_2 spacetimes observed in [38]. If we consider the back-reaction from adding some energy to the spacetime at one boundary, the spacetime will fail to be asymptotically AdS_2 to the future of the point where the energy is inserted. So we will not be able to impose asymptotically AdS_2 boundary conditions on the part of the other boundary causally connected to the point where the energy is inserted. This suggests that we cannot consider correlation functions like (6.3.19), so the fact that a *linearised* bulk analysis predicts a value for this correlation function which is inconsistent with our entanglement description does not lead to actual inconsistencies in the full theory. This possibility thus seems plausible, but it would be valuable to understand the restrictions on which correlation functions we can consider in detail. It would be particularly useful to understand this from the CFT point of view.

An alternative resolution is that there might not be any propagating states in the AdS_2 spacetime. The authors of [37] argued for such a picture by noting that the DLCQ field theory dual to AdS_2 does not have physical states charged under this $SL(2,\mathbb{R})$ group, so there should be no bulk states charged under the $SL(2,\mathbb{R})$ isometries of the spacetime. In the absence of such propagating degrees of freedom, there can be no causal physical interaction between the two AdS_2 boundaries, again suggesting that the two boundary CFTs are non-interacting. From the CFT point of view, this would correspond to a claim that local operators like $\mathcal{O}_1$ do not create well-defined states in the chiral CFT dual to the self-dual orbifold.

6.4 Asymptotically Self-dual orbifold spacetimes

We want to identify the self-dual orbifold geometry with a particular entangled state in a chiral CFT. The CFT should presumably have other states, and it is important to try to construct other geometries dual to these states. In this section, we discuss such constructions. We first discuss the boundary conditions defining what we mean by asymptotically self-dual orbifold. We then note that we can obtain another quotient geometry with a single boundary, which can be interpreted as the dual of the ground state of a single copy of the CFT. We then consider more general geometries constructed from the self-dual orbifold, first attempting a perturbative approach and then applying Garfinkle-Vachaspati solution-generating transformations. The more general solutions we construct have singularities in the bulk.

6.4.1 Boundary conditions

Before looking for solutions, we must first specify the asymptotic boundary conditions we want to impose. We consider boundary conditions on a single conformal boundary, defining states in a single copy of the CFT. If we wanted to consider spacetimes dual to two copies of the CFT, they should have two conformal boundaries and satisfy these boundary conditions on each of them.

Since the spacetime is locally AdS_3, we can impose the standard Brown-Henneaux boundary conditions [82] using the near horizon limit coordinates (u, v, r). These boundary conditions are

$$g_{uv} \sim e^{2r} + \mathcal{O}(1), \quad g_{uu}, g_{vv} \sim \mathcal{O}(1), \quad g_{ru}, g_{rv} \sim \mathcal{O}(e^{-2r}), \quad g_{rr} \sim 1 + \mathcal{O}(e^{-2r}). \tag{6.4.20}$$

The leading term gives us the null cylinder metric on the conformal boundary if we assume the coordinate u is periodically identified as in the self-dual orbifold solution. The subleading terms will then be interpreted as determining the stress tensor of the dual field theory. The subleading part of g_{uv}, which gives the trace of the stress tensor, vanishes when we satisfy the bulk equations of motion, so on-shell solutions actually have $g_{uv} \sim e^{2r} + \mathcal{O}(e^{-2r})$.

However, in [37], a more restrictive boundary condition for asymptotically self-dual orbifold spacetimes was proposed, requiring the $\mathcal{O}(1)$ part of g_{vv} to vanish as well. This corresponds in the field theory to saying that the stress tensor component $T_{vv} = 0$. This was motivated by the chiral nature of the dual CFT. As reviewed in section 3.3, taking the near-horizon limit corresponds to a DLCQ limit in the field theory, which sets the right-movers to their vacuum state. If we interpret this as saying that the limiting theory dual to the self-dual orbifold has no right-moving excitations, it would be inconsistent to have a non-zero right-moving stress tensor. It is therefore appropriate to require that $T_{vv} = 0$ as part of the boundary conditions. We therefore propose that the dual description of a chiral CFT on the boundary is spacetimes with the boundary condition

$$g_{uv} \sim e^{2r} + \mathcal{O}(1), \quad g_{uu} \sim \mathcal{O}(1), \quad g_{ru}, g_{rv}, g_{vv} \sim \mathcal{O}(e^{-2r}), \quad g_{rr} \sim 1 + \mathcal{O}(e^{-2r}). \tag{6.4.21}$$

