Topological BPS charges in 10- and 11-dimensional supergravity

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A Thesis presented for the degree of
Doctor of Philosophy

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England

November 2010

Dedicated to

Mum and Dad

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Abstract

In this thesis we construct closed expressions that correspond to the topological charges of the various BPS branes of the IIA, IIB and $D = 11$ supergravity theories. These expressions are related to the structure of the SUSY algebras in curved spacetimes. We consider charges for all the M-, NS- and D-branes as well as the Kaluza-Klein monopoles. Additionally we consider the $SL(2,\mathbb{R})$ symmetry that exists in the IIB theory and $D = 11$ theory in a double M9-brane background, and determine the charges for the remaining branes that fill up the $SL(2,\mathbb{R})$ multiplets. These include the charges corresponding to the multiplets of 7 and 9-branes in IIB as well as several new types of branes in $D = 11$. We find that examining the possible multiplet structures of the charges provides another tool for exploring the spectrum of BPS branes that appear in these theories. Furthermore, we demonstrate how these charges map between theories. As a prerequisite to constructing some of the charges we determine the field equations and multiplet structure of the $D = 11$ gauge potentials, extending previous results on the subject. The massive gauge transformations of the fields are also discussed, and we demonstrate how they are compatible with the construction of an $SL(2,\mathbb{R})$ covariant kinetic term in the $D = 11$ Kaluza-Klein monopole worldvolume action.

Declaration

The work in this thesis is based on research carried out at the Centre for Particle Theory, the Department of Mathematical Sciences, University of Durham, England. No part of this thesis has been submitted elsewhere for any other degree or qualification and it is all my own work unless referenced to the contrary in the text.

Acknowledgements

I would like to thank my supervisor Douglas Smith for his constant guidance, advice and assistance throughout the course of my research. I am also grateful to the many friends and colleagues I have been fortunate to have over the last few years who have provided me with many fond memories and moral support when I needed it. Finally I wish to thank my family and above all my parents for their continued encouragement, belief and patience.

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Part I

Chapter 1

Introduction

The principal concern of this thesis is with the investigation and construction of a series of differential form expressions which determine the topological charges of the different p-branes associated with the supergravity theories. For the sake of clarity, we will loosely refer to these expressions as generalised charges, or simply charges, although strictly speaking they are charge densities. It is a requirement that these be closed for arbitrary on-shell field configurations, and as such their interpretation as charge (densities) is valid in curved, supersymmetric background spacetimes. This idea was first investigated in [1] for the M2 and M5-branes in $D = 11$ supergravity, and was primarily motivated with the aim of generalising the relationship between the brane charges and the supersymmetry (SUSY) algebras to curved spacetimes. Although this relation was also our initial and remains as our main motivation for the study of these generalised charges, our central focus is on the charges themselves and not the curved SUSY algebras which can be derived from them.

In this thesis we substantially build upon the work of [1]. We focus on the IIA, IIB and $D = 11$ supergravity theories and set about determining the generalised charges for the various branes that appear in these theories. Furthermore, we investigate how the charges map under the duality relations that exist between these theories. We do this not only for the 'conventional' spectra of branes, which consists of those branes whose flatspace charges are straight forwardly represented in the flatspace SUSY algebras as discussed in [2–4], but also for more exotic branes whose relations with the flatspace algebras are less clear. It is necessary to consider these 'exotic' branes as they fill up the $SL(2,\mathbb{R})$ multiplets in the IIB theory as well as in the $D = 11$ theory compactified on T^2.

Before outlining the contents of the research in detail, we will first give an overview of the current state of particle physics and discuss some of the relevant background material. This will provide the foundation and motivation for the main bulk of this thesis.

1.1 Current state of Theoretical Physics

Over the last century much of the progress in the search for a fundamental theory of nature has taken place across two frontiers and has resulted in the development of two revolutionary theories. The first of these is Einstein's theory of General Relativity (GR) which describes gravity, one of the four known fundamental forces of nature. Here space and time are united into spacetime, a dynamical entity whose curvature describes the gravitational field felt by all matter. The second revolutionary theory is the Standard Model (SM) which describes the microscopic world of elementary particles. This is a single Quantum Field theory (QFT) with gauge group $SU(3) \times SU(2) \times U(1)$ that successfully describes the three remaining fundamental forces, namely the electromagnetic, and the Strong and Weak nuclear forces.

Both these theories have met with unprecedented successes having repeatedly made accurate predictions that have been experimentally verified to a high degree of precision. Despite this however there is little doubt that neither provides the correct framework for a truly fundamental theory. The list of unsatisfying issues with the SM is well known, amongst these is its somewhat fragmented structure and the fact it contains some 21 parameters whose seemingly arbitrary values can only be determined experimentally; this is hardly a feature one would expect from a fundamental theory and suggests the existence of a deeper, richer theory. GR on the other hand, whilst arguably more elegant, is a classical theory and it is a non-trivial matter to quantise it due to the fact it is non-renormalisable.

It seems therefore that the two theories are incompatible which is in essence the strongest indication that neither provides an exact representation of nature. For many practical purposes however this incompatibility is not disastrous since each theory is applicable to a different regime of nature, with many physical systems of theoretical interest falling exclusively into one regime or the other. Broadly speaking the SM provides an accurate description of the universe on the atomic scale where gravity usually has an insignificant effect and so can be neglected, whereas GR describes objects that are very massive such as stars and galaxies where gravity plays a prominent role. However, two important systems that fall into both these regimes are the centers of black holes and the first few moments of the Big Bang. It is in instances such as these where neither SM or GR provide an accurate descriptions of nature, and an unknown deeper, unifying theory must apply, so called Quantum Gravity. Due to the stark differences in structure between the SM and GR, and the various problems each suffers from, it is possible that Quantum Gravity will be a radical departure from each.

Of the numerous approaches that have been explored over the years to construct a fundamental theory, currently it is the developments from string theory that are widely believed to be the most promising and have been intensely studied by theorists in recent times. The central idea of string theory is that, instead of being point-like, all the elementary particles that exist in the universe actually have the structure of a microscopic string. This stringy nature leads to a vast array of phenomena which are not present in conventional theories. An example of this is that the various properties of the particles such as the mass, spin and charges are now determined by the different vibrational modes of the string rather than taking arbitrary values.

The only length scale in the theory is determined from the square root of the string tension parameter α' and is expected to have a value of approximately 10^{-33}cm. This must therefore also determine the typical length scale of the strings. Such a small distance is well beyond the scope of any modern day experiments which is why we would not expect to have directly observed the string-like nature of the elementary particles. Unfortunately, this also limits the extent to which we can experimentally verify such a theory. This has led to its criticism from those that strictly adhere to the principle that all meaningful scientific theories should be experimentally verifiable. However, this is a problem of practicality rather than principle, and for many its rich mathematical structure and consistency are motivation enough for its study.

The introduction of fermionic states is achieved by incorporating supersymmetry into the model, yielding superstrings. Doing this gives rise to two central characteristic features of string theory. Firstly, for quantum consistency the background spacetime must be 10-dimensional. Therefore in order to provide a realistic model of the observed physical world six of the spatial dimensions must be compactified so that they are sufficiently small and beyond the resolution of current experiments. Secondly, one finds that there are actually five different varieties of theory that can be produced known as Type I, Type IIA, Type IIB, Heterotic $SO(32)$ and Heterotic $E_8 \times E_8$. Each of these are ultraviolet convergent and contain a massless particle which is identified as the graviton and so are possible candidates for a quantum theory of gravity. The existence of five such theories does seem problematic however since it is not immediately clear how this is compatible with the idea of having a preferred, unique fundamental theory. Another initial issue is that each of these theories is only defined in terms of asymptotic expansions of the string coupling constant g_s. One is therefore limited to the small coupling regimes and perturbative methods to perform calculations.

The solution to both these problems was realised in the [5] where it was argued that all five string theories were actually different perturbative expansions of a single underlying non-perturbative 11-dimensional theory named M-theory. Because of this each string theory can be thought of as being equivalent in a certain sense. Whilst many advances have been made in our understanding of M-theory it is safe to stay that there is still some way to go largely due to the difficulties associated with studying non-perturbative theories. Much of the progress that has been made has resulted from its connections with the string theories and their non-perturbative extensions, as well as through its low energy effective action which corresponds to $D = 11$ supergravity [6] (see Section 1.2). The modern day image that has emerged for M-theory is summarised in Figure 1.1.

Whereas strings play a central role in perturbation theory, non-perturbative

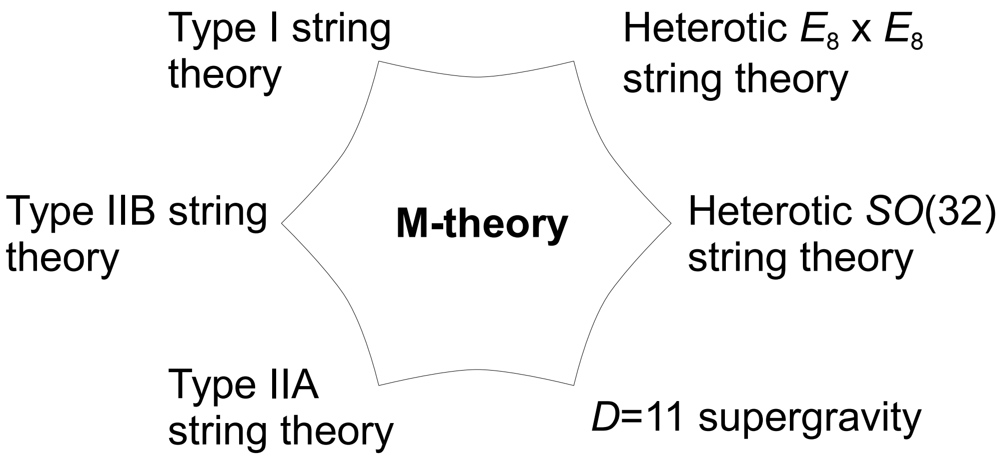

Figure 1.1: Schematic diagram showing the various faces of M-theory.

string theory is known to contain a host of multi-dimensional extended objects, known as branes. In this scheme strings are merely viewed as 1-branes, a brane with a $1+1$-dimensional worldvolume. The situation here is more democratic as each type of brane is considered to be on equal footing with none playing a fundamental role. The same is true for 11-dimensional M-theory which does not in fact contain string-like branes but rather is a theory of 2-branes and 5-branes (amongst others). Analysis of these branes is one of the key methods of investigating the non-perturbative nature of the theories and has been the subject of intense study over the past decade or so, and is the main focus of this thesis.

1.2 Supergravity and p-branes

Supergravity theories have undergone a resurgence in popularity in recent times due to their connection with M-theory and string theory, and the prominent role they play in our understanding of many of the central features of these theories. As previously mentioned the low energy effective action for M-theory corresponds to $D=11$ supergravity. Similarly the effective actions of each of the (uncompactified) string theories describing the massless fields turn out to correspond to a $D=10$ supergravity theory (coupled to a Yang-Mills sector in some cases). Specifically, the supergravity equations of motion of the massless bosonic field corresponds to the conditions required for anomaly cancellation when conformal invariance is imposed on the relevant string theory. The correspondence between each theory and its low energy limit is summarised in Table 1.1.

Quantum theory	Effective theory
Type I string theory	$D = 10$, $\mathcal{N} = 1$ supergravity + Yang Mills with $SO(32)$ gauge group
Type IIA string theory	Non-chiral $D = 10$, $\mathcal{N} = 2$ IIA supergravity
Type IIB string theory	Chiral $D = 10$, $\mathcal{N} = 2$ IIB supergravity
Heterotic $SO(32)$ string theory	$D = 10$, $\mathcal{N} = 1$ supergravity + Yang Mills with $SO(32)$ gauge group
Heterotic $E_8 \times E_8$ string theory	$D = 10$, $\mathcal{N} = 1$ supergravity + Yang Mills with $E_8 \times E_8$ gauge group
M-theory	$D = 11$, $\mathcal{N} = 1$ supergravity

Table 1.1: Table showing the correspondence between the string theories, M-theory and the supergravity theories.

Supergravity theories themselves are essentially the result of incorporating supersymmetry into the framework of gravitational models. Put simply, they are supersymmetric field theories which are invariant under general co-ordinate and local supersymmetry transformations. This requires the introduction of the vielbein ${e_\mu}^a$ and gravitino Ψ_μ^α as dynamic fields respectively. The latter must transform as $\delta\Psi_\mu^\alpha = \partial_\mu\epsilon^\alpha + \ldots$ where ϵ^α is the local SUSY parameter.

These theories were originally investigated in the hope that they would solve the ultraviolet divergence problems which plagued the QFT's of the day. However it was soon realised that these divergences were not completely eradicated, but merely tamed [7], and so initially there was uncertainty about the importance of these theories. Nonetheless their mathematical elegance was appreciated due to the strong constraints resulting from their symmetries and the restrictions these place on the possible terms that can appear in the Lagrangian.

While these stringent restrictions prohibit the coupling of any external fieldtheoretic matter, it is still possible to couple appropriate source terms that support the existence of extended objects that sweep out hypersurfaces in the background spacetime, known as p-branes. These are solitonic solutions to the classical equations of motion and correspond to the branes which exist in non-perturbative string theory. These branes are an active area of research and exhibit many intriguing phenomena, for example there exists an interrelation between the supersymmetric Yang-Mills theories that live on the brane worldvolumes' and the gravity-based physics of the background spacetime.

For each supergravity theory there are a number of different types of branes that can be coupled, these make up the brane spectrum and is unique for each theory. In some cases the string theory interpretation is well understood such as for the D-branes that occur in the IIA and IIB theories which are defined as the surfaces to which the end points of open strings are constrained [8]. In other cases though their precise role in string theory is not fully understood, this can be the case with the more exotic types of branes that occur. In any case, studying the branes via the supergravity theories is often a useful way to understand their general features and dynamics.

1.3 Duality symmetries and strong coupling limits

Another important feature of the string theories (and their various compactifications) is that a web of duality symmetries exists between them. These may relate two different theories or alternatively relate a given theory to itself and are often categorised into two main varieties. The first of these is known as T-duality where a theory compactified on a large volume is equivalent to a theory compactified on a small volume [9,10]. These types of duality map between perturbative regimes, and have been shown to hold to all orders in string perturbation theory [11]. The second category is known as S-duality which typically relates a theory at weak coupling to one at strong coupling, [12]. Additionally there is U-duality which roughly speaking is a mixture of T-duality and S-duality [13].

Since our current mathematical techniques are limited to performing calculations in the perturbative regime, dualities which relate weak and strong coupling regimes are of particular interest since they provide a useful tool for exploring the non-perturbative features of the theories and demonstrate the importance of dualities. Testing these types of duality requires the use of non-renormalisation theorems which guarantee that certain features of the theories must be protected from quantum corrections, thus allowing results found at weak coupling to be extrapolated to strong coupling. These essentially arise from supersymmetry, the most powerful of which occurring for theories with at least 16 SUSY generators.

One such example involves the effective supergravity actions which are completely determined from supersymmetry and so cannot get renormalised by string loop corrections. A second example concerns the BPS states. These are states which saturate a 'Bogomolny'i bound' derived from the SUSY algebra which takes the form

$$M \geq Z \tag{1.1}$$

where M is the mass (or mass density), and Z is the charge (or charge density). They typically preserve half the supersymmetry of the background spacetime and belong to supermultiplets that undergo a shortening compared to the general massive states in a theory. Since quantum corrections are not expected to produce new states, it is believed that BPS spectrum must be independent of the coupling strength [14]. The branes that exist in the string and supergravity theories may carry a charge, in which case will satisfy a bound of the form (1.1) (with the mass being equivalent to the brane tension). These branes are loosely referred to as BPS branes since there will exist certain configurations where the bound is saturated.

It follows that for the duality symmetries of the parent theories to be valid they must also apply to the effective actions and BPS spectra. Under the different duality transformations the BPS states are mapped between one another forming an intricate web. Consistency between the dualities and this web of states not only validates the dualities but can also serve as a useful tool in determining the characteristics of lesser known states from the better understood ones.

In this thesis we will specifically be interested in the duality relations that exist between the IIA, IIB and $D = 11$ supergravities with particular focus on their applications to the spectra of p-branes. We will therefore now give a brief overview of these relations which also determine the strong coupling limits of the type IIA and IIB string theories. We summarise these dualities in Figure 1.2. For more details on the following discussion see [15].

It is well known that, with the exception of IIB supergravity, all maximal supergravity theories (i.e. those that contain 32 SUSY charges) can be obtained from dimensional reduction of $D = 11$ supergravity. Notably IIA supergravity is obtained from compactification on a circle [16–18]. The Kaluza-Klein scalar associated with the compactification is related to the only massless scalar field in the IIA theory namely the dilaton, ϕ. Furthermore, it is a general feature of string theory that the dilaton is related to the string coupling constant through the relation $g_s = e^{<\phi>}$. This therefore leads to the interpretation of the coupling as a measure of the radius R_{11} of the compact 11th dimension. The general relationship between these is found to be of the form [5]

$$R_{11} \sim g_s^{\frac{2}{3}}. \tag{1.2}$$

This shows that IIA perturbative string theory is an expansion about $R_{11} = 0$ and explains why the 11th dimension is invisible. On the other hand, in the strong coupling limit we have $R_{11} \to \infty$ which demonstrates that the strong coupling limit of IIA string theory is $D = 11$ M-theory.

The IIB supergravity is brought into the picture due to an equivalence with IIA supergravity when both are compactified on a circle, in which case they both yield the same (unique) $D = 9$ supergravity. Specifically the IIA theory compactified on a circle of radius R_{10} can be shown to be equivalent to the IIB theory compactified on a circle of radius of radius $1/R_{10}$, and so the two theories exhibit a T-duality relationship. The type IIA and IIB string theories are found to be similarly related when compactified on a circle [19,20] and so consistency between the duality relations of the parent and effective theories is observed.

It follows that IIB supergravity compactified on a circle can be interpreted as $D = 11$ supergravity compactified on a torus. In the limit that the volume of this torus shrinks to zero one obtains the uncompactified IIB theory. The IIB string coupling constant is determined by the (fixed) ratio of the two radii as they tend to zero and so is of the general form

$$g_s \sim \frac{R_{11}}{R_{10}}. \tag{1.3}$$

Furthermore the IIB theory is known to possess a non-geometric $SL(2,\mathbb{R})$ symmetry [21–23], which from the $D = 11$ perspective corresponds from the global reparametrisations of the torus. A similar relationship also exists between M-theory and IIB string theory, however upon quantisation this symmetry group is reduced to the discrete $SL(2,\mathbb{Z})$ subgroup. Since R_{11} and R_{10} are interchanged by a reparametrisation of the torus we see from (1.3) that a subgroup of $SL(2,\mathbb{Z})$ must exchange between weak and strong coupling. This symmetry is therefore an example of S-duality, and leads to the well known result that the IIB theory is self dual with an equivalence between weak and strong coupling.

1.4 Brane charges

It is a general feature that p-branes are electrically charged with respect to $p+1$-form gauge potentials. The situation is essentially the higher dimensional generalisation of the Maxwell charge of the electron. Classically the electron is considered as a point-like object with a one dimensional worldvolume, and so it naturally couples to a one-index vector potential $A^{(1)}$. It follows that a general p-brane with $p+1$ dimensional worldvolume is supported by a $p+1$-form potential $A^{(p+1)}$. These charges are an important feature of branes so we will now outline the general mechanism by which they occur.

In order for a brane to arise as a solitonic solution to the supergravity equations of motion one must add a worldvolume Wess-Zumino term to the supergravity action which typically takes the form

$$\int d^{p+1}\sigma \frac{\partial X^{\mu_1}}{\partial \sigma^0} \cdots \frac{\partial X^{\mu_{p+1}}}{\partial \sigma^p} A^{(p+1)}_{\mu_1 \cdots \mu_{p+1}}(X(\sigma^m)) \equiv \int_\Sigma A^{(p+1)}(\sigma^m). \tag{1.4}$$

The integrand here is simply just the pullback of the $p+1$-form potential $A^{(p+1)}$ from the embedding D-dimensional background spacetime to the brane worldvolume Σ. X^μ and σ^m are the background and worldvolume co-ordinates respectively.

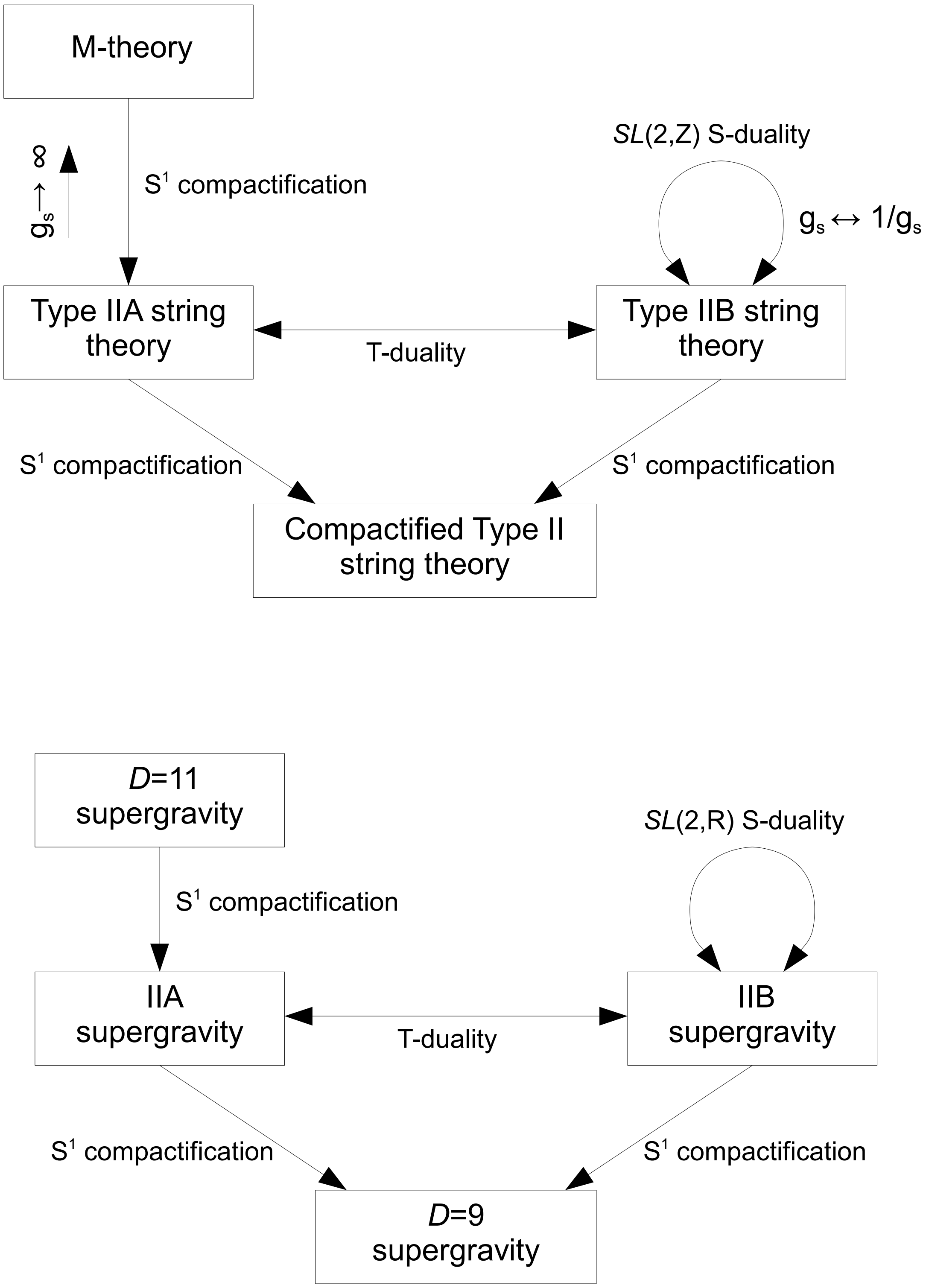

Figure 1.2: Duality relations between M-theory, IIA and IIB string theory as well as their effective supergravity theories.

It is useful to re-express this is in terms of an integral over the entire embedding spacetime so that it can share the integral with the main supergravity action. We therefore write

$$\int d^D x A^{(p+1)}_{\mu_1 \ldots \mu_{p+1}}(x) j^{\mu_1 \ldots \mu_{p+1}}(x) \tag{1.5}$$

where

$$j^{\mu_1 \ldots \mu_{p+1}}(x) = \int d^{p+1}\sigma \delta^D(x - X(\sigma^m)) \frac{\partial X^{[\mu_1}}{\partial \sigma^0} \ldots \frac{\partial X^{\mu_{p+1}]}}{\partial \sigma^p} \tag{1.6}$$

is interpreted as a current density that, due to the presence of the δ function, is only non-zero on the brane. Analogously to the Maxwell case where j^0 is the charge density, one defines the charge density tensor here as

$$z^{i_1 \ldots i_p} = j^{0 i_1 \ldots i_p} \tag{1.7}$$

which obviously has only spatial components, and these can be shown to always lie parallel to the brane. It then follows that the total charge tensor is given by

$$Z^{i_1 \ldots i_p} = \int d^d x \; z^{i_1 \ldots i_p} \tag{1.8}$$

where the integral is taken to be over a $d = D - 1$ spatial hyperplane. Substituting (1.6) into (1.8) and carrying out spacetime integrals to eliminate the delta function one obtains for Z

$$Z^{i_1 \ldots i_p} = Q_{(p)} \int dX^{i_1} \wedge \ldots \wedge dX^{i_p} \tag{1.9}$$

where the integral is taken over the p-cycle occupied by the brane in the background spacetime. Since the integrand is a closed form this expression only depends on the homology class of this p-cycle and is therefore topological. The charge $Q_{(p)}$ is a measure of the number of times the brane wraps a given p-cycle, or equivalently the number of branes that wrap the p-cycle once. This quantity can be calculated by integrating the current density over the $D - p - 1$ dimensional transverse volume, which is conveniently expressed as the following integral using the Hodge dual operator

$$Q_{(p)} = \frac{1}{\Omega_{D-p-2}} \int_{V^{D-p-1}} *j \tag{1.10}$$

where the normalisation factor Ω_{D-p-2} is the volume of the unit $D-p-2$ sphere. Varying the action with respect to the $p+1$-form potential A gives rise to equations of motion that typically take the form

$$d\tilde{F} = *j \tag{1.11}$$

where $\tilde{F}$ is the $p+2$-form field strength of A (possibly containing additional Chern-Simons terms). Using Stoke's theorem we can re-express (1.10) as a flux integral over the $D-p-2$ dimensional boundary to this volume, which is S^{D-p-2} for infinitely extended p-branes and so takes the form

$$Q_{(p)} = \int_{S^{D-p-2}} \tilde{F}. \tag{1.12}$$

Due to its topological nature this quantity vanishes unless the p-cycle occupied by the brane is non-contractible. The simplest example is the string winding number in string theory. In the case where one or more of the brane directions is infinitely extended then the brane can be considered to be wrapped around an circle with infinite R in each of those directions. In this case some components of Z will be infinite but $Q_{(p)}$ will remain finite.

1.5 Extended SUSY algebras

It is a general feature that, in the presence of branes, the flatspace SUSY algebras of the supergravity theories are modified beyond the super Poincaré algebra by including terms which contain the topological charges of the branes (1.9). For individual brane configurations these terms typically take the form [2]

$$\frac{1}{p!}(C\Gamma_{\mu_1\ldots\mu_p})_{\alpha\beta}Z^{\mu_1\ldots\mu_p}. \tag{1.13}$$

Here C is the charge conjugation matrix, X^μ are spacetime co-ordinates and $\Gamma_{\mu_1\ldots\mu_p}$ is an antisymmetric combination of p Dirac Γ matrices that occur in the particular supergravity theory. The charges here are often referred to as being central despite the fact that they do not commute with the Lorentz generators. The values of p for which the terms (1.13) occur in a given algebra is constrained by matching the degrees of freedom with the SUSY generators. The ranks of charges allowed from this consideration corresponds to the dimensionality of the various branes that exist in the given theory highlighting a relationship between the SUSY algebras and the p-brane spectra.

We will illustrate this idea by considering the example of $D = 11$ supergravity. In this case the theory contains the M2-brane and M5-brane which have 2-form and 5-form charges respectively. The flatspace SUSY algebra therefore contains the terms (1.13) for $p = 2, 5$ and takes the following form [3, 4]

$$\{Q_\alpha, Q_\beta\} = (C\Gamma^\mu)_{\alpha\beta}P_\mu + \frac{1}{2}(C\Gamma_{\mu_1\mu_2})_{\alpha\beta}Z^{\mu_1\mu_2} + \frac{1}{5!}(C\Gamma_{\mu_1\dots\mu_5})_{\alpha\beta}Z^{\mu_1\dots\mu_5} \tag{1.14}$$

where P_μ is the momentum. It is a simple matter to show that these charges are sufficient to account for all the degrees of freedom. Since in this instance there are 32 SUSY generators, the LHS is a 32×32 symmetric matrix and therefore contains $32(32+1)/2 = 528$ independent components. On the RHS, taking into account that the charges are anti-symmetric, we have $11+(11\times10)/2+(11\times10\times9\times8\times7)/5! = 528$ independent components, and so we see that the presence of 2-form and 5-form charges is precisely what is required for the degrees of freedom to match. Note also that the fact that the above equation is symmetric in the spinor indices fixes the combinations of Γ matrices appearing on the RHS.

An apparent discrepancy does arise here since in order to match degrees of freedom we must sum over all the indices of the charges. However it was shown in the previous section that it only makes sense to interpret the spatial components of (1.9) as brane charges. This creates a dilemma of how to interpret the time components of the charges in the SUSY algebra. It was suggested in [3, 4] that for the flatspace case these should be interpreted in terms of their Hodge duals. In other words, the time components of a p-form charge should be thought of as actually being the spatial components of a $D-p$-form charge. It then follows that for every p-brane that occurs in a given theory, there must also exist a $D-p$-brane associated with this $D-p$-form charge. For the $D = 11$ supergravity example we therefore conclude from (1.14) that a 6-brane and 9-brane must also exist. These correspond to the Kaluza-Klein (KK) monopole [24–28] and the M9-brane [29] respectively. This idea of examining the SUSY algebra to determine the brane spectra is well known and was carried out in [3, 4] for the IIA, IIB and $D = 11$ theories.

In [2] it was further shown that the origin of these charges appearing in the SUSY algebra was due to the fact that the Wess-Zumino term in the worldvolume actions is only quasi-invariant under global SUSY transformations. Subsequent work carried out in [30] showed that the presence of a worldvolume gauge potential in the worldvolume action, with non-trivial SUSY transformations, also gives rise to topological terms in the SUSY algebras. The M5-brane example was considered and in this instance a further result showed that the M5-brane algebra not only included a 5-form charge but also the M2-charge, describing mixed brane configurations. In such a case (1.9) with $p = 5$ can be thought of as some 'core' charge for the M5-brane but another term also appears in the algebra involving the 2-form worldvolume gauge potential and the M2-brane charge.[1]

This effect was further investigated in [31] where the general formulation of the method by which a worldvolume gauge potential can give rise to a topological extension to the SUSY algebra was presented. The cases of the D-branes were examined and a similar result to the M5-brane algebra was found. In this case the D-brane algebras not only contained the 'core' D-brane charges but also terms with lower rank charges corresponding to the lower dimensional D-branes, once again reflecting mixed brane configurations.

These results demonstrate that, due to the possibility of mixed brane configurations, in general the SUSY algebras include additional terms that do not take the form of (1.13), even with flat spacetime backgrounds as assumed in these works. Specific cases of curved backgrounds have also been investigated in [32–34]. Here various examples of branes immersed in backgrounds sourced by other branes were considered and the worldvolume superalgebras were constructed for these cases. The same general effect as for flat backgrounds was found with the added feature that the non-zero target space gauge fields now also appear in the algebras. See also [35–39] for extensions of AdS spacetime superalgebras. The generalised charges considered in this thesis provide an elegant way of extending these results to general asymptotically flat, curved backgrounds with fluxes by determining precisely which terms

[1] This effectively amounts to a redefinition of $Z^{\mu_1 \cdots \mu_5}$ in the background superalgebra given by (1.14) and so does not effect the matching of the degrees of freedom.

should appear in the SUSY algebras.[2]

1.6 Outline of Research

We now discuss the main research topic of this thesis, namely the investigation and construction of the generalised charges in the $D = 11$, IIA and IIB supergravity theories. As already mentioned, these generalise the relationship between the topological charges of the p-branes and the SUSY algebras to curved spacetimes. While this fact represents the primary motivation for their study, in this thesis we predominantly focus on the generalised charges themselves and not the curved space SUSY algebras.

While the structures of the flatspace SUSY algebras are useful for determining the various brane spectra it is well known that this process has a number of shortcomings. For example, there is no way to infer from the flatspace SUSY algebra that certain branes contain isometries in their spacetime solutions (or equivalently that certain branes contain gauged isometries in their worldvolume actions), the most well known examples being the KK-monopoles and M9-brane. This problem was acknowledged in [41] and led the authors to conclude that the usefulness of the flatspace SUSY algebra in determining the brane spectra had been overstated. Another problem with the flatspace SUSY algebra is that in some instances the same charge corresponds to more than one type of brane. An example of this is the triplet of 7-branes that exist in the IIB theory [42, 43]. Relating to this is the further problem that in these IIB cases the branes and their flatspace charges transform under different representations of the $SL(2,\mathbb{R})$ symmetry group. A further aim of this thesis is to demonstrate that the generalised charges do not suffer from these shortcomings.

The generalised charges are comprised of a series of terms involving combinations of the background gauge potentials and bilinear forms made out of products of a Killing spinor and antisymmetric combinations of Γ matrices. They therefore im-

[2]An alternative method for the construction of the charges in arbitrary supersymmetric curved backgrounds has been given in [40] which involves finding a worldvolume action which is invariant under any isometries of the background. The action relevant for the D-branes was explicitly given.

plicitly contain information about the preserved supersymmetry of the background through the presence of the Killing spinor and also the background curvature through the gauge fields and Γ matrices. In order to establish the relation between the generalised charges and the SUSY algebras one considers the anti-commutator of a specific SUSY transformation. For curved spacetimes the generalised charges then appear on the RHS of the algebra as a central extension. In the flatspace instance the charges simplify since the background gauge fields vanish and the flatspace algebras discussed in the previous section are produced.

Due to their topological interpretation, the generalised charges are required to be closed for arbitrary on-shell field configurations and this puts strong constraints on their possible constituent terms. In fact it turns out that this restriction is largely enough to uniquely determine their structure. The problem of formulating the generalised charges then effectively reduces to determining the exterior derivatives of the constituent terms and then finding a combination which is generally closed. This is therefore the main method adopted in this thesis.

The first investigation of the generalised charges was carried out in [1] where $D = 11$ supergravity was considered and the charges for the M2 and M5-branes were determined. This work was followed in [44] and [45] where partial results for the IIA and IIB supergravities were given respectively. The principle aim of this thesis is to provide a more complete treatment of these three supergravity theories.

We divide the research presented in this thesis into two parts based on the original works of the author [46] and [47]. In Part I we consider the charges of the 'conventional' spectra of branes. By 'conventional' we roughly mean those branes which have a well understood charge in the flatspace SUSY algebras, and that are usually listed as comprising the brane spectra from these considerations [3, 4]. Specifically, these consist of the M-branes in $D = 11$ supergravity, the D-branes, NS5-branes and F-strings in the $D = 10$ theories as well as the KK-monopoles that are found in each theory. In Part II we move on to the more exotic branes whose relation with the flatspace algebras is less well understood. We essentially define these as the additional branes required to fill up the $SL(2, \mathbb{R})$ multiplets in IIB supergravity, as well as those branes they map to on the IIA side via T-duality.

We begin the main body of this thesis in Chapter 2 where we review the work of [1] by considering the M2 and M5-brane charges in $D = 11$ supergravity. These are the simplest examples of charges and serve as a useful introduction. We treat these cases in detail, using them as a chance to discuss some of the main principles at work and explaining the steps to go about determining the structure of the charges. Furthermore, we relate the results to previous work in the literature on SUSY algebras and demonstrate how the modifications in curved spacetimes can be found.

From this point on the work presented in this thesis is predominantly original work carried out by the author. The first of which is a treatment of the $D = 10$ supergravities in Chapters 3 and 4 where the charges of the D-branes, NS5-branes and F-strings are considered. The process here is largely the same as the $D = 11$ case though more complicated due to the increased number of fields in these theories. At this point it becomes appropriate to discuss some further features of the charges that are not apparent in the $D = 11$ case. When dealing with the IIA we consider Romans' massive version [48] in order to properly formulate the D8-brane charge.