Imposing the standard Brown-Henneaux boundary conditions would correspond to considering a non-chiral CFT on the null cylinder, where we retain some right-moving excitations. This is not the theory obtained in the strict near-horizon limit, but it remains interesting to consider it. It may be useful to consider situations where we do not take the strict near-horizon limit, and use the self-dual orbifold as an approximation to a region of the BTZ black hole spacetime [83, 84]. Since there are still some right-moving excitations, it may be that the Brown-Henneaux boundary conditions are then the appropriate ones to use to model the matching of the near-horizon region to the rest of the spacetime in this case.

The choice of boundary conditions determines the asymptotic symmetries of the spacetime. For the standard Brown-Henneaux boundary conditions, the analysis of [82] tells us that the asymptotic symmetries are diffeomorphisms depending on two arbitrary functions $\xi^+(u)$, $\xi^-(v)$. The vector field generating the diffeomorphism is

$$\xi^u = 2\xi^+(u) + \frac{1}{2}e^{-2r}\partial_v^2\xi^-(v) + \mathcal{O}(e^{-4r}), \tag{6.4.22}$$

$$\xi^v = 2\xi^-(v) + \frac{1}{2}e^{-2r}\partial_u^2\xi^+(u) + \mathcal{O}(e^{-4r}), \tag{6.4.23}$$

$$\xi^r = -\partial_u\xi^+(u) - \partial_v\xi^-(v) + \mathcal{O}(e^{-2r}). \tag{6.4.24}$$

Since u is a compact coordinate, $\xi^+(u)$ is a periodic function, and can be expanded in terms of modes which satisfy a Virasoro algebra. This includes the left-moving $U(1)$ symmetry of the self-dual orbifold given by ∂_u. However, as v is non-compact, $\xi^-(v)$ is not periodic, and the right-moving symmetry here is not simply a Virasoro algebra. Its interpretation from the CFT point of view is somewhat unclear.

For the boundary conditions (6.4.21), the asymptotic isometries are restricted. As shown in [37], only the diffeomorphisms with $\xi^-(v) = A + Bv + Cv^2$ survive. These correspond to the $SL(2,\mathbb{R})$ Killing vectors (3.3.48). Thus, for the boundary conditions (6.4.21), the asymptotic isometries are $SL(2,\mathbb{R})\times$ Virasoro, where the Virasoro contains the left-moving $U(1)$ symmetry.

Note that the diffeomorphisms which transform between the u, v, r coordinates we are using and the orbifold global coordinates t, ϕ, z or black hole coordinates $\tilde{t}, \tilde{\phi}, \bar{r}$ do not satisfy the boundary condition (6.4.21). Rather the boundary conditions are also transformed by these diffeomorphisms and must be expressed in the new coordinate frame. From the boundary point of view, the coordinate transformations to global and black hole coordinates corresponded to conformal rescalings of the v coordinate. But with the boundary conditions (6.4.21), the CFT only has a conformal symmetry acting on the u coordinate. The conformal transformations of the v coordinate are a part of the symmetries in $\xi^-(v)$ which is not in the $SL(2,\mathbb{R})$ subgroup we retain. Thus, while we are free to make such conformal transformations, the theory will not be invariant under the change of variables. In view of this, when we look for asymptotically self-dual orbifold solutions satisfying the boundary conditions (6.4.21), we will study them in the analogue of the (u, v, r) coordinates only.