Next in Chapter 5 we consider the T-duality relations between the charges we have thus far constructed. Our motivation for doing this is to confirm that they map to one another in the same fashion as the branes themselves, which also serves as a consistency check on their structures. Showing that the charges T-dualise appropriately is important since in later chapters we use T-duality as a way of determining new charges from previously constructed ones.

We end Part I of the thesis in Chapter 6 where we consider the charges for the KK-monopoles. It turns out that there is a general difference between these charges compared with those previously considered due to the presence of the Taub-NUT isometry direction in their spacetime solutions. The process for constructing the charges in these cases is more complicated due to the need to incorporate these isometries. As a result the Killing vectors describing these isometries end up explicitly appearing in the charges. We discuss this point as well as providing details of the calculation for the $D = 11$ case before going on to present the $D = 10$ cases.

In Part II we set about determining the remaining charges for the exotic branes.

This requires us to consider a massive extension to the $D = 11$ theory and for this reason we delay presenting the M9-brane charge until this point. During this part of the investigation we emphasize the $SL(2, \mathbb{R})$ symmetry not only in the IIB theory but also in the $D = 11$ theory. We determine the various brane multiplets that exist and demonstrate how they map between theories. In order to construct these charges it is necessary to consider several gauge potentials that are often neglected when discussing these supergravity theories. We therefore spend some time investigating these potentials and determining their field strength equations and it is for this reason that we delay considering the exotic branes until this point.

Chapter 2

Generalised charges in $D = 11$ supergravity

We begin our formulation of the generalised charges by considering the M2 and M5-branes of $D = 11$ supergravity. This chapter is essentially an expanded review of [1] and serves as a useful introduction to some of the concepts and methods which will be used throughout the remainder of this thesis. Furthermore, we will expand upon some specific ideas in this chapter that we will not generally consider in the later chapters.

We start by first giving a short, non-rigorous review of $D = 11$ supergravity focusing on the relevant details that will be required later. These largely consists of the field content, flatspace SUSY algebra and the spectrum of branes. We then go on to discuss the relevant details necessary to formulate the generalised charges explaining the types of calculation that are performed. Finally we determine the generalised charges for the M2 and M5-brane and relate them to other results in the literature.

2.1 Review of $D = 11$ supergravity

The $D = 11$ supergravity was first constructed in [6]. In its most basic formulation the field content comprises of[1] an elfbein field $\hat{e}_{\hat{\mu}}{}^{\hat{A}}$, a Majorana gravitino $\hat{\Psi}^{\alpha}_{\hat{\mu}}$ and a 3-form gauge field $\hat{A}_{\hat{\mu}_1\hat{\mu}_2\hat{\mu}_3}$. A notable feature of this theory is the absence of a scalar field which we shall find simplifies the analysis of the generalised charges.

In our conventions the bosonic part of the action is given by

$$\hat{S} = \frac{1}{2}\int d^{11}\hat{x}\sqrt{-\hat{g}}\left(\hat{R} - \frac{1}{2.4!}|\hat{F}|^2\right) \tag{2.1}$$

together with a Chern-Simons term

$$-\frac{1}{2}\int \frac{1}{6}\hat{F}\wedge\hat{F}\wedge\hat{A} \tag{2.2}$$

where $\hat{F}$ is the 4-form field strength defined by $\hat{F} = d\hat{A}$. We use the mostly plus signature metric $(-, +, +, \ldots, +)$.

Varying this action with respect to $\hat{A}$ yields the sourceless equation of motion

$$d\hat{F}^{(7)} + \frac{1}{2}\hat{F}\wedge\hat{F} = 0 \tag{2.3}$$

where we have defined $\hat{F}^{(7)} = \hat{*}\hat{F}$. The convention we adopt for the Hodge dual operation on a p-form in a $D = d + 1$ dimensional spacetime is as follows

$$(*F)_{\mu_1\ldots\mu_{D-p}} = \frac{\sqrt{|g|}}{p!}\epsilon_{\mu_1\ldots\mu_{D-p}}{}^{\nu_1\ldots\nu_p}F_{\nu_1\ldots\nu_p} \tag{2.4}$$

with the antisymmetric symbol satisfying

$$\epsilon_{01\ldots d} = +1. \tag{2.5}$$

Equation (2.3) allows for the definition of the 6-form magnetic dual potential $\hat{C}$, given by

$$d\hat{C} = \hat{F}^{(7)} + \frac{1}{2}\hat{A}\wedge\hat{F}. \tag{2.6}$$

[1] We adopt the conventions of using Greek characters from the middle of the alphabet to denote curved spacetime indices, uppercase Roman characters from the beginning of the alphabet to denote orthonormal indices and Greek characters from the beginning of the alphabet to denote spinor indices. Furthermore, $D = 11$ objects and indices are denoted by a hat.

Despite the fact that only $\hat{A}$ makes an explicit appearance in the action, for our purposes both $\hat{A}$ and $\hat{C}$ are considered on an equal footing.

As discussed in the introduction, the spectrum of 'conventional' branes consists of the M2 and M5-branes as well as the KK-monopole [24–26] (with 7 dimensional worldvolume) and M9-brane [29]. It therefore follows that the flatspace SUSY algebra receives modifications to the right hand side from 2-form and 5-form charges

$$\begin{aligned}\{\hat{Q}_\alpha, \hat{Q}_\beta\} &= (\hat{C}\hat{\Gamma}^{\mu})_{\alpha\beta}\hat{P}_{\hat{\mu}} + \frac{1}{2}(\hat{C}\hat{\Gamma}_{\hat{\mu}_1\hat{\mu}_2})_{\alpha\beta}\hat{Z}^{\hat{\mu}_1\hat{\mu}_2} \\ &\quad + \frac{1}{5!}(\hat{C}\hat{\Gamma}_{\hat{\mu}_1\ldots\hat{\mu}_5})_{\alpha\beta}\hat{Z}^{\hat{\mu}_1\ldots\hat{\mu}_5}\end{aligned} \tag{2.7}$$

where $\hat{Q}_\alpha$ are real 32 component Majorana spinors, the charges are given by (1.9), $\hat{P}_\mu$ is the momentum and $\hat{C}$ is the charge conjugation matrix (not to be confused with the 6-form gauge potential). As already mentioned the spatial components of the charges are associated with the M2 and M5-brane charges, whereas those components that include a time index are interpreted by taking the Hodge dual and represent the charges of the KK-monopole and M9-brane.

To demonstrate the nature of the typical spacetime solutions of branes (and anti-branes) we will now give the explicit solutions for the M2 and M5-branes. These are given by [49]

$$\begin{aligned} d\hat{s}^2_{(11)} &= \hat{H}^{-\frac{2}{3}}d\hat{x}^2_{(1,2)} + \hat{H}^{\frac{1}{3}}d\hat{x}^2_{(8)} \\ \hat{F} &= \pm d(\hat{H}^{-1}) \wedge \epsilon_{(1,2)} \\ \hat{H} &= 1 + \frac{c_{(2)}N}{\hat{r}^6} \end{aligned} \tag{2.8}$$

and [50]

$$\begin{aligned} d\hat{s}^2_{(11)} &= \hat{H}^{-\frac{1}{3}}d\hat{x}^2_{(1,5)} + \hat{H}^{\frac{2}{3}}d\hat{x}^2_{(5)} \\ \hat{F} &= \pm * d(\hat{H}) \\ \hat{H} &= 1 + \frac{c_{(5)}N}{\hat{r}^3} \end{aligned} \tag{2.9}$$

respectively. Here $\hat{r}$ is the radial coordinate on the transverse Euclidean space and the Hodge dual operator acts within the transverse space only. $\epsilon_{(1,2)}$ is the volume form parallel to the M2-brane worldvolume and the constants $c_{(p)}$ are related to the branes' tensions. These solutions are each interpreted as N infinite, flat and coincident branes located at $r = 0$. In each case the entire solution depends only on the form of the harmonic function $\hat{H}$. This function can also be multi-centered where the solution becomes that of several non-coincident parallel branes.

One point to notice about these solutions is that they exhibit an $SO(p, 1)$ invariance along their worldvolumes, as well as an $SO(D - p - 1)$ invariance in the transverse space. This feature however is not fully exhibited by either the KK-monopole or M9-brane solutions due to the presence in these cases of compact isometry directions. In the case of the KK-monopole solution the isometry in question is that associated with the 4-dimensional Taub-NUT transverse space and therefore lies transverse to the worldvolume and breaks the $SO(4)$ symmetry to a local $SO(3) \times U(1)$ symmetry. The M9-brane on the other hand exists in the massive version of $D = 11$ supergravity presented in [51] which we will discuss in Chapter 8. A characteristic feature of this theory is the presence of a compact, space-like isometry, dimensional reduction over which yields Romans' massive IIA theory [48]. In fact this isometry direction actually arises from the M9-brane solution which sources the mass parameter of the theory. In contrast to the KK-monopole case, here the isometry lies parallel to the M9-brane worldvolume thus breaking the worldvolume symmetry from $SO(9, 1)$ to $SO(8, 1) \times U(1)$. For the sake of brevity we will refrain from giving the spacetime solutions for these states since their explicit form will not be required for our analysis.

The SUSY algebras in curved, supersymmetric backgrounds sourced by the various branes such as (2.8) and (2.9) have been investigated in the literature [32, 34]. Obviously the structure of these algebras depends on the specific background being considered. The generalised charges considered in this thesis provide a general method for determining the background SUSY algebras as a function of a given brane configuration that is acting as the source.

It is worth noting that the notion of reinterpreting the time components of the charges appearing in (2.7) in terms of the spatial components of the Hodge duals of the charges only makes sense in flat backgrounds. For curved backgrounds time and space components can be mixed when the charges are Hodge dualised. Therefore it is worth keeping in mind that a more democratic treatment of the charges is generally required where the algebra is written in terms of all the charges (i.e. also the KK-monopole and M9-brane charges) with the understanding that they are only evaluated on space-like hypersurfaces. This type of treatment of the SUSY algebra will be implied throughout the remainder of this thesis when considering all the generalised charges.

We will now discuss the method of constructing the generalised charges explicitly for the M2 and M5-brane cases. For the KK-monopole and M9-brane cases the presence of the isometries in their spacetime solutions is reflected in the structure of their generalised charges; this however, together with the fact that the M9-brane appears in massive $D = 11$ supergravity, creates additional complications in formulating these generalised charges. For this reason it is instructive to return to these cases separately after we have discussed the basic methodology. We discuss the KK-monopole case in Section 6.1 and the M9-brane case in Section 8.4.2.

Additionally one may attempt to apply these ideas to the M-wave solution [52] that is also found in this theory. However in this case the associated 'charge' appearing in the SUSY algebra is the momentum and therefore this case is qualitatively different from the others. An additional complicating factor is that the spacetime solution in this instance contains a null isometry direction. For these reasons we do not investigate the M-wave states in this thesis.[2]

2.2 Generalised charges for the M2 and M5-branes

The starting point for the formulation of the generalised charges is to consider the bilinear forms that consist of products of Γ matrices and spinors. Specifically, the general structure of a p-form bilinear is given by

$$K_{(p)} = \bar{\epsilon}\Gamma_{(p)}\epsilon \tag{2.10}$$

where $\Gamma_{(p)}$ is an antisymmetric product of p Γ matrices defined by

$$\{\hat{\Gamma}_{\hat{\mu}}, \hat{\Gamma}_{\hat{\nu}}\} = 2\hat{g}_{\hat{\mu}\hat{\nu}}. \tag{2.11}$$

[2] Furthermore, in the case of the momentum $\hat{P}_{\hat{\mu}}$ the time component is the Hamiltonian and as such we would not expect the 10-form dual 'charge' to be associated with a brane.

We work in the Majorana representation where due to our choice of signature the $\hat{\Gamma}$ are real 32×32 component matrices. ϵ is a spinor of the type found in the specific supergravity theory being considered. For the $D = 11$ case here $\hat{\epsilon}$ is a real Majorana spinor. We use the notation

$$\bar{\epsilon} = \epsilon^{\dagger}\Gamma^{\underline{0}} = \epsilon^{T}\Gamma^{\underline{0}}. \tag{2.12}$$

The underline of the index here signifies an orthonormal frame. In principle bilinears can of course be constructed from 2 different spinors, however here we only consider the case where a single spinor is used. In this case the bilinears are only non-zero for certain values of p due to the transpose properties of the Γ matrices in the particular supergravity theory one is dealing with. For $D = 11$ supergravity we have the identity[3]

$$(\hat{C}\hat{\Gamma}_{\hat{A}_1\ldots\hat{A}_p})^T = (-1)^{\frac{(p-1)(p-2)}{2}}(\hat{C}\hat{\Gamma}_{\hat{A}_1\ldots\hat{A}_p}). \tag{2.13}$$

Then by taking the transpose of the bilinears and using $\hat{C} = \hat{\Gamma}^{\underline{0}}$ it is trivial to show that they identically vanish for $p = 0, 3, 4$. We label the remaining non-zero forms for $p = 1, 2, 5$ as follows[4]

$$\begin{aligned} \hat{K}_{\mu} &= \hat{\bar{\epsilon}}\hat{\Gamma}_{\mu}\hat{\epsilon} && (2.14) \\ \hat{\omega}_{\mu_1\mu_2} &= \hat{\bar{\epsilon}}\hat{\Gamma}_{\mu_1\mu_2}\hat{\epsilon} && (2.15) \\ \hat{\Sigma}_{\mu_1\ldots\mu_5} &= \hat{\bar{\epsilon}}\hat{\Gamma}_{\mu_1\ldots\mu_5}\hat{\epsilon}. && (2.16) \end{aligned}$$

Such combinations of Γ matrices are assumed antisymmetrised

$$\Gamma_{\mu_1\ldots\mu_p} = \Gamma_{[\mu_1\ldots\mu_p]} \tag{2.17}$$

where there is a factor of $\frac{1}{p!}$ in our definition for anti-symmetrisation.

The bilinears with rank greater than 5 are related to these through Hodge duality. Adopting the convention

$$\hat{\Gamma}_{0123456789(10)} = +1 \tag{2.18}$$

[3]For a discussion of the properties of Γ matrices and the corresponding Clifford algebras they form in various dimensions see for example [53]

[4]Note that bilinears such as these are not fully independent, but rather satisfy certain algebraic relations due to the underlying Clifford algebra.

allows us to define the $p = 6, 9, 10$ bilinears as

$$\hat{\Lambda}_{(6)} = \hat{\bar{\epsilon}}\hat{\Gamma}_{(6)}\hat{\epsilon} = \hat{*}\hat{\Sigma}_{(5)} \tag{2.19}$$

$$\hat{\Pi}_{(9)} = \hat{\bar{\epsilon}}\hat{\Gamma}_{(9)}\hat{\epsilon} = -\hat{*}\hat{\omega}_{(2)} \tag{2.20}$$

$$\hat{\Upsilon}_{(10)} = \hat{\bar{\epsilon}}\hat{\Gamma}_{(10)}\hat{\epsilon} = \hat{*}\hat{K}_{(1)} \tag{2.21}$$

where the ranks of the forms have been temporarily indicated for convenience. Even though these higher rank bilinears are not independent from the lower rank ones we will see that it is natural to include them in our analysis when we consider the KK-monopole and M9-brane charges. This 'democratic' methodology will be adhered to throughout the course of this thesis. Note that these bilinears only vanish when $\hat{\epsilon}$ does [54].

The significance of these bilinears originates from the fact that they act as calibrations [55] for the branes in flat backgrounds [56, 57]. Simply put, a calibration is a closed p-form φ that satisfies an inequality

$$\varphi|_\xi \leq vol_\xi \tag{2.22}$$

for all oriented tangent p-planes ξ on the background manifold. It then follows that any submanifold where this inequality is saturated has a minimum volume (known as a minimal surface) compared to all other homologically equivalent submanifolds, i.e. submanifolds who share the same boundary and can be continuously deformed into one another. The branes are then said to be calibrated and have minimum volume density and are therefore minimum energy configurations.

This connection between the bilinears and the branes was shown in [57] to arise from the SUSY algebra. In addition to this, based on previous work carried out in [58] concerning adS spacetimes, the idea of calibrations was expanded upon and the idea of a 'generalised' calibration was introduced. These essentially amount to a reinterpretation of the ordinary calibrations for the branes in curved spacetimes by relaxing the requirement that they be closed. Rather, they are taken in conjunction with other terms that involve the background fields such that the overall expression is closed; these resulting expressions are the generalised charges we consider in this thesis. We will return to this idea when we consider the M2-brane generalised charge in the next subsection.

We summarise the connection between the bilinears, branes and charges in a flat background in Table 2.1. For the sake of completeness we include the 1-form bilinear $\hat{K}$, which naively should act as a calibrating form for M-waves. We will see however that this bilinear turns out to be Killing, presumably due to its connection with the momentum in the SUSY algebra, and plays a unique role in the structure of the generalised charges. Note that we have neglected the 10-form bilinear $\hat{\Upsilon}$ since it is not associated with any known brane. The potentials $\hat{N}^{(8)}$ and $\hat{A}^{(10)}$ which couple to the KK-monopole and M9-brane respectively will be discussed when we discuss these branes whereas $\hat{k}$ is the Killing vector describing the null isometry associated with the M-wave.

Brane	Charge	Bilinear	Potential
M-wave	$\hat{P}_i$	$\hat{K}$	$\hat{k}$
M2	$\hat{Z}_{i_1 i_2}$	$\hat{\omega}$	$\hat{A}$
M5	$\hat{Z}_{i_1 \ldots i_5}$	$\hat{\Sigma}$	$\hat{C}$
KK-monopole	$\hat{*}(\hat{Z}_{\underline{0} i_1 \ldots i_4})$	$\hat{\Lambda}$	$\hat{N}^{(8)}$
M9	$\hat{*}(\hat{Z}_{\underline{0} i_1})$	$\hat{\Pi}$	$\hat{A}^{(10)}$

Table 2.1: Branes, their calibrating bilinears and their flatspace charges in $D = 11$ supergravity. Note that the indices on the charges are purely spatial. Also included are the potentials that minimally couple to the branes.

In order to construct the generalised charges it will be crucial to first determine the exterior derivatives of these bilinears in curved spacetimes as a general function of the background field strengths. To do this we use the fact that the backgrounds we are considering here are purely bosonic and as a result the supersymmetry transformations of the Rarita-Schwinger fermion $\hat{\psi}_\mu$ must be zero in order for the solution to remain purely bosonic. Therefore any SUSY transformation parameter $\hat{\epsilon}$ must satisfy the Killing spinor equation

$$\delta_{\hat{\epsilon}} \hat{\psi}_{\hat{\mu}} = \hat{\tilde{D}}_{\hat{\mu}} \hat{\epsilon} = 0 \tag{2.23}$$

where

$$\hat{\tilde{D}}_{\hat{\mu}} = \hat{\nabla}_{\hat{\mu}} + \frac{1}{288} \left[\hat{\Gamma}_{\hat{\mu}}{}^{\hat{\nu}_1 \ldots \hat{\nu}_4} - 8 \delta_{\hat{\mu}}^{\hat{\nu}_1} \hat{\Gamma}^{\hat{\nu}_2 \hat{\nu}_3 \hat{\nu}_4} \right] \hat{F}_{\hat{\nu}_1 \ldots \hat{\nu}_4} . \tag{2.24}$$

Spinors which satisfy (2.23) are known as Killing spinors and we use them to construct the bilinear forms. The Killing spinor equation can then be used to calculate the derivatives of the bilinears in terms of the background field strengths and other bilinear forms. For a general p-form bilinear $\hat{K}_{(p)}$ we write its derivative according to

$$\begin{aligned}\hat{\nabla}_{\hat{\mu}}\hat{K}_{\hat{\nu}_1\ldots\hat{\nu}_p} &= \hat{\nabla}_{\hat{\mu}}(\hat{\bar{\epsilon}}\hat{\Gamma}_{\hat{\nu}_1\ldots\hat{\nu}_p}\hat{\epsilon}) \\ &= (\overline{\hat{\nabla}_{\hat{\mu}}\hat{\epsilon}})\hat{\Gamma}_{\hat{\nu}_1\ldots\hat{\nu}_p}\hat{\epsilon} + \hat{\bar{\epsilon}}\hat{\Gamma}_{\hat{\nu}_1\ldots\hat{\nu}_p}(\hat{\nabla}_{\hat{\mu}}\hat{\epsilon}) \qquad (2.25)\end{aligned}$$

where we have used the fact that the $\hat{\Gamma}$ matrices are covariantly constant. The derivative in the second term on the RHS of the lower line here is straight forward enough to replace using the Killing spinor equation, however the first term requires some slight manipulation using the following property of the $\hat{\Gamma}$ matrices

$$(\hat{\Gamma}_{\hat{\mu}})^{\dagger} = (\hat{\Gamma}_{\hat{\mu}})^{T} = \hat{\Gamma}_{\hat{0}}\hat{\Gamma}_{\hat{\mu}}\hat{\Gamma}_{\hat{0}} \qquad (2.26)$$

to produce the result

$$\overline{\hat{\nabla}_{\hat{\mu}}\hat{\epsilon}} = \frac{1}{288}\hat{\bar{\epsilon}}\left[\hat{\Gamma}_{\hat{\mu}}{}^{\hat{\nu}_1\ldots\hat{\nu}_4} + 8\delta_{\hat{\mu}}^{\hat{\nu}_1}\hat{\Gamma}^{\hat{\nu}_2\hat{\nu}_3\hat{\nu}_4}\right]\hat{F}_{\hat{\nu}_1\ldots\hat{\nu}_4}. \qquad (2.27)$$

Determining the derivatives then becomes a combinatoric exercise in combining the $\hat{\Gamma}$ according to their defining property (2.11). It is practical to work in an orthonormal basis to carry out this procedure in order to make use of the simpler anticommuting properties. We will refrain from giving the full details of each of the calculations since they are somewhat messy and not particularly enlightening. However the following is an example of the typical calculation that is required

$$\begin{aligned}\hat{\Gamma}^{\hat{C}_1\hat{C}_2\hat{C}_3}\hat{\Gamma}^{\hat{B}_1\ldots\hat{B}_p}\hat{F}_{\hat{C}_1\hat{C}_2\hat{C}_3\hat{A}} &= \hat{\Gamma}^{\hat{C}_1\hat{C}_2\hat{C}_3\hat{B}_1\ldots\hat{B}_p}\hat{F}_{\hat{C}_1\hat{C}_2\hat{C}_3\hat{A}} \\ &+ (-1)^{p-1}3p\hat{\Gamma}^{\hat{C}_1\hat{C}_2[\hat{B}_1\ldots\hat{B}_{p-1}}\hat{F}^{\hat{B}_p]}{}_{\hat{C}_1\hat{C}_2\hat{A}} \\ &- \frac{3.2p(p-1)}{2}\hat{\Gamma}^{\hat{C}_1[\hat{B}_1\ldots\hat{B}_{p-2}}\hat{F}^{\hat{B}_{p-1}\hat{B}_p]}{}_{\hat{C}_1\hat{A}} \\ &+ (-1)^{p}\frac{3.2p(p-1)(p-2)}{3!}\hat{\Gamma}^{[\hat{B}_1\ldots\hat{B}_{p-3}}\hat{F}^{\hat{B}_{p-2}\hat{B}_{p-1}\hat{B}_p]}{}_{\hat{A}}.\end{aligned} \qquad (2.28)$$

One sees that combining products of $\hat{\Gamma}$ matrices like this produces a variety of different types of term, in this case 4. Once the SUSY parameters are included in the calculation approximately half of these terms disappear since they would correspond to bilinears that identically vanish. Carrying out this procedure for the 1-form bilinear $\hat{K}$ yields

$$\hat{\nabla}_{\hat{\mu}}\hat{K}_{\hat{\nu}} = \frac{1}{3}i_{\hat{\omega}}\hat{F}_{\hat{\mu}\hat{\nu}} + \frac{1}{6}i_{\hat{F}}\hat{\Lambda}_{\hat{\mu}\hat{\nu}} \tag{2.29}$$

where we have employed the following inner product notation

$$i_A B_{\mu_1\mu_2} = \frac{1}{p!}A^{\nu_1\ldots\nu_p}B_{\nu_1\ldots\nu_p\mu_1\mu_2}. \tag{2.30}$$

The exterior derivative is then trivially found by antisymmetrising the free indices. The process to find the exterior derivatives for the other bilinears is essentially the same although slightly more complicated due to the larger number of $\hat{\Gamma}$ matrices involved. Ultimately we produce the following set of relations

$$\begin{aligned}
d\hat{K} &= \frac{2}{3}i_{\hat{\omega}}\hat{F} + \frac{1}{3}i_{\hat{\Sigma}}\hat{F}^{(7)} && (2.31)\\
d\hat{\omega} &= i_{\hat{K}}\hat{F} && (2.32)\\
d\hat{\Sigma} &= i_{\hat{K}}\hat{F}^{(7)} - \hat{\omega}\wedge\hat{F} && (2.33)\\
d\hat{\Lambda}_{\hat{\mu}_1\ldots\hat{\mu}_7} &= \frac{14}{3}\hat{\omega}^{\hat{\nu}}{}_{[\hat{\mu}_1}\hat{F}^{(7)}_{\hat{\mu}_2\ldots\hat{\mu}_7]\hat{\nu}} - \frac{35}{3}\hat{\Sigma}^{\hat{\nu}}{}_{[\hat{\mu}_1\ldots\hat{\mu}_4}\hat{F}_{\hat{\mu}_5\hat{\mu}_6\hat{\mu}_7]\hat{\nu}} && (2.34)\\
d\hat{\Pi} &= -\frac{1}{3}\hat{F}\wedge\hat{\Lambda} && (2.35)\\
d\hat{\Upsilon} &= 0 && (2.36)
\end{aligned}$$

where the first three relations have been derived previously in [54, 59]. From (2.29) it can be trivially read off that $\hat{\nabla}_{(\mu}\hat{K}_{\nu)} = 0$ and therefore $\hat{K}$ is a Killing vector.[5] Furthermore, using the following identity[6] for the Lie derivative of a p-form α with respect to a given Killing vector field X

$$\mathcal{L}_X\alpha = d(i_X\alpha) + i_X d\alpha \tag{2.37}$$

it is straight forward to show from the Bianchi identity for $\hat{F}$ and (2.32) that

$$\mathcal{L}_{\hat{K}}\hat{F} = 0 \tag{2.38}$$

and so $\hat{K}$ generates a symmetry of the solution.

[5] Additionally it turns out that $\hat{K}$ must be either time-like or null [54, 59]

[6] We will be making extensive use of this identity throughout the course of this thesis.

In expressing the relations (2.31)-(2.36) the following identity was used to rewrite some terms

$$\begin{aligned} & \frac{1}{m!} A^{\nu_1 \ldots \nu_m}{}_{[\mu_1 \ldots \mu_{(p-m)}} B_{\mu_{(p-m+1)} \cdots \mu_{(p+q-2m)}]\nu_1 \ldots \nu_m} \\ = & \frac{(-1)^{(D-q)(q-m)+m(p-m)+1}}{(D-p-q+m)!} \times \\ & (*B)^{\nu_1 \ldots \nu_{(D-p-q+m)}}{}_{[\mu_1 \ldots \mu_{(p-m)}} (*A)_{\mu_{(p-m+1)} \cdots \mu_{(p+q-2m)}]\nu_1 \ldots \nu_{(D-p-q+m)}} . \end{aligned} \tag{2.39}$$

This identity re-expresses a given term through the Hodge duals of the sub terms. In some cases this can convert a term involving both summed and anti-symmetrised indices into a 'neater' wedge product term or inner product term (2.30). This process will prove very useful particularly when considering the $D = 10$ supergravity theories since these 'neater' terms are more easily manipulated.

As a point of interest, in deriving the relations (2.31)-(2.36) it turned out that both derivative terms on the RHS of the lower line of (2.25) provided equal contributions to the RHS of these relations. Considering the more general case where the bilinear forms are each constructed from two distinct spinors requires the inclusion of a greater number of bilinears, namely those for $p = 0, 3, 4$ (and their Hodge duals) that are identically zero in the current case. One might then ask what additional terms containing these bilinears would appear in the relations (2.31)-(2.36). Once again both derivative terms on the RHS of the lower line of (2.25) produce the same set of terms involving these bilinears, however in these instances they have opposite signs and cancel. Therefore the form of (2.31)-(2.36) remains essentially unaltered in this more general case as can be seen in [54, 59].

2.2.1 M2-brane charge

It is a simple matter to demonstrate how the connection between the bilinears and branes arises from the flatspace SUSY algebra. For the case of the M2-brane we can consider an M2-brane probe in a flat background in which case the SUSY transformations are parametrised by constant commuting Majorana spinor fields $\hat{\epsilon}^{\alpha}$.

The M2-brane truncation of (2.7) then leads to

$$\begin{aligned}\{\hat{\epsilon}^\alpha \hat{Q}_\alpha, \hat{\epsilon}^\beta \hat{Q}_\beta\} = 2(\hat{\epsilon}\hat{Q})^2 &= (\hat{\epsilon}^T \hat{C}\hat{\Gamma}^{\hat{\mu}}\hat{\epsilon})\hat{P}_{\hat{\mu}} + \frac{1}{2}(\hat{\epsilon}^T \hat{C}\hat{\Gamma}_{\hat{\mu}_1\hat{\mu}_2}\hat{\epsilon})\hat{Z}^{\hat{\mu}_1\hat{\mu}_2} \\ &= \hat{K}^{\hat{\mu}}\hat{P}_{\hat{\mu}} + \frac{1}{2}\hat{\omega}_{\hat{\mu}_1\hat{\mu}_2}\hat{Z}^{\hat{\mu}_1\hat{\mu}_2} \end{aligned} \tag{2.40}$$

where we have used $\hat{C} = \hat{\Gamma}^{\underline{0}}$ and the definitions for $\hat{K}$ and $\hat{\omega}$ in the second line. The pullback of this to the M2-brane worldvolume gives the M2-brane worldvolume SUSY algebra as

$$2(\hat{\epsilon}\hat{Q})^2 = \hat{K}^{\hat{\mu}} \int_{M2} d^2\sigma \hat{p}_{\hat{\mu}} + \frac{1}{2}\hat{T}_2\ \hat{\omega}_{\hat{\mu}_1\hat{\mu}_2} \int_{M2} d\hat{X}^{\hat{\mu}_1} \wedge d\hat{X}^{\hat{\mu}_2} \tag{2.41}$$

thus

$$2(\hat{\epsilon}\hat{Q})^2 = \int_{M2} d^2\sigma \hat{K}^{\hat{\mu}}\hat{p}_{\hat{\mu}} + \hat{T}_2 \int_{M2} \hat{\omega} \tag{2.42}$$

where $\hat{T}_2$ is the M2-brane tension and σ_i are the worldvolume co-ordinates. We have used the definition of $\hat{Z}^{\mu_1\mu_2}$ given by (1.9), and the fact that the momentum $\hat{P}_\mu$ is defined by the integration of the momentum density $\hat{p}_\mu(\sigma)$ over the spatial worldvolume of the brane. Note that we have replaced $\hat{Q}_{(2)}$ by the brane tension $\hat{T}_2$ since these are BPS states and we have performed a pullback to the brane. We have also moved the constant bilinears into the integrals so that the expression is in the same form as we will find for curved backgrounds. $\hat{\omega}$ in (2.42) represents a central extension to the SUSY algebra and as a result must be topological. We therefore require $\hat{\omega}$ to be closed. As can be seen from (2.32) this is the case for flat backgrounds where the 4-form field strength vanishes.

The situation is not so straight forward when one considers curved supersymmetric backgrounds however. As discussed above, for the bosonic solutions we consider here the SUSY parameter $\hat{\epsilon}^\alpha$ is no longer constant but must satisfy the Killing spinor equation (2.23) which leads to the relation (2.32). As a results an extra term must be added to the RHS of (2.42) to form an expression that is closed generally. This expression will be our generalised charge (density) for the M2-brane.

With minimal effort one finds that the 2-form expression

$$\hat{L}^{(2)} = \hat{\omega} + i_{\hat{K}}\hat{A} \tag{2.43}$$

is generally closed if one chooses a gauge where

$$\mathcal{L}_{\hat{K}}\hat{A} = 0 \tag{2.44}$$

and uses the identity (2.37). However we must consider whether such a gauge condition is possible. From (2.38) we see that $\mathcal{L}_{\hat{K}}\hat{A}$ is closed and therefore, at least locally, exact. In fact we can show this directly by realising that for a general gauge choice for $\hat{A}$ we have $d\hat{L}^{(2)} = \mathcal{L}_{\hat{K}}\hat{A}$. Then considering the gauge transformation $\hat{A} \to \hat{A} + d\hat{\chi}$, where $\hat{\chi}$ is a 2-form gauge parameter, we find $\mathcal{L}_{\hat{K}}\hat{A} \to \mathcal{L}_{\hat{K}}\hat{A} + d(\mathcal{L}_{\hat{K}}\hat{\chi})$, in other words $\mathcal{L}_{\hat{K}}\hat{A}$ is shifted by an exact amount, which can be shown to be arbitrary by considering the degrees of freedom of $\hat{\chi}$. Therefore it must be possible in this instance to always satisfy the condition (2.44).

Therefore, if one fixes the gauge to satisfy (2.44), $\hat{L}^{(2)}$ is the generalised charge for the M2-brane. As written though, (2.43) is not gauge invariant. However, a modified version that is gauge invariant is easily found and is given by

$$\hat{L}^{(2)} = \hat{\omega} + i_{\hat{K}}\hat{A}_g - i_{\hat{K}}d\hat{\hat{\chi}}. \tag{2.45}$$

Here $\hat{A}_g$ represents the 3-form potential in a general gauge and $\hat{\hat{\chi}}$ is a gauge dependent parameter defined such that $\mathcal{L}_{\hat{K}}d\hat{\hat{\chi}} = \mathcal{L}_{\hat{K}}\hat{A}_g$. It is then trivial to show that (2.45) is generally closed. Note that we have $\hat{A}_g = \hat{A} + d\hat{\hat{\chi}}$ where $\hat{A}$ satifies (2.44) and so $\hat{\hat{\chi}}$ is merely a gauge parameter relating a general gauge to this special gauge. For simplicity we will always work with a potential that satisfies (2.44) and therefore work with (2.43) when discussing $\hat{L}^{(2)}$. For the other charges presented later in this thesis we will make analogous simplifications, giving the relevant gauge conditions each time.

The commutator of the SUSY transformations for curved spaces that are asymptotically Minkowski should generalise then from (2.42) to

$$2(\hat{\epsilon}\hat{Q})^2 = \int_{M2} d^2\sigma \hat{K}^M \hat{p}_M + \hat{T}_2 \int_{M2} (\hat{\omega} + i_{\hat{K}}\hat{A}). \tag{2.46}$$

Unlike for the flatspace case, the $\hat{\Gamma}$ matrices and spinors here are not constant and must be brought inside the integral.

We now demonstrate how to deduce worldvolume superalgebras from (2.46) by reproducing the result found in [32] where the M2-brane worldvolume algebra was determined in an M2-brane sourced background. To do this we recall that for M2-brane backgrounds the Killing spinors take the form $\hat{\epsilon} = \hat{H}^{-\frac{1}{6}}\hat{\epsilon}_0$ where $\hat{\epsilon}_0$ is a constant spinor and $\hat{H}$ is the harmonic function appearing in the M2-brane space-time, (2.8). We then simply strip off the constant spinors $\hat{\epsilon}_0$ and absorb the factors of $\hat{H}$ on the LHS into the SUSY generators which converts them from target space to worldvolume SUSY generators. Then we convert $\hat{\Gamma}^0$, used in the definition of the curved space $\hat{\bar{\epsilon}}$'s present in the bilinears, to the charge conjugation matrix $\hat{\Gamma}^{\underline{0}}$. The result will precisely coincide with the algebra given in [32] if we define our potential $\hat{A}$ so that its non-zero components are proportional to $\hat{H}^{-1} - 1$, so that they vanish asymptotically.