6.4.2 Ground state of a chiral CFT and the null orbifold

In [85], a general classification of causally well-behaved quotients of AdS_3 was given. There was one other quotient which had the same type of boundary metric as the self-dual orbifold, namely the quotient by

$$\xi = (V - Y)(\partial_U + \partial_X) - (U - X)(\partial_V + \partial_Y), \tag{6.4.25}$$

where U, V, X, Y are the coordinates on the $\mathbb{R}^{2,2}$ embedding space. This Killing vector has $||\xi||^2 = 0$, but the Killing vector never vanishes, so the quotient has no fixed points in the spacetime. However, when we consider the action just on AdS_3, the resulting quotient space will contain closed null curves. This Killing vector lies in one of the two $SL(2,\mathbb{R})$ factors in the $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ isometry group, so the quotient has an $SL(2,\mathbb{R}) \times U(1)$ isometry, as for the self-dual orbifold. A coordinate system which covers the whole spacetime is [85]

$$U + X = e^{-\rho} \sin v + 2ue^{\rho} \cos v, \tag{6.4.26}$$

$$U - X = e^{\rho} \sin v, \tag{6.4.27}$$

$$V + Y = e^{-\rho} \cos v - 2ue^{\rho} \sin v, \tag{6.4.28}$$

$$V - Y = e^{\rho} \cos v. \tag{6.4.29}$$

In these coordinates, $\xi = \partial_u$ and the metric takes the form

$$ds^2 = -dv^2 + d\rho^2 - 2e^{2\rho} du dv. \tag{6.4.30}$$

The quotient in these coordinates is an identification $u \sim u + 2\pi$. Since the quotient has no fixed points, the resulting quotient spacetime is smooth. The spacetime has a single boundary at $\rho \to \infty$. The metric on this boundary is a null cylinder, as in the self-dual orbifold. In terms of the global coordinates on the boundary of AdS_3, the quotient has a single line of fixed points at $\tau + \theta = \pi$. The supersymmetry of this solution was analysed in [86], where it was shown that taking this quotient of AdS preserves 3/4 of the supersymmetry (SUSY), including 1/2 of the left-moving SUSY. Note that although the geometry has an $SL(2,\mathbb{R}) \times U(1)$ isometry, it does not have an AdS_2 factor.

This quotient can also be viewed as an identification along a null direction in Poincaré coordinates. That is, if we introduce the standard Poincaré coordinates x^+, x^-, Z on AdS_3, in terms of which the metric is

$$ds^2 = \frac{-2dx^+ dx^- + dZ^2}{Z^2}, \tag{6.4.31}$$

then $\xi = \partial_{x^+}$. This Poincaré coordinate system only covers a region of the spacetime, but it has the advantage that the spacetime written in these coordinates satisfies the more restrictive boundary conditions of section 6.4.1. This solution and the self-dual orbifold are the only locally AdS_3 spacetimes whose boundary metrics are null cylinders.

This geometry can also be obtained by taking the near-horizon limit of the $M = 0$ BTZ black hole: if we start with the Poincaré coordinates (6.4.31), the $M = 0$ BTZ black hole is obtained by making the identifications $(x^+, x^-) \sim (x^+ + 2\pi, x^- - 2\pi)$. We take the near-horizon limit by defining $x^+ = \tilde{x}^+$, $x^- = e^{-2\rho_0}\tilde{x}^-$, $Z = e^{\rho_0}\tilde{Z}$, and take $\rho_0 \to -\infty$ for fixed $\tilde{x}^+, \tilde{x}^-, \tilde{Z}$. This gives a metric of the same form, but with $\tilde{x}^+ \sim \tilde{x}^+ + 2\pi$, giving the null quotient (6.4.31).

Since the geometry has a single boundary, we would interpret this spacetime as the dual description of a single copy of the CFT living on the null cylinder in some pure state. We can identify the appropriate state by proceeding as in the self-dual orbifold, taking the $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ invariant vacuum state on the boundary of AdS_3 and considering the quotient action on it. The appropriate coordinate system in this case is just the Poincaré coordinates (6.4.31). We know that the state dual to AdS_3 in Poincaré coordinates is a ground state for both the left- and right-moving modes, so we propose that the dual of this quotient spacetime is the same ground state with the momentum for left-moving excitations quantised, breaking the symmetry of the state to $SL(2,\mathbb{R}) \times U(1)$. This is consistent with obtaining the quotient as the near-horizon limit of the $M = 0$ BTZ black hole; as the black hole mass goes to zero, the temperature for both left- and right-moving modes vanishes. Thus, the dual CFT interpretation of (6.4.30) is as a saddle-point for a single copy of the CFT on the null cylinder in a ground state. (See the related discussion in [68].)