The process described thus far of replacing the flatspace charges with the generalised charges for curved backgrounds is essentially the same as that discussed in [57] where ordinary calibrations were replaced with generalised calibrations. This demonstrates the close relation between the generalised calibrations and the generalised charges discussed in this thesis. This viewpoint leads to the observation that the p-branes describe surfaces that minimise a particular energy functional consisting of the brane volume density and also other contributions depending on the background gauge fields. Such surfaces are termed minimal worldspaces. We will now demonstrate the essential details of this idea for the present case of the M2-brane and $\hat{L}^{(2)}$. In this case $\hat{\omega}$ acts as a generalised calibration so we have the defining inequality

$$\hat{\omega}|_{M2} \leq vol_{M2}. \tag{2.47}$$

We then consider another 2-dimensional spatial hypersurface U in the same homology class as the M2-brane such that $\partial M2 = \partial U$. We then find that

$$\int_{M2} d^2\sigma\sqrt{det\ \hat{g}} + \int_{M2} i_{\hat{K}}\hat{A} = \int_{M2}\hat{L}^{(2)} = \int_U \hat{L}^{(2)} \leq \int_U d^2\sigma\sqrt{det\ \hat{g}} + \int_U i_{\hat{K}}\hat{A} \tag{2.48}$$

where $\hat{g}$ is the background metric and pullbacks to a spatial hypersurface of the brane worldvolume are implied for the background fields. The first equality arises since the M2-brane is calibrated by $\hat{\omega}$ and so the inequality (2.47) is saturated. The second equality follows from the fact that $\hat{L}^{(2)}$ is closed. The final equality results from (2.47). We thus end up with the relation $E(M2) \leq E(U)$ where E is an energy whose contributions are described by $\hat{L}^{(2)}$. Generalising this argument to the other p-branes shows that the generalised charges provide a method for determining the precise nature of the energy contributions from the background gauge fields.

2.2.2 M5-brane charge

Applying the procedure described above leads to the following expression derived from the flatspace SUSY algebra (2.7) for the M5-brane worldvolume SUSY algebra in a flat background

$$2(\hat{\epsilon}\hat{Q})^2 = \int_{M5} d^2\sigma \hat{K}^{\hat{\mu}} \hat{p}_{\hat{\mu}} + \hat{T}_5 \int_{M5} \hat{\Sigma} \tag{2.49}$$

where once again a pullback of the background fields to the brane worldvolume is implied and $\hat{T}_5$ is the M5-brane tension. In curved backgrounds however it can be seen from (2.33) that $\hat{\Sigma}$ is not closed and so additional terms must be added to (2.49). From looking at the form of the M2-brane charge (2.43) we would expect to add a term of the form $i_{\hat{K}}\hat{C}$ since it is this field that minimally couples to the M5-brane. Adding such a term to $\hat{\Sigma}$ on its own does not however lead to a closed expression and so other terms must be added also.

As mentioned in the introduction, in [30] it was shown that (2.7) is too simplistic an algebra even in flat backgrounds and in fact the M5-brane algebra included not only a term of the form of $\hat{Z}^{\hat{\mu}_1\cdots\hat{\mu}_5}$, but also a term of the form

$$\frac{1}{2}(\hat{C}\hat{\Gamma}_{\mu_1\mu_2})_{\alpha\beta}\hat{\tilde{Z}}^{\mu_1\mu_2} \tag{2.50}$$

where

$$\hat{\tilde{Z}}^{\mu_1\mu_2} = \int_{M5} d\hat{X}^{\mu_1} \wedge d\hat{X}^{\mu_2} \wedge d\mathcal{B}. \tag{2.51}$$

Here $\mathcal{B}$ is the 2-form worldvolume gauge potential of the M5-brane and $d\mathcal{B} = \mathcal{H} + \hat{A}$ where $\mathcal{H}$ is the modified worldvolume field strength and $\hat{A}$ is the pullback of the 3-form background gauge field to the brane worldvolume. This term arises due to the possibility of having M2-branes contained entirely within the M5-brane worldvolume, so-called mixed brane configurations, and will be non-zero in such instances. We do not consider the worldvolume field $\mathcal{B}$ in this thesis,[7] however the presence of

[7]Its appearance in the generalised M5-brane charge is proposed in [1] however.

the pullback of $\hat{A}$ suggests that the generalised charge should contain a term of the form $\hat{\omega} \wedge \hat{A}$. Indeed we find that this is the case.

After some consideration we find that the expression

$$\hat{L}^{(5)} = \hat{\Sigma} + i_{\hat{K}}\hat{C} + \hat{L}^{(2)} \wedge \hat{A} - \frac{1}{2}\hat{A} \wedge i_{\hat{K}}\hat{A} \tag{2.52}$$

is closed if we once again fix the gauge according to (2.44) as well as

$$\mathcal{L}_{\hat{K}}\hat{C} = 0. \tag{2.53}$$

The argument used to show that (2.44) is an allowed gauge choice can be used here for the above gauge condition also.

From the generalised calibration interpretation the structure of $\hat{L}^{(5)}$ can be used to determine the form of the energy functional that is minimised for calibrated M5-branes. Note that the structure of (2.52) shows that the generalised M5-brane charge takes a more complicated form than the M2-brane charge. Furthermore, this shows that generalised calibrations can take a more complicated form than the expression given in [57] which essentially takes the same form as (2.45). This is due to the presence of the worldvolume field strength and the possibility of having mixed brane configurations which results in the extra terms present in $\hat{L}^{(5)}$ that have no analogues in the $\hat{L}^{(2)}$ case. These were not accounted for in [57] because in that reference only worldvolume scalar fields were considered with higher rank ones being set to zero.

Once again it would be fairly straightforward to produce the worldvolume algebras in curved backgrounds, albeit with the worldvolume field $\mathcal{B}$ set to zero. Some examples were found in [32], for M2 and M5-brane backgrounds. We see the same general structure as the charge presented here, except for those backgrounds the term $\hat{A} \wedge i_{\hat{K}}\hat{A}$ vanishes so is not present in the algebra.

2.2.3 Other $D = 11$ supergravity charges

We could now proceed by formulating the generalised charges for the KK-monopole and M9-brane. As already mentioned though, the process here is more complicated and the reason for this can be traced back to the presence of the compact isometry directions in their spacetime solutions. Their flatspace calibrations are the 6-form bilinear $\hat{\Lambda}$ and the 9-form $\hat{\Pi}$ respectively. Considering the KK-monopole case first, one sees that from a mathematical standpoint the problem arises from the complicated index structure present in the relation (2.34). These types of terms prevent the formulation of generalised charges since it is a non-trivial matter to integrate them. In some cases the index structure can be simplified by making use of the identity (2.39) however in this case no simplification occurs. The same issue also effects the construction of the M9-brane generalised charge. Even though in this instance the relevant relation (2.35) has a simple index structure, one finds that the generalised charge must contain $\hat{\Lambda}$ in one of its constituent terms and so we inevitably run into the same problem.

A further obstacle is that so far we have neglected to discuss in any detail the relevant potentials $\hat{N}^{(8)}$ and $\hat{A}^{(10)}$ which minimally couple to these branes. We would expect these to feature in these generalised charges so it should come as no surprise that we cannot proceed without them.

We will discuss how to formulate the generalised charges for the KK-monopole in Chapter 6. Here a general method for considering branes with an extra isometry direction will be discussed which will therefore also be applicable to the M9-brane case. However the M9-brane itself won't be considered until Chapter 8 where we consider the massive version of $D = 11$ supergravity. Before that we explore the simpler cases of the branes in the $D = 10$ supergravities.

Chapter 3

Generalised charges in IIA supergravity

The first $D = 10$ theory we consider is the IIA supergravity. This theory was originally constructed in [16–18] and later extended to Romans' IIA theory [48] (see also [60]) which includes a scalar mass parameter that acts as a cosmological constant and is non-zero in D8-brane backgrounds [61, 62]. We consider Romans' version of the theory in this thesis so that we can consider the complete spectrum of branes. However when formulating the generalised charges we initially set the mass parameter to zero before generalising to non-zero masses later due to a slight complication which arises in massive backgrounds. A partial analysis of the massless IIA theory was carried out in [44], here we extend those results.

The bulk of the analysis here follows on from the $D = 11$ case. As already discussed in the introduction, the IIA theory is determined from the $D = 11$ theory by performing a Kaluza-Klein dimensional reduction over a circle. However, here we opt to study the IIA theory in its own right since we feel this approach is more instructional. Considering the $D = 11$ origin however will play a crucial role in later chapters and so we provide the general rules for performing the dimensional reduction in Appendix A where we also give the relations between the $D = 11$ charges and those formulated in this chapter.

We begin by reviewing the essential details of the IIA supergravity theory that will be required for our purposes. We then move on to consider the types of bilinear forms that can be constructed from the IIA spinors before calculating the differential relations for these bilinears and explaining their importance in formulating the generalised charges. We then go on to formulate the D-brane and NS-brane charges in massless backgrounds, offering an interpretation of our results and comparisons with other results in the literature. Finally we consider the effect of having massive backgrounds and show that with a slight alteration the charges are equally valid in those cases.

3.1 Review of IIA supergravity

The basic field content of IIA supergravity consists of a vielbein ${e_\mu}^A$, the Ramond-Ramond (RR) n-form gauge potentials $C^{(n)}$ for $n = 1, 3$, the Neveu-Schwarz-Neveu-Schwarz (NS) 2-form gauge potential B, the NS scalar dilaton ϕ, the IIA gravitino ψ_μ^α and dilatino λ^α. In our conventions the bosonic part of the string frame IIA action is given by

$$S_{IIA} = \frac{1}{2}\int d^{10}x\sqrt{-g}\left[e^{-2\phi}\left(R + 4|\nabla\phi|^2 - \frac{1}{2.3!}|H|^2\right) - \frac{1}{2}\sum_n \frac{1}{n!}|F^{(n)}|^2\right] \tag{3.1}$$

where $n = 0, 2, 4$, plus a Chern-Simons term

$$-\frac{1}{2}\int \frac{1}{2}dC^{(3)} \wedge dC^{(3)} \wedge B + \frac{1}{6}mdC^{(3)} \wedge (B)^3 + \frac{1}{40}m^2(B)^5. \tag{3.2}$$

Once again we work with a mostly plus signature metric $(-, +, +, \ldots, +)$. $F^{(2)}$, $F^{(4)}$ and H are the field strengths for $C^{(1)}$, $C^{(3)}$ and B respectively. The mass parameter m is viewed as being a scalar RR field strength $F^{(0)}$ and we freely use both notations. Since we must follow a democratic approach it is necessary to also consider the magnetic duals of these fields. We define these as

$$F^{(6)} = - * F^{(4)} \qquad F^{(8)} = *F^{(2)} \qquad F^{(10)} = - * m \qquad H^{(7)} = e^{-2\phi} * H. \tag{3.3}$$

The field strength equations are defined as

$$F^{(2n)} = dC^{(2n-1)} - H \wedge C^{(2n-3)} + \frac{1}{n!}m(B)^n \tag{3.4}$$

$$H = dB \tag{3.5}$$

$$\begin{aligned} H^{(7)} = {} & dB^{(6)} + C^{(1)} \wedge F^{(6)} - \frac{1}{2}C^{(3)} \wedge (F^{(4)} + H \wedge C^{(1)}) \\ & -m(C^{(7)} - C^{(5)} \wedge B + \frac{1}{4}C^{(3)} \wedge (B)^2) \end{aligned} \tag{3.6}$$

for $n = 1, 2, 3, 4, 5$. These lead to the following Bianchi identities

$$\begin{aligned} dF^{(2n)} &= H \wedge F^{(2n-2)} && (3.7) \\ dH &= 0 && (3.8) \\ dH^{(7)} &= F^{(6)} \wedge F^{(2)} - \frac{1}{2} F^{(4)} \wedge F^{(4)} - mF^{(8)} && (3.9) \end{aligned}$$

which agree with those derived from the action by considering the variations of the gauge potentials.

The gauge potentials $C^{(5)}$, $C^{(7)}$ and $B^{(6)}$ are the magnetic duals of $C^{(3)}$, $C^{(1)}$ and B respectively. The 9-form potential $C^{(9)}$ on the other hand is not related to a lower rank field through duality. This results from the fact that its field strength $F^{(10)}$ is the Hodge dual of the mass parameter which obviously has no associated gauge potential since it is a scalar. $C^{(9)}$ can be introduced to the massive IIA action as a non-dynamical auxiliary field [61] and minimally couples to the D8-brane. Varying the action with respect to this field provides the constraint on the mass parameter $dm = 0$ showing it to be constant albeit with a possible discontinuity across D8-branes which act as domain walls.

The flatspace SUSY algebra of IIA reads

$$\begin{aligned} \{Q_\alpha, Q_\beta\} &= (C\Gamma^\mu)_{\alpha\beta} P_\mu + (C\Gamma_{11})_{\alpha\beta} Z_{11} + (C\Gamma^\mu \Gamma_{11})_{\alpha\beta} Z_\mu \\ &\quad + \frac{1}{2} (C\Gamma^{\mu_1\mu_2})_{\alpha\beta} Z_{\mu_1\mu_2} + \frac{1}{4!} (C\Gamma^{\mu_1\ldots\mu_4}\Gamma_{11})_{\alpha\beta} Z_{\mu_1\ldots\mu_4} \\ &\quad + \frac{1}{5!} (C\Gamma^{\mu_1\ldots\mu_5})_{\alpha\beta} Z_{\mu_1\ldots\mu_5} && (3.10) \end{aligned}$$

where Q_α are real 32 component Majorana spinors and Γ_{11} is the chirality operator defined below. From this we can deduce the spectrum of 'conventional' branes in IIA. These consist of the D0, D2, D4, D6 and D8-branes which couple to RR fields, and NS1(F-string) and NS5-brane which couple to the NS fields. The gravitational wave and IIA KK-monopole with 6-dimensional worldvolume are also included and are purely gravitational.

Finally the flatspace SUSY algebra reveals the existence of a spacetime filling 9-brane referred to as the NS9-brane in [60]. In this reference the supersymmetry transformations of the fields were considered and it was shown that additional gauge potentials to the ones listed above could be consistently included in the theory. These consist of two non-propagating 10-form potentials which we label $A^{(10)}$ and $\overline{A}^{(10)}$, as well as the 8-form dual to the dilaton which we label $\phi^{(8)}$. The 9-brane deducible from the flatspace SUSY algebra minimally couples to $\overline{A}^{(10)}$. Further considerations are required before we can study the branes to which these fields couple and formulate their generalised charges. We delay this part of the investigation until the second half of the thesis when we consider the exotic varieties of branes that exist.

3.2 Bilinears in IIA

Once again our starting point for the formulation of the generalised charges is to consider the types of bilinear forms that can be constructed. As for the $D = 11$ case the IIA theory contains Majorana spinors ϵ with 32 real components. However, here we can also have Majorana-Weyl spinors $\epsilon^{\pm}$ which have only 16 real components. These can either be chiral or anti-chiral reflecting the fact that IIA supergravity is a non-chiral theory. Specifically we have

$$\Gamma_{11}\epsilon^{\pm} = \pm\epsilon^{\pm} \tag{3.11}$$

where Γ_{11} is the chirality operator defined as

$$\Gamma_{11} = \Gamma_{0123456789}. \tag{3.12}$$

The two spinor representations are related by

$$\epsilon = \epsilon^{+} + \epsilon^{-}. \tag{3.13}$$

We ultimately choose to work with the Majorana spinors since this simplifies matters when relating the IIA and $D = 11$ results. However we will temporarily work with the Majorana-Weyl spinors in order to determine which of the bilinear forms are non-trivial. The general form for the bilinear is then written as $\overline{\epsilon}^{i}\Gamma_{(p)}\epsilon^{j}$ where the indices i, j take the values $+, -$ depending on the chirality of the spinor. We have the same transpose identity as in $D = 11$ given by (2.13), which translates to the bilinear identity

$$(\overline{\epsilon}^{i}\Gamma_{(p)}\epsilon^{j})^{T} = (-1)^{\frac{(p-1)(p-2)}{2}}(\overline{\epsilon}^{j}\Gamma_{(p)}\epsilon^{i}). \tag{3.14}$$

Unlike the $D = 11$ case however this relation now does not eliminate any values of p since the bilinears can be constructed from two different spinors, but rather determines whether the bilinear is symmetric or anti-symmetric in the indices i, j.

There is now a second identity originating from the chirality of the spinors. Using the chirality condition (3.11) one obtains the relation

$$\bar{\epsilon}^i \Gamma_{(p)} \epsilon^j = (-1)^{(i+j)} (\epsilon^i)^T \Gamma_{11}^T \Gamma^{\underline{0}} \Gamma_{(p)} \Gamma_{11} \epsilon^j \tag{3.15}$$

where

$$(-1)^i = \begin{cases} +1 & \text{if } i = + \\ -1 & \text{if } i = -. \end{cases}$$

Then from using the following properties of Γ_{11}

$$\{\Gamma_A, \Gamma_{11}\} = 0 \qquad \Gamma_{11}^T = \Gamma_{11} \qquad (\Gamma_{11})^2 = \mathbb{1} \tag{3.16}$$

the following identity is trivially obtained

$$\bar{\epsilon}^i \Gamma_{(p)} \epsilon^j = \begin{cases} (-1)^{p+1} \bar{\epsilon}^i \Gamma_{(p)} \epsilon^j & \text{if } i = j \\ (-1)^p \bar{\epsilon}^i \Gamma_{(p)} \epsilon^j & \text{if } i \neq j. \end{cases} \tag{3.17}$$

From the two identities (3.14) and (3.17) we find the following set of non-trivial bilinears

$$\begin{aligned} &\bar{\epsilon}^i \Gamma_{(p)} \epsilon^i && \text{for } p = 1, 5, 9 \\ &\bar{\epsilon}^- \Gamma_{(p)} \epsilon^+ = +\bar{\epsilon}^+ \Gamma_{(p)} \epsilon^- && \text{for } p = 2, 6, 10 \\ &\bar{\epsilon}^- \Gamma_{(p)} \epsilon^+ = -\bar{\epsilon}^+ \Gamma_{(p)} \epsilon^- && \text{for } p = 0, 4, 8. \end{aligned}$$

Recasting these in terms of Majorana spinors results in the following set of bilinears

$$\begin{array}{lll} \Psi^{(0)} = \bar{\epsilon} \Gamma_{11} \epsilon & K_A = \bar{\epsilon} \Gamma_A \epsilon & \tilde{K}_A = \bar{\epsilon} \Gamma_A \Gamma_{11} \epsilon \\ \Psi^{(2)}_{A_1 A_2} = \bar{\epsilon} \Gamma_{A_1 A_2} \epsilon & \Psi^{(4)}_{A_1 \ldots A_4} = \bar{\epsilon} \Gamma_{A_1 \ldots A_4} \Gamma_{11} \epsilon & \Sigma_{A_1 \ldots A_5} = \bar{\epsilon} \Gamma_{A_1 \ldots A_5} \epsilon \end{array}$$

together with their Hodge duals which are defined as

$$\tilde{\Sigma}^{(5)} = \bar{\epsilon} \Gamma_{(5)} \Gamma_{11} \epsilon = - * \Sigma^{(5)} \tag{3.18}$$

$$\Psi^{(6)} = \bar{\epsilon} \Gamma_{(6)} \epsilon = * \Psi^{(4)} \tag{3.19}$$

$$\Psi^{(8)} = \bar{\epsilon} \Gamma_{(8)} \Gamma_{11} \epsilon = - * \Psi^{(2)} \tag{3.20}$$

$$\Pi^{(9)} = \bar{\epsilon} \Gamma_{(9)} \epsilon = - * \tilde{K}^{(1)} \tag{3.21}$$

$$\tilde{\Pi}^{(9)} = \bar{\epsilon} \Gamma_{(9)} \Gamma_{11} \epsilon = - * K^{(1)} \tag{3.22}$$

$$\Psi^{(10)} = \bar{\epsilon} \Gamma_{(10)} \epsilon = * \Psi^{(0)} \tag{3.23}$$

where the ranks of the bilinears have been included for convenience.

As in the $D = 11$ case the majority of these bilinears act as calibrations for the p-branes in flat backgrounds. We summarise the connection in Table 3.1.

Brane	Charge	Bilinear	Potential
D0	Z_{11}	$\Psi^{(0)}$	$C^{(1)}$
D2	$Z_{i_1 i_2}$	$\Psi^{(2)}$	$C^{(3)}$
D4	$Z_{i_1 \ldots i_4}$	$\Psi^{(4)}$	$C^{(5)}$
D6	$*(Z_{\underline{0} i_1 i_2 i_3})$	$\Psi^{(6)}$	$C^{(7)}$
D8	$*(Z_{\underline{0} i_1})$	$\Psi^{(8)}$	$C^{(9)}$
F-string	Z_i	$\tilde{K}$	B
NS5	$Z_{i_1 \ldots i_5}$	Σ	$B^{(6)}$
9-brane	$*(Z_{\underline{0}})$	Π	$\overline{A}^{(10)}$
KK-monopole	$*(Z_{\underline{0} i_1 \ldots i_4})$	$\tilde{\Sigma}$	$i_\alpha N^{(7)}$

Table 3.1: Branes, their calibrating bilinears and their flatspace charges in IIA supergravity. Note that the indices on the charges are purely spatial. Also included are the potentials that minimally couple to the branes.

3.3 D-brane generalised charges

We now set about formulating the generalised charges by first calculating the differential relations for the bilinears. Once again this is achieved by considering the supersymmetry transformations of the fermions which must vanish for the bosonic backgrounds we are considering. Since there are now two fermions we have two

Killing spinor equations given by [60][1]

$$\begin{aligned}\delta\psi_\mu &= \nabla_\mu\epsilon - \frac{1}{8}H_{\mu\nu_1\nu_2}\Gamma^{\nu_1\nu_2}\Gamma_{11}\epsilon \\ &\quad +\frac{1}{8}e^{\phi}\left[\frac{1}{4!}F^{(4)}_{\nu_1\ldots\nu_4}\Gamma^{\nu_1\ldots\nu_4}\Gamma_\mu - \frac{1}{2}F^{(2)}_{\nu_1\nu_2}\Gamma^{\nu_1\nu_2}\Gamma_\mu\Gamma_{11} + m\Gamma_\mu\right]\epsilon \\ &= 0 \end{aligned} \tag{3.24}$$

$$\begin{aligned}\delta\lambda &= \left[\partial_\nu\phi\Gamma^\nu - \frac{1}{12}H_{\nu_1\nu_2\nu_3}\Gamma^{\nu_1\nu_2\nu_3}\Gamma_{11}\right]\epsilon \\ &\quad +\frac{1}{8}e^{\phi}\left[\frac{1}{12}F^{(4)}_{\nu_1\ldots\nu_4}\Gamma^{\nu_1\ldots\nu_4} - 3F^{(2)}_{\nu_1\nu_2}\Gamma^{\nu_1\nu_2}\Gamma_{11} + 10m\right]\epsilon \\ &= 0. \end{aligned} \tag{3.25}$$

The first of these can be used to produce differential relations for the bilinears in an analogous fashion to the $D = 11$ case. Carrying this process out for K yields

$$\nabla_\mu K_\nu = \frac{1}{2}i_{\tilde{K}}H_{\mu\nu} + \frac{1}{4}e^{\phi}\left[i_{F^{(4)}}\Psi^{(6)} + i_{\Psi^{(2)}}F^{(4)} - i_{F^{(2)}}\Psi^{(4)} - \Psi^{(0)}F^{(2)} + m\Psi^{(2)}\right]_{\mu\nu} \tag{3.26}$$

from which we deduce the important fact that $\nabla_{(\mu}K_{\nu)} = 0$ and therefore K is a Killing vector.

Unlike the $D = 11$ case, here we have a second Killing spinor equation (3.25) which can be used to produce a set of algebraic bilinear relations. This is done by hitting (3.25) from the left with $\bar{\epsilon}\Gamma_{(p)}$ for any p, or alternatively one can produce the Hodge dual of these by using $\bar{\epsilon}\Gamma_{(p)}\Gamma_{11}$.

The goal here is to determine the exterior derivatives of the calibrating bilinears. These are not only required in order to prove that a given generalised charge is closed, but more crucially we actually use them as the principal tool for formulating the generalised charges in the first place. Whilst it is possible to calculate a set of such relations from using (3.24) alone we find that these are insufficient to formulate the charges with and that it is necessary to supplement these with the second set of relations produced from (3.25).

We will illustrate this point in more detail by considering the case of $\Psi^{(0)}$ and the D0-brane generalised charge $M^{(0)}$. In this instance (3.24) yields

$$d\Psi^{(0)} = \frac{1}{2}i_{\Psi^{(2)}}H + \frac{1}{4}e^{\phi}\left[i_{F^{(4)}}\tilde{\Sigma} - i_K F^{(2)} + m\tilde{K}\right]. \tag{3.27}$$

[1] Note that our conventions differ from those in [60] by $B \to -B_{(2)}$, $C^{(1)} \to -C_{(1)}$, $C^{(5)} \to -C_{(5)}$ and $C^{(9)} \to -C_{(9)}$.

A differential relation such as this poses a problem when it comes to formulating a generalised charge since roughly speaking it is not obvious how to integrate some of the terms on the RHS due to their index structure. In other words there does not appear to be any terms that could be added to $\Psi^{(0)}$ such that the resulting expression is closed and therefore it does not seem possible to formulate a generalised charge with $\Psi^{(0)}$ as the lead term.

That such a problem arises is perhaps not too surprising since the majority of terms that might appear on the RHS of these differential relations in general will be non-integrable due to their index structure. The types of terms that can be integrated, and therefore from which we can determine the structure of the generalised charges, can be categorised into the following 3 varieties

- terms where all the indices of the sub-terms are anti-symmetrised and none are contracted, i.e. terms in the form of wedge products.
- terms that involve a contraction between K and a field strength.
- terms involving the mass parameter.

The first group has the appropriate index structure for integration. The second and third groups can be integrated by using the fact that the Lie derivatives of the gauge potentials with respect to K will either vanish or contain terms proportional to m. We will discuss this further below. Note that sometimes it will be necessary to use the identity (2.39) to re-express a given term that looks non-integrable into one of these groups.

The first step in formulating the generalised charges is then to find a differential relation for the calibrating bilinear that includes only terms of the type listed above. The relation (3.27) does not satisfy this requirement since it contains the problematic terms

$$i_{\Psi^{(2)}}H \text{ and } i_{F^{(4)}}\tilde{\Sigma}$$

which cannot be integrated.

With these considerations in mind we can produce an alternative differential relation by considering the algebraic relation obtained by hitting (3.25) with $\bar{\epsilon}\Gamma_A\Gamma_{11}$

from the left, given by

$$0 \quad = \quad -\Psi^{(0)}d\phi + \frac{1}{2}i_{\Psi^{(2)}}H + e^{\phi}\left[\frac{1}{4}i_{F^{(4)}}\tilde{\Sigma} + \frac{3}{4}i_K F^{(2)} + \frac{5}{4}m\tilde{K}\right]. \qquad (3.28)$$

Subtracting this from (3.27) and multiplying through by a factor of $e^{-\phi}$ yields

$$d(e^{-\phi}\Psi^{(0)}) \quad = \quad -i_K F^{(2)} - m\tilde{K} \qquad (3.29)$$

and we have thus found a differential relation that contains none of the problematic terms discussed above. This relation indicates that the leading term of the D0-brane generalised charge should be $e^{-\phi}\Psi^{(0)}$ instead of just $\Psi^{(0)}$. Considering the massless case where we set $m = 0$ it is then a simple matter to deduce the full structure of the D0-brane generalised charge as being

$$M^{(0)} = e^{-\phi}\Psi^{(0)} - i_K C^{(1)} \qquad (3.30)$$

where we work in a gauge where $\mathcal{L}_K C^{(1)} = 0$. For the massless case this gauge choice can be shown to be possible by following the argument given when the analogous gauge for the $D = 11$ 3-form potential was discussed. Similar gauge conditions in the massless theories will be used throughout the course of this thesis and in each instance the same argument can be given for their validity. For the sake of brevity we will not repeat this argument in each case. We will however note that as a consistency check we require the condition $\mathcal{L}_K F^{(2)} = 0$ in this instance. It is simple to show that this is satisfied by taking the exterior derivative of (3.29), and using the Bianchi identity (3.7) and differential relation (3.32) given below.

Formulating the generalised charges for the other D-branes follows essentially the same steps as carried out above with it being necessary to use relations derived from both (3.24) and (3.25). We will refrain from giving the details of these calculations since they offer no new insights but ultimately we find the following set of differential relations for the D-brane calibrating bilinears

$$\begin{aligned} d(e^{-\phi}\Psi^{(2n)}) \quad &= \quad -e^{-\phi}\Psi^{(2n-2)} \wedge H + (-1)^{n+1}i_K F^{(2n+2)} \\ &\quad\; +(-1)^{n+1}\tilde{K} \wedge F^{(2n)} \end{aligned} \qquad (3.31)$$

for $n = 0, 1, 2, 3, 4$. From this we conclude that the D-brane generalised charges must contain terms of the form $e^{-\phi}\Psi^{(2n)}$ rather than just $\Psi^{(2n)}$. In order to formulate the charges we need the further differential relation

$$d\tilde{K} \;=\; i_K H. \tag{3.32}$$

With these relations and the field strength equations at our disposal we can go about determining the form of the generalised charges by using the fact they are closed. Unlike for the D0-charge case, the charges are generally too complicated to deduce by inspection so a more systematic method is required. Essentially since we take $e^{-\phi}\Psi^{(2n)}$ as the lead terms, we start by considering the set of terms that match the RHS of the relations (3.31) but where the field strengths have been replaced by their respective gauge potentials. After choosing appropriate signs, the exterior derivatives of these terms will cancel with those on the RHS of (3.31), however there will be some new terms left over involving the field strengths. We then cancel these left over terms by following the same steps. We repeat the process as many times as necessary, canceling the remaining terms each time until none are left. The overall set of terms required to do this then determines the structure of the generalised charges.

Note that following this method does not guarantee that a closed expression, and therefore a generalised charge, will ultimately be found. In principle one would usually expect to reach a point where the remaining terms simply cannot be canceled. For this not to be the case a delicate cancellation between the terms must occur. The form of the field strength equations, (3.4) and (3.5) in the current case, play a crucial role in these cancellations. We find as a general result that it is not possible to find a closed expression corresponding to the generalised charges unless the field strength equations are defined in a consistent manner with the Bianchi identities. In other words, it is manifest in the formulation of the generalised charges that they can only be defined for on-shell field configurations.

A further point is that the method above breaks down for non-zero m since it has no associated gauge potential. For this reason we first consider the massless case where $m = 0$.

We now present the generalised charges for the D($2n$)-branes $M^{(2n)}$ which are

given by

$$M^{(2n)} = \sum_{j=0}^{n} \frac{1}{(n-j)!}\Big[e^{-\phi}\Psi^{(2j)} +(-1)^{j+1}(i_K C^{(2j+1)} + \tilde{K}\wedge C^{(2j-1)})\Big]\wedge (B)^{n-j} \qquad (3.33)$$

for $n = 0, 1, 2, 3, 4$ and where we have chosen gauges for the potentials that satisfy

$$\mathcal{L}_K C^{(2n+1)} = 0. \qquad (3.34)$$

These conditions are consistent with the result $\mathcal{L}_K F^{(2n)} = 0$ which is straightforwardly deducible from taking the exterior derivative of (3.31).

Note that we have formally included the D8-brane charge despite the fact that the D8-brane is a solution of the massive version of the theory. When we discuss the charges in massive backgrounds below we find that their general structure is unaltered and so it makes sense to include the D8-charge at this point.

We will now offer an interpretation of the general structure of these charges. Firstly, the factor multiplying the leading bilinears $e^{-\phi} = g_s^{-1}$ also appears multiplying the Dirac-Born-Infeld term in the D-brane worldvolume action [63] and can be interpreted as part of the branes' tension. As we saw from (1.9), the tension also appears in the flatspace charges (since these are BPS states we can equate the tension with the charge $Q_{(p)}$). For curved backgrounds where the tension is not constant it makes sense that the tension should appear as part of the integrand in the definition of the charge and this explains the appearance of the tension factor $e^{-\phi}$ in the leading term in (3.33).

The term involving a contraction between the Killing vector K and the RR potential to which the branes' minimally couple also appeared analogously in the $D = 11$ charges (2.43) and (2.52). These terms have the same general structure as those which appeared in [57] when discussing generalised calibrations, and appear to be a general feature of the charges.

As with the M5-brane charge, we see here a charge within charge structure. This can be made explicit by rewriting (3.33) as

$$\begin{aligned} M^{(2n)} &= e^{-\phi}\Psi^{(2n)} + (-1)^{n+1}(i_K C^{(2n+1)} + \tilde{K}\wedge C^{(2n-1)}) \\ &\quad + \sum_{j=0}^{n-1} \frac{(-1)^{n-j+1}}{(n-j)!} M^{(2j)} \wedge (B)^{n-j}. \end{aligned} \tag{3.35}$$

This nested structure arises due to the possibility of having branes within branes as discussed in [64]. The sub-charges will be non-zero when a lower dimensional brane wraps a topologically non-trivial submanifold. A similar line of thought leads to the interpretation of the terms involving $\tilde{K}$ as relating to the possibility of configurations with fundamental strings within branes. The situation is not quite as as straightforward since the full string charge (see (3.39) below) is not present in (3.35). However this is merely a result of our definition of the gauge potentials and is rectified by, for example, making the following redefinition $C^{(2n+1)} \to C^{(2n+1)} + C^{(2n-1)} \wedge B$. However, making this redefinition creates additional terms which partly cancel with the lower rank D-brane sub-charges. One then observes that it does not seem possible to define the potentials such that both the F-string charge and all the lower dimensional D-brane sub-charges simultaneously appear in (3.33) in their entirety. The interpretation of this is not clear to us but perhaps is a reflection of having certain restrictions on strings and branes within branes configurations.

We now compare these expressions to the results in [31] specifically equation (233). In this reference the worldvolume SUSY algebras on the IIA D-branes in flat backgrounds were investigated. It was found that additional terms to those induced from (3.10) were present which arose due to the non-trivial SUSY transformations of the worldvolume Born-Infeld 1-form V.

A complete comparison with these results is not possible since here we have considered the role of the background fields on the algebra and have neglected the role of the worldvolume fields. In [31] however, the converse case was investigated where the effect of the worldvolume fields were considered and the background fields were set to zero. Despite this, similarities between the two results are observed. We find that the combinations of Γ matrices determined in [31] are also present implicitly in (3.33) in the definitions of the bilinears $\Psi^{(2n)}$ and $\tilde{K}$. In the expressions here the $\Psi^{(2n)}$ bilinears are always wedged with the 2-form gauge field B some number of times. In [31] the same structure occurs except in that instance dV is present instead of B. Since we have not formally considered the worldvolume fields in our analysis we cannot concretely know the worldvolume field structure in the charges. However we note that the Wess-Zumino term in the D-brane worldvolume action typically takes the form of the complex (see for example [63])

$$S_{WZ} \sim \int C \wedge e^{(dV-B)} \tag{3.36}$$

where

$$C = \sum_{n=0}^{4} C^{(2n+1)} \tag{3.37}$$

and where all the terms of the appropriate rank are understood to be present and pullbacks on the background fields are implied. Due to the relationship between the SUSY algebras and the Wess-Zumino term this suggests that V and B should appear in the worldvolume generalised charges in the combination $(dV - B)$. This can also be argued on the grounds of gauge invariance. In other words we should make the substitution $B \rightarrow B - dV$ in (3.33). We note that after performing this substitution the charges remain closed due to the specific nature of the way B appears. Furthermore, complete agreement between the $\Psi^{(2n)}$ terms and the corresponding terms in [31] would be achieved.

Next we consider the terms involving $\tilde{K}$. Here we find that $\tilde{K}$ always appears wedged with a single RR potential as well as B a number of times. The relevant Γ matrices in [31] on the other hand are wedged with the worldvolume Hodge dual of the canonical conjugate of V. It is not a simple matter to relate these two expressions. However we note that the charges above remain closed if we make the substitution $C \rightarrow C + d\mathcal{A} \wedge e^B$ and assume $\mathcal{L}_K d\mathcal{A} = 0$ where

$$\mathcal{A} = \sum_{n=0}^{4} \mathcal{A}^{(2n)} \tag{3.38}$$

is a complex of additional worldvolume gauge potentials which were not considered in [31]. These fields were considered in [40, 51] where alternative forms of the D-brane actions were considered. We expect that if the analysis of [31] was applied to one of these more democratic actions then the results would match with the results here after making the above substitution. We note that after making this substitution the RR fields would appear in the charges in combinations equivalent to the worldvolume field strengths defined in the bottom line of equation (2.13) in [51] for $m = 0$.

The remaining terms in (3.33) which involve a contraction between the Killing vector K and an RR potentials are not produced from the method used in [31] but in principle they would, at least partially, arise from the Wess-Zumino term when calculating the conjugate momentum for non-zero background fields.

3.4 NS-brane generalised charges

Next we consider the charges for the IIA F-string and NS5-brane. To formulate these charges we will need the differential relations for $\tilde{K}$ and Σ as well as (3.31). The former is straightforward to calculate and was already given above (3.32). It is then a simple matter to determine the generalised charge for the F-string as being

$$M^{(F1)} = \tilde{K} + i_K B \tag{3.39}$$

where we have chosen a gauge where $\mathcal{L}_K B = 0$. This charge has a basic structure essentially due to its low rank. Since the tension of the F-string is independent of g_s we see no additional factor appearing in the lead term.

Moving on to the NS5-brane charge requires the differential relation for Σ. When calculating this from (3.24) we find that we run into the same problems as described above for $\Psi^{(2n)}$. We thus have to use an algebraic relation found by hitting (3.25) from the left with $\bar{\epsilon}\Gamma_{A_1 \ldots A_6}$. We then produce the following relation

$$\begin{aligned} d(e^{-2\phi}\Sigma) &= i_K H^{(7)} + e^{-\phi}\Big[-\Psi^{(0)}F^{(6)} - \Psi^{(2)} \wedge F^{(4)} - \Psi^{(4)} \wedge F^{(2)} \\ &\quad -m\Psi^{(6)}\Big]. \end{aligned} \tag{3.40}$$

We thus conclude that the leading term of the NS5-brane charge must be $e^{-2\phi}\Sigma$. This is in agreement with the general observation that the tension of the NS5-brane scales as $e^{-2\phi}$. The NS5-charge is then found for the massless case by following the method outlined above when discussing the D-brane charges. The result is

$$\begin{aligned} M^{(NS5)} &= e^{-2\phi}\Sigma + e^{-\phi}(\Psi^{(4)}\wedge C^{(1)} + \Psi^{(2)}\wedge C^{(3)} + \Psi^{(0)}C^{(5)}) + i_K B^{(6)} \\ &\quad + \tilde{K}\wedge C^{(1)}\wedge C^{(3)} - i_K C^{(1)}C^{(5)} + \frac{1}{2} i_K C^{(3)}\wedge C^{(3)} \end{aligned} \tag{3.41}$$

where we have chosen a gauge where $\mathcal{L}_K B^{(6)} = 0$ along with the previous RR gauge choices. These gauge conditions are consistent with the result $\mathcal{L}_K H^{(7)} = 0$ which is determined by taking the exterior derivative of (3.40). We see that there is not as clear a charge within charge structure compared to the D-brane charges although the necessary calibrating bilinears are present. Therefore the lower rank charges could appear in their entirety by making suitable redefinitions of the gauge potentials.

3.5 Massive charges

Next we consider the effect of having a massive background. Since the charges are a function of the gauge potentials rather than the field strengths we would not expect the mass parameter to make an explicit appearance in their structure. Consideration of the massive background therefore amounts to simply determining whether or not the charges constructed so far remain closed for non-zero m.

We begin by reconsidering the D0-brane charge (3.30) as a first example. For non-zero m we find

$$\begin{aligned} dM^{(0)} &= -m\tilde{K} - m i_K B - \mathcal{L}_K C^{(1)} \\ &= -m M^{(F1)} - \mathcal{L}_K C^{(1)} \end{aligned} \tag{3.42}$$

where we have left the gauge choice for $C^{(1)}$ undetermined. If we were to pick a gauge where $\mathcal{L}_K C^{(1)} = 0$ as in the massless theory, then $M^{(0)}$ would obviously not be closed. In the massive theory however, such a gauge choice is not generally possible. To see this consider for the moment the massless gauge transformation $C^{(1)} \to C^{(1)} + d\lambda^{(0)}$. This has the effect of shifting $\mathcal{L}_K C^{(1)}$ by an exact term but we see from (3.42) that, unlike the massless case, $\mathcal{L}_K C^{(1)}$ is now no longer exact so $\mathcal{L}_K C^{(1)} = 0$ cannot be achieved generally. This argument is essentially the same as we presented in Section 2.2.1 for the gauge choice of $\hat{A}$, where there it was used to show $\mathcal{L}_{\hat{K}}\hat{A} = 0$ was viable. It follows that in the current case we obtain the gauge condition

$$\mathcal{L}_K C^{(1)} = -mM^{(F1)} \tag{3.43}$$

for which $M^{(0)}$ is closed. Since $M^{(F1)}$ is closed this condition is consistent with the result $\mathcal{L}_K F^{(2)} = 0$ which still holds for non-zero m. For consistency with the field equations (3.4) we require that the higher rank RR gauge potentials satisfy

$$\mathcal{L}_K C^{(2n-1)} = -\frac{1}{(n-1)!} mM^{(F1)} \wedge (B)^{(n-1)} \tag{3.44}$$

which can be viewed as the generalisation of (3.34) to massive backgrounds. With these gauge conditions satisfied all the D-brane charges (3.33) are found to be closed in the massive version of the theory. Note that we still have $\mathcal{L}_K F^{(2n)} = 0$ as a general result.

Looking next at the NS-branes, we find for the F-string that the charge (3.39) is still closed for $\mathcal{L}_K B = 0$ since neither the exterior derivative of B or $\tilde{K}$ receive massive corrections. For the NS5-charge (3.41) we now find

$$dM^{(NS5)} = -mM^{(6)} + \mathcal{L}_K B^{(6)} - \mathcal{L}_K C^{(1)} \wedge C^{(5)} + \frac{1}{2}\mathcal{L}_K C^{(3)} \wedge C^{(3)}. \tag{3.45}$$

Therefore we see that $M^{(NS5)}$ is closed if we impose the following condition on $B^{(6)}$

$$\mathcal{L}_K B^{(6)} \;\; = \;\; mM^{(6)} - mM^{(F1)} \wedge C^{(5)} + \frac{1}{2} mM^{(F1)} \wedge C^{(3)} \wedge B \tag{3.46}$$

together with (3.44). Consistency between these conditions can be checked using (3.6) together with the fact that the result $\mathcal{L}_K H^{(7)} = 0$ still holds in massive backgrounds.

We therefore conclude that all the IIA charges derived previously are valid for massive backgrounds by virtue of a generalisation of the gauge conditions imposed on the potentials.

Chapter 4

Generalised charges in IIB supergravity

In this chapter we will formulate the generalised charges for the conventional branes in the IIB supergravity theory [21–23]. As discussed in the introduction, the IIB theory is related to the IIA theory via T-duality. We could therefore analyse the IIB theory via its T-duality relation from IIA. However, we initially opt to study the IIB theory in its own right since we feel this approach is more instructional. We will explore the T-duality aspects however in Chapter 5. In later chapters when we consider the exotic branes, T-duality will prove to be an essential tool in formulating their charges. Furthermore, in this chapter we will not consider the global $SL(2,\mathbb{R})$ symmetry that is known to exist in the IIB theory, nor will we consider the multiplets to which each type of brane belongs. We will deal with this subject in Chapter 12.

The bulk of the analysis here follows on analogously from the IIA case. We begin by reviewing the essential details of the IIB supergravity required for our purposes. We then move on to consider the types of bilinear forms that can be constructed from the IIB spinors before calculating their differential relations. We then use these relations to formulate the generalised charges for the D-branes and finally the NS-branes.

4.1 Review of IIB supergravity

The basic field content of IIB supergravity consists of a vielbein $e_\mu{}^A$, the RR n-form gauge potentials $\mathcal{C}^{(n)}$ for $n = 0, 2, 4$, the NS 2-form potential $\mathcal{B}$, the NS scalar dilaton φ, the IIB gravitino ψ_μ^α and dilatino λ^α. In our conventions the bosonic part of the string frame IIB action is given by

$$\begin{aligned} S_{IIB} &= \frac{1}{2}\int d^{10}x\sqrt{-g}\Big[e^{-2\varphi}\Big(R + 4|\nabla\varphi|^2 - \frac{1}{2.3!}|\mathcal{H}|^2\Big) \\ &\quad -\frac{1}{2}\sum_{n=1,3}\frac{1}{n!}|\mathcal{F}^{(n)}|^2 - \frac{1}{4.5!}|\mathcal{F}^{(5)}|^2\Big] \end{aligned} \tag{4.1}$$

plus a Chern-Simons term

$$-\frac{1}{2}\int \mathcal{C}^{(4)} \wedge \mathcal{H} \wedge \mathcal{F}^{(3)}. \tag{4.2}$$

Once again we work with a mostly plus signature metric $(-, +, +, \ldots, +)$. $\mathcal{F}^{(1)}$, $\mathcal{F}^{(3)}$, $\mathcal{F}^{(5)}$ and $\mathcal{B}$ are the field strengths for l, $\mathcal{C}^{(2)}$, $\mathcal{C}^{(4)}$ and $\mathcal{H}$ respectively. Note that we employ both notations $\mathcal{C}^{(0)}$ and l to denote the scalar RR potential, or axion. Since we must follow a democratic approach it is necessary to also consider the magnetic duals of these fields. We define these as

$$\mathcal{F}^{(7)} = *\mathcal{F}^{(3)} \qquad \mathcal{F}^{(9)} = - * \mathcal{F}^{(1)} \qquad \mathcal{H}^{(7)} = e^{-2\varphi} * \mathcal{H} \tag{4.3}$$

with $\mathcal{F}^{(5)} = - * \mathcal{F}^{(5)}$ being anti-self-dual in our conventions. We follow the usual approach where this self-duality condition is only imposed at the level of the equations of motion derived from (4.1).

The field strength equations are defined as

$$\mathcal{F}^{(2n+1)} = d\mathcal{C}^{(2n)} - \mathcal{C}^{(2n-2)} \wedge \mathcal{H} \tag{4.4}$$

$$\mathcal{H} = d\mathcal{B} \tag{4.5}$$

$$\mathcal{H}^{(7)} = d\mathcal{B}^{(6)} - \mathcal{C}^{(2)} \wedge \mathcal{F}^{(5)} - \frac{1}{2}\mathcal{C}^{(2)} \wedge \mathcal{C}^{(2)} \wedge \mathcal{H} + l\mathcal{F}^{(7)} \tag{4.6}$$

for $n = 0, 1, 2, 3, 4, 5$. These lead to the following Bianchi identities

$$d\mathcal{F}^{(2n+1)} = \mathcal{H} \wedge \mathcal{F}^{(2n-1)} \tag{4.7}$$

$$d\mathcal{H} = 0 \tag{4.8}$$

$$d\mathcal{H}^{(7)} = \mathcal{F}^{(1)} \wedge \mathcal{F}^{(7)} - \mathcal{F}^{(3)} \wedge \mathcal{F}^{(5)} \tag{4.9}$$

which are in agreement with those derived from the action by considering the variations of the gauge potentials.

The gauge potentials $\mathcal{C}^{(6)}$, $\mathcal{C}^{(8)}$ and $\mathcal{B}^{(6)}$ are the magnetic duals of $\mathcal{C}^{(2)}$, l and $\mathcal{B}$ respectively, with $\mathcal{C}^{(4)}$ being self-dual. Note that we have formally included the 10-form potential $\mathcal{C}^{(10)}$ whose 11-form field strength trivially vanishes and which is not the magnetic dual of a lower rank potential. The existence of this potential can be argued on the grounds of T-duality and is known to couple to the D9-brane [65–68].

The flatspace SUSY algebra reads

$$\begin{aligned}\{Q^i_\alpha, Q^j_\beta\} \;=\;& (C\mathcal{P}^+\Gamma^\mu)_{\alpha\beta}(\delta^{ij}P_\mu + \sigma_3^{ij}Z_\mu + \sigma_1^{ij}\tilde{Z}_\mu) \\ &+\frac{1}{3!}i\sigma_2^{ij}(C\mathcal{P}^+\Gamma^{\mu_1\mu_2\mu_3})_{\alpha\beta}Z_{\mu_1\mu_2\mu_3} + \frac{1}{5!}\delta^{ij}(C\mathcal{P}^+\Gamma^{\mu_1\ldots\mu_5})_{\alpha\beta}Z_{\mu_1\ldots\mu_5} \\ &+\frac{1}{5!}(C\mathcal{P}^+\Gamma^{\mu_1\ldots\mu_5})_{\alpha\beta}(\sigma_3^{ij}\tilde{Z}_{\mu_1\ldots\mu_5} + \sigma_1^{ij}\overline{Z}_{\mu_1\ldots\mu_5}) \qquad (4.10)\end{aligned}$$

where the Q^i_α are real 16 component Majorana-Weyl spinors, $\mathcal{P}^+ = \frac{1}{2}(1+\Gamma_{11})$ is a chiral projector, σ_i are the Pauli matrices and the 5-form charges are anti-self-dual.

From this we can deduce the spectrum of conventional branes in IIB. These consist of the D1, D3, D5, D7 and D9-branes which couple to RR fields, and NS1(F-string) and NS5-branes which couple to NS fields. The gravitational wave and IIB KK-monopole with 6-dimensional worldvolume are also included and are purely gravitational.

The existence of a second 9-brane can also be deduced from (4.10). This was studied in [68] and couples to an additional 10-form potential whose existence (along with that of other 10-form potentials) can be inferred by considering the SUSY transformations of all the fields [67]. We class this brane as exotic and therefore don't consider it until the second half of this thesis.

4.2 Bilinears in IIB

Following the previous cases we begin the formulation of the generalised charges by first considering the types of bilinear forms that can be constructed in the IIB theory. In this instance we work with two real 16 component Majorana-Weyl spinors ϵ^i where $i = 1, 2$. Both of these are chiral reflecting the fact that IIB supergravity is a chiral theory. Specifically we have

$$\Gamma_{11}\epsilon^i = \epsilon^i. \tag{4.11}$$

The general form of the bilinears is then written as $\bar{\epsilon}^i\Gamma_{(p)}\epsilon^j$. To determine which of these are non-trivial we repeat the analysis used for the IIA case. In this instance however we have to take into account that the theory is chiral. Because of this the identity (3.17) becomes

$$\bar{\epsilon}^i\Gamma_{(p)}\epsilon^j = (-1)^{p+1}\bar{\epsilon}^i\Gamma_{(p)}\epsilon^j \quad \text{for all } i,j \tag{4.12}$$

and we conclude that only bilinears with odd p are non-trivial. Combining this with (3.14) we find the following set of non-trivial bilinears

$$\begin{aligned} \bar{\epsilon}^i\Gamma_{(p)}\epsilon^i \qquad\qquad & \text{for } p = 1,5,9 \\ \bar{\epsilon}^1\Gamma_{(p)}\epsilon^2 = +\bar{\epsilon}^2\Gamma_{(p)}\epsilon^1 \quad & \text{for } p = 1,5,9 \\ \bar{\epsilon}^1\Gamma_{(p)}\epsilon^2 = -\bar{\epsilon}^2\Gamma_{(p)}\epsilon^1 \quad & \text{for } p = 3,7. \end{aligned}$$

We label these as follows

$$\begin{gathered} \Phi^{(1)}_A = \bar{\epsilon}^1\Gamma_A\epsilon^2 \qquad\qquad K^{11}_A = \bar{\epsilon}^1\Gamma_A\epsilon^1 \qquad\qquad K^{22}_A = \bar{\epsilon}^2\Gamma_A\epsilon^2 \\ \Phi^{(3)}_{A_1A_2A_3} = \bar{\epsilon}^1\Gamma_{A_1A_2A_3}\epsilon^2 \\ \Phi^{(5)}_{A_1\ldots A_5} = \bar{\epsilon}^1\Gamma_{A_1\ldots A_5}\epsilon^2 \quad \Sigma^{11}_{A_1\ldots A_5} = \bar{\epsilon}^1\Gamma_{A_1\ldots A_5}\epsilon^1 \quad \Sigma^{22}_{A_1\ldots A_5} = \bar{\epsilon}^2\Gamma_{A_1\ldots A_5}\epsilon^2 \end{gathered}$$

together with their Hodge duals

$$\begin{aligned} \Phi^{(7)} = \;& \bar{\epsilon}^1\Gamma_{(7)}\epsilon^2 && = + * \Phi^{(3)} \\ \Phi^{(9)} = \;& \bar{\epsilon}^1\Gamma_{(9)}\epsilon^2 && = - * \Phi^{(1)} \\ \Omega^{11} = \;& \bar{\epsilon}^1\Gamma_{(9)}\epsilon^1 && = - * K^{11} \\ \Omega^{22} = \;& \bar{\epsilon}^2\Gamma_{(9)}\epsilon^2 && = - * K^{22}. \end{aligned}$$

The 5-form bilinears are all anti-self-dual in our conventions. For later convenience we define

$$\begin{aligned} K^+ &= \frac{1}{2}(K^{11} + K^{22}) \qquad & K^- &= \frac{1}{2}(K^{11} - K^{22}) \\ \Sigma^+ &= \frac{1}{2}(\Sigma^{11} + \Sigma^{22}) \qquad & \Sigma^- &= \frac{1}{2}(\Sigma^{11} - \Sigma^{22}) \\ \Omega^+ &= \frac{1}{2}(\Omega^{11} + \Omega^{22}) \qquad & \Omega^- &= \frac{1}{2}(\Omega^{11} - \Omega^{22}). \end{aligned}$$

Once again the majority of these bilinears act as calibrations for the p-branes in flat backgrounds. We summarise the connection in Table 4.1. The correspondence between the branes and the potentials is given in [68] for example.

Brane	Charge	Bilinear	Potential
D1	$\tilde{Z}_i$	$\Phi^{(1)}$	$\mathcal{C}^{(2)}$
D3	$Z_{i_1 i_3 i_2}$	$\Phi^{(3)}$	$\mathcal{C}^{(4)}$
D5	$\overline{Z}_{i_1 \ldots i_5}$	$\Phi^{(5)}$	$\mathcal{C}^{(6)}$
D7	$*(Z_{0 i_1 i_2})$	$\Phi^{(7)}$	$\mathcal{C}^{(8)}$
D9	$*(\tilde{Z}_0)$	$\Phi^{(9)}$	$\mathcal{C}^{(10)}$
F-string	Z_i	K^-	$\mathcal{B}$
NS5	$\tilde{Z}_{i_1 \ldots i_5}$	Σ^-	$\mathcal{B}^{(6)}$
9-brane	$*(Z_0)$	Ω^-	$\overline{\mathcal{A}}^{(10)}$
KK-monopole	$Z_{i_1 \ldots i_5}$	Σ^+	$i_\alpha \mathcal{N}^{(7)}$

Table 4.1: Branes, their calibrating bilinears and their flatspace charges in IIB supergravity. Note that the indices on the charges are purely spatial and that the 5-form charges are self-dual. Also included are the potentials that minimally couple to the branes.

4.3 D-brane generalised charges

Next we set about formulating the generalised charges for the D-branes by first calculating the differential relations for the calibrating bilinears. As with the previous cases this is achieved by considering the supersymmetry transformations of the fermions which must vanish for the bosonic backgrounds we are considering. This produces the following Killing spinor equations

$$\delta_{\epsilon^i} \psi_\mu = (\mathcal{D}_\mu \epsilon)^i = 0 \tag{4.13}$$

$$\delta_{\epsilon^i} \lambda = (\mathcal{P} \epsilon)^i = 0 \tag{4.14}$$

where

$$\begin{aligned} D_\mu &= \nabla_\mu + \frac{1}{8} H_{\mu\nu_1\nu_2}\Gamma^{\nu_1\nu_2}\otimes\sigma_3 \\ &+\frac{1}{16}e^{\varphi}\sum_{n=1}^{5}\frac{(-1)^{n-1}}{(2n-1)!}\mathcal{F}^{(2n-1)}_{\nu_1\ldots\nu_{2n-1}}\Gamma^{\nu_1\ldots\nu_{2n-1}}\Gamma_\mu\otimes\lambda_n \qquad (4.15)\\ \mathcal{P} &= \Gamma^\nu\partial_\nu\varphi + \frac{1}{12}H_{\nu_1\ldots\nu_3}\Gamma^{\nu_1\ldots\nu_3}\otimes\sigma_3 \\ &+\frac{1}{4}e^{\varphi}\sum_{n=1}^{5}\frac{(-1)^{n-1}(n-3)}{(2n-1)!}\mathcal{F}^{(2n-1)}_{\nu_1\ldots\nu_{2n-1}}\Gamma^{\nu_1\ldots\nu_{2n-1}}\otimes\lambda_n \qquad (4.16)\end{aligned}$$

and

$$\lambda_n = \begin{cases} \sigma_1 & \text{if } n \text{ even} \\ i\sigma_2 & \text{if } n \text{ odd.}\end{cases}$$

From using (4.13) we can calculate the differential relation for K^+ which reads as

$$\nabla_\mu K^+_\nu = -\frac{1}{2}i_{K^-}\mathcal{H}_{\mu\nu} + \frac{1}{4}e^{\varphi}\Big[-i_{\mathcal{F}^{(1)}}\Phi^{(3)} + i_{\mathcal{F}^{(3)}}\Phi^{(5)} + i_{\Phi^{(1)}}\mathcal{F}^{(3)} - i_{\Phi^{(3)}}\mathcal{F}^{(5)}\Big]_{\mu\nu} \qquad (4.17)$$

from which we conclude $\nabla_{(\mu}K^+_{\nu)} = 0$ and thus K^+ is a Killing vector. Note that neither of the other two 1-form bilinears K^- and $\Phi^{(1)}$ share this property.

When we determine the exterior derivatives of the calibrating bilinears we find that in general they contain non-integrable terms as described in the IIA case. In order to remove these terms we make use of the algebraic Killing spinor equation (4.14). This process is in complete analogy to that of the IIA case. As an example of this calculation we will give the details for the case of $\Phi^{(1)}$. Using (4.13) we find

$$d\Phi^{(1)}_{A_1A_2} = \frac{1}{2}\mathcal{H}^{B_1B_2}{}_{[A_1}\Phi^{(3)}_{A_2]B_1B_2} + e^{\phi}\left[\frac{1}{2}i_{K^+}\mathcal{F}^{(3)} + \frac{1}{2}i_{\mathcal{F}^{(3)}}\Sigma^+\right]_{A_1A_2} \qquad (4.18)$$

and we see that the RHS contains terms that cannot be easily integrated. In order to produce an alternative relation we use (4.14) to calculate $\bar{\epsilon}^1\Gamma_{A_1A_2}(\mathcal{P}\epsilon)^2$ and $\bar{\epsilon}^2\Gamma_{A_1A_2}(\mathcal{P}\epsilon)^1$ to obtain

$$\begin{aligned}\frac{1}{2}(\bar{\epsilon}^1\Gamma_{A_1A_2}(\mathcal{P}\epsilon)^2 - \bar{\epsilon}^2\Gamma_{A_1A_2}(\mathcal{P}\epsilon)^1) &= (\Phi^{(1)}\wedge d\varphi)_{A_1A_2} + \frac{1}{2}\mathcal{H}^{B_1B_2}{}_{[A_1}\Phi^{(3)}_{A_2]B_1B_2} \\ &+e^{\phi}\left[K^-\wedge\mathcal{F}^{(1)} - \frac{1}{2}i_{K^+}\mathcal{F}^{(3)} + \frac{1}{2}i_{\mathcal{F}^{(3)}}\Sigma^+\right]_{A_1A_2} \\ &= 0. \qquad (4.19)\end{aligned}$$

We then subtract this algebraic relation from (4.18) and multiply through by a factor of $e^{-\varphi}$ to produce the following differential relation

$$d(e^{-\varphi}\Phi^{(1)}) = i_{K^+}\mathcal{F}^{(3)} - K^-\wedge\mathcal{F}^{(1)} \qquad (4.20)$$

which is free from the problematic terms on the RHS.

The same process can be used to determine all the differential relations for the D-brane calibrating bilinears. These turn out to be given by[1]

$$d(e^{-\varphi}\Phi^{(2n+1)}) = e^{-\varphi}\Phi^{(2n-1)} \wedge \mathcal{H} + (-1)^n i_{K^+}\mathcal{F}^{(2n+3)} + (-1)^{n+1}K^- \wedge \mathcal{F}^{(2n+1)} \quad (4.21)$$

for $n = 0, 1, 2, 3, 4$. Using these relations together with

$$dK^- = -i_{K^+}\mathcal{H} \quad (4.22)$$

and also the field strength equations (4.4) and (4.5) we can set about formulating the D-brane generalised charges $N^{(2n+1)}$. We follow the same process we outlined when formulating the IIA D-brane charges. We ultimately find that they are given by

$$\begin{aligned} N^{(2n+1)} &= \sum_{j=0}^{n} \frac{1}{(n-j)!}\Big[e^{-\varphi}\Phi^{(2j+1)} + (-1)^j i_{K^+}\mathcal{C}^{(2j+2)} \\ &\quad +(-1)^{j+1}K^- \wedge \mathcal{C}^{(2j)}\Big] \wedge (\mathcal{B})^{n-j} \end{aligned} \quad (4.23)$$

for $n = 0, 1, 2, 3, 4$ where we have imposed the gauge condition $\mathcal{L}_{K^+}\mathcal{C}^{(2n+2)} = 0$. These charges can be rewritten to emphasize their charge within charge structure as follows

$$\begin{aligned} N^{(2n+1)} &= e^{-\varphi}\Phi^{(2n+1)} + (-1)^n i_{K^+}\mathcal{C}^{(2n+2)} + (-1)^{n+1}K^- \wedge \mathcal{C}^{(2n)} \\ &\quad + \sum_{j=0}^{n-1} \frac{(-1)^{n-j+1}}{(n-j)!} N^{(2j+1)} \wedge (\mathcal{B})^{n-j}. \end{aligned} \quad (4.24)$$

It is clear that the general structure of these charges is similar to those of the IIA D-branes as one would expect, and the discussion in reference to those charges is equally as valid here. There are however a few additional comments specific to the current charges which we will now discuss.

In analogy to the IIA D-branes we find that here the leading bilinears in each case are multiplied by a factor of $e^{-\varphi}$ which arises due to the dependence of the branes' tensions on the dilaton. There is however a discrepancy with the D1-brane

[1]The set of differential relations derived purely from (4.13) as well as the algebraic relations derived from (4.14) were given in [45].

since in this case the tension also depends on the axion. Specifically the D1-brane tension scales as $\sqrt{e^{-2\varphi}+l^2}$, [68]. Examining the generalised charge for the D1-brane, (4.23) with $n=0$, we see that it is unique amongst the charges we have considered so far in that it contains two bilinears that are of the same rank as the charge itself, namely $\Phi^{(1)}$ and K^-. Although it is only $\Phi^{(1)}$ that calibrates the D1-branes in flat backgrounds, in the context of the D1-brane generalised charge it seems natural that both $\Phi^{(1)}$ and K^- should be considered on an equal footing.

From their interpretation as generalised calibrations, the pullback of both these bilinears to some 1-dimensional line will obey some bound related to the length of the line. This is given schematically by $\Phi^{(1)}|_{line} \leq length$ for example, where '*length*' will depend on the norm of the Killing spinors which of course vary in curved backgrounds. Assuming that the bilinears are orthogonal in the sense that for a line where one of the bilinears satisfies the bound the other will vanish, then the leading bilinear terms in the D1-brane charge will obey a bound of the form

$$(e^{-\varphi}\Phi^{(1)} - lK^-)|_{line} \leq length\sqrt{e^{-2\varphi}+l^2}. \tag{4.25}$$

For BPS states this bound is saturated and the brane tension will scale as $\sqrt{e^{-2\varphi}+l^2}$ which is in agreement with the tension factor for the D1-brane. This general effect will come into play whenever a generalised charge contains more than one term that involves a calibrating bilinear of rank equal to that of the charge itself.

We note that we have included the term $i_{K^+}\mathcal{C}^{(10)}$ in the D9-brane charge even though its presence is not required for the charge to be closed since this term is trivially closed itself. From examining the structure of the charges constructed thus far it does however seem as though this term should be included. It is a general feature that for spacetime filling branes such as this the requirement that the charge is closed does not uniquely define its structure due to its high rank. A more robust method of formulating the charges is to use the duality relations that exist between these branes and other non-spacetime filling branes. We examine the T-duality relationship between the IIB D9-brane charge and the IIA D8-brane charge in Chapter 5 and find that the term $i_{K^+}\mathcal{C}^{(10)}$ is required in the D9-brane charge. As a point of interest, we see that the $\mathcal{N}=1$ truncation required to rectify the gauge anomalies and charge non-conservation [65, 66, 74] associated with spacetime filling branes is not required in order for the D9-brane charge to be closed.

4.4 NS-brane generalised charges

Next we consider the charges for the IIB F-string and NS5-brane. To formulate these charges we will need the differential relations for K^- and Σ^- as well as (4.21). The former is straight forward to calculate and was already given above (4.22). It is then a simple matter to determine the generalised charge for the F-string as being

$$N^{(F1)} = K^- - i_{K^+}\mathcal{B} \tag{4.26}$$

where we have imposed the gauge condition $\mathcal{L}_{K^+}\mathcal{B} = 0$. This charge has an analogous structure to the IIA F-string. Since the F-string tension is independent of the scalars of the theory we see no factor appearing in the lead term.

As an aside one can determine the tension of a (p, q)-string bound state by considering the D1-brane charge (4.23) together with (4.26). Adopting the convention that $(1, 0)$ denotes an F-string and $(0, 1)$ a D1-brane, we find that the leading terms for a (p, q)-string charge are

$$pN^{(F1)} + qN^{(1)} \sim (p - lq)K^- + qe^{-\varphi}\Phi^{(1)}. \tag{4.27}$$

Following the argument given when discussing the D1-brane tension above, we determine the tension here to scale as

$$\sqrt{(p - lq)^2 + e^{-2\varphi}q^2} \tag{4.28}$$

which is in agreement with [68] once we make the redefinition $l \to -l$.

Finally we consider the NS5-brane charge. In this case as well as the relation for $d\Sigma^-$ obtained from (4.13) we require the algebraic relation for

$$\frac{1}{2}\left[\bar{\epsilon}^1\Gamma_{A_1\ldots A_6}(\mathcal{P}\epsilon)^1 - \bar{\epsilon}^2\Gamma_{A_1\ldots A_6}(\mathcal{P}\epsilon)^2\right]$$

obtained from (4.14). Combining these we deduce the following relation

$$\begin{aligned} d(e^{-2\varphi}\Sigma^-) &= -i_{K^+}H^{(7)} + e^{-\varphi}\Big[\Phi^{(1)} \wedge \mathcal{F}^{(5)} + \Phi^{(3)} \wedge \mathcal{F}^{(3)} \\ &\quad +\Phi^{(5)} \wedge \mathcal{F}^{(1)}\Big]. \end{aligned} \tag{4.29}$$

From this and the differential relations for the other bilinears (4.21) and (4.22), together with the field equations (4.4), (4.5) and(4.6), we determine that the NS5-brane charge is given by

$$\begin{aligned} N^{(NS5)} &= e^{-2\varphi}\Sigma^{-} + e^{-\varphi}(l\Phi^{(5)} + \Phi^{(3)} \wedge \mathcal{C}^{(2)} + \Phi^{(1)} \wedge \mathcal{C}^{(4)}) \\ &\quad - i_{K^+}\mathcal{B}^{(6)} + \frac{1}{2}K^{-} \wedge \mathcal{C}^{(2)} \wedge \mathcal{C}^{(2)} - lK^{-} \wedge \mathcal{C}^{(4)} \\ &\quad + i_{K^+}\mathcal{C}^{(2)} \wedge \mathcal{C}^{(4)} \end{aligned} \tag{4.30}$$

where we have imposed the condition $\mathcal{L}_{K^+}\mathcal{B}^{(6)} = 0$ in addition to our previous gauge choices which is consistent with the result $\mathcal{L}_{K^+}\mathcal{H}^{(7)} = 0$ which can be determined by taking the exterior derivative of (4.29).

The presence of the lower rank bilinears suggests an implicit charge within charge structure here that can be made explicit by suitable redefinitions of the potentials. Looking at the above charge we see that it contains two 5-form bilinears Σ^{-} and $\Phi^{(5)}$, with relative factors $e^{-2\varphi}$ and $le^{-\varphi}$. Assuming that the bilinears calibrate orthogonal hypersurfaces we conclude that the IIB NS5-brane tension scales as $e^{-\varphi}\sqrt{e^{-2\varphi} + l^2}$ which is in agreement with [68]. It is straight forward to extend this idea to determine the tension of (p, q)-5-brane bound states.

Chapter 5

Generalised charges and T-duality

In this chapter we determine the relationships under T-duality between the IIA and IIB generalised charges previously given. One would expect that the charges map to one another in an analogous way to how the branes themselves map to one another. Our motivation here is to confirm that this is indeed the case, which also serves as a consistency check on the charge structures we have so far established.

The usual T-duality procedure maps between the massless IIA and IIB theories, however since we have so far considered Romans' IIA theory we must also consider a generalisation of this procedure. There are two approaches to do this. The first was considered in [61] (see also [28]). Since the IIB theory has no standard massive deformation, here the mass parameter is produced by performing a Scherk-Schwarz dimensional reduction [69] on the IIB side using the subgroup of the global $SL(2,\mathbb{R})$ symmetry that involves shifts of the axion. In the IIB theory the mass parameter is therefore expressed by the potentials having a linear dependence on the T-duality isometry direction. The usual massless T-duality rules are then modified in order to account for this dependence and are then referred to as the 'massive' T-duality rules.

The second approach was presented in [43] where the IIB potentials remain *independent* of the T-duality isometry direction. Instead the mass parameter is encoded on the IIB side through a modification of the derivative operator along the isometry direction. This can be viewed as a pseudo-reformulation of IIB which we loosely refer to as 'massive' IIB by which we simply mean a IIB theory which is

T-dual to a massive extension of IIA. The mass terms in IIA now explicitly map to mass terms in IIB that arise from the modification of the derivative operator. The advantage of this scheme is that the usual massless T-duality rules for the gauge potentials remain valid.

At the level of the field equations and action both these schemes are equivalent since in both approaches the field strengths remain independent of the isometry direction. The equivalence however is less clear when one considers expressions that involve only the potentials since in the former scheme it is not clear to us how to interpret the isometry dependence on the IIA side. Such a situation arises when considering the generalised charges and for this reason we adopt the latter approach when T-dualising the charges. A complete account of the T-duality relations must therefore include the 'massive' reformulation of IIB. This however is more easily expressed in the $SL(2,\mathbb{R})$ covariant formulation of the theory and so we present the details of this in Chapter 12. In the current chapter we will be satisfied with simply determining how the charges relate to each other when mapping between theories.

5.1 T-duality rules for the background fields

The general T-duality rules for the metric and background gauge potentials have been determined previously in the literature [70, 71]. We now restate these rules in terms of the definitions of the potentials adopted in this thesis. For the purposes of the T-duality we split the co-ordinates into $\{\bar{\mu}_i, y\}$ where y parametrises the compact isometry direction over which the T-duality is being performed and the $\bar{\mu}_i$ represent the other 9 directions. We denote the IIA/B metrics by $g^{(A/B)}_{\mu\nu}$ and in both theories the Killing vector describing the T-duality isometry by β. In our adapted co-ordinate system we have $\beta^\mu = \delta^{\mu y}$, $|\beta|^2 = \beta_y = R^2$ in the IIA case and $|\beta|^2 = \beta_y = \mathcal{R}^2$ in the IIB case where R and $\mathcal{R}$ are the radii of the compact isometries in IIA and IIB respectively. Furthermore, the metrics take the form

$$g^{(A)}_{\mu\nu} = \begin{pmatrix} g^{(9)}_{\bar{\mu}\bar{\nu}} + \beta_{\bar{\mu}}\beta_{\bar{\nu}}/R^2 & \beta_{\bar{\mu}} \\ \beta_{\bar{\nu}} & R^2 \end{pmatrix} \tag{5.1}$$

with a similar form for $g^{(B)}_{\mu\nu}$ where $R \to \mathcal{R}$. $g^{(9)}_{\bar\mu\bar\nu}$ is interpreted as the $D = 9$ supergravity metric.