The main problem with this discussion is that the spacetime contains closed null curves (CNCs), so one might not expect the spacetime (6.4.30) to be a good description of the boundary field theory state. In particular, winding string modes wrapping this compact direction will be light and could produce important corrections to the geometry. The unbroken supersymmetry in this spacetime may protect the geometry from such corrections, however. This problem is highly reminiscent of the Schrödinger spacetimes [69, 70], which similarly contain closed null curves (as remarked in [71]). The self-dual orbifold would then be thought of as analogous to the finite-temperature versions of Schrödinger spacetimes [71, 87, 88], in that the circle becomes spacelike everywhere in the bulk. The situation is actually slightly better than in the Schrödinger case, as the circle becomes constant size in the bulk, whereas it was asymptotically null in the finite-temperature Schrödinger spacetimes. It is interesting that while the metric (6.4.30) has CNCs, adding a tiny temperature on the left or right moving side of the dual field theory seems to regulate the CNCs. A purely left or right-moving temperature corresponds to extremal rotation in the BTZ black hole whose near-horizon limit we are examining. This is reminiscent of the "desingularization by rotation" in [89, 90].

In [85], the problem with the causal structure of (6.4.30) was formally resolved by combining the quotient action on AdS_3 with an action on the S^3 to obtain an action which was everywhere spacelike. This removes the closed null curves, but the resulting spacetime is not stably causal, as a $\mathbb{Z}$ identification on a compact space like S^3 will identify pairs of points which are arbitrarily close together. As a result, even if we include such an action on the S^3, we still need to worry about light states associated with strings winding around the circle: there will be winding sectors where these strings are arbitrarily light.

Nonetheless, as in the Schrödinger case, this quotient spacetime provides an interesting simple example of the dual of a pure state, and it is worth considering as at least a formal dual of the ground state. The quotients with actions on the sphere are also interesting, as they should correspond to ensembles where in addition to the temperature we are turning on a chemical potential for some R-charge.

6.4.3 Perturbative excitations

From the CFT point of view, we would expect to be able to consider arbitrary states for the left-movers. These might not all have a geometrical interpretation, but small excitations around the state corresponding to the self-dual orbifold might be expected to correspond to perturbations around the self-dual orbifold geometry. We are most interested in understanding chiral excitations, which will satisfy the boundary conditions of [37], and account for the entropy of the original black hole. In this section, we consider such chiral excitations on the full extremal BTZ black hole geometry. We find that surprisingly, excitations which keep the right-movers in their ground states cannot be consistently described by small perturbations around the black hole spacetime.[6]

We consider linearised fields on the extremal BTZ black hole background. We will start by considering scalar fields. We consider a scalar field Φ of mass μ^2, and write the field in Fourier modes as

$$\Phi = e^{i\omega t + im\phi} f_{\omega m}(r), \tag{6.4.32}$$

where we are working in the BTZ black hole coordinates defined in (3.3.56). For the extremal BTZ black hole, $r_+ = r_-$, chiral excitations from the CFT point of view correspond to considering co-rotating modes with $\omega = -m$ in the bulk spacetime.

[6] Since the self-dual orbifold is a near-horizon limit of BTZ, this immediately implies that such chiral excitations also cannot be described perturbatively on the self-dual orbifold.

It turns out that precisely these modes are not regular on the horizon. The field equation is $\Box\Phi - \mu^2\Phi = 0$, and on the BTZ black hole (3.3.56), if $\Phi = e^{i\omega t + im\phi} f(r)$,

$$\Box\Phi = \frac{1}{r^2 h}[r^2\omega^2 - (r^2 - r_+^2 - r_-^2)m^2 + 2r_+ r_- \omega m] e^{i\omega t + im\phi} f(r) + e^{i\omega t + im\phi}\frac{1}{r}\partial_r(rh\partial_r f(r)), \tag{6.4.33}$$

where $h(r) = (r^2 - r_+^2)(r^2 - r_-)^2/r^2$. Hence, if $r_+ = r_-$, and $\omega = -m$, the first term disappears and $\Box\Phi = e^{i\omega t + im\phi}\frac{1}{r}\partial_r(rh\partial_r f(r))$. The solution of the radial equation which satisfies the boundary conditions at infinity in this case is then just

$$f_{\omega=-m}(r) = c_m(r^2 - r_+^2)^{-h_+/2}, \tag{6.4.34}$$

where $h_+ = \frac{1}{2}(1 + \sqrt{1 + \mu^2})$. The surprising feature of this solution is that it blows up as we approach the black hole horizon at $r \to r_+$. This indicates that if we want to consider chiral modes on the BTZ boundary, this perturbative analysis will break down.