In this co-ordinate system the T-duality rules for the metric going from IIA to IIB are

$$\begin{aligned}
g^{(A)}_{\bar\mu\bar\nu} &\to g^{(B)}_{\bar\mu\bar\nu} - (g^{(B)}_{\bar\mu y} g^{(B)}_{\bar\nu y} - \mathcal{B}_{\bar\mu y}\mathcal{B}_{\bar\nu y})/g^{(B)}_{yy} \\
g^{(A)}_{\bar\mu y} &\to -\mathcal{B}_{\bar\mu y}/g^{(B)}_{yy} \\
g^{(A)}_{yy} &\to 1/g^{(B)}_{yy}
\end{aligned} \tag{5.2}$$

and similarly from IIB to IIA

$$\begin{aligned}
g^{(B)}_{\bar\mu\bar\nu} &\to g^{(A)}_{\bar\mu\bar\nu} - (g^{(A)}_{\bar\mu y} g^{(A)}_{\bar\nu y} - B_{\bar\mu y}B_{\bar\nu x})/g^{(A)}_{yy} \\
g^{(B)}_{\bar\mu y} &\to -B_{\bar\mu y}/g^{(A)}_{yy} \\
g^{(B)}_{yy} &\to 1/g^{(A)}_{yy}.
\end{aligned} \tag{5.3}$$

From these we deduce the transformations of β_μ. Transforming from IIA to IIB we have

$$\beta_{\bar\mu} \to \mathcal{R}^{-2} i_\beta \mathcal{B}_{\bar\mu}. \tag{5.4}$$

Similarly, going from IIB to IIA is given by

$$\beta_{\bar\mu} \to R^{-2} i_\beta B_{\bar\mu}. \tag{5.5}$$

Finally R and $\mathcal{R}$ are mapped according to

$$R \leftrightarrow \mathcal{R}^{-1}. \tag{5.6}$$

In addition to this we will need to consider the mappings for the inverse metrics. These can be determined from considering that their general structures are given by

$$g^{\mu\nu}_{(A)} = \begin{pmatrix} g^{\bar\mu\bar\nu}_{(9)} & -R^{-2} g^{\bar\mu\bar\rho}_{(9)}\beta_{\bar\rho} \\ -R^{-2} g^{\bar\nu\bar\rho}_{(9)}\beta_{\bar\rho} & R^{-2} + R^{-4} g^{\bar\rho\bar\lambda}_{(9)}\beta_{\bar\rho}\beta_{\bar\lambda} \end{pmatrix} \tag{5.7}$$

for IIA, with a similar structure for $g^{\mu\nu}_{(B)}$ where $R \to \mathcal{R}$. We then conclude

$$g^{\bar\mu\bar\nu}_{(A)} \to g^{\bar\mu\bar\nu}_{(B)} \tag{5.8}$$

$$g^{\bar\mu y}_{(A)} \to -g^{\bar\mu\bar\rho}_{(B)} i_\beta \mathcal{B}_{\bar\rho} \tag{5.9}$$

$$g^{yy}_{(A)} \to \mathcal{R}^2 + g^{\bar\rho\bar\lambda}_{(B)} i_\beta \mathcal{B}_{\bar\rho} i_\beta \mathcal{B}_{\bar\lambda} \tag{5.10}$$

and

$$g^{\bar\mu\bar\nu}_{(B)} \rightarrow g^{\bar\mu\bar\nu}_{(A)} \tag{5.11}$$

$$g^{\bar\mu y}_{(B)} \rightarrow -g^{\bar\mu\bar\rho}_{(A)} i_\beta B_{\bar\rho} \tag{5.12}$$

$$g^{yy}_{(B)} \rightarrow R^2 + g^{\bar\rho\bar\lambda}_{(A)} i_\beta B_{\bar\rho} i_\beta B_{\bar\lambda}. \tag{5.13}$$

The T-duality transformations for the potentials going from IIA to IIB are

$$\begin{aligned}
\phi &\rightarrow \varphi - \frac{1}{2}\log(\mathcal{R}^2) \\
i_\beta B_{\bar\mu} &\rightarrow \mathcal{R}^{-2}\beta_{\bar\mu} \\
B_{\bar\mu_1\bar\mu_2} &\rightarrow (\mathcal{B} + \mathcal{R}^{-2} i_\beta \mathcal{B} \wedge \beta)_{\bar\mu_1\bar\mu_2} \\
i_\beta C^{(2n+1)}_{\bar\mu_1\ldots\bar\mu_{2n}} &\rightarrow (\mathcal{C}^{(2n)} + \mathcal{R}^{-2} i_\beta \mathcal{C}^{(2n)} \wedge \beta)_{\bar\mu_1\ldots\bar\mu_{2n}} \\
C^{(2n+1)}_{\bar\mu_1\ldots\bar\mu_{2n+1}} &\rightarrow (-i_\beta\mathcal{C}^{(2n+2)} + \mathcal{C}^{(2n)} \wedge i_\beta\mathcal{B} - \mathcal{R}^{-2} i_\beta\mathcal{C}^{(2n)} \wedge i_\beta\mathcal{B} \wedge \beta)_{\bar\mu_1\ldots\bar\mu_{2n+1}} \\
i_\beta B^{(6)}_{\bar\mu_1\ldots\bar\mu_5} &\rightarrow (i_\beta\mathcal{B}^{(6)} - \frac{1}{2} i_\beta\mathcal{C}^{(4)} \wedge \mathcal{C}^{(2)} - \frac{1}{2}\mathcal{R}^{-2} i_\beta\mathcal{C}^{(4)} \wedge i_\beta\mathcal{C}^{(2)} \wedge \beta)_{\bar\mu_1\ldots\bar\mu_5}
\end{aligned}$$

whilst going from IIB to IIA we have

$$\begin{aligned}
\varphi &\rightarrow \phi - \frac{1}{2}\log(R^2) \\
i_\beta \mathcal{B}_{\bar\mu} &\rightarrow R^{-2}\beta_{\bar\mu} \\
\mathcal{B}_{\bar\mu_1\bar\mu_2} &\rightarrow (B + R^{-2} i_\beta B \wedge \beta)_{\bar\mu_1\bar\mu_2} \\
i_\beta \mathcal{C}^{(2n)}_{\bar\mu_1\ldots\bar\mu_{2n-1}} &\rightarrow (-C^{(2n-1)} + R^{-2} i_\beta C^{(2n-1)} \wedge \beta)_{\bar\mu_1\ldots\bar\mu_{2n-1}} \\
\mathcal{C}^{(2n)}_{\bar\mu_1\ldots\bar\mu_{2n}} &\rightarrow (i_\beta C^{(2n+1)} + C^{(2n-1)} \wedge i_\beta B + R^{-2} i_\beta C^{(2n-1)} \wedge i_\beta B \wedge \beta)_{\bar\mu_1\ldots\bar\mu_{2n}} \\
i_\beta \mathcal{B}^{(6)}_{\bar\mu_1\ldots\bar\mu_5} &\rightarrow (i_\beta B^{(6)} - \frac{1}{2} i_\beta C^{(3)} \wedge C^{(3)} + \frac{1}{2} R^{-2} i_\beta C^{(3)} \wedge i_\beta C^{(3)} \wedge \beta)_{\bar\mu_1\ldots\bar\mu_5}.
\end{aligned}$$

Next we consider the T-duality rules for the bilinears. To determine these we first consider how the Γ matrices transform. This can be determined from their defining property, the $D = 10$ analogue of (2.11), as well as the transformation rules of the metrics given above. We then find that they transform according to

$$\Gamma^{(A)}_y \rightarrow \mathcal{R}^{-2}\Gamma^{(B)}_y \tag{5.14}$$

$$\Gamma^{(A)}_{\bar\mu} \rightarrow \Gamma^{(B)}_{\bar\mu} + \mathcal{R}^{-2}\Gamma^{(B)}_y(-\beta_{\bar\mu} + i_\beta\mathcal{B}_{\bar\mu}) \tag{5.15}$$

$$\Gamma^{(B)}_y \rightarrow R^{-2}\Gamma^{(A)}_y \tag{5.16}$$

$$\Gamma^{(B)}_{\bar\mu} \rightarrow \Gamma^{(A)}_{\bar\mu} + R^{-2}\Gamma^{(A)}_y(-\beta_{\bar\mu} + i_\beta B_{\bar\mu}) \tag{5.17}$$

where the superscript on the Γ matrices refers to which of the theories they are in.

We need also the T-duality rules for the spinors which, up to normalisation, are given by

$$\epsilon^+ \leftrightarrow \frac{1}{\sqrt{2}}\epsilon^2 \qquad \epsilon^- \leftrightarrow \frac{1}{\sqrt{2}}\mathcal{R}^{-1}\Gamma_y^{(B)}\epsilon^1 \tag{5.18}$$

where we have chosen our normalisation for convenience.

From these results we can determine the transformation rules for the bilinears under T-duality. Going from IIA to IIB we find

$$\begin{aligned}
i_\beta K &\rightarrow -\mathcal{R}^{-2} i_\beta K^- \\
K_{\bar\mu} &\rightarrow K^+_{\bar\mu} - \mathcal{R}^{-2} i_\beta K^+ \wedge \beta_{\bar\mu} - \mathcal{R}^{-2} i_\beta K^- \wedge i_\beta \mathcal{B}_{\bar\mu} \\
i_\beta \tilde{K} &\rightarrow \mathcal{R}^{-2} i_\beta K^+ \\
\tilde{K}_{\bar\mu} &\rightarrow -K^-_{\bar\mu} + \mathcal{R}^{-2} i_\beta K^- \wedge \beta_{\bar\mu} + \mathcal{R}^{-2} i_\beta K^+ \wedge i_\beta \mathcal{B}_{\bar\mu} \\
i_\beta \Psi^{(2n)}_{\bar\mu_1\ldots\bar\mu_{2n-1}} &\rightarrow \mathcal{R}^{-1}(-\Phi^{(2n-1)} + \mathcal{R}^{-2} i_\beta \Phi^{(2n-1)} \wedge \beta)_{\bar\mu_1\ldots\bar\mu_{2n-1}} \\
\Psi^{(2n)}_{\bar\mu_1\ldots\bar\mu_{2n}} &\rightarrow \mathcal{R}^{-1}(-i_\beta \Phi^{(2n+1)} + \Phi^{(2n-1)} \wedge i_\beta \mathcal{B} \\
&\quad +\mathcal{R}^{-2} i_\beta \Phi^{(2n-1)} \wedge i_\beta \mathcal{B} \wedge \beta)_{\bar\mu_1\ldots\bar\mu_{2n}}.
\end{aligned} \tag{5.19}$$

We find that bilinears with the same structure up to the number of Γ matrices mod 2 transform analogously. The transformation rules for the remaining IIA bilinears therefore follow from the first four lines here. So for example the transformation rule of Σ is the same as for K but with $K^+ \rightarrow \Sigma^+$ and $K^- \rightarrow \Sigma^-$.

The rules for mapping from IIB to IIA are given by

$$\begin{aligned}
i_\beta K^+ &\rightarrow R^{-2} i_\beta \tilde{K} \\
K^+_{\bar\mu} &\rightarrow K_{\bar\mu} - R^{-2} i_\beta K \wedge \beta_{\bar\mu} + R^{-2} i_\beta \tilde{K} \wedge i_\beta B_{\bar\mu} \\
i_\beta K^- &\rightarrow -R^{-2} i_\beta K \\
K^-_{\bar\mu} &\rightarrow -\tilde{K}_{\bar\mu} + R^{-2} i_\beta \tilde{K} \wedge \beta_{\bar\mu} - R^{-2} i_\beta K \wedge i_\beta B_{\bar\mu} \\
i_\beta \Phi^{(2n+1)}_{\bar\mu_1\ldots\bar\mu_{2n}} &\rightarrow R^{-1}(-\Psi^{(2n)} - R^{-2} i_\beta \Psi^{(2n)} \wedge \beta)_{\bar\mu_1\ldots\bar\mu_{2n}} \\
\Phi^{(2n+1)}_{\bar\mu_1\ldots\bar\mu_{2n+1}} &\rightarrow R^{-1}(-i_\beta \Psi^{(2n+2)} - \Psi^{(2n)} \wedge i_\beta B \\
&\quad +R^{-2} i_\beta \Psi^{(2n)} \wedge i_\beta B \wedge \beta)_{\bar\mu_1\ldots\bar\mu_{2n+1}}
\end{aligned} \tag{5.20}$$

where once again the transformation rules for the remaining IIB bilinears follow the same structure as the top four lines here.

In addition to these we need the mappings for the Killing vectors K and K^+ with raised indices. For K these are calculated from the above rules as follows

$$\begin{aligned}
K^{\bar\mu} &= g_{(A)}^{\bar\mu\bar\nu}K_{\bar\nu} + g_{(A)}^{\bar\mu y}K_y \\
&\to g_{(B)}^{\bar\mu\bar\nu}(K^+_{\bar\nu} - \mathcal{R}^{-2}i_\beta K^+\beta_{\bar\nu} - \mathcal{R}^{-2}i_\beta K^- i_\beta\mathcal{B}_{\bar\nu}) + \mathcal{R}^{-2}g_{(B)}^{\bar\mu\bar\nu}i_\beta\mathcal{B}_{\bar\nu}i_\beta K^- \\
&= g_{(B)}^{\bar\mu\bar\nu}K^+_{\bar\nu} + g_{(B)}^{\bar\mu y}K^+_y \\
&= K^{+\bar\mu} \qquad (5.21)
\end{aligned}$$

and

$$\begin{aligned}
K^y &= g_{(A)}^{y\bar\nu}K_{\bar\nu} + g_{(A)}^{yy}K_y \\
&\to -g_{(B)}^{\bar\rho\bar\nu}i_\beta\mathcal{B}_{\bar\rho}(K^+_{\bar\nu} - \mathcal{R}^{-2}i_\beta K^+\beta_{\bar\nu} - \mathcal{R}^{-2}i_\beta K^- i_\beta\mathcal{B}_{\bar\nu}) \\
&\quad -(1 + \mathcal{R}^{-2}g_{(B)}^{\bar\rho\bar\nu}i_\beta\mathcal{B}_{\bar\rho}i_\beta\mathcal{B}_{\bar\nu})i_\beta K^- \qquad (5.22) \\
&= -i_\beta K^- - K^{+\bar\rho}i_\beta\mathcal{B}_{\bar\rho} \\
&= -i_\beta N^{(F1)}. \qquad (5.23)
\end{aligned}$$

Finally the rules for K^+ are given by

$$K^{+\bar\mu} \to K^{\bar\mu} \qquad (5.24)$$

$$K^{+y} \to i_\beta M^{(F1)}. \qquad (5.25)$$

5.2 T-duality of the generalised charges

We now use the above rules to examine the T-duality relations of the generalised charges and compare these to the T-duality transformations of the branes themselves which have been discussed in the literature many times and which we summarise in Figure 5.1. There are two parts to this analysis; first we must consider the mappings of the charge structures, and second we must consider the mappings of the gauge conditions we impose so that the charges are closed. We begin with the former.

We consider the D-branes first. When T-dualising the D-brane charges (3.33) and (4.23) the following relations derived from the mappings given above prove

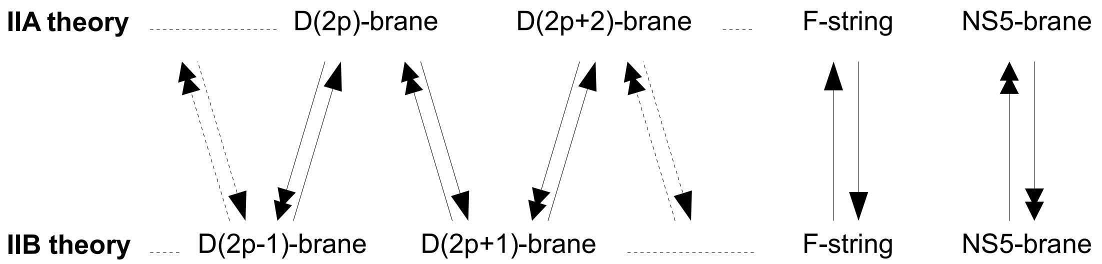

Figure 5.1: T-duality rules between the conventional branes in the IIA and IIB theories. The single headed arrows signify direct T-duality transformations whereas the double headed arrows signify double T-duality transformations.

useful

$$
\begin{aligned}
i_K C^{(2n+1)}_{\bar\mu_1\dots\bar\mu_{2n}} &\to \Big[-i_{K^+}(i_\beta \mathcal{C}^{(2n+2)}) - i_\beta K^-(\mathcal{C}^{(2n)} + \mathcal{R}^{-2} i_\beta \mathcal{C}^{(2n)} \wedge \beta) \\
&\quad + i_{K^+}(\mathcal{C}^{(2n)} + \mathcal{R}^{-2} i_\beta \mathcal{C}^{(2n)} \wedge \beta) \wedge i_\beta B\Big]_{\bar\mu_1\dots\bar\mu_{2n}} \\
i_\beta(i_K C^{(2n+1)})_{\bar\mu_1\dots\bar\mu_{2n-1}} &\to -i_{K^+}(\mathcal{C}^{(2n)} + \mathcal{R}^{-2} i_\beta \mathcal{C}^{(2n)} \wedge \beta)_{\bar\mu_1\dots\bar\mu_{2n-1}} \\
i_{K^+}\mathcal{C}^{(2n)}_{\bar\mu_1\dots\bar\mu_{2n-1}} &\to \Big[i_K(i_\beta C^{(2n+1)}) + i_\beta \tilde{K}(-C^{(2n-1)} + R^{-2} i_\beta C^{(2n-1)} \wedge \beta) \\
&\quad + i_K(C^{(2n-1)} - R^{-2} i_\beta C^{(2n-1)} \wedge \beta) \wedge i_\beta B\Big]_{\bar\mu_1\dots\bar\mu_{2n-1}} \\
i_\beta(i_{K^+}\mathcal{C}^{(2n)})_{\bar\mu_1\dots\bar\mu_{n-2}} &\to i_K(C^{(2n-1)} - R^{-2} i_\beta C^{(2n-1)} \wedge \beta)_{\bar\mu_1\dots\bar\mu_{2n-2}}.
\end{aligned}
\tag{5.26}
$$

Ultimately we find that the D-brane charges obey the following T-duality relations

$$
\begin{aligned}
M^{(2n)}_{\bar\mu_1\dots\bar\mu_{2n}} &\leftrightarrow -i_\beta N^{(2n+1)}_{\bar\mu_1\dots\bar\mu_{2n}} \\
i_\beta M^{(2n)}_{\bar\mu_1\dots\bar\mu_{2n-1}} &\leftrightarrow -N^{(2n-1)}_{\bar\mu_1\dots\bar\mu_{2n-1}}.
\end{aligned}
$$

The general rule for the D-branes is that T-duality maps between the direct dimensional reduction of a Dp-brane and the double dimensional reduction of a D$(p+1)$-brane in the dual theory [8,72,73]. From the above relations we see that this general mapping is also obeyed between the D-brane charges after appropriate normalisation.

Next we consider the F-strings which are related through direct T-duality transformations.[1] When T-dualising the F-string charges (3.39) and (4.26) the following

[1]By 'direct' T-duality transformation we mean a T-duality transformation that occurs trans-

transformation rules prove useful

$$
\begin{aligned}
i_K B_{\bar{\mu}} &\rightarrow i_{K^+}\mathcal{B}_{\bar{\mu}} - \mathcal{R}^{-2} i_\beta K^+ i_\beta \mathcal{B}_{\bar{\mu}} - \mathcal{R}^{-2} i_\beta K^- \beta_{\bar{\mu}} && (5.27)\\
i_{K^+}\mathcal{B}_{\bar{\mu}} &\rightarrow i_K B_{\bar{\mu}} - R^{-2} i_\beta K i_\beta B_{\bar{\mu}} + R^{-2} i_\beta \tilde{K} \beta_{\bar{\mu}}. && (5.28)
\end{aligned}
$$

We then find that the charges are related via

$$M^{(F1)}_{\bar{\mu}} \leftrightarrow -N^{(F1)}_{\bar{\mu}} \tag{5.29}$$

which matches the mappings between the F-strings themselves after appropriate normalisation.

Finally we consider the NS5-branes which are related through double T-duality transformations [71]. We find that the NS5-brane charges (3.41) and (4.30) are related under T-duality by

$$i_\beta M^{(NS5)}_{\mu_1\ldots\mu_4} \quad \leftrightarrow \quad -i_\beta N^{(NS5)}_{\mu_1\ldots\mu_4} \tag{5.30}$$

which also matches the mappings between NS5-branes themselves after appropriate normalisation.

We thus conclude that that all of the $D = 10$ generalised charge expressions so far presented map under T-duality in the same fashion as the branes themselves. However, for consistency we must also consider the mappings of the gauge conditions we impose on these charges which take the form of a condition on the Lie derivatives of the gauge potentials with respect to the relevant Killing vector K or K^+. As a general rule we find that the Lie derivatives in one theory map exclusively to a set of Lie derivatives in the dual theory. To demonstrate this we consider a general p-form in IIA $Y^{(p)}$ with transformation rules to IIB given by

$$
\begin{aligned}
i_\beta Y^{(p)}_{\bar{\mu}_1\ldots\bar{\mu}_{p-1}} &\rightarrow \mathcal{Y}^{(p-1)}_{\bar{\mu}_1\ldots\bar{\mu}_{p-1}} && (5.31)\\
Y^{(p)}_{\bar{\mu}_1\ldots\bar{\mu}_p} &\rightarrow \left[\mathcal{Y}^{(p+1)} + (-1)^{p+1}\mathcal{Y}^{(p-1)} \wedge i_\beta \mathcal{B}\right]_{\bar{\mu}_1\ldots\bar{\mu}_p}. && (5.32)
\end{aligned}
$$

Using the definition of the Lie derivative (2.37) together with the rules for K given

verse to the brane worldvolume as opposed to a 'double' T-duality transformation by which we mean a T-duality that occurs parallel to the brane worldvolume.

previously it is straightforward to show that the Lie derivatives map as follows

$$\begin{aligned} \mathcal{L}_K i_\beta Y^{(p)}_{\bar{\mu}_1\ldots\bar{\mu}_{p-1}} &\rightarrow \mathcal{L}_{K^+}\mathcal{Y}^{(p-1)}_{\bar{\mu}_1\ldots\bar{\mu}_{p-1}} \\ \mathcal{L}_K Y^{(p)}_{\bar{\mu}_1\ldots\bar{\mu}_p} &\rightarrow \Big[\mathcal{L}_{K^+}\mathcal{Y}^{(p+1)} + (-1)^{p+1}\mathcal{L}_{K^+}(\mathcal{Y}^{(p-1)}) \wedge i_\beta \mathcal{B}\Big]_{\bar{\mu}_1\ldots\bar{\mu}_p} . \end{aligned} \tag{5.33}$$

The converse rules going from IIB to IIA take the same form as these but with each field being swapped for its analogue in the dual theory. Therefore we conclude that the gauge conditions in the massless IIA theory and IIB theory will trivially map to one another since they amount to the vanishing of the Lie derivatives. Recall however that in Romans' theory the Lie derivatives in question are generally non-zero and proportional to m. In the 'massive' reformulation of IIB discussed in the beginning of this chapter it turns out that the IIB Lie derivatives are also non-zero. These are given in Section 12.3. Using the general rule (5.33) it is a straightforward exercise to confirm that these conditions map to those already given in Section 3.5.

Chapter 6

Kaluza-Klein-monopole generalised charges

In this chapter we formulate the generalised charges for the KK-monopoles in each of the three supergravity theories we consider in this thesis. The formulation of these charges is more involved than the charges previously considered due to the presence of the Taub-NUT isometry direction in the KK-monopole backgrounds [24–26] which appears as a gauged isometry in the worldvolume action [26]. This isometry plays an active role in their construction and we find that, unlike for the flatspace charges present in the flatspace SUSY algebras, it makes an explicit appearance in the generalised charge structures.

We will first consider the $D = 11$ case where we explain the method required to formulate the charge and give details of the required calculations. We then consider the $D = 10$ cases. For the sake of brevity we do not repeat the calculations in these latter cases which essentially follow the same method but are more long winded due to the larger number of fields present in these theories. Instead we obtain the IIA KK-monopole charge from dimensional reduction of the $D = 11$ charge, and for consistency check this maps to the IIB NS5-brane charge by T-duality. Then finally we obtain the IIB KK-monopole charge by T-dualising the IIA NS5-brane charge. We summarise the duality relations between the KK-monopoles and NS5-branes in Figure 6.1 [28, 71].

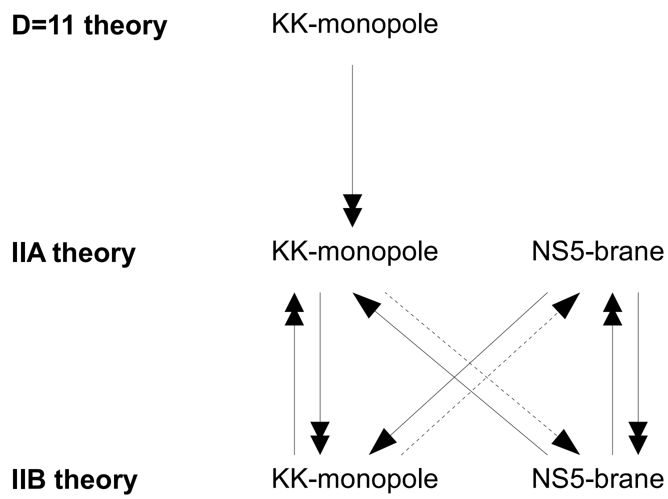

Figure 6.1: Dimensional reduction and T-duality rules between the KK-monopoles and NS5-branes. The single headed arrows signify direct T-duality transformations whereas the double headed arrows signify double T-duality tranformations or double dimensional reductions. The dotted arrows indicate the T-duality is being performed along the Taub-NUT isometry direction.

6.1 $D = 11$ KK-monopole generalised charge

In Chapter 2 we calculated the exterior derivative of the KK-monopole 6-form calibrating bilinear $\hat{\Lambda}$ (2.34) and noted that the RHS could not be integrated which prevented the formulation of a generalised charge. This situation is essentially the same as that encountered for the charges in the $D = 10$ theories where it was found that the problem was solved by the fact that the leading bilinear terms in the charges contained a factor reflecting the branes' tension. In the case of the KK-monopoles the factor $\hat{R}^2$, where $\hat{R}$ is the radius of the compact Taub-NUT isometry, appears in the worldvolume action multiplying the Born-Infeld term [26] and therefore also in the brane tension [29, 74]. We would therefore expect the leading term in this case to be $\hat{R}^2\hat{\Lambda}$ and so the first task in formulating this charge is to determine the exterior derivative of this term.

We realise that owing to the presence of the isometry in the KK-monopole background, we can obtain an additional Killing spinor equation from (2.23) by considering the component that lies parallel to the isometry in isolation. We assume all background fields are independent of this direction and as such the partial derivative vanishes yielding an algebraic Killing spinor equation. We can then use this equation in conjunction with the usual Killing spinor equation to yield the differential relation for $\hat{R}^2\hat{\Lambda}$. We now present the details of this calculation.

For the purposes of this calculation it will be necessary to split the co-ordinates into $\{\mu_i, z\}$ where z parametrises the Taub-NUT isometry direction and the μ_i represent the other 10 directions. We denote the Killing vector describing the isometry by $\hat{\alpha}$. In these adapted co-ordinates we have $\hat{\alpha}^{\hat{\mu}} = \delta^{\hat{\mu}z}$ and $|\hat{\alpha}|^2 = \hat{\alpha}_z = \hat{R}^2$. It will be useful to work in the orthonormal frame defined by the elfbeins

$$\hat{e}_{\hat{\mu}}{}^{\hat{A}} = \begin{pmatrix} \hat{e}_\mu{}^A & \hat{R}^{-1}\hat{\alpha}_\mu \\ 0 & \hat{R} \end{pmatrix} \qquad \hat{e}_{\hat{A}}{}^{\hat{\mu}} = \begin{pmatrix} \hat{e}_A{}^\mu & -\hat{R}^{-2}\hat{e}_A{}^\nu\hat{\alpha}_\nu \\ 0 & \hat{R}^{-1} \end{pmatrix} \tag{6.1}$$

where the orthonormal indices split into $\{A_i, \underline{z}\}$.

We now consider the z component of the Killing spinor equation (2.23). The covariant derivative in this equation is defined as

$$\hat{\nabla}_{\hat{\mu}} = \partial_{\hat{\mu}} + \frac{1}{4}\hat{\Omega}_{\hat{\mu}}{}^{\hat{A}\hat{B}}\hat{\Gamma}_{\hat{A}\hat{B}} \tag{6.2}$$

where $\hat{\Omega}_{\hat{\mu}}{}^{\hat{A}\hat{B}}$ is the affine spin connection 1-form. It will be necessary for us to calculate the explicit form of the z components. In order to do this we use the *Cartan structure equation* (see for example [75])

$$\hat{T}^{\hat{A}}{}_{\hat{\mu}\hat{\nu}} = \partial_{[\hat{\mu}}\hat{e}_{\hat{\nu}]}{}^{\hat{A}} + \hat{e}_{[\hat{\mu}}{}^{\hat{B}}\hat{\Omega}_{\hat{\nu}]\hat{B}}{}^{\hat{A}} = 0 \tag{6.3}$$

where $\hat{T}^{\hat{A}}{}_{\hat{\mu}\hat{\nu}}$ is the torsion tensor which vanishes in this case. From the $\hat{T}^{\underline{z}}{}_{\mu z}$ component we have

$$\partial_\mu\hat{e}_z{}^{\underline{z}} + \hat{e}_\mu{}^A\hat{\Omega}_{zA}{}^{\underline{z}} = 0 \tag{6.4}$$

which we can contract with $\hat{e}_B{}^\mu$ and rearrange to obtain the expression for $\hat{\Omega}_z{}^{A\underline{z}}$. Next we consider the $\hat{T}^A{}_{\mu z}$ and $\hat{T}^{\underline{z}}{}_{\mu\nu}$ components which are given by

$$\hat{e}_\mu{}^B\hat{\Omega}_{zB}{}^A + \hat{e}_\mu{}^{\underline{z}}\hat{\Omega}_{z\underline{z}}{}^A - \hat{e}_z{}^{\underline{z}}\hat{\Omega}_{\mu\underline{z}}{}^A = 0 \tag{6.5}$$

and

$$\partial_{[\mu}\hat{e}_{\nu]}{}^{\underline{z}} + \hat{e}_{[\mu}{}^A\hat{\Omega}_{\nu]A}{}^{\underline{z}} = 0 \tag{6.6}$$

respectively. We then rearrange (6.5) for $\hat{\Omega}_\mu{}^{A\underline{z}}$ and substitute this into (6.6), then after some slight manipulation one can determine the form of $\hat{\Omega}_z{}^{AB}$. The results are

$$\hat{\Omega}_{zA\underline{z}} = -\frac{1}{2}\hat{R}^{-1}\partial_A \hat{R}^2 \tag{6.7}$$

$$\hat{\Omega}_{zAB} = -\frac{1}{2}d\hat{\alpha}_{AB}. \tag{6.8}$$

Given these results we then write the z component of (2.23) as

$$\begin{aligned}\hat{\hat{D}}_z\hat{\epsilon} &= \hat{\nabla}_z\hat{\epsilon} + \frac{1}{288}\hat{e}_z{}^{\underline{z}}\hat{\Gamma}_{\underline{z}}{}^{ABCD}\hat{F}_{ABCD}\hat{\epsilon} - \frac{1}{36}\hat{e}_z{}^{\underline{z}}\hat{\Gamma}^{ABC}\hat{F}_{\underline{z}ABC}\hat{\epsilon} \\ &= \left[-\frac{1}{8}d\hat{\alpha}_{AB}\hat{\Gamma}^{AB} - \frac{1}{4}\hat{R}^{-1}\partial_A(\hat{R}^2)\hat{\Gamma}^{A\underline{z}}\right]\hat{\epsilon} \\ &\quad + \left[\frac{1}{288}\hat{R}\hat{\Gamma}_{\underline{z}}{}^{ABCD}\hat{F}_{ABCD} - \frac{1}{36}\hat{R}\hat{\Gamma}^{ABC}\hat{F}_{\underline{z}ABC}\right]\hat{\epsilon} = 0\end{aligned} \tag{6.9}$$

which is interpreted as a $D = 11$ algebraic Killing spinor equation. From this point we proceed just as in the $D = 10$ cases where we supplemented the relations obtained purely from the differential Killing spinor equation with those obtained from the algebraic Killing spinor equation. In fact (6.9) reduces to the IIA algebraic Killing spinor equation (3.25) when a dimensional reduction along z is performed.

The required algebraic relation in this instance is obtained by hitting (6.9) from the left with $\hat{\bar{\epsilon}}\hat{\Gamma}_{A_1\ldots A_7\underline{z}}$. After using (2.39) to 'tidy up' some terms and multiplying through by a factor of $-4\hat{R}$ one obtains

$$\begin{aligned}0 &= \Big[i_{\hat{K}}(i_{\hat{\alpha}}\hat{G}^{(9)}) + d\hat{\alpha}\wedge i_{\hat{\alpha}}\hat{\Lambda} - d(\hat{R}^2)\wedge\hat{\Lambda} \\ &\quad + \frac{1}{3}i_{\hat{\alpha}}\hat{\omega}\wedge i_{\hat{\alpha}}\hat{F}^{(7)} + \frac{2}{3}i_{\hat{\alpha}}\hat{\Sigma}\wedge i_{\hat{\alpha}}\hat{F}\Big]_{A_1\ldots A_7} - \frac{14}{3}\hat{R}^2\hat{\omega}^B{}_{[A_1}\hat{F}^{(7)}_{A_2\ldots A_7]B} \\ &\quad + \frac{35}{3}\hat{R}^2\hat{\Sigma}_{B[A_1\ldots A_4}\hat{F}_{A_5A_6A_7]}{}^B.\end{aligned} \tag{6.10}$$

We consider $\hat{\alpha}$ as a 1-form potential and have defined the 9-form field strength $\hat{G}^{(9)} = \hat{*}\hat{G}^{(2)} = \hat{*}d\hat{\alpha}$ [76, 77] with 8-form potential $\hat{N}^{(8)}$ [51] which is the magnetic dual of $\hat{\alpha}$. Note that these fields are not intrinsic to the $D = 11$ supergravity theory, but appear in spacetime solutions of the equations of motion that contain an isometry direction.

We can then combine (6.10) with (2.34) (multiplied by $\hat{R}^2$) to eliminate the unwanted terms and produce the following relation

$$\begin{aligned}\hat{R}^2 d\hat{\Lambda}_{A_1\dots A_7} &= \Big[i_{\hat{K}}(i_{\hat{\alpha}}\hat{G}^{(9)}) + d\hat{\alpha}\wedge i_{\hat{\alpha}}\hat{\Lambda} - d(\hat{R}^2)\wedge\hat{\Lambda} \\ &\qquad i_{\hat{\alpha}}\hat{\omega}\wedge i_{\hat{\alpha}}\hat{F}^{(7)} + i_{\hat{\alpha}}\hat{\Sigma}\wedge i_{\hat{\alpha}}\hat{F}\Big]_{A_1\dots A_7}. \end{aligned} \tag{6.11}$$

As it stands, this relation is not fully tensorial since it has only been calculated for a subset of components. However we can convert these to a co-ordinate basis using the following general rule for a p-form $\hat{Y}$

$$\begin{aligned}\hat{Y}_{\mu_1\dots\mu_p} &= \hat{e}_{\mu_1}{}^{A_1}\dots\hat{e}_{\mu_p}{}^{A_p}\hat{Y}_{A_1\dots A_p} + p\hat{e}_{[\mu_1}{}^{\underline{z}}\hat{e}_{\mu_2}{}^{A_1}\dots\hat{e}_{\mu_p]}{}^{A_{p-1}}\hat{Y}_{\underline{z}A_1\dots A_{p-1}} \\ &= \hat{e}_{\mu_1}{}^{A_1}\dots\hat{e}_{\mu_p}{}^{A_p}\hat{Y}_{A_1\dots A_p} + \hat{R}^{-2}(\hat{\alpha}\wedge i_{\hat{\alpha}}\hat{Y})_{\mu_1\dots\mu_p}. \end{aligned} \tag{6.12}$$

Contracting (6.11) with $\hat{e}_{\mu_1}{}^{A_1}\dots\hat{e}_{\mu_7}{}^{A_7}$ and using the above conversion rule we ultimately find

$$d(\hat{R}^2\hat{\Lambda} + i_{\hat{\alpha}}\hat{\Lambda}\wedge\hat{\alpha})_{\hat{\mu}_1\dots\hat{\mu}_7} = \Big[i_{\hat{K}}(i_{\hat{\alpha}}\hat{G}^{(9)}) + i_{\hat{\alpha}}\hat{\omega}\wedge i_{\hat{\alpha}}\hat{F}^{(7)} + i_{\hat{\alpha}}\hat{\Sigma}\wedge i_{\hat{\alpha}}\hat{F}\Big]_{\hat{\mu}_1\dots\hat{\mu}_7}. \tag{6.13}$$

Note that we have restored the free indices here to run over all values. Although this relation was only calculated for those components transverse to the isometry direction, it is straight forward to see that it also (trivially) holds for the parallel components and therefore is fully tensorial. Furthermore the structure of this relation is sufficient for the formulation of a generalised charge.

From the presence of the term involving $\hat{G}^{(9)}$ in (6.13) we conclude that the KK-monopole minimally couples to $i_{\hat{\alpha}}\hat{N}^{(8)}$ and therefore need the explicit form of its field strength equation. This was given in [51, 76, 77] as

$$\begin{aligned} i_{\hat{\alpha}}\hat{G}^{(9)} &= -d(i_{\hat{\alpha}}\hat{N}^{(8)}) - d(i_{\hat{\alpha}}\hat{A})\wedge i_{\hat{\alpha}}\hat{C} - \frac{1}{6}d(\hat{i_\alpha}\hat{A})\wedge i_{\hat{\alpha}}\hat{A}\wedge\hat{A} \\ &\quad + \frac{1}{6}d\hat{A}\wedge i_{\hat{\alpha}}\hat{A}\wedge i_{\hat{\alpha}}\hat{A} \end{aligned} \tag{6.14}$$

and is defined so that after dimensional reduction over $\hat{\alpha}$ it produces the field strength equation for $F^{(8)}$ in IIA (3.4) which can be checked using the reduction rules given in Appendix A together with (8.15). Note that only the components of $\hat{G}^{(9)}$ parallel to the isometry are relevant here, however we will eventually need to calculate the full field strength equation and do so in Section 8.2.

Using these results we then calculate the KK-monopole generalised charge following the usual method and find it to be given by

$$\begin{aligned}\hat{L}^{(KK)} &= \hat{R}^2\hat{\Lambda} + i_{\hat{\alpha}}\hat{\Lambda}\wedge\hat{\alpha} - i_{\hat{K}}(i_{\hat{\alpha}}\hat{N}^{(8)} - \frac{1}{3!}\hat{A}\wedge(i_{\hat{\alpha}}\hat{A})^2) - i_{\hat{\alpha}}\hat{\omega}\wedge(i_{\hat{\alpha}}\hat{C}\\ &\quad + \frac{1}{2}\hat{A}\wedge i_{\hat{\alpha}}\hat{A}) + i_{\hat{\alpha}}\hat{L}^{(5)}\wedge i_{\hat{\alpha}}\hat{A} - \frac{1}{2}\hat{L}^{(2)}\wedge(i_{\hat{\alpha}}\hat{A})^2 \end{aligned} \tag{6.15}$$

where we have imposed the gauge condition $\mathcal{L}_{\hat{K}} i_{\hat{\alpha}}\hat{N}^{(8)} = 0$ as well as the previous gauge conditions (2.44) and (2.53). In addition to this we assume that $\mathcal{L}_{\hat{\alpha}}$ of each field vanishes; it is stated in [28] that this is always possible in the IIA case and so it must also be achievable in the $D = 11$ case.

The explicit appearance of $\hat{\alpha}$ in the above charge occurs as a result of the active role this isometry played in its formulation. Because of this the structure of the charge is representative of the nature of the KK-monopole background. The form of the leading term $\hat{R}^2\hat{\Lambda}$ was already expected and reflects the brane tension however the second term involving $\hat{\Lambda}$ has no analogues in any of the charges we have considered thus far. The interpretation of this term is not completely clear to us but seems to arise due to the fact that the isometry lies transverse to the brane worldvolume. Due to its presence we find the following identity

$$i_{\hat{\alpha}}\hat{L}^{(KK)} = 0 \tag{6.16}$$

which naively could also be interpreted as representing the transverse nature of the isometry. However this interpretation is not justified since we find a similar result for the M9-brane charge (8.56) where the isometry lies parallel to the worldvolume. The structure of the terms involving the leading bilinears are different for the M9-brane charge however, and for this reason we conclude that the nature of the isometry direction relative to the worldvolume is encoded in these leading terms.

6.2 $D = 10$ KK-monopole generalised charges

We now present the $D = 10$ KK-monopole generalised charges beginning with the IIA case. The method here is essentially the same as that described for the $D = 11$ case with the slight complication that an algebraic Killing spinor equation is already present in the theory. Therefore one needs to combine three relations in a specific way in order to remove any unwanted terms and produce the appropriate differential relation for the KK-monopole calibrating bilinear $\tilde{\Sigma}$, which can then be used to determine the structure of the charge. However, we instead opt to take the more straight forward approach and derive the charge via dimensional reduction of the $D = 11$ KK-monopole charge (6.15). The IIA KK-monopole is related to the $D = 11$ KK-monopole via a double dimensional reduction [28] and so one would expect that the charges are related similarly. We stress that the reduction direction in this instance is distinct from the Taub-NUT isometry direction. Performing this reduction on (6.15) using the rules given in Appendix A yields the IIA KK-monopole charge as

$$\begin{aligned} M^{(KK)} &= e^{-2\phi}R^2\tilde{\Sigma} - e^{-2\phi}i_\beta\tilde{\Sigma}\wedge\beta + e^{-\phi}i_\beta C^{(1)}i_\beta\Psi^{(6)} \\ &\quad -i_K(i_\beta N^{(7)}) + \frac{1}{2}i_K(B\wedge(i_\beta C^{(3)})^2) \\ &\quad +i_\beta(e^{-\phi}\Psi^{(2)} + \tilde{K}\wedge C^{(1)})\wedge(i_\beta C^{(5)} - i_\beta C^{(3)}\wedge B) \\ &\quad +i_\beta\tilde{K}(i_\beta B^{(6)} + \frac{1}{2}C^{(3)}\wedge i_\beta C^{(3)}) + i_\beta M^{(NS5)}\wedge i_\beta B \\ &\quad +i_\beta M^{(4)}\wedge i_\beta C^{(3)} - M^{(2)}\wedge i_\beta C^{(3)}\wedge i_\beta B \\ &\quad -\frac{1}{2}M^{(F1)}\wedge(i_\beta C^{(3)})^2. \end{aligned} \tag{6.17}$$

Here β is the vector describing the Taub-NUT isometry, R is the radius of the isometry and $N^{(7)}$ is the 7-form magnetic dual potential of β with 8-form field strength $G^{(8)} = e^{-2\phi} * G^{(2)} = e^{-2\phi} * d\beta$. We see here an analogous structure to the $D = 11$ KK-monopole charge and the comments made for that charge are also relevant here. The tension of the IIA KK-monopole is known to scale as [27]

$$e^{-2\phi}R^2\sqrt{1 + e^{2\phi}R^{-2}(i_\beta C^{(1)})^2}. \tag{6.18}$$

This can be read of from the above charge by noting that in addition to the first term involving $\tilde{\Sigma}$ we also have the term involving $i_\beta\Psi^{(6)}$ which acts as a second 5-form bilinear. Therefore the situation is similar to that encountered with the IIB D1-brane and NS5-brane charges. Here though we realise that $i_\beta\Psi^{(6)}$ is actually calibrating a 6-dimensional surface where one of the directions is the compact Taub-NUT isometry direction, and so a discrepancy of a factor of R will arise when interpreting this term as calibrating the remaining 5-dimensional surface. We therefore must multiply the factor appearing with $i_\beta\Psi^{(6)}$ by R when calculating the tension. We then find that the tension inferred from the structure of (6.17) is in agreement with (6.18).

In order to check that (6.17) is closed we need the field strength equation for $i_\beta N^{(7)}$ as well as the differential relation for $e^{-2\phi}R^2\tilde{\Sigma}$. In order to determine the massive versions of these relations we must use results from later on in the thesis. Firstly, the field strength equation is determined from (9.2) and is given by

$$\begin{aligned} i_\beta G^{(8)} &= -d(i_\beta N^{(7)}) - i_\beta C^{(1)} i_\beta F^{(8)} - i_\beta H \wedge i_\beta B^{(6)} \\ &\quad - i_\beta(F^{(4)} + H \wedge C^{(1)}) \wedge i_\beta C^{(5)} - \frac{1}{2} i_\beta (H \wedge C^{(3)}) \wedge i_\beta C^{(3)} \\ &\quad + m(i_\beta C^{(7)} \wedge i_\beta B + \frac{1}{2} i_\beta C^{(3)} \wedge (B)^2 \wedge i_\beta B). \end{aligned} \tag{6.19}$$

Secondly, the differential relation for $\tilde{\Sigma}$ is obtained via a double dimensional reduction of (6.13) which yields

$$\begin{aligned} d(e^{-2\phi}R^2\tilde{\Sigma} - e^{-2\phi} i_\beta \tilde{\Sigma} \wedge \beta) &= +i_K(i_\beta G^{(8)}) + e^{-2\phi} i_\beta \Sigma \wedge i_\beta H + i_\beta \tilde{K} i_\beta H^{(7)} \\ &\quad - e^{-\phi}\left[i_\beta \Psi^{(6)} \wedge i_\beta F^{(2)} + i_\beta \Psi^{(4)} \wedge i_\beta F^{(4)} + i_\beta \Psi^{(2)} \wedge i_\beta F^{(6)} \right] \end{aligned} \tag{6.20}$$

which turns out to have no explicit mass terms, although of course there are mass terms implicit in the definitions of the field strengths.

Taking the exterior derivative of (6.17) we find, after imposing the massive gauge conditions given previously

$$\begin{aligned} dM^{(KK)} &= -\mathcal{L}_K i_\beta N^{(7)} - m i_\beta M^{(6)} \wedge i_\beta B - m i_\beta M^{(F1)} \wedge i_\beta C^{(7)} \\ &\quad - \frac{1}{2} m i_\beta (M^{(F1)} \wedge (B)^2) \wedge i_\beta C^{(3)} \end{aligned} \tag{6.21}$$

and therefore conclude that the required massive gauge condition for $i_\beta N^{(7)}$ is

$$\begin{aligned} \mathcal{L}_K i_\beta N^{(7)} &= -m i_\beta M^{(6)} \wedge i_\beta B - m i_\beta M^{(F1)} \wedge i_\beta C^{(7)} \\ &\quad - \frac{1}{2} m i_\beta (M^{(F1)} \wedge (B)^2) \wedge i_\beta C^{(3)}. \end{aligned} \tag{6.22}$$

Note that just as in the $D = 11$ case we have assumed all fields to be independent of the isometry direction and therefore have the condition that $\mathcal{L}_\beta$ vanishes for each field.

Next we consider the T-duality relations between the KK-monopoles and the NS5-branes. The general rule is that T-duality of the KK-monopoles along the Taub-NUT direction maps to the direct dimensional reduction of the NS5-brane in the dual theory. Therefore, as a consistency check on the IIA KK-monopole charge (6.17) we may check that it maps appropriately to the IIB NS5-brane charge (4.30). In addition to the T-duality rules already given in Chapter 5 we require the following rules relating the IIA potential $i_\beta N^{(7)}$ with the IIB potential $\mathcal{B}^{(6)}$ [71]

$$\begin{aligned}
\mathcal{B}^{(6)}_{\bar{\mu}_1\ldots\bar{\mu}_6} &\rightarrow (-i_\beta N^{(7)} - i_\beta B^{(6)} \wedge i_\beta B + i_\beta C^{(3)} \wedge i_\beta C^{(5)} \\
&\quad + \frac{1}{2} i_\beta C^{(3)} \wedge C^{(3)} \wedge i_\beta B + \frac{1}{2} R^{-2} i_\beta C^{(3)} \wedge i_\beta C^{(3)} \wedge i_\beta B \wedge \beta)_{\bar{\mu}_1\ldots\bar{\mu}_6} \\
i_\beta N^{(7)}_{\bar{\mu}_1\ldots\bar{\mu}_6} &\rightarrow (-\mathcal{B}^{(6)} - \mathcal{R}^{-2} i_\beta \mathcal{B}^{(6)} \wedge \beta + \mathcal{C}^{(4)} \wedge \mathcal{C}^{(2)} \\
&\quad + \mathcal{R}^{-2} i_\beta \mathcal{C}^{(4)} \wedge \mathcal{C}^{(2)} \wedge \beta + \mathcal{R}^{-2} \mathcal{C}^{(4)} \wedge i_\beta \mathcal{C}^{(2)} \wedge \beta)_{\bar{\mu}_1\ldots\bar{\mu}_6}. \qquad (6.23)
\end{aligned}$$

We then ultimately find the following relation between the charges

$$M^{(KK)}_{\bar{\mu}_1\ldots\bar{\mu}_5} \leftrightarrow -N^{(NS5)}_{\bar{\mu}_1\ldots\bar{\mu}_5} \qquad (6.24)$$

thus confirming they are related through T-duality as we would expect.

The simplest way to determine the IIB KK-monopole charge is to use the T-duality relationship between the IIB KK-monopole and the IIA NS5-brane. From this we expect this charge to be related to the IIA NS5-brane charge via

$$M^{(NS5)}_{\bar{\mu}_1\ldots\bar{\mu}_5} \leftrightarrow N^{(KK)}_{\bar{\mu}_1\ldots\bar{\mu}_5} \qquad (6.25)$$

where the T-duality is performed over the Taub-NUT isometry on the IIB side. We can then T-dualise (3.41) to produce the IIB KK-monopole charge. In order to carry out this procedure we require the following T-duality rules relating the IIA field $B^{(6)}$ to the IIB field $i_\beta \mathcal{N}^{(7)}$ which is the magnetic dual of the Taub-NUT isometry β on the IIB side

$$\begin{aligned}
B^{(6)}_{\bar{\mu}_1\ldots\bar{\mu}_6} &\rightarrow (-i_\beta \mathcal{N}^{(7)} - i_\beta \mathcal{B}^{(6)} \wedge i_\beta \mathcal{B} + \frac{1}{2} i_\beta \mathcal{C}^{(4)} \wedge \mathcal{C}^{(2)} \wedge i_\beta \mathcal{B} \\
&\quad - \frac{1}{2} \mathcal{R}^{-2} i_\beta \mathcal{C}^{(4)} \wedge i_\beta \mathcal{C}^{(2)} \wedge i_\beta \mathcal{B} \wedge \beta)_{\bar{\mu}_1\ldots\bar{\mu}_6} \\
i_\beta \mathcal{N}^{(7)}_{\bar{\mu}_1\ldots\bar{\mu}_6} &\rightarrow (-B^{(6)} - R^{-2} i_\beta B^{(6)} \wedge \beta)_{\bar{\mu}_1\ldots\bar{\mu}_6}. \qquad (6.26)
\end{aligned}$$

We ultimately find the IIB KK-monopole generalised charge to be given by

$$\begin{aligned} N^{(KK)} &= e^{-2\varphi}\mathcal{R}^2\Sigma^+ - e^{-2\varphi}i_\beta\Sigma^+\wedge\beta - i_{K^+}(i_\beta\mathcal{N}^{(7)}) + e^{-\varphi}i_\beta\Phi^{(5)}\wedge i_\beta\mathcal{C}^{(2)} \\ &\quad + e^{-\varphi}i_\beta\Phi^{(3)}\wedge i_\beta\mathcal{C}^{(4)} + e^{-\varphi}i_\beta\Phi^{(1)}\wedge i_\beta\mathcal{C}^{(6)} - i_\beta N^{(NS5)}\wedge i_\beta\mathcal{B} \\ &\quad + i_\beta K^-(-i_\beta\mathcal{B}^{(6)} + \mathcal{C}^{(2)}\wedge i_\beta\mathcal{C}^{(4)} - l i_\beta\mathcal{C}^{(6)}) \\ &\quad + K^-\wedge i_\beta\mathcal{C}^{(4)}\wedge i_\beta\mathcal{C}^{(2)} + \frac{1}{2}i_{K^+}(i_\beta\mathcal{C}^{(4)})\wedge i_\beta\mathcal{C}^{(4)} \\ &\quad - i_{K^+}(i_\beta\mathcal{C}^{(2)})\wedge i_\beta\mathcal{C}^{(6)}. \end{aligned} \tag{6.27}$$

We observe that this charge has an analogous structure to the $D = 11$ and IIA cases. The IIB KK-monopole tension factor is simply read off as being $e^{-2\varphi}\mathcal{R}^2$ in agreement with [71].

In order to show that this charge is closed we require the differential relation for $e^{-2\varphi}\mathcal{R}^2\Sigma^+$ as well as the field strength equation for $i_\beta\mathcal{N}^{(7)}$. The former can be obtained by T-dualising (3.40) and is found to be

$$\begin{aligned} d(e^{-2\varphi}\mathcal{R}^2\Sigma^+ - e^{-2\varphi}i_\beta\Sigma^+\wedge\beta) &= i_{K^+}(i_\beta\mathcal{G}^{(8)}) - i_\beta K^- i_\beta\mathcal{H}^{(7)} - e^{-2\varphi}i_\beta\Sigma^-\wedge i_\beta\mathcal{H} \\ &\quad + e^{-\varphi}\Big[i_\beta\Phi^{(7)}i_\beta\mathcal{F}^{(1)} + i_\beta\Phi^{(5)}\wedge i_\beta\mathcal{F}^{(3)} \\ &\quad + i_\beta\Phi^{(3)}\wedge i_\beta\mathcal{F}^{(5)} + i_\beta\Phi^{(1)}i_\beta\mathcal{F}^{(7)}\Big]. \end{aligned} \tag{6.28}$$

The field strength equation is obtained by T-dualising (3.6) and is found to be

$$\begin{aligned} i_\beta\mathcal{G}^{(8)} &= -d(i_\beta\mathcal{N}^{(7)}) - i_\beta\mathcal{B}^{(6)}\wedge i_\beta\mathcal{H} - i_\beta\mathcal{C}^{(2)}\wedge i_\beta\mathcal{F}^{(7)} \\ &\quad - \frac{1}{2}i_\beta\mathcal{C}^{(2)}\wedge i_\beta\mathcal{C}^{(4)}\wedge\mathcal{H} + \frac{1}{2}\mathcal{C}^{(2)}\wedge i_\beta\mathcal{C}^{(4)}\wedge i_\beta\mathcal{H} + \frac{1}{2}i_\beta\mathcal{C}^{(4)}\wedge i_\beta\mathcal{F}^{(5)} \\ &\quad + i_\beta\mathcal{F}^{(1)}(i_\beta\mathcal{C}^{(8)} - i_\beta\mathcal{C}^{(6)}\wedge\mathcal{B} + \frac{1}{4}i_\beta\mathcal{C}^{(4)}\wedge(\mathcal{B})^2) \end{aligned} \tag{6.29}$$

where $\mathcal{G}^{(8)} = e^{-2\varphi} * \mathcal{G}^{(2)} = e^{-2\varphi} * d\beta$. Note the inclusion of terms containing $i_\beta\mathcal{F}^{(1)}$ which will usually be identically zero since we assume independence of the isometry direction, however it will be non-zero in the 'massive' IIB reformulation that we present in Section 12.3 and so we include it here for completeness.[1]

[1] In fact the terms involving $i_\beta\mathcal{F}^{(1)}$ in (6.29) differ slightly depending upon which of the two T-duality schemes, discussed at the beginning of Chapter 5, one adopts. Equation (6.29) is acheived by adopting the first of these schemes where the potentials linearly depend on the T-duality isometry direction.

Using these relations it is then a straight forward task to show that (6.27) is closed as long as we impose the gauge condition $\mathcal{L}_{K+}i_\beta\mathcal{N}^{(7)} = 0$. As discussed in Chapter 5 this is trivially shown to be consistent with the gauge condition on $B^{(6)}$ in the massless IIA theory. This condition does however generalise in the 'massive' IIB reformulation to (12.76) which is the T-dual of the gauge condition on $B^{(6)}$ in Romans' IIA (3.46).

The KK-monopoles are mapped to one another via a double T-duality transformation, so we could at this point perform a consistency check on their charges by checking that they obey the relation

$$i_\beta M^{(KK)}_{\bar{\mu}_1\ldots\bar{\mu}_4} \quad \leftrightarrow \quad i_\beta N^{(KK)}_{\bar{\mu}_1\ldots\bar{\mu}_4}. \tag{6.30}$$

However there is a subtlety in performing this calculation which we will now explain. The T-duality rules for the 7-form potentials $N^{(7)}$ and $\mathcal{N}^{(7)}$ differ depending on whether or not the T-duality is being performed along the isometry direction to which they are the magnetic duals of. So far in this chapter this has always been the case, and the T-duality rules (6.23) and (6.26) apply only to this scenario. This is however not the case when performing a double T-duality transformation on the KK-monopoles since the isometry direction and the Taub-NUT directions (the isometries which the 7-form potentials are currently defined as the magnetic duals of) are now distinct. Therefore we need a different set of T-duality rules for the 7-form potentials in order to confirm the relation (6.30). We do not consider this type of T-duality transformation in this thesis and therefore cannot explicitly confirm this relation. However from examining the T-duality rules of the relevant bilinears we find that the leading terms do match up appropriately and we have no doubt that the relation (6.30) will be satisfied.

Part II

Chapter 7

Introduction to Part II

So far in this thesis we have considered the branes which are naively deducible from the flatspace SUSY algebras via a straight forward scan of the flatspace charges that are present [3, 4]. It is however well known that additional branes also exist which are not as easily inferred from this method. Perhaps the most well known examples of these 'exotic' branes are the 7-branes that exist in IIB and form a triplet under the classical $SL(2,\mathbb{R})$ symmetry group.[1] It is often stated that each of these, which includes the D7-brane, corresponds to the same 7-form flatspace SUSY charge which is assumed to be invariant under the $SL(2,\mathbb{R})$ symmetry [42,43]. This example therefore highlights a breakdown of the one-to-one correspondence which is usually thought to occur, as well as a discrepancy between the $SL(2,\mathbb{R})$ representations of the branes compared with the flatspace charges.

A similar situation occurs with the 9-brane multiplets in IIB [67,68]. There exist six 9-branes in total, four of which transform as a quadruplet and two as a doublet. The flatspace IIB SUSY algebra however only contains two 9-form charges and so once again the relationship between the flatspace SUSY charges and branes contains discrepancies. In the remainder of this thesis we will construct the generalised charges for these additional branes as well as their IIA and $D = 11$ dualisations. We show that the degenerate relationship exhibited between the branes and the charges is a feature only of the flatspace charges and does not extend to the generalised

[1] In this thesis we only consider the classical solutions and therefore do not consider the quantisation of this group to $SL(2,\mathbb{Z})$.

charges which retain a one-to-one correspondence with the branes. Furthermore, in the IIB instance we consider the $SL(2,\mathbb{R})$ representations of the generalised charges and show that they form the same set of multiplets as the branes themselves. Analysis of the types of generalised charge that can be formulated therefore serves as a more powerful tool for exploring the brane spectra than consideration of the flatspace SUSY algebras alone.

The IIB 7-branes have been studied many times in the literature (see [78–80] for some recent work, and references therein). In [43], extending the earlier works of [61,81], it was shown that performing a Scherk-Schwarz [69] dimensional reduction of the IIB theory in a background containing the 7-branes leads to a triplet of $SL(2,\mathbb{R})$ covariant massive $D=9$ supergravity theories, with the 7-branes becoming domain walls. It was also shown that on the IIA side the same $D=9$ theories could be obtained from an $SL(2,\mathbb{R})$ covariant, massive version of $D=11$ supergravity containing two M9-branes and a symmetric 2×2 mass matrix. The Killing isometries associated with the M9-branes make the theory inherently non-covariant (in the spacetime sense) and define a 2-torus, and dimensional reduction over this submanifold leads to the massive $D=9$ theories. The $SL(2,\mathbb{R})\subset GL(2,\mathbb{R})$ symmetry associated with this reduction corresponds to the IIB symmetry group [70, 82–84]. Dimensional reduction over just an S^1 on the other hand yields a non-covariant massive IIA theory (which can be viewed as a further massive extension of Romans' IIA), which is mapped to the IIB theory using an extension of the 'massive' T-duality rules.

Our approach in the remainder of this thesis will be to initially focus on the branes and charges in the $SL(2,\mathbb{R})$ covariant massive $D=11$ supergravity, and then relate these to the charges in IIA and IIB by performing dimensional reductions and T-dualities, emphasizing the $SL(2,\mathbb{R})$ structure. We find that the IIB charges of the 7-branes and 9-brane quadruplet map to certain variations of the $D=11$ KK-monopole and M9-brane charges, which form the appropriate multiplets in $D=11$. For example, carrying out a double dimensional reduction on the standard M9-brane gives either a D8-brane or KK8-monopole depending upon whether or not the reduction is along the 'massive' isometry associated with the M9-brane [29].

Therefore reducing $D = 11$ supergravity in a background of two non-parallel M9-branes over a single isometry direction will produce a non-covariant massive IIA theory containing both a D8-brane and KK8-monopole, which are the T-duals of the D7-brane and its S-dual partner[2], the NS7-brane, respectively [43]. However the triplet of IIB 7-branes contains a third member which in this thesis we refer to as the 'r7'-brane. One would therefore expect there to be a corresponding third brane in $D = 11$ along with the two M9-branes, and that these should transform as a triplet. We find that the charge for such a brane does exist and it has structure that is essentially the same as the M9-brane but explicitly requires two distinct Killing vectors. Furthermore, we also find that there exists a triplet of 10-form gauge potentials which these states couple to.

Alternatively, one can consider the $D = 11$ KK-monopole. Performing a direct dimensional reduction on the KK-monopole gives either the D6-brane or KK6-monopole [42] depending upon whether the reduction is along the Taub-NUT direction or not. These states also map to the D7 and NS7-branes in IIB respectively. As with the M9-branes we would therefore expect there to be a third type of brane present in $D = 11$ related to the KK-monopole, and which maps to the IIB r7-brane forming a triplet of $D = 11$ states. Once again we find the charge of such a state, which has a structure similar to that of the KK-monopole charge but depends explicitly on two Killing vectors.

A similar story exists for the quadruplet of 9-branes in IIB whose charges are mapped to variations of the M9-brane charge. The IIB 9-brane doublet on the other hand seems to follow a separate path and is unrelated to the M9-brane charges.

As before, we will initially construct the generalised charges by demanding that they are closed, and therefore require the field equations of the potentials to which these branes couple. The required field equations in IIB are known [67, 85], however their $D = 11$ counterparts have not yet been fully given. The first task is to therefore calculate these equations. It is a tricky matter to determine all the field equations in $D = 11$ supergravity directly so we employ an an indirect approach here by first considering the massive $D = 11$ supergravity presented in [51]. This

[2]By 'S-dual partner' we mean the branes related by the discrete S-duality transformation.

theory contains only a single M9-brane and therefore only a single 'massive' Killing isometry. Dimensional reduction over the this isometry yields Romans' IIA theory. We will however introduce a second 'non-massive' Killing isometry into the theory, such as that which occurs in the KK-monopole solution. The reason for doing this is that we can determine the field equations that intrinsically depend upon this Killing vector, for example that associated with its 8-form magnetic dual potential, by considering the associated field equations in Romans' theory which must have standard gauge properties. Once we have these equations it is then a simple matter to 'promote' this Killing vector to a 'massive' Killing vector and in doing so we produce an $SL(2,\mathbb{R})$ covariant set of field equations.

For the sake of clarity, in the remainder of this thesis when we are discussing one of the theories specifically we will refer to the $D = 11$ supergravity containing a single M9-brane as simply massive $D = 11$ supergravity, and that containing two M9-branes as $SL(2,\mathbb{R})$ covariant $D = 11$ supergravity. The IIA theories obtained by the dimensional reductions of these will be referred to as Romans' IIA and non-covariant massive IIA respectively.

The organisation of the remainder of this thesis is as follows: we start by considering the massive $D = 11$ supergravity in Chapter 8. We give a brief overview of the differences between this theory and the usual $D = 11$ theory, focusing on the role of the 'massive' isometry and how this differs from that of an ordinary isometry. Next we determine the field equations for the Killing vectors describing these isometries as well as those of their 8-form magnetic dual potentials. We then reconsider the $D = 11$ charges so far constructed and determine the effect of the massive background on them before going on to formulate the M9-brane charge.

In Chapter 9 we map the newly determined $D = 11$ equations to Romans' IIA and IIB. This not only determines the structure of the $D = 10$ equations but also acts as a consistency check on some of the $D = 11$ equations. Then in Chapter 10, using the previous results, we consider the $SL(2,\mathbb{R})$ covariant $D = 11$ theory. We place particular emphasis on the $SL(2,\mathbb{R})$ representations of the fields and determine a general form of the field equations. Using this result we then construct the generalised charges in this theory and discuss how to determine the different multiplet structures that exist.

Next we map these charges to the $D = 10$ theories beginning in Chapter 11 where we perform dimensional reductions to obtain the charges in the non-covariant massive IIA theory. We follow this with Chapter 12 where we revisit the IIB theory. We begin by producing the exotic brane charges by T-dualising the IIA results of the previous chapter. We then go on to reformulate the IIB theory in an $SL(2,\mathbb{R})$ covariant manner and recast all the charges into their $SL(2,\mathbb{R})$ multiplets. Finally we consider how the masses on the IIA side are represented in the IIB theory by discussing a 'massive' pseudo-reformulation of the IIB theory. We end in Chapter 13 with a brief overall summary of the thesis and propose a possible extension to the work presented.

Chapter 8

Massive $D = 11$ supergravity

8.1 Review of massive $D = 11$ supergravity

We begin by briefly reviewing the massive $D = 11$ supergravity presented in [51] focusing on the key characteristics that are relevant for our purposes. Since this theory is essentially the usual $D = 11$ supergravity in a single M9-brane background it retains predominantly the same structure. The exception is that we now have a scalar mass parameter $\hat{m}$ and compact isometry direction described by the Killing vector $\hat{\alpha}$, both of which arise directly from the presence of the M9-brane [29]. The isometry direction in the M9-brane background is required for consistency and lies parallel to the brane worldvolume. Its presence makes the theory inherently non-covariant and in this way circumvents the no-go theorem of constructing a massive $D = 11$ supergravity presented in [86]. This 'massive' Killing vector appears as a gauged isometry not just in the M9-brane worldvolume action, but in all the brane worldvolume actions that are coupled to the theory [28, 51, 87, 88]. It is assumed that no fields depend on this direction and so we make use of relation (2.37) with respect to $\hat{\alpha}$ in what follows. Note that by setting $\hat{m} = 0$ and restoring the field dependencies along $\hat{\alpha}$ we regain the usual massless $D = 11$ supergravity.

The mass parameter itself can be introduced to the action via an auxiliary 10-form gauge potential $\hat{A}^{(10)}$ [76]. This potential minimally couples to the M9-brane and has an 11-form field strength $\hat{F}^{(11)}$ which is related to $\hat{m}$ via a Hodge duality relation. In this way the M9-brane sources the mass parameter. The equation of motion for $\hat{m}$ restricts it to being piece-wise constant, with a possible discontinuity across the M9-brane.

After integrating out this potential the action takes the same form as (2.1) but with an additional cosmological constant-type term, explicitly we have

$$\hat{S} = \frac{1}{2}\int d^{11}\hat{x}\sqrt{-\hat{g}}\left(\hat{R} - \frac{1}{2.4!}|\hat{F}|^2 - \frac{1}{2}\hat{m}^2|\hat{\alpha}^2|^2\right). \tag{8.1}$$

Furthermore, the Chern-Simons term (2.2) is modified to

$$-\frac{1}{2}\int \frac{1}{6}(d\hat{A})^2 \wedge \hat{A} + \frac{1}{8}\hat{m}d\hat{A}\wedge\hat{A}\wedge(i_{\hat{\alpha}}\hat{A})^2 + \frac{1}{40}\hat{m}^2\hat{A}\wedge(i_{\hat{\alpha}}\hat{A})^4. \tag{8.2}$$

Notice that $\hat{\alpha}$ appears explicitly in the action which demonstrates its intrinsic role in the theory.

In order to produce Romans' massive IIA supergravity the dimensional reduction must specifically be performed over $\hat{\alpha}$ (see Appendix A). If the reduction occurs over a different isometry direction then a different massive IIA theory is produced. We will not consider this latter option in the upcoming sections since this massive IIA theory is merely a truncation of the non-covariant massive IIA theory obtained from dimensionally reducing the $SL(2,\mathbb{R})$ covariant $D = 11$ supergravity. Note that the mass parameter is defined such that it obeys the simple reduction rule $\hat{m} \to m$ to IIA.

A further example of the special role played by $\hat{\alpha}$ is its presence in the massive terms of the field equations. For example, the field equations for the 3-form potential $\hat{A}$ and its dual 6-form potential $\hat{C}$ were given in [51] as[1]

$$\hat{F} = d\hat{A} + \frac{1}{2}\hat{m}(i_{\hat{\alpha}}\hat{A})^2 \tag{8.3}$$

$$\begin{aligned}\hat{F}^{(7)} &= d\hat{C} - \frac{1}{2}\hat{F}\wedge\hat{A} + \hat{m}i_{\hat{\alpha}}\hat{A}\wedge i_{\hat{\alpha}}\hat{C} + \frac{1}{12}\hat{m}\hat{A}\wedge(i_{\hat{\alpha}}\hat{A})^2 \\ &\quad + \hat{m}i_{\hat{\alpha}}\hat{N}^{(8)}\end{aligned} \tag{8.4}$$

where $\hat{N}^{(8)}$ is the 8-form dual potential to $\hat{\alpha}$. We return to this field in the next subsection.

These field equations give rise to the following Bianchi identities

$$d\hat{F} = -\hat{m}i_{\hat{\alpha}}\hat{F}\wedge i_{\hat{\alpha}}\hat{A} \tag{8.5}$$

$$d\hat{F}^{(7)} = -\frac{1}{2}\hat{F}\wedge\hat{F} - \hat{m}i_{\hat{\alpha}}\hat{G}^{(9)} - \hat{m}i_{\hat{\alpha}}\hat{F}^{(7)}\wedge i_{\hat{\alpha}}\hat{A} \tag{8.6}$$

[1] Our definition of the mass parameter differs from [51] by a factor of 2.

where in determining the second equation we have used the field equation for $\hat{N}^{(8)}$ given below (8.18), and where $\hat{G}^{(9)}$ is its 9-form field strength. Notice the presence of the 3-form gauge potential $\hat{A}$ in the above Bianchi identities. This is a consequence of the fact that the field strengths are no longer completely gauge invariant but rather transform 'covariantly'[2] under the massive gauge transformations, i.e. those gauge transformations that are proportional to $\hat{m}$. The covariant transformation is defined as follows for a p-form field $\hat{Y}^{(p)}$

$$\delta\hat{Y}^{(p)} = \hat{m}\hat{\lambda} \wedge i_{\hat{\alpha}}\hat{Y}^{(p)} \tag{8.7}$$

where $\hat{\lambda} = i_{\hat{\alpha}}\hat{\chi}$ and $\hat{\chi}$ is the standard 2-form gauge parameter of $\hat{A}$. Using this rule as well as that given below for $\hat{A}$ (8.8) it is a simple matter to check that both the Bianchi identities stated above transform consistently.

It is stated in [51] that all the p-form fields with the exception of $\hat{A}$ and $\hat{C}$ transform according to (8.7). However in this reference the higher rank potentials were not considered. We find that generally the rule (8.7) does not apply to gauge potentials, which usually undergo more complicated transformations. Furthermore, we give an example in the next subsection of field strengths that do not transform simply according to (8.7). The situation is clearer in the $SL(2,\mathbb{R})$ covariant theory, where we propose that the general rules depend on the $SL(2,\mathbb{R})$ representation of the field.

The full gauge transformation of $\hat{A}$ is given by

$$\delta\hat{A} = d\hat{\chi} + \hat{m}\hat{\lambda} \wedge i_{\hat{\alpha}}\hat{A} \tag{8.8}$$

which acts as a connection-field for the massive gauge transformations. The massless connection given in (6.2) is now modified to

$$\hat{\Omega} \to \hat{\Omega} + \hat{\hat{K}} \tag{8.9}$$

where $\hat{\hat{K}}$ is the contorsion tensor defined as

$$\hat{\hat{K}}_{\hat{\mu}_1\hat{\mu}_2\hat{\mu}_3} = \frac{1}{2}\left[\hat{\hat{T}}_{\hat{\mu}_1\hat{\mu}_3\hat{\mu}_2} + \hat{\hat{T}}_{\hat{\mu}_2\hat{\mu}_3\hat{\mu}_1} - \hat{\hat{T}}_{\hat{\mu}_1\hat{\mu}_2\hat{\mu}_3}\right] \tag{8.10}$$

[2]The use of the term 'covariant' here is completely separate from its use in describing $SL(2,\mathbb{R})$ 'covariant' theories, and also 'non-covariant' (in the spacetime sense) supergravity theories.

with the torsion tensor given by

$$\hat{\hat{T}}_{\hat{\mu}_1\hat{\mu}_2}{}^{\hat{\mu}_3} = -\hat{m}(i_{\hat{\alpha}}\hat{A})_{\hat{\mu}_1\hat{\mu}_2}\hat{\alpha}^{\hat{\mu}_3}. \tag{8.11}$$

Next we define the field equation for the 10-form potential $\hat{A}^{(10)}$, which from [46] is given by

$$\begin{aligned} i_{\hat{\alpha}}\hat{F}^{(11)} &= -d(i_{\hat{\alpha}}\hat{A}^{(10)}) - i_{\hat{\alpha}}\hat{F} \wedge i_{\hat{\alpha}}\hat{N}^{(8)} + \frac{1}{4!}i_{\hat{\alpha}}(\hat{F} \wedge \hat{A}) \wedge (i_{\hat{\alpha}}\hat{A})^2 \\ &\quad -\frac{1}{80}\hat{m}(i_{\hat{\alpha}}\hat{A})^5 \end{aligned} \tag{8.12}$$

where $\hat{F}^{(11)}$ is related to $\hat{m}$ through $\hat{F}^{(11)} = -\hat{*}|\hat{\alpha}|^4\hat{m}$. Equation (8.12) was defined so that it reduces to the field strength equation of the 9-form RR potential in Romans' IIA (3.4) after performing a double dimensional reduction. Note that since $\hat{F}^{(11)}$ is of maximum rank it always has a component parallel to $\hat{\alpha}$, which has been made explicit above by the overall contraction with $\hat{\alpha}$. While this is correct, the gauge algebras and field equations seem to have a structure that is independent of the spacetime dimension [67, 89]. This explicit contraction masks the full structure of the field equation and it is possible to write $\hat{F}^{(11)}$ without it. The full structure can be inferred from demanding that $\hat{F}^{(11)}$ be gauge covariant without making use of the dimensionality of the background, and will eventually be required to construct the M9-brane charges in Section 10.2. In doing this we find that it is only possible to construct such a field equation if we introduce a 12-form potential, which is included in an analogous fashion to $\hat{N}^{(8)}$ in (8.4). One can then consider the field equation and gauge transformations of this 12-form potential, which in turn leads to the introduction of a 14-form potential. Repeating the process uncovers an infinite tower of gauge potentials. We will delay giving the general structure to these equations until Section 10.1 where they are dealt with more systematically in an $SL(2, \mathbb{R})$ covariant fashion.