In fact, this failure is analogous to the "no-hair" theorem for the non-rotating black hole, which says that there is no regular solution for the scalar field with $\omega = 0$. The connection can be seen more clearly by considering the general BTZ black hole, with $r_+ \neq r_-$. Then if we define $\omega_c = \omega + \frac{r_-}{r_+}m$,

$$r^2\omega^2 - (r^2 - r_+^2 - r_-^2)m^2 + 2r_+ r_- \omega m = r^2\omega_c^2 - 2\frac{r_-}{r_+}(r^2 - r_+^2)\omega_c m - \frac{r_+^2 - r_-^2}{r_+^2}(r^2 - r_+^2)m^2. \tag{6.4.35}$$

So if $\omega_c = 0$, this factor vanishes on the horizon. The radial equation can be rewritten in terms of an effective potential by introducing a tortoise coordinate r_* such that $dr_* = h^{-1}dr$; then writing $f(r) = r^{-1/2}\psi(r)$, the radial equation becomes

$$\partial_{r_*}^2\psi + \omega_c^2\psi - h(r)v_{eff}(r)\psi = 0, \tag{6.4.36}$$

where $v_{eff}(r) > 0$ for all r. Because of the overall $h(r)$ factor, the effective potential contribution vanishes near the horizon, so for $\omega_c \neq 0$, the solutions near the horizon will look like $e^{\pm i\omega_c r_*}$, giving the usual ingoing and outgoing modes on the horizon. But if $\omega_c = 0$, the solutions will grow or decay near the horizon, as $v_{eff}(r) > 0$. The solution which is regular at infinity will (at least generically) include a growing part near the horizon.

For $r_- = 0$, this is the usual argument that there are no static scalar hairs on the black hole. A similar interpretation in our case would be that the black hole cannot support a chiral perturbation of the scalar field.

This calculation can trivially be extended to the vector case by observing that since we are in $2+1$ dimensions, a vector field is dual to a scalar. Thus, if we want a solution for a Maxwell field with field strength given by F, we can find it by writing $F = \star d\Phi$ for a scalar Φ satisfying the massless wave equation. The scalar solution of (6.4.34) gives a field strength which blows up on the horizon.

6.4.4 Non-thermal states of chiral CFT and traveling wave solutions

Having failed to construct more general geometries perturbatively, we will now consider applying a solution-generating transformation to obtain new solutions of the full equations of motion. Both the extremal BTZ black hole and the self-dual orbifold have a null Killing vector field, given by ∂_v in the near horizon limit (u, v, r) coordinates. We can therefore apply the Garfinkle-Vachaspati solution generating transformation [72] to add a travelling wave, as was done for asymptotically flat black string solutions in [91]. The null Killing vector is $k = \partial_v$, so with the index lowered $k = e^{2r} du$. This satisfies $\nabla_{[\mu} k_{\nu]} = k_{[\mu} \nabla_{\nu]} S$ with $S = -2r$. The Garfinkle-Vachaspati technique tells us that we can generate a new solution $\tilde{g}_{\mu\nu}$ by choosing a function Ψ satisfying $\partial_v \Psi = 0$ and $\nabla^2 \Psi = 0$, and defining

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + e^S \Psi k_\mu k_\nu. \tag{6.4.37}$$

That is,

$$\tilde{ds}^2 = (1 + e^{2r}\Psi) du^2 + 2e^{2r} du dv + dr^2. \tag{6.4.38}$$

This spacetime will be asymptotically AdS_3 or asymptotically self-dual orbifold depending on whether we make the spacelike direction $u + v$ or the null direction u compact.

If we consider just the three-dimensional spacetime, then $\nabla^2 \Psi = e^{-2r} \partial_r (e^{2r} \partial_r \Psi)$, and $\Psi = f_0(u) + f_1(u) e^{-2r}$. To preserve the asymptotics of the spacetime, we should set $f_0(u) = 0$; the solution is then

$$\tilde{ds}^2 = (1 + f_1(u))du^2 + 2e^{2r} du dv + dr^2. \tag{6.4.39}$$

However, this transformation is trivial; the Garfinkle-Vachaspati transformation in general adds a gravitational wave to the previous solution, but in three dimensions, there is no gravitational radiation. That is, any solution of the vacuum equations of motion in $2 + 1$ dimensions is locally AdS_3. Thus, this solution is just a locally AdS_3 spacetime written in an unfamiliar coordinate system.[7]

To obtain something non-trivial we need to introduce some additional directions and allow Ψ to depend them. Consider for example taking the product of our geometry with an S^3, as in the simplest embeddings in string theory, and allowing Ψ to depend on the coordinates on the S^3 factor in the geometry. For simplicity, assume Ψ is in a particular spherical harmonic on the sphere, so $\Psi = Y_{lm}(\theta, \phi, \psi) g(r, u)$. Then

$$e^{-2r} \partial_r (e^{2r} \partial_r g) - l(l + 2)g = 0, \tag{6.4.40}$$

with solutions $g(r, u) = f_0(u) e^{lr} + f_1(u) e^{-(l+2)r}$. As previously, take $f_0(u) = 0$ to preserve the boundary conditions, and for each spherical harmonic we have one functions worth of solutions. For example, if we take the harmonic with $l = 2, m = 0$, we have a solution

$$\tilde{ds}^2 = (1+f_1(u) \cos 2\theta e^{-2r})du^2+2e^{2r} du dv+dr^2+d\theta^2+\sin^2\theta d\phi^2+\cos^2\theta d\psi^2. \tag{6.4.41}$$

This solution has a non-vanishing Weyl tensor, indicating the presence of a gravitational wave, and showing explicitly that this is a non-trivial example of an asymptotically self-dual orbifold spacetime. As in [91], these geometries are singular: they have diverging Riemann tensor components on the would-be horizon at $r \to -\infty$, although the curvature invariants are finite. Thus they have no extension to global coordinates and only satisfy the self-dual orbifold boundary conditions on one boundary, in the region corresponding to the (u, v, r) coordinates. The deformed spacetime

[7] Note that the situation here is different from the full asymptotically flat solution considered in [91], where adding the $l = 0$ mode produces a physically different solution. This implies that these different solutions all have the same near-horizon BTZ region.

breaks the $SL(2,\mathbb{R})$ symmetry, and does not have an AdS_2 factor, although if we made a dimensional reduction to two dimensions, the geometry would be asymptotically AdS_2 in Poincaré coordinates.

It is very satisfying that we finally have some examples of asymptotically self-dual orbifold spacetimes, even if they have mild singularities in the bulk. It would be interesting to understand their dual description. We expect them to correspond to more general states for the left-movers, where the operators dual to excitations on the sphere have non-zero expectation values, and a ground state for the right-movers. However, the travelling wave breaks the symmetry corresponding to L_{-1} in $SL(2,\mathbb{R})$. We think of this symmetry as acting on the right-movers, so these geometries do not correspond precisely to the same right-moving ground state as in the self-dual orbifold. (Since the geometry is still invariant under L_0 and L_1, it still corresponds to a ground state. This is also clear from the fact that it satisfies the boundary conditions of [37].) It's surprising that the Garfinkle-Vachaspati transformation breaks this symmetry; we would naively have thought of it as acting just on the left-movers. It would be interesting to understand this in more detail.

Other AdS_2 cases

This solution-generating transformation provides a useful way to generate new solutions. It is therefore interesting to ask if it is special to the self-dual orbifold, or can be applied in other contexts where the geometry has an AdS_2 factor. The NHEK geometry does not have a null Killing vector, so it cannot be applied in that case (the analogue of the Killing vector considered here would be ∂_t, which is not everywhere null because of the fibration over θ). Thus, there are no such solutions in NHEK, as we might have expected given the results of [66, 67].

In the context of Reissner-Nordström AdS black holes, the near-horizon geometry in for example AdS_4 is

$$ds^2 = \frac{l_2^2}{\rho^2}(-dt^2 + d\rho^2) + d\vec{x}^2, \tag{6.4.42}$$

where $l_2 = l/\sqrt{6}$, with a vector field $A = \frac{g_F}{\sqrt{12}\rho}dt$. Clearly there is no null Killing vector here, but one might hope to find one in the uplift to the full string or M theory geometry. In the self-dual orbifold case, the v direction is timelike in the

AdS_2 factor, but becomes null when we uplift it to the three-dimensional geometry.