In the above discussion we have tried to emphasize the special role $\hat{\alpha}$ plays in the theory. It is therefore important to distinguish between the 'massive' isometry and any other isometry that may be present in a given spacetime solution, for example the Taub-NUT isometry associated with the KK-monopole. In the following we will denote the Killing vector describing such a 'non-massive' isometry by $\hat{\beta}$.

When considering solutions where both types of isometry are in principal present, there are two possibilities: either both isometries are completely distinct and should therefore be treated separately, or they coincide in which case the massive isometry plays both roles. This distinction is important for example when one wants to perform a dimensional reduction of the KK-monopole in massive $D = 11$ supergravity to find a solution to Romans' IIA as in [27, 28]. If one wants to produce the D6-brane solution then one must consider the configuration where both the Taub-NUT and massive isometries coincide, however if one wants to produce either the IIA KK-monopole or KK6-monopole by performing the reduction over a worldvolume or standard transverse direction respectively, then one must distinguish between the two isometries.

As we discussed when considering the KK-monopoles in Chapter 6, when an isometry is present the associated Killing vector plays a similar role to a gauge potential with the KK-monopole minimally coupling to its magnetic dual. Due to the intrinsic role the massive isometry plays in the theory the field strength equation of its Killing vector (as well as that of its magnetic dual) will differ from those associated with a non-massive isometry. An understanding of these differences is important for example when we consider the generalised charges of the KK-monopole since one must consider both possibilities of whether the massive and Taub-NUT isometries coincide or not. In the following subsections we will determine these field equations by comparing the $D = 11$ equations with the equations in Romans' IIA produced after dimensionally reducing along $\hat{\alpha}$ using the rules stated in Appendix A. We stress that knowledge of the non-massive isometry case is important for constructing the field equations in $SL(2, \mathbb{R})$ covariant $D = 11$ supergravity.

8.2 Massive Killing vector field equations

The field strength $\hat{G}^{(2)}$ of the massive Killing vector $\hat{\alpha}$ has been previously considered in the literature [76, 77] and is given by

$$\hat{G}^{(2)} \;=\; d\hat{\alpha} + \hat{m}\hat{R}^2_{\hat{\alpha}} i_{\hat{\alpha}}\hat{A} \tag{8.13}$$

where $\hat{R}_{\hat{\alpha}}^2 = |\hat{\alpha}|^2$ is the square of the radius of the massive compact isometry. By applying the massive gauge transformation rule (8.7) and realising that $\hat{\alpha}$ transforms as $\delta\hat{\alpha} = \hat{m}\hat{R}_{\hat{\alpha}}^2\hat{\lambda}$, we confirm that both sides of (8.13) transform consistently.

Dimensional reducing (8.13) over $\hat{\alpha}$ allows us to calculate the reduction rule for $\hat{G}^{(2)}$ as

$$\begin{aligned}\hat{G}^{(2)} &\rightarrow e^{\frac{4}{3}\phi}dC^{(1)} + e^{\frac{4}{3}\phi}mB + \frac{4}{3}e^{\frac{4}{3}\phi}d\phi \wedge (C^{(1)} + dz) \\ &= e^{\frac{4}{3}\phi}F^{(2)} + \frac{4}{3}e^{\frac{4}{3}\phi}d\phi \wedge (C^{(1)} + dz)\end{aligned} \tag{8.14}$$

where we have used an adapted co-ordinate system where $\hat{\alpha}^{\hat{\mu}} = \hat{\delta}^{\hat{\mu}z}$, and have substituted in $F^{(2)}$ using (3.4).

Next we define the 9-form field strength $\hat{G}^{(9)} = \hat{*}\hat{G}^{(2)}$ with potential $\hat{N}^{(8)}$. Note that we reuse the same symbols as in the massless theory considered in Chapter 6. For the remainder of this thesis these symbols will be understood to apply to the massive case considered in this chapter. The field equation for $i_{\hat{\alpha}}\hat{N}^{(8)}$ has been given previously [51, 76, 77] and is sufficient for constructing the generalised charge for the massive KK-monopole in the instance where $\hat{\alpha}$ and the Taub-NUT isometry coincide. However, for later analysis in this thesis we will require the full equation which is found by considering its dimensional reduction to IIA.

The reduction rule for $\hat{G}^{(9)}$ can be determined from its relation to $\hat{G}^{(2)}$ and the reduction rule (8.14). We find it to be given by

$$\hat{G}^{(9)} \rightarrow \frac{4}{3}H^{(9)} + F^{(8)} \wedge (C^{(1)} + dz) \tag{8.15}$$

where $H^{(9)}$ is the dual 9-form field strength of the dilaton 'field strength', explicitly $H^{(9)} = e^{-2\phi} * d\phi$. The equation of motion for this field strength can be found by varying Romans' IIA action (3.1) with respect to ϕ which yields[3] [89]

$$dH^{(9)} = \frac{1}{2}H^{(7)} \wedge H + \frac{1}{4}F^{(4)} \wedge F^{(6)} - \frac{3}{4}F^{(2)} \wedge F^{(8)} + \frac{5}{4}mF^{(10)}. \tag{8.16}$$

Using this relation and the Bianchi identity for $F^{(8)}$, given by (3.7), we can uplift and determine the Bianchi identity for $\hat{G}^{(9)}$ as being

$$d\hat{G}^{(9)} = -\frac{2}{3}\hat{F}^{(7)} \wedge i_{\hat{\alpha}}\hat{F} - \frac{1}{3}\hat{F} \wedge i_{\hat{\alpha}}\hat{F}^{(7)} + \frac{5}{3}\hat{m}i_{\hat{\alpha}}\hat{F}^{(11)} - \hat{m}i_{\hat{\alpha}}\hat{G}^{(9)} \wedge i_{\hat{\alpha}}\hat{A}. \tag{8.17}$$

[3]Note that the most straightforward way of doing this is to first express the action in the Einstein frame by Weyl rescaling the metric $g \rightarrow e^{\frac{1}{2}\phi}g_E$.

It is once again a simple matter to apply the covariant gauge transformation rule (8.7) as well as (8.8) for $\hat{A}$ to check that both sides transform consistently. We may use (8.17) to determine the field equation for the 8-form potential $\hat{N}^{(8)}$ as

$$\begin{aligned} \hat{G}^{(9)} &= d\hat{N}^{(8)} - \frac{2}{3} i_{\hat{\alpha}} \hat{F} \wedge \hat{C} + \frac{1}{3} \hat{F} \wedge i_{\hat{\alpha}} \hat{C} + \frac{1}{3!} \hat{F} \wedge i_{\hat{\alpha}} \hat{A} \wedge \hat{A} \\ &\quad - \frac{5}{3} \hat{m} i_{\hat{\alpha}} \hat{A}^{(10)} + \hat{m} i_{\hat{\alpha}} \hat{A} \wedge i_{\hat{\alpha}} \hat{N}^{(8)} - \frac{1}{4!} \hat{m} (i_{\hat{\alpha}} \hat{A})^3 \wedge \hat{A}. \end{aligned} \tag{8.18}$$

For the sake of brevity we delay giving the full gauge transformations of the potentials until (10.23) where we do so in an $SL(2,\mathbb{R})$ covariant fashion. In the mean time we determine the structure of the field strength equations purely from considering the Bianchi identities, as done here.

Contracting $\hat{\alpha}$ with (8.18) yields the field equation

$$\begin{aligned} i_{\alpha} \hat{G}^{(9)} &= -d(i_{\hat{\alpha}} \hat{N}^{(8)}) + i_{\hat{\alpha}} \hat{F} \wedge i_{\hat{\alpha}} \hat{C} + \frac{1}{3!} i_{\hat{\alpha}} (\hat{F} \wedge \hat{A}) \wedge i_{\hat{\alpha}} \hat{A} \\ &\quad - \frac{1}{4!} \hat{m} (i_{\hat{\alpha}} \hat{A})^4 \end{aligned} \tag{8.19}$$

which agrees with the previous definitions in the literature [51,76,77]. Note however that (8.18) is not uniquely determinable from (8.19). Some terms are only present in the full equation whilst the term involving $\hat{C}$ actually splits into two terms in the full equation. This is a characteristic that applies to many of the fields considered in the remainder of this thesis. A proposal for the full $\hat{G}^{(9)}$ field equation was given in [51] but disagrees with (8.18) for this reason.

8.3 Non-massive Killing vector field equations

We now repeat the method of the previous subsection for a non-massive Killing vector $\hat{\beta}$ with the assumption that all Lie derivatives with respect to $\hat{\beta}$ vanish. Here we have the added complication that the $D = 11$ field equations now involve both $\hat{\alpha}$ and $\hat{\beta}$. We do know however that the structure of the field equations here should only differ from the analogous ones given above by their massive terms and that these should simplify to those given previously when we take $\hat{\alpha}$ and $\hat{\beta}$ to coincide. In determining these equations we will also find it necessary to introduce a second 10-form potential.

In order to deduce the field equation for $\hat{\beta}$ we must first determine its gauge transformations. These can be inferred from the gauge transformations of the IIA fields that $\hat{\beta}$ dimensionally reduces to. The reduction rules for $\hat{\beta}$ differ from those of $\hat{\alpha}$ and are given by

$$\hat{\beta} \rightarrow e^{-\frac{2}{3}\phi}\beta + e^{\frac{4}{3}\phi} i_\beta C^{(1)}(C^{(1)} + dz). \tag{8.20}$$

This is easily inferred by considering the reduction rule of the metric (A.1) in a co-ordinate system adapted to $\hat{\beta}$. In Romans' IIA theory β and ϕ do not have massive gauge transformations while $C^{(1)}$ transforms according to[4] $\delta C^{(1)} = -m\lambda$ [60] where λ is obtained from the dimensional reduction of $\hat{\lambda}$ (specifically we have $\hat{\lambda}_{\hat{\mu}} \rightarrow -\lambda_\mu$) and is the gauge parameter of the NS-NS 2-form potential, $\delta B = d\lambda$. We therefore conclude

$$\delta\hat{\beta} = \hat{m}\hat{\lambda}\ i_{\hat{\alpha}}\hat{\beta} + \hat{m} i_{\hat{\beta}}\hat{\lambda}\ \hat{\alpha}. \tag{8.21}$$

We note that $\hat{\beta}$ does not transform covariantly according to (8.7) but rather has an extra term present involving $i_{\hat{\beta}}\hat{\lambda}$. Therefore we would not necessarily expect the field strength of $\hat{\beta}$, which we denote $\hat{S}^{(2)}$, to transform covariantly either. In fact, given (8.21) we were unable to define $\hat{S}^{(2)}$ such that it did transform covariantly. We therefore propose the following definition[5]

$$\hat{S^{(2)}} = d\hat{\beta} + \hat{m} i_{\hat{\beta}}\hat{\alpha}\ i_{\hat{\alpha}}\hat{A} - \hat{m} i_{\hat{\beta}\hat{\alpha}}\hat{A} \wedge \hat{\alpha} \tag{8.22}$$

which transforms as

$$\delta\hat{S}^{(2)} = \hat{m}\hat{\lambda} \wedge i_{\hat{\alpha}}\hat{S}^{(2)} + \hat{m} i_{\hat{\beta}}\hat{\lambda}\ \hat{G}^{(2)} \tag{8.23}$$

which is the field strength generalisation of (8.21). Note that we have taken $\hat{\alpha}^{\hat{\mu}}$ and $\hat{\beta}^{\hat{\mu}}$ to be gauge invariant which can also be shown to be correct from considering their dimensional reduction to Romans' IIA.

We mention that in [28] it was proposed that $\hat{\beta}$ should transform according to (8.7) with there being a compensating transformation for $\hat{\beta}^{\hat{\mu}}$. It was argued that these transformations were necessary to construct a gauge invariant kinetic term in

[4] We neglect the non-massive gauge transformations.

[5] We use the shorthand notation $i_{\hat{\beta}\hat{\alpha}}\hat{A}_{\hat{\mu}}$ to express the double contraction $i_{\hat{\beta}}(i_{\hat{\alpha}}\hat{A})_{\hat{\mu}} = \hat{\beta}^{\hat{\rho}}\hat{\alpha}^{\hat{\nu}}\hat{A}_{\hat{\nu}\hat{\rho}\hat{\mu}}$.

the worldvolume action of the massive KK-monopole. However, we have found that it is also possible to construct an appropriate term using the rule (8.21). We give the explicit details of how to do this in Appendix B. Furthermore, when considering the higher rank potentials below we were only able to construct field equations with consistent gauge properties[6] if transformations analogous to (8.21) were used. We will return to this point in Chapter 10 when we consider the $SL(2,\mathbb{R})$ covariant theory.

The reduction rule for $\hat{S}^{(2)}$ is calculated as

$$\begin{aligned}
\hat{S}^{(2)} \;\rightarrow\;& d(e^{-\frac{2}{3}\phi}\beta + e^{\frac{4}{3}\phi}i_\beta C^{(1)}C^{(1)}) + me^{\frac{4}{3}\phi}i_\beta C^{(1)}B \\
& -me^{\frac{4}{3}\phi}i_\beta B\wedge C^{(1)} + (d(e^{\frac{4}{3}\phi}i_\beta C^{(1)}) - me^{\frac{4}{3}\phi}i_\beta B)\wedge dz \\
=\;& e^{-\frac{2}{3}\phi}G^{(2)} - \frac{2}{3}e^{-\frac{2}{3}\phi}d\phi\wedge\beta + e^{\frac{4}{3}\phi}i_\beta C^{(1)}F^{(2)} \\
& +e^{\frac{4}{3}\phi}(\frac{4}{3}i_\beta C^{(1)}d\phi + X^{(1)})\wedge(C^{(1)}+dz)
\end{aligned} \qquad (8.24)$$

where we have defined $G^{(2)} = d\beta$ and $X^{(1)} = d(i_\beta C^{(1)}) - mi_\beta B$. Due to our assumption that no fields depend on $\hat{\beta}$ in $D = 11$ it follows that we have $\mathcal{L}_\beta C^{(1)} = 0$ and therefore $X^{(1)} = -i_\beta F^{(2)}$, which we treat as an independent field. Such a field is not intrinsically present in Romans' theory but is sourced by the KK6-monopole which magnetically couples to the scalar potential $i_\beta C^{(1)}$. The situation is essentially the same as the KK-monopole magnetically coupling to the Taub-NUT Killing isometry which also is not an intrinsic field of the theory. Note that $X^{(1)}$ is gauge invariant since $F^{(2)}$ is gauge invariant.

Following the previous case we define the dual 9-form field strength $\hat{S}^{(9)} = \hat{*}\hat{S}^{(2)}$ and use (8.24) to determine how it dimensionally reduces. We find the following result

$$\begin{aligned}
\hat{S}^{(9)} \;\rightarrow\;& X^{(9)} + \frac{4}{3}i_\beta C^{(1)}H^{(9)} \\
& +(G^{(8)} + i_\beta C^{(1)}F^{(8)} + \frac{2}{3}i_\beta H^{(9)})\wedge(C^{(1)}+dz)
\end{aligned} \qquad (8.25)$$

where we have defined the field strength $X^{(9)} = e^{-2\phi} * X^{(1)}$.

[6] In Chapter 10 we generalise the idea of gauge covariance to different representations of the $SL(2,\mathbb{R})$ symmetry group and find that these field equations then transform covariantly according to this definition.

We now set about determining the field equations for the three unknown field strengths in play: $G^{(8)}$ and $X^{(9)}$ in IIA, and $\hat{S}^{(9)}$ in $D = 11$. Finding these equations is more problematic than for the analogous fields in the previous subsection. This stems from the fact that it is not straight forward to determine the Bianchi identities of the IIA fields by using a variational principal as was done for $H^{(9)}$, since in this case we are not dealing with intrinsic fields of the theory.

One can attempt to construct the equations directly be demanding gauge invariance but this proves insufficient. Firstly, for $G^{(8)}$ gauge invariance alone does not fully determine the field equation since there is effectively another gauge invariant 8-form field, namely $i_\beta H^{(9)}$, and so any combination of these two fields is gauge invariant. Secondly, when considering $X^{(9)}$ it appears impossible to construct a gauge invariant field strength using just the IIA fields we have so far considered. A similar situation also occurs for $\hat{S}^{(9)}$ irrespective of whether we assume the gauge properties to follow (8.7) or (8.21). To solve this problem it is necessary to introduce a second 10-form potential in the $D = 11$ theory which appears in the definition of $\hat{S}^{(9)}$ and reduces to a 9-form potential in IIA that appears in the field equation for $X^{(9)}$.

We now outline a method which can be used to determine these equations. This method determines the field equations piece by piece and is fairly cumbersome, we therefore do not give the explicit results of the intermediate calculations.

Since $\hat{G}^{(9)}$ is fully known, the massless equation for $\hat{S}^{(9)}$ is also known, since these fields can only differ in their massive terms. Using this fact and the reduction rule (8.25), we see that a double dimensional reduction of $\hat{S}^{(9)}$ will give us the massless terms of $G^{(8)}$ since the equations for $F^{(8)}$ and $i_\beta H^{(9)}$ are already known. The massive terms are then uniquely determined by demanding gauge invariance under the massive gauge transformations of IIA. Equivalently we find that the only possible massive Bianchi identity is given by

$$\begin{aligned} dG^{(8)} &= \frac{1}{2} i_\beta F^{(2)} \wedge F^{(8)} - \frac{1}{2} F^{(2)} \wedge i_\beta F^{(8)} - i_\beta H \wedge H^{(7)} + \frac{1}{2} F^{(4)} \wedge i_\beta F^{(6)} \\ &\quad - \frac{1}{2} i_\beta F^{(4)} \wedge F^{(6)} + \frac{1}{2} m i_\beta F^{(10)}. \end{aligned} \tag{8.26}$$

From this we can determine the equation for $G^{(8)}$ which we give later on (9.2) in Chapter 9 where we consider the $D = 10$ fields in more detail.

Given $G^{(8)}$ we can uplift to $D = 11$ and determine the full structure of $i_{\hat{\alpha}}\hat{S}^{(9)}$, i.e. those components parallel to $\hat{\alpha}$. This puts constraints on the massive terms in $\hat{S}^{(9)}$ but it does not determine them fully. We then proceed by considering the components parallel to $\hat{\beta}$, i.e. $i_{\hat{\beta}}\hat{S}^{(9)}$, and their dimensional reduction to IIA which we see from (8.25) will uniquely determine the massless terms in the field equation for $i_\beta X^{(9)}$. We can then follow the example of $G^{(8)}$ and try to determine the massive terms by using the massless terms and demanding gauge invariance under the massive gauge transformations of IIA. However in this case it seems impossible to achieve gauge invariance using only the fields we have introduced so far. In terms of the Bianchi identity this is illustrated by the presence of non-gauge invariant massive terms which cannot be removed no matter how $i_\beta X^{(9)}$ is defined. The solution to this problem is to introduce a new 9-form potential into the definition of $i_\beta X^{(9)}$ which uplifts to a 10-form potential in $D = 11$. We then find that it is possible to write a suitable Bianchi identity which must be given by

$$\begin{aligned} i_\beta dX^{(9)} &= 2mi_\beta H^{(10)} - 2i_\beta F^{(2)} \wedge i_\beta H^{(9)} - i_\beta(G^{(8)} \wedge F^{(2)}) \\ &\quad -i_\beta F^{(4)} \wedge i_\beta H^{(7)} \end{aligned} \tag{8.27}$$

where $H^{(10)}$ is the field strength of the new 9-form potential that has been introduced. Although we were forced to include the 9-form potential for the sake of gauge invariance, the inclusion of $H^{(10)}$ at this point is somewhat formal since it vanishes in Romans' theory. It is in fact only non-zero in the non-covariant massive IIA theory discussed previously but we include it here for the sake of completeness. It is related by Hodge duality to the extra mass parameters that exist in that theory and we discuss it further in Appendix C. In the current analysis however it plays no significant role.

Uplifting (8.27) to $D = 11$ using (8.25) along with the other appropriate IIA equations will give the form of $i_{\hat{\beta}}d\hat{S}^{(9)}$. Taking into account the terms we have already calculated for $i_{\hat{\alpha}}\hat{S}^{(9)}$ one determines the following Bianchi identity

$$\begin{aligned} d\hat{S}^{(9)} &= -\frac{1}{3}\hat{F} \wedge i_{\hat{\beta}}\hat{F}^{(7)} + \frac{2}{3}i_{\hat{\beta}}\hat{F} \wedge \hat{F}^{(7)} - \frac{1}{3}\hat{m}i_{\hat{\beta}}\hat{F}^{(11)} \\ &\quad +2\hat{m}i_{\hat{\alpha}}\hat{H}^{(11)} - \hat{m}i_{\hat{\alpha}}\hat{A} \wedge i_{\hat{\alpha}}\hat{S}^{(9)} - \hat{m}i_{\hat{\alpha}\hat{\beta}}\hat{A} \wedge \hat{G}^{(9)}. \end{aligned} \tag{8.28}$$

In order to achieve consistency with (8.27) we have been forced to introduce the new

11-form field strength $\hat{H}^{(11)}$ which is defined so that it obeys the following reduction rule to IIA

$$i_{\hat{\alpha}}\hat{H}^{(11)} \rightarrow i_{\beta}C^{(1)}F^{(10)} + H^{(10)} \tag{8.29}$$

where we have included $H^{(10)}$ for completeness although it should be remembered that it vanishes in Romans' theory. From this we can conclude that $\hat{H}^{(11)}$ must be related to the $D = 11$ mass parameter via $\hat{H}^{(11)} = -(|\hat{\alpha}|^2 i_{\hat{\beta}}\hat{\alpha})\hat{*}\hat{m}$, and therefore seems to be some kind of generalisation of the standard 11-form field strength $\hat{F}^{(11)}$. We will show in Section 10.1 that these two fields along with a third field given by (9.42) form a triplet in the $SL(2,\mathbb{R})$ covariant $D = 11$ supergravity. Note that in the instance where $\hat{\alpha}$ and $\hat{\beta}$ coincide we have $\hat{H}^{(11)} \rightarrow \hat{F}^{(11)}$ and $\hat{S}^{(9)} \rightarrow \hat{G}^{(9)}$ and then (8.28) becomes equivalent to (8.17) which is a non-trivial check on the massive terms.

It is straight forward to show that both sides of (8.28) transform consistently under the massive gauge transformations if $\hat{S}^{(9)}$ and $\hat{H}^{(11)}$ transform in analogy to (8.21) as

$$\begin{aligned} \delta\hat{S}^{(9)} &= \hat{m}\hat{\lambda}\wedge i_{\hat{\alpha}}\hat{S}^{(9)} + \hat{m}i_{\hat{\beta}}\hat{\lambda}\ \hat{G}^{(9)} & (8.30)\\ \delta\hat{H}^{(11)} &= \hat{m}\hat{\lambda}\wedge i_{\hat{\alpha}}\hat{H}^{(11)} + \hat{m}i_{\hat{\beta}}\hat{\lambda}\ \hat{F}^{(11)}. & (8.31)\end{aligned}$$

The first term on the RHS of (8.31) is identically zero since it is a maximum rank equation and must therefore have a free index tangential to $\hat{\alpha}$, however we have included it to emphasize the full structure of the transformations.

Using (8.28) and the correspondence between $\hat{S}^{(9)}$ and $\hat{G}^{(9)}$ as well as that between $\hat{H}^{(11)}$ and $\hat{F}^{(11)}$ we can define $\hat{S}^{(9)}$ in terms of an 8-form potential $\hat{T}^{(8)}$ as

$$\begin{aligned} \hat{S}^{(9)} &= d\hat{T}^{(8)} + \frac{1}{3}\hat{F}\wedge i_{\hat{\beta}}\hat{C} - \frac{2}{3}i_{\hat{\beta}}\hat{F}\wedge\hat{C} + \frac{1}{6}\hat{F}\wedge i_{\hat{\beta}}\hat{A}\wedge\hat{A}\\ &\quad +\frac{1}{3}\hat{m}i_{\hat{\beta}}\hat{A}^{(10)} - 2\hat{m}i_{\hat{\alpha}}\hat{B}^{(10)} + \hat{m}i_{\hat{\alpha}\hat{\beta}}\hat{A}\wedge\hat{N}^{(8)}\\ &\quad +\hat{m}i_{\hat{\alpha}}\hat{A}\wedge i_{\hat{\alpha}}\hat{T}^{(8)} - \frac{1}{4!}\hat{m}i_{\hat{\beta}}\hat{A}\wedge(i_{\hat{\alpha}}\hat{A})^2\wedge\hat{A} \end{aligned} \tag{8.32}$$

where $\hat{B}^{(10)}$ is the gauge potential of $\hat{H}^{(11)}$ which we define below. Due to the way we determined this equation it is still possible that there could also be massive terms present of the type $\hat{m}i_{\hat{\alpha}\hat{\beta}}$(11-form) which would have been missed by our method. However, adding any such term would appear to spoil the massive gauge transformations of (8.32) and we therefore do not think that any such terms should be present. This is further supported when we construct the generalised charges in Section 10.2 where (8.32) is precisely the field equation necessary for the charges to be closed.

We stress that in determining (8.32) the 10-form potential $\hat{B}^{(10)}$ has only been partially defined, specifically only the $i_{\hat{\alpha}\hat{\beta}}\hat{B}^{(10)}$ components of its 11-form field equation. No additional information on its structure can be inferred from (8.32) even though $\hat{B}^{(10)}$ appears here only contracted with $\hat{\alpha}$. This is because of the presence of the term involving $i_{\hat{\beta}}\hat{A}^{(10)}$ whose field strength components we have not yet determined; we stress that these cannot be uniquely determined from (8.12) since in that case there was an overall contraction with $\hat{\alpha}$ and so the full structure of the 11-form equation is not determinable. From (8.32) we could however determine the 'joint' field equations for the specific combination of components of the 10-form potentials that appear, however this would serve little purpose. At this point we will therefore only state the field equation for $i_{\hat{\beta}\hat{\alpha}}\hat{B}^{(10)}$. This is sufficient for most purposes since in practice the equation will always have two components parallel to the isometry directions. It is given by

$$\begin{aligned} i_{\hat{\beta}\hat{\alpha}}\hat{H}^{(11)} &= d(i_{\hat{\beta}\hat{\alpha}}\hat{B}^{(10)}) - \frac{1}{2}i_{\hat{\beta}}(i_{\hat{\alpha}}\hat{F} \wedge i_{\hat{\alpha}}\hat{T}^{(8)}) + \frac{1}{2}i_{\hat{\alpha}}(i_{\hat{\beta}}\hat{F} \wedge i_{\hat{\beta}}\hat{N}^{(8)}) \\ &\quad + \frac{1}{4!}i_{\hat{\beta}\hat{\alpha}}(\hat{F} \wedge i_{\hat{\alpha}}\hat{A} \wedge i_{\hat{\beta}}\hat{A} \wedge \hat{A}) - \hat{m} i_{\hat{\beta}\hat{\alpha}}\hat{A} \wedge i_{\hat{\beta}\hat{\alpha}}\hat{A}^{(10)} \\ &\quad - \frac{1}{16}\hat{m} i_{\hat{\beta}}\hat{A} \wedge (i_{\hat{\alpha}}\hat{A})^3 \wedge i_{\hat{\beta}\hat{\alpha}}\hat{A}. \end{aligned} \tag{8.33}$$

We determine the full structure of the 11-form field equations in an $SL(2, \mathbb{R})$ covariant fashion in Chapter 10.

8.4 Generalised charges in massive $D = 11$ supergravity

We will now reconsider the $D = 11$ generalised charges already given to see how to adapt them to this theory. Essentially this exercise simply amounts to rechecking that the previously formulated charges remain closed once the massive terms in the field equations and bilinear differential relations are included. In addition to this we consider for the first time the M9-brane and formulate its generalised charge.

The first task in this process is to reconsider the Killing spinor equation (2.23). The supercovariant derivative operator (2.24) now becomes

$$\begin{aligned} \hat{\tilde{D}}_{\hat{\mu}} &= \hat{\nabla}_{\hat{\mu}} + \frac{1}{288}\left[\hat{\Gamma}_{\hat{\mu}}{}^{\hat{\nu}_1\ldots\hat{\nu}_4} - 8\delta_{\hat{\mu}}^{\hat{\nu}_1}\hat{\Gamma}^{\hat{\nu}_2\hat{\nu}_3\hat{\nu}_4}\right]\hat{F}_{\hat{\nu}_1\ldots\hat{\nu}_4} - \frac{1}{12}\hat{m}\hat{R}_{\hat{\alpha}}^2\hat{\Gamma}_{\hat{\mu}} + \frac{1}{2}\hat{m}\hat{\alpha}_{\hat{\mu}}\hat{\alpha}_{\hat{\nu}}\hat{\Gamma}^{\hat{\nu}} \\ &\quad + \frac{1}{8}\hat{m}\left[2i_{\hat{\alpha}}\hat{A}_{\hat{\mu}\hat{\nu}_1}\hat{\alpha}_{\hat{\nu}_2} - i_{\hat{\alpha}}\hat{A}_{\hat{\nu}_1\hat{\nu}_2}\hat{\alpha}_{\hat{\mu}}\right]\hat{\Gamma}^{\hat{\nu}_1\hat{\nu}_2}. \end{aligned} \quad (8.34)$$

Note that the covariant derivative operator here is the massless one and we have explicitly written the terms arising from the contorsion tensor (8.10) which involve the 3-form potential $\hat{A}$. This expression can be derived by demanding that it dimensionally reduces to the Killing spinor equations (3.24) and (3.25) in Romans' theory. Using this expression we then determine the massive differential relations for the bilinears.

We first consider the charges for the massive M2 and M5-branes in which case we require the differential relations for the 2-form and 5-form bilinears $\hat{\omega}$ and $\hat{\Sigma}$. We find that (2.32) and (2.33) are modified to

$$d\hat{\omega} = i_{\hat{K}}\hat{F} - \hat{m}i_{\hat{\alpha}}\hat{A}\wedge i_{\hat{\alpha}}\hat{\omega} \quad (8.35)$$

$$d\hat{\Sigma} = i_{\hat{K}}\hat{F}^{(7)} - \hat{\omega}\wedge\hat{F} - \hat{m}(\hat{R}_{\hat{\alpha}}^2\hat{\Lambda} + i_{\hat{\alpha}}\hat{\Lambda}\wedge\hat{\alpha} + i_{\hat{\alpha}}\hat{A}\wedge i_{\hat{\alpha}}\hat{\Sigma}). \quad (8.36)$$

Taking into account the field equations (8.3) and (8.4) we now find the exterior derivatives of the M2 and M5-brane charges (2.43) and (2.52) are given by

$$d\hat{L}^{(2)} = -\hat{m}i_{\hat{\alpha}}\hat{L}^{(2)}\wedge i_{\hat{\alpha}}\hat{A} + \mathcal{L}_{\hat{K}}\hat{A} \quad (8.37)$$

$$\begin{aligned} d\hat{L}^{(5)} &= -\hat{m}\hat{L}^{(KK)}_{(\hat{\alpha})} - \hat{m}i_{\hat{\alpha}}\hat{L}^{(2)}\wedge i_{\hat{\alpha}}\hat{C} - \frac{1}{2}\hat{m}i_{\hat{\alpha}}\hat{L}^{(2)}\wedge\hat{A}\wedge i_{\hat{\alpha}}\hat{A} \\ &\quad + \mathcal{L}_{\hat{K}}\hat{C} + \frac{1}{2}\mathcal{L}_{\hat{K}}\hat{A}\wedge\hat{A}. \end{aligned} \quad (8.38)$$

where $\hat{L}^{(KK)}_{(\hat{\alpha})}$ is the generalised charge of the KK-monopole in the instance where the Taub-NUT and massive isometries coincide which we discuss further in the next subsection. We therefore conclude that these charges are closed and therefore remain valid in the massive theory if the following gauge conditions are satisfied

$$\mathcal{L}_{\hat{K}}\hat{A} = \hat{m}i_{\hat{\alpha}}\hat{L}^{(2)} \wedge i_{\hat{\alpha}}\hat{A} \tag{8.39}$$

$$\mathcal{L}_{\hat{K}}\hat{C} = \hat{m}\hat{L}^{(KK)}_{(\hat{\alpha})} + \hat{m}i_{\hat{\alpha}}\hat{L}^{(2)} \wedge i_{\hat{\alpha}}\hat{C}. \tag{8.40}$$

Such conditions are analogous to those which are required in Romans' theory as we discussed in Section 3.5 and which are merely the dimensional reductions of the conditions here. Consistency of the above conditions with the field equations (8.3) and (8.4) requires the following conditions on the Lie derivatives to hold true

$$\mathcal{L}_{\hat{K}}\hat{F} = \hat{m}i_{\hat{\alpha}}\hat{L}^{(2)} \wedge i_{\hat{\alpha}}\hat{F} \tag{8.41}$$

$$\mathcal{L}_{\hat{K}}\hat{F}^{(7)} = \hat{m}i_{\hat{\alpha}}\hat{L}^{(2)} \wedge i_{\hat{\alpha}}\hat{F}^{(7)} \tag{8.42}$$

together with the vanishing of $\mathcal{L}_{\hat{K}}i_{\hat{\alpha}}\hat{N}^{(8)}$ which we will find to be the case below. The conditions (8.41) and (8.42) can be confirmed to be true by calculating the exterior derivatives of (8.35) and (8.36) and using the general definition of the Lie derivative (2.37). From these conditions we conclude that $\hat{K}$ no longer generates a symmetry of the background. Furthermore from (8.34) we now find the covariant derivative of $\hat{K}$ to be given by

$$\begin{aligned}\hat{\nabla}_{\hat{\mu}}\hat{K}_{\hat{\nu}} &= \frac{1}{3}i_{\hat{\omega}}\hat{F}_{\hat{\mu}\hat{\nu}} + \frac{1}{6}i_{\hat{F}}\hat{\Lambda}_{\hat{\mu}\hat{\nu}} - \frac{1}{6}\hat{m}\hat{R}^2_{\hat{\alpha}}\hat{\omega}_{\hat{\mu}\hat{\nu}} + \hat{m}\hat{\alpha}_{\hat{\mu}}i_{\hat{\alpha}}\hat{\omega}_{\hat{\nu}} - \frac{1}{2}\hat{m}i_{\hat{\alpha}}\hat{A}_{\hat{\mu}\hat{\nu}}i_{\hat{\alpha}}\hat{K} \\ &\quad -\frac{1}{2}\hat{m}i_{\hat{K}\hat{\alpha}}\hat{A}_{\hat{\mu}}\hat{\alpha}_{\hat{\nu}} - \frac{1}{2}\hat{m}i_{\hat{K}\hat{\alpha}}\hat{A}_{\hat{\nu}}\hat{\alpha}_{\hat{\mu}}.\end{aligned} \tag{8.43}$$

Hence we find

$$\hat{\nabla}_{(\hat{\mu}}\hat{K}_{\hat{\nu})} = \hat{m}i_{\hat{\alpha}}\hat{L}^{(2)}_{(\hat{\mu}}\hat{\alpha}_{\hat{\nu})} \tag{8.44}$$

and therefore conclude that $\hat{K}$ is no longer Killing in certain M2-brane backgrounds.