Consider for definiteness the uplift to eleven dimensions given in [92]. If we work in units where the S^7 has unit radius, $l = 1/2$, $g_F = 1/2$, and the eleven-dimensional metric is

$$ds_{11}^2 = ds_4^2 + ds_{CP^3}^2 + (\eta + A)^2, \tag{6.4.43}$$

where η is the one-form dual to the Reeb vector in the writing of S^7 as a Hopf fibration over $\mathbb{C}P^3$. Hence in eleven dimensions there is a partial cancellation between the two factors as in the self-dual orbifold case, but

$$g_{tt} = -\frac{l_2^2}{\rho^2} + \frac{g_F^2}{12\rho^2} = -\frac{1}{48\rho^2}, \tag{6.4.44}$$

so the Killing vector ∂_t remains timelike, and we can't apply the Garfinkle-Vachaspati transformation to this solution. The situation for Reissner-Nordström AdS_5 black holes is the same; it seems to be only when we are uplifting from AdS_2 to a three-dimensional solution that the factors work out so that we get a null isometry in the higher-dimensional geometry.

This suggests that there is something a little special about the AdS_2 from the dual CFT point of view in the self-dual orbifold case; the null structure that is responsible for the chiral CFT interpretation here isn't obviously present in higher-dimensional cases.

6.5 Discussion

We have studied the description of the self-dual orbifold from the point of view of the dual CFT, and constructed examples of asymptotically self-dual orbifold spacetimes, which should be dual to other states of the CFT. We have proposed that the full spacetime can be described as an entangled state in two copies of the CFT, living on the two boundaries. This description appears to have problems with the causal connection between the two boundaries, which would lead to predictions for bulk correlation functions which cannot be reproduced by considering an entangled state. However, the special nature of AdS_2 suggests that there will be restrictions on the correlation functions which can be consistently considered. Acting with an operator in the field theory to throw in some energy from the boundary will cause a back-reaction which violates the boundary conditions on the boundary after the operator insertion. We have suggested that the problematic correlations involving timelike separated operators on the two boundaries may simply not be legitimate observables. This issue needs further exploration.

We discussed the asymptotic boundary conditions for the spacetime. Following [37], we argued that considering a chiral CFT on the boundary is associated with a boundary condition that is more restrictive that the ones imposed by Brown and Henneaux. We constructed examples of asymptotically self-dual orbifold spacetimes satisfying this boundary condition. This is interesting as it can be challenging to construct asymptotically AdS_2 spacetimes; in the Kerr-CFT context it was shown in [66, 67] that the only spacetimes satisfying the relevant boundary condition are diffeomorphic to the background. Note however that the solutions we construct are singular in the bulk. Other examples of asymptotically AdS_2 spacetimes which are regular were recently obtained in higher-dimensional contexts in [93] by considering RG flows from one AdS_2 to another. We also identified a geometry corresponding to the ground state of the chiral CFT which is dual at finite temperature to the self-dual orbifold. This new geometry is obtained as a near-horizon limit of the $M = 0$ BTZ black hole, just as the self-dual orbifold is the near-horizon limit of the $M > 0$ extremal BTZ black holes.

In the simplest embedding in string theory, we would consider the self-dual orbifold geometry $\times S^3 \times T^4$. As discussed in [85], in this context we can generalise the quotients considered here by adding an action on the S^3. This corresponds to introducing a chemical potential for an R-charge. For the orbifold of section 6.4.2, corresponding to the CFT in a ground state for both the left- and right-moving excitations, it is meaningful to introduce such a chemical potential because we are considering the theory in a Ramond sector, so there is a degenerate set of ground states, which carry different R-charge.

We have assumed throughout this work that we were considering the field theory in the Ramond sector, so there is some unbroken supersymmetry in the solution. Since the circle is of finite size everywhere in the spacetime, we can change our choice of spin structure, which corresponds to considering the field theory in the Neveu-Schwarz sector. If the compact circle in the interior is smaller than the string scale, this solution will then have a tachyon. Since this circle has the same size everywhere, we would expect the condensation of this tachyon to destroy the whole spacetime.

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