8.4.1 Massive KK-monopole generalised charge

Next we consider the generalised charge for the KK-monopole which we found to have a general form given by (6.15). In the massive theory it is important to distinguish whether the isometry appearing here corresponds to either $\hat{\alpha}$ or $\hat{\beta}$ in our current notation. These two possibilities correspond physically to whether the massive and Taub-NUT isometries coincide or are distinct. Here we will consider the latter of these two possibilities since this is the more general circumstance. The former case can be found by merely making the identification $\hat{\alpha} = \hat{\beta}$ which has the effect of simplifying the equations given below.

We therefore need to determine the differential relation for $\hat{R}_{\hat{\beta}}^2\hat{\Lambda}$ where $\hat{R}_{\hat{\beta}} = |\hat{\beta}|$ is the radius of the compact Taub-NUT isometry. To achieve this we repeat the method discussed in Chapter 6. The first step in this process is to redetermine the differential relation for $\hat{\Lambda}$ using the massive operator (8.34). This is now found to be given by

$$\begin{aligned} d\hat{\Lambda}_{\hat{A}_1\ldots\hat{A}_7} &= \frac{14}{3}\hat{\omega}^{\hat{B}}{}_{[\hat{A}_1}\hat{F}^{(7)}_{\hat{A}_2\ldots\hat{A}_7]\hat{B}} - \frac{35}{3}\hat{\Sigma}^{\hat{B}}{}_{[\hat{A}_1\ldots\hat{A}_4}\hat{F}_{\hat{A}_5\hat{A}_6\hat{A}_7]\hat{B}} \\ &\quad -\hat{m}(i_{\hat{\alpha}}\hat{A}\wedge i_{\hat{\alpha}}\hat{\Lambda})_{\hat{A}_1\ldots\hat{A}_7}. \end{aligned} \tag{8.45}$$

Next we determine the algebraic Killing spinor equation which arises from considering the projection of (8.34) along $\hat{\beta}$. For this we adopt a co-ordinate system adapted to $\hat{\beta}$. This is analogous to that used in Chapter 6 with the exception that we now parametrise the isometry direction by x. Furthermore, we will once again work in an orthonormal basis analogous to (6.1). After splitting the components of the covariant derivative operator we find that the algebraic Killing spinor equation (6.9) now becomes

$$\begin{aligned} \hat{\hat{D}}_x\hat{\epsilon} &= -\frac{1}{8}d\hat{\beta}_{AB}\hat{\Gamma}^{AB}\hat{\epsilon} - \frac{1}{4}\hat{R}_{\hat{\beta}}^{-1}\partial_A(\hat{R}_{\hat{\beta}}^2)\hat{\Gamma}^{A\underline{x}}\hat{\epsilon} \\ &\quad +\frac{1}{288}\hat{R}_{\hat{\beta}}\hat{\Gamma}_{\underline{x}}{}^{ABCD}\hat{F}_{ABCD}\hat{\epsilon} - \frac{1}{36}\hat{R}_{\hat{\beta}}\hat{\Gamma}^{ABC}\hat{F}_{\underline{x}ABC}\hat{\epsilon} \\ &\quad -\frac{1}{12}\hat{m}\hat{R}_{\hat{\alpha}}^2\hat{R}_{\hat{\beta}}\hat{\Gamma}_{\underline{x}}\hat{\epsilon} + \frac{1}{2}\hat{m}i_{\hat{\beta}}\hat{\alpha}\ \hat{\alpha}_A\hat{\Gamma}^A\hat{\epsilon} + \frac{1}{2}\hat{m}\hat{R}_{\hat{\beta}}^{-1}(i_{\hat{\beta}}\hat{\alpha})^2\ \hat{\Gamma}^{\underline{x}}\hat{\epsilon} \\ &\quad +\frac{1}{8}\hat{m}\left[2i_{\hat{\beta}\hat{\alpha}}\hat{A}_A\hat{\alpha}_B - i_{\hat{\beta}}\hat{\alpha}\ i_{\hat{\alpha}}\hat{A}_{AB}\right]\hat{\Gamma}^{AB}\hat{\epsilon} \\ &\quad +\frac{1}{2}\hat{m}\hat{R}_{\hat{\beta}}^{-1}i_{\hat{\beta}}\hat{\alpha}\ i_{\hat{\beta}\hat{\alpha}}\hat{A}_A\hat{\Gamma}^{A\underline{x}}\hat{\epsilon} = 0. \end{aligned} \tag{8.46}$$

Hitting this from the left with $\bar{\hat{\epsilon}}\hat{\Gamma}_{A_1\ldots A_7\underline{x}}$ determines that the algebraic relation (6.10) is modified to

$$\begin{aligned}
0 &= \left[i_{\hat{K}}(i_{\hat{\beta}}\hat{S}^{(9)}) + d\hat{\beta}\wedge i_{\hat{\beta}}\hat{\Lambda} - d(\hat{R}^2_{\hat{\beta}})\wedge\hat{\Lambda} + \frac{1}{3}i_{\hat{\beta}}\hat{\omega}\wedge i_{\hat{\beta}}\hat{F}^{(7)} + \frac{2}{3}i_{\hat{\beta}}\hat{\Sigma}\wedge i_{\hat{\beta}}\hat{F}\right]_{A_1\ldots A_7} \\
&\quad -\frac{14}{3}\hat{R}^2_{\hat{\beta}}\hat{\omega}^B{}_{[A_1}\hat{F}^{(7)}_{A_2\ldots A_7]B} + \frac{35}{3}\hat{R}^2_{\hat{\beta}}\hat{\Sigma}_{B[A_1\ldots A_4}\hat{F}_{A_5A_6A_7]}{}^B \\
&\quad +\hat{m}\Big[2i_{\hat{\beta}}\hat{\alpha}\; i_{\hat{\beta}\hat{\alpha}}\hat{\Pi} + i_{\hat{\beta}\hat{\alpha}}\hat{A}\wedge i_{\hat{\beta}}\hat{\Lambda}\wedge\hat{\alpha} + 2i_{\hat{\beta}}\hat{\alpha}\; i_{\hat{\beta}\hat{\alpha}}\hat{A}\wedge\hat{\Lambda} \\
&\quad + i_{\hat{\beta}}\hat{\alpha}\; i_{\hat{\alpha}}\hat{A}\wedge i_{\hat{\beta}}\hat{\Lambda}\Big]_{A_1\ldots A_7}
\end{aligned} \tag{8.47}$$

where we have multiplied through by a factor of $-4\hat{R}_{\hat{\beta}}$. Finally we combine this with (8.45) multiplied by $\hat{R}^2_{\hat{\beta}}$, and after converting to the co-ordinate basis we obtain the following differential relation

$$\begin{aligned}
d(\hat{R}^2_{\hat{\beta}}\hat{\Lambda} + i_{\hat{\beta}}\hat{\Lambda}\wedge\hat{\beta}) &= i_{\hat{K}\hat{\beta}}\hat{S}^{(9)} + i_{\hat{\beta}}\hat{\omega}\wedge i_{\hat{\beta}}\hat{F}^{(7)} + i_{\hat{\beta}}\hat{\Sigma}\wedge i_{\hat{\beta}}\hat{F}^{(4)} \\
&\quad +\hat{m}\Big[2i_{\hat{\beta}}\hat{\alpha}\; i_{\hat{\beta}\hat{\alpha}}\hat{A}\wedge\hat{\Lambda} - i_{\hat{\alpha}\hat{\beta}}\hat{A}\wedge i_{\hat{\beta}}\hat{\Lambda}\wedge\hat{\alpha} \\
&\quad + i_{\hat{\beta}}\hat{\alpha}\; i_{\hat{\beta}}\hat{\Lambda}\wedge i_{\hat{\alpha}}\hat{A} + i_{\hat{\alpha}\hat{\beta}}\hat{A}\wedge\hat{\beta}\wedge i_{\hat{\alpha}}\hat{\Lambda} \\
&\quad - i_{\hat{\alpha}}\hat{A}\wedge i_{\hat{\alpha}\hat{\beta}}\hat{\Lambda}\wedge\hat{\beta} - \hat{R}^2_{\hat{\beta}} i_{\hat{\alpha}}\hat{A}\wedge i_{\hat{\alpha}}\hat{\Lambda} + 2i_{\hat{\beta}}\hat{\alpha}\; i_{\hat{\beta}\hat{\alpha}}\hat{\Pi}\Big].
\end{aligned} \tag{8.48}$$

Now that we have this relation together with the field equation (8.32) we can check the KK-monopole charge (6.15) to see whether it remains closed in massive backgrounds. For convenience we restate this charge here

$$\begin{aligned}
\hat{L}^{(KK)}_{(\hat{\beta})} &= \hat{R}^2_{\hat{\beta}}\hat{\Lambda} + i_{\hat{\beta}}\hat{\Lambda}\wedge\hat{\beta} - i_{\hat{K}}(i_{\hat{\beta}}\hat{T}^{(8)} - \frac{1}{3!}\hat{A}\wedge(i_{\hat{\beta}}\hat{A})^2) - i_{\hat{\beta}}\hat{\omega}\wedge(i_{\hat{\beta}}\hat{C} \\
&\quad + \frac{1}{2}\hat{A}\wedge i_{\hat{\beta}}\hat{A}) + i_{\hat{\beta}}\hat{L}^{(5)}\wedge i_{\hat{\beta}}\hat{A} - \frac{1}{2}\hat{L}^{(2)}\wedge(i_{\hat{\beta}}\hat{A})^2.
\end{aligned} \tag{8.49}$$

Explicit calculation reveals that this charge remains closed if we fix the gauges so that the conditions (8.39) and (8.40) are satisfied together with

$$\begin{aligned}
\mathcal{L}_{\hat{K}} i_{\hat{\beta}}\hat{T}^{(8)} &= \hat{m}\Big[i_{\hat{\beta}}(i_{\hat{\alpha}}\hat{L}^{(2)}\wedge i_{\hat{\alpha}}\hat{T}^{(8)}) + i_{\hat{\alpha}\hat{\beta}}\hat{L}^{(2)}\wedge i_{\hat{\beta}}\hat{N}^{(8)} - i_{\hat{\beta}}\hat{L}^{(KK)}_{(\hat{\alpha})}\wedge i_{\hat{\beta}}\hat{A} \\
&\quad + 2i_{\hat{\beta}\hat{\alpha}}\hat{L}^{(9)}_{12}\Big].
\end{aligned} \tag{8.50}$$

Here $\hat{L}^{(9)}_{12}$ is a variation of the M9-brane charge which we will discuss in Section 10.2.2. Due to the appearance of $\hat{B}^{(10)}$ in (8.32), showing that these gauge conditions are consistent with one another requires knowledge of $\mathcal{L}_{\hat{K}}\hat{B}^{(10)}$. Considering this at this point leads to unnecessary complications, we therefore delay discussion of this topic until Chapter 10 where we show that these gauge conditions are indeed consistent with one another.

The analysis of generalised charge for the KK-monopole configuration where the massive and Taub-NUT isometries coincide is somewhat simpler than the more general case considered above. In this case the charge $\hat{L}^{(KK)}_{(\hat{\alpha})}$ is merely (8.49) with the substitution $\hat{\beta} \to \hat{\alpha}$ (and of course $\hat{T}^{(8)} \to \hat{N}^{(8)}$), and many of the relevant relations simplify after this substitution. It follows that if $\hat{L}^{(KK)}_{(\hat{\beta})}$ is closed then $\hat{L}^{(KK)}_{(\hat{\alpha})}$ is guaranteed to be closed as well, so explicit calculations to show the latter are not required. The gauge condition on $\hat{N}^{(8)}$ is determined from (8.50) after making the identification $\hat{\alpha} = \hat{\beta}$ and is given by

$$\mathcal{L}_{\hat{K}} i_{\hat{\alpha}} \hat{N}^{(8)} = 0. \tag{8.51}$$

8.4.2 M9-brane generalised charge

We end this chapter by formulating the generalised charge of the M9-brane. The calibrating bilinear in this case is the 9-form $\hat{\Pi}$. Using the massive Killing spinor equation (8.34) we find that the differential relation (2.35) is now modified to

$$d\hat{\Pi} \;\; = \;\; -\frac{1}{3}\hat{F} \wedge \hat{\Lambda} + \hat{m}\Big[-\frac{5}{3}\hat{R}^2_{\hat{\alpha}}\hat{\Upsilon} - i_{\hat{\alpha}}\hat{\Upsilon} \wedge \hat{\alpha} - i_{\hat{\alpha}}\hat{\Pi} \wedge i_{\hat{\alpha}}\hat{A}\Big]. \tag{8.52}$$

In analogy to the KK-monopole case we find that it is not possible to formulate a generalised charge with leading term given by just $\hat{\Pi}$ due to the structure of (8.52). Once again the physical reason for this is the presence of the compact isometry found in the M9-brane solution and the fact that the brane tension is a function of the compact radius [29]. We therefore require an additional relation from the algebraic Killing spinor equation (8.46) which will allow for the formation of a suitable differential relation for $\hat{\Pi}$.

The M9-brane isometry is in fact identified as the massive isometry $\hat{\alpha}$. The situation here is therefore different to that of the KK-monopole since it does not make sense to think of these isometries as being distinct in the current theory. In fact treating them as distinct isometries leads physically to a double M9-brane configuration and therefore only makes sense in the $SL(2,\mathbb{R})$ covariant theory. We therefore do not consider this possibility until Section 10.2.2 and for the remainder of this section we make the identification $\hat{\alpha} = \hat{\beta}$.

For the following calculation we work in the same co-ordinate system as previously and note that after identifying the two isometries (8.46) simplifies. The required algebraic relation here is obtained by hitting (8.46) from the left with $\hat{\Gamma}_{A_1 \dots A_9}$ which, after multiplication by $4\hat{R}^2_{\hat{\alpha}}$, yields

$$\begin{aligned} 0 \;\; = \;\; & \Big[i_{\hat{\alpha}}\hat{\omega} \wedge i_{\hat{\alpha}}\hat{G}^{(9)} - d(\hat{R}^2_{\hat{\alpha}}) \wedge i_{\hat{\alpha}}\hat{\Pi} - \frac{1}{3}\hat{R}^2_{\hat{\alpha}} i_{\hat{\alpha}}\hat{\Lambda} \wedge \hat{F} + \frac{2}{3}\hat{R}^2_{\hat{\alpha}}\hat{\Lambda} \wedge i_{\hat{\alpha}}\hat{F} \\ & - \frac{5}{3} i_{\hat{K}\hat{\alpha}}\hat{F}^{(11)} \Big]_{A_1 \dots A_9} . \end{aligned} \tag{8.53}$$

We then contract (8.52) with $\hat{\alpha}$ and multiply by $\hat{R}^2_{\hat{\alpha}}$ and combine with the above relation to produce

$$d(\hat{R}^2_{\hat{\alpha}} i_{\hat{\alpha}}\hat{\Pi})_{A_1 \dots A_9} = \Big[\hat{R}^2_{\hat{\alpha}}\hat{\Lambda} \wedge i_{\hat{\alpha}}\hat{F} + i_{\hat{\alpha}}\hat{\omega} \wedge i_{\hat{\alpha}}\hat{G}^{(9)} - i_{\hat{K}\hat{\alpha}}\hat{F}^{(11)} \Big]_{A_1 \dots A_9} . \tag{8.54}$$

The final step is to convert this to the co-ordinate basis which yields the following fully tensorial equation

$$\begin{aligned} d(\hat{R}^2 i_{\hat{\alpha}}\hat{\Pi}) \;\; = \;\; & \hat{R}^2_{\hat{\alpha}}\hat{\Lambda} \wedge i_{\hat{\alpha}}\hat{F} + i_{\hat{\alpha}}\hat{\Lambda} \wedge \hat{\alpha} \wedge i_{\hat{\alpha}}\hat{F} + i_{\hat{\alpha}}\hat{\omega} \wedge i_{\hat{\alpha}}\hat{G}^{(9)} \\ & - i_{\hat{K}\hat{\alpha}}\hat{F}^{(11)} . \end{aligned} \tag{8.55}$$

From this relation we then determine the M9-brane generalised charge using the usual method. We ultimately find this to be given by

$$\begin{aligned} i_{\hat{\alpha}}\hat{L}^{(9)} \;\; = \;\; & \hat{R}^2_{\hat{\alpha}} i_{\hat{\alpha}}\hat{\Pi} + i_{\hat{K}}(i_{\hat{\alpha}}\hat{A}^{(10)} - \frac{1}{4!}\hat{A} \wedge (i_{\hat{\alpha}}\hat{A})^3) \\ & - i_{\hat{\alpha}}\hat{\omega} \wedge (i_{\hat{\alpha}}\hat{N}^{(8)} - \frac{1}{3!}\hat{A} \wedge (i_{\hat{\alpha}}\hat{A})^2) + \hat{L}^{(KK)}_{(\hat{\alpha})} \wedge i_{\hat{\alpha}}\hat{A} \\ & - \frac{1}{2} i_{\hat{\alpha}}\hat{L}^{(5)} \wedge (i_{\hat{\alpha}}\hat{A})^2 + \frac{1}{3!}\hat{L}^{(2)} \wedge (i_{\hat{\alpha}}\hat{A})^3 . \end{aligned} \tag{8.56}$$

Explicit calculation reveals that this charge is closed if we fix the gauges so that the conditions (8.39), (8.40) and (8.51) are satisfied together with

$$\mathcal{L}_{\hat{K}}(i_{\hat{\alpha}}\hat{A}^{(10)}) = 0. \tag{8.57}$$

It is straightforward to show that these conditions are consistent with each other by considering the field equations and calculating $\mathcal{L}_{\hat{K}}$ of the field strengths.

Like the KK-monopole charge (6.15) we find that the isometry makes an explicit appearance in the M9-brane charge. Note however that the structure of the leading bilinear term here takes a different form than that found in the KK-monopole case. We conclude that the structure of the leading term encodes whether or not the isometry lies parallel or transverse to the brane worldvolume. Due to the contraction of $\hat{\alpha}$ with $\hat{\Pi}$ here we find that the M9-brane charge is actually an 8-form. We have however written this in terms of a 9-form charge $\hat{L}^{(9)}$ since the worldvolume is 9-dimensional. We have not shown that the full 9-form is closed though and the interpretation of such a charge is not fully clear to us. We discuss this expression further in the context of the $SL(2,\mathbb{R})$ covariant theory in Section 10.2.2 where we show that it is closed and see that it plays an important role in determining the different types of M9-brane charge multiplets that can be formed. Finally we observe the scalar $\hat{R}^2_{\hat{\alpha}}$ in the leading term in (8.56) however we note that the M9-brane tension scales with $\hat{R}^3_{\hat{\alpha}}$. The discrepancy here arises due to the contraction between $\hat{\alpha}$ with $\hat{\Pi}$ and is analogous to that encountered with the IIA KK-monopole charge (6.17).

Chapter 9

Additional field equations in the $D = 10$ supergravity theories

We now consider the mappings of the field equations obtained in the previous chapter to the $D = 10$ supergravity theories. Our primary motivation for doing this is to determine the structure of the 9-form field equations in IIB which are required for the formulation of the exotic 7-brane generalised charges. Additionally, this process acts as a non-trivial check on some of the field equations derived previously as well as allowing us to determine the structure of the final field equation for the triplet of 10-form $D = 11$ potentials. This will be important when we consider the $SL(2, \mathbb{R})$ covariant $D = 11$ theory.

9.1 Field equations in IIA from $D = 11$

We start by giving the IIA equations produced from dimensionally reducing the equations of the last chapter. The required reduction rules for the potentials are given in Appendix A.

From (8.15) and (8.18) we find the equation for $H^{(9)}$ is given by

$$\begin{aligned} H^{(9)} &= d\phi^{(8)} + \frac{1}{2}H \wedge B^{(6)} + \frac{1}{4}(F^{(4)} + H \wedge C^{(1)}) \wedge C^{(5)} - \frac{3}{4}F^{(8)} \wedge C^{(1)} \\ &\quad + \frac{5}{4}mC^{(9)} - m\frac{3}{4}C^{(7)} \wedge B + \frac{1}{4!}mC^{(3)} \wedge (B)^3 \end{aligned} \tag{9.1}$$

where $\phi^{(8)}$ is the 8-form magnetic dual of the dilaton. This equation has been previously given in [60, 89].

From (8.25) and (8.32) we find the equation for $G^{(8)}$ is given by

$$\begin{aligned} G^{(8)} \;=\;& dN^{(7)} - \frac{1}{2} i_\beta C^{(1)} \wedge F^{(8)} + \frac{1}{2} i_\beta F^{(8)} \wedge C^{(1)} - i_\beta H \wedge B^{(6)} \\ & - \frac{1}{2}(F^{(4)} + H \wedge C^{(1)}) \wedge i_\beta C^{(5)} + \frac{1}{2} i_\beta (F^{(4)} + H \wedge C^{(1)}) \wedge C^{(5)} \\ & - \frac{1}{2} H \wedge i_\beta C^{(3)} \wedge C^{(3)} - \frac{1}{2} m i_\beta B \wedge C^{(7)} + \frac{1}{2} m B \wedge i_\beta C^{(7)} - \frac{1}{2} m i_\beta C^{(9)} \\ & - \frac{1}{4} m i_\beta B \wedge (B)^2 \wedge C^{(3)} + \frac{1}{12} m i_\beta C^{(3)} \wedge (B)^3 \end{aligned} \tag{9.2}$$

where $N^{(7)}$ is the 7-form magnetic dual of the Killing vector β.

From (8.25) and (8.32) we find the equation for $i_\beta X^{(9)}$ is given by

$$\begin{aligned} i_\beta X^{(9)} \;=\;& -d(i_\beta N^{(8)}) - 2 i_\beta C^{(1)} i_\beta H^{(9)} - i_\beta (G^{(8)} \wedge C^{(1)}) - i_\beta C^{(1)} i_\beta (F^{(8)} \wedge C^{(1)}) \\ & + i_\beta (F^{(4)} + H \wedge C^{(1)}) \wedge (i_\beta B^{(6)} + \frac{1}{6} i_\beta C^{(3)} \wedge C^{(3)}) \\ & + \frac{1}{6}(F^{(4)} + H \wedge C^{(1)}) \wedge (i_\beta C^{(3)})^2 + 2 m i_\beta B \wedge i_\beta \phi^{(8)} \\ & + 2 m i_\beta B^{(9)} - m i_\beta (N^{(7)} \wedge B) + \frac{1}{6} m (i_\beta C^{(3)})^2 \wedge (B)^2 \\ & - \frac{1}{3} m C^{(3)} \wedge i_\beta C^{(3)} \wedge B \wedge i_\beta B \end{aligned} \tag{9.3}$$

where $N^{(8)}$ is the 8-form magnetic dual of the scalar potential $i_\beta C^{(1)}$. We only state this equation with a contraction with β here. We do not require the full equation for $X^{(9)}$ at this point and calculating it requires considering a 10-form potential resulting from the term $i_{\hat{\beta}} \hat{A}^{(10)}$ in (8.32) which leads to unnecessary complications at this point. Note the presence of the new potential $B^{(9)}$ which originates from $i_{\hat{\alpha}} \hat{B}^{(10)}$ in $D = 11$.

Finally, we formally give the field equation for $i_\beta H^{(10)}$ which, as already mentioned, vanishes in Romans' theory. This is obtained from reducing (8.33)

$$\begin{aligned} i_\beta H^{(10)} \;=\;& -d(i_\beta B^{(9)}) - i_\beta C^{(1)} i_\beta F^{(10)} - \frac{1}{2} i_\beta (H \wedge N^{(7)}) + i_\beta \phi^{(8)} \wedge i_\beta H \\ & - \frac{1}{2} i_\beta (F^{(4)} + H \wedge C^{(1)}) \wedge i_\beta C^{(7)} + m i_\beta C^{(9)} \wedge i_\beta B \\ & + \frac{1}{12} m i_\beta C^{(3)} \wedge i_\beta B \wedge (B)^3. \end{aligned} \tag{9.4}$$

Once again we only state this equation contracted with β since it serves our current purposes and considering the full equation leads to unnecessary complications at this point.

9.2 Field equations from T-duality

We now T-dualise the IIA fields to the IIB theory. It is to be understood that we always perform the T-duality transformation along the isometry described by β. This is important since this vector is used in the definition of the fields $N^{(7)}$ and $N^{(8)}$. They would therefore exhibit different T-duality transformations if we were using a different isometry and we do not consider this option in this thesis. Performing the T-duality produces the IIB field equations for the two 8-form potentials that transform as part of a triplet along with the standard RR 8-form potential. These equations were given in a $SU(1,1)$ covariant form in [67, 85]. This triplet maps to a triplet of 10-form potentials in the $D = 11$ theory by performing direct T-duality transformations to IIA and then uplifting. The potentials $\hat{A}^{(10)}$ and $\hat{B}^{(10)}$ both belong to this triplet and we can deduce the third via its relation to one of the IIB 8-forms.

Since we are not considering the generalised charges here we will follow the first of the two approaches discussed at the beginning of Chapter 5 when performing the following T-duality transformations. The advantage of doing this is that we will not have to worry about the reformulation of IIB when calculating the field strength equations. Instead the massive terms on the IIA side are encoded implicitly on the IIB side through the Lie derivatives of the potentials with respect to β. Therefore for the remainder of this chapter it is to be understood that the IIB potentials are *dependent* on the T-duality isometry direction.

The axion has a linear dependence which is expressed as $\mathcal{L}_\beta l = m$ [61]. Since the field strengths remain independent of the isometry direction the dependence of the axion fixes the dependence of all the other potentials. From analysing the IIB field equations we determine these to be given by

$$\mathcal{L}_\beta \mathcal{C}^{(2n)} = \frac{1}{n!} m(\mathcal{B})^n \tag{9.5}$$

$$\mathcal{L}_\beta \mathcal{B} = 0 \tag{9.6}$$

$$\mathcal{L}_\beta \mathcal{B}^{(6)} = m(-\mathcal{C}^{(6)} + \mathcal{C}^{(4)} \wedge \mathcal{B}) \tag{9.7}$$

$$\mathcal{L}_\beta \varphi = 0. \tag{9.8}$$

The dependency of the dilaton was given in [61].

9.2.1 T-dualising from IIA to IIB

We will now T-dualise the IIA fields $\phi^{(8)}$, $N^{(8)}$ and $N^{(7)}$. We restrict ourselves to the case where all of the IIA potentials are independent of the T-duality isometry. We can determine the T-duality rules for the potentials by first finding how their field strengths transform. This can be done by using essentially the same technique as used in the previous chapter, where the dimensional reductions of the higher rank field strengths were determined by considering the reduction rules of their lower rank Hodge duals. For the purposes of the T-duality we split the co-ordinate system into $\{\bar{\mu}_i, y\}$ where y parameterises the compact direction and the $\bar{\mu}_i$ represent the other 9 directions.

We start by considering $N^{(7)}$. Using the rules given in Chapter 5 we determine that

$$
\begin{aligned}
i_\beta G^{(2)}_{\bar{\mu}} &\rightarrow -\mathcal{R}^{-4} i_\beta \mathcal{G}^{(2)}_{\bar{\mu}} && (9.9)\\
G^{(2)}_{\bar{\mu}_1\bar{\mu}_2} &\rightarrow (-\mathcal{R}^{-2} i_\beta \mathcal{H} + \mathcal{R}^{-4} i_\beta \mathcal{G}^{(2)} \wedge i_\beta \mathcal{B})_{\bar{\mu}_1\bar{\mu}_2} && (9.10)
\end{aligned}
$$

from which we determine

$$
\begin{aligned}
i_\beta G^{(8)}_{\bar{\mu}_1\ldots\bar{\mu}_7} &\rightarrow (\mathcal{H}^{(7)} - \mathcal{R}^{-2}\mathcal{H}^{(7)} \wedge \beta)_{\bar{\mu}_1\ldots\bar{\mu}_7} && (9.11)\\
G^{(8)}_{\bar{\mu}_1\ldots\bar{\mu}_8} &\rightarrow \Big[-\mathcal{G}^{(8)} + \mathcal{R}^{-2} i_\beta \mathcal{G}^{(8)} \wedge \beta \\
&\qquad +(-\mathcal{H}^{(7)} + \mathcal{R}^{-2} i_\beta \mathcal{H}^{(7)} \wedge \beta) \wedge i_\beta \mathcal{B}\Big]_{\bar{\mu}_1\ldots\bar{\mu}_8}. && (9.12)
\end{aligned}
$$

We then use these to T-dualise (9.2) and deduce the rules for $N^{(7)}$. The rules for $i_\beta N^{(7)}_{\bar{\mu}_1\ldots\bar{\mu}_6}$ have already been given by (6.23) so we do not repeat them here. We find that in order for the equation for $\mathcal{G}^{(8)}$ to be well formed $N^{(7)}_{\bar{\mu}_1\ldots\bar{\mu}_7}$ must transform as

$$
\begin{aligned}
N^{(7)}_{\bar{\mu}_1\ldots\bar{\mu}_7} \rightarrow \Big[&-\mathcal{N}^{(7)} + \mathcal{R}^{-2} i_\beta \mathcal{N}^{(7)} \wedge \beta - \frac{1}{2} i_\beta \mathcal{C}^{(6)} \wedge \mathcal{C}^{(2)} \\
&-\frac{1}{2}\mathcal{R}^{-2} i_\beta \mathcal{C}^{(6)} \wedge i_\beta \mathcal{C}^{(2)} \wedge \beta + (-\mathcal{B}^{(6)} - \mathcal{R}^{-2} i_\beta \mathcal{B}^{(6)} \wedge \beta \\
&+\mathcal{C}^{(4)} \wedge \mathcal{C}^{(2)} + \mathcal{R}^{-2} i_\beta \mathcal{C}^{(4)} \wedge \mathcal{C}^{(2)} \wedge \beta \\
&+\mathcal{R}^{-2}\mathcal{C}^{(4)} \wedge i_\beta \mathcal{C}^{(2)} \wedge \beta) \wedge i_\beta \mathcal{B}\Big]_{\bar{\mu}_1\ldots\bar{\mu}_7}. \qquad (9.13)
\end{aligned}
$$

For completeness we state the inverse rule as well which is given by

$$\begin{aligned}\mathcal{N}^{(7)}_{\bar{\mu}_1\ldots\bar{\mu}_7} \;\;\rightarrow\;\; & \Big[-N^{(7)} + R^{-2} i_\beta N^{(7)} \wedge \beta + \frac{1}{2} C^{(5)} \wedge i_\beta C^{(3)} - \frac{1}{2} R^{-2} i_\beta C^{(5)} \wedge i_\beta C^{(3)} \wedge \beta \\ & -(B^{(6)} + R^{-2} i_\beta B^{(6)} \wedge \beta) \wedge i_\beta \mathcal{B}\Big]_{\bar{\mu}_1\ldots\bar{\mu}_7}. \qquad (9.14)\end{aligned}$$

The field equation for $\mathcal{G}^{(8)}$ is then determined as

$$\begin{aligned}\mathcal{G}^{(8)} \;=\;& d\mathcal{N}^{(7)} + \frac{1}{2}\mathcal{C}^{(0)} i_\beta \mathcal{F}^{(9)} - \frac{1}{2}\mathcal{F}^{(7)} \wedge i_\beta \mathcal{C}^{(2)} - \frac{1}{2} i_\beta \mathcal{F}^{(7)} \wedge \mathcal{C}^{(2)} \\ & + \frac{1}{2} i_\beta \mathcal{F}^{(5)} \wedge \mathcal{C}^{(4)} - \frac{1}{2}\mathcal{H} \wedge i_\beta \mathcal{C}^{(2)} \wedge \mathcal{C}^{(4)} + \frac{1}{2} i_\beta \mathcal{H} \wedge \mathcal{C}^{(4)} \wedge \mathcal{C}^{(2)} \\ & - i_\beta \mathcal{H} \wedge \mathcal{B}^{(6)} + \frac{1}{2} i_\beta \mathcal{F}^{(1)} (\mathcal{C}^{(8)} - \mathcal{C}^{(6)} \wedge \mathcal{B}) \qquad (9.15)\end{aligned}$$

which agrees with (6.29) after contraction with β and taking into account (9.31). Note that in the current scheme we have $i_\beta \mathcal{F}^{(1)} = \mathcal{L}_\beta l = m$. Given this definition the Bianchi identity reads

$$\begin{aligned}d\mathcal{G}^{(8)} \;=\;& \frac{1}{2} i_\beta \mathcal{F}^{(1)} \mathcal{F}^{(9)} + \frac{1}{2}\mathcal{F}^{(1)} \wedge i_\beta \mathcal{F}^{(9)} - \frac{1}{2}\mathcal{F}^{(7)} \wedge i_\beta \mathcal{F}^{(3)} \\ & - \frac{1}{2} i_\beta \mathcal{F}^{(7)} \wedge \mathcal{F}^{(3)} + \frac{1}{2} i_\beta \mathcal{F}^{(5)} \wedge \mathcal{F}^{(5)} - i_\beta \mathcal{H} \wedge \mathcal{H}^{(7)} \qquad (9.16)\end{aligned}$$

which confirms that the above definition of $\mathcal{G}^{(8)}$ is gauge invariant.

Next we consider the potentials $\phi^{(8)}$ and $N^{(8)}$. Both of these are magnetically dual to scalars and since we are assuming independence of y we have $i_\beta d\phi = 0$ and $i_\beta X^{(1)} = 0$. Because of this the IIB field strengths that $X^{(9)}$ and $H^{(9)}$ map to under direct T-duality transformations vanish.[1] For this reason we only consider double T-duality transformations, namely that of $N^{(8)}_{y\bar{\mu}_1\ldots\bar{\mu}_7} = i_\beta N^{(8)}_{\bar{\mu}_1\ldots\bar{\mu}_7}$ etc. We start by consider the 2-form field strengths using the rules given in Chapter 5

$$d\phi_{\bar{\mu}} \;\rightarrow\; d\varphi_{\bar{\mu}} + \frac{1}{2}\mathcal{R}^{-2} i_\beta \mathcal{G}^{(2)}_{\bar{\mu}} \qquad (9.17)$$

$$X^{(1)}_{\bar{\mu}} \;\rightarrow\; \mathcal{F}^{(1)}_{\bar{\mu}} - \mathcal{R}^{-2} i_\beta \mathcal{F}^{(1)} \beta_{\bar{\mu}} \qquad (9.18)$$

from which we determine

$$i_\beta H^{(9)}_{\bar{\mu}_1\ldots\bar{\mu}_8} \;\rightarrow\; (i_\beta \mathcal{H}^{(9)} + \frac{1}{2}\mathcal{G}^{(8)} + \frac{1}{2}\mathcal{R}^{-2} i_\beta \mathcal{G}^{(8)} \wedge \beta)_{\bar{\mu}_1\ldots\bar{\mu}_8} \qquad (9.19)$$

$$i_\beta X^{(9)}_{\bar{\mu}_1\ldots\bar{\mu}_8} \;\rightarrow\; -e^{-2\varphi} i_\beta \mathcal{F}^{(9)}_{\bar{\mu}_1\ldots\bar{\mu}_8} \qquad (9.20)$$

[1] The situation is different if the global $SO(1,1)$ symmetry is used to perform a Scherk-Schwarz reduction from the IIA side [78, 90] which would allow for a dependence on y. This then maps to a non-covariant form of IIB in which such field strengths are non-zero. We consider this option further in Appendix C.

where $\mathcal{H}^{(9)}$ is the dual of the IIB dilaton 'field strength', explicitly $\mathcal{H}^{(9)} = e^{-2\varphi} * d\varphi$.

We then use these to T-dualise (9.1) and (9.3) and deduce the rules for $i_\beta\phi^{(8)}$ and $i_\beta N^{(8)}$. We find that in order for the IIB field equations to be well formed we must have the following transformation rules for the 8-form potentials

$$
\begin{aligned}
i_\beta\phi^{(8)}_{\bar{\mu}_1\ldots\bar{\mu}_7} &\rightarrow (i_\beta\varphi^{(8)} - \frac{1}{2}\mathcal{N}^{(7)} + \frac{1}{2}\mathcal{R}^{-2}i_\beta\mathcal{N}^{(7)}\wedge\beta + \frac{1}{4}i_\beta\mathcal{C}^{(6)}\wedge\mathcal{C}^{(2)} \\
&\quad + \frac{1}{4}\mathcal{R}^{-2}i_\beta\mathcal{C}^{(6)}\wedge i_\beta\mathcal{C}^{(2)}\wedge\beta)_{\bar{\mu}_1\ldots\bar{\mu}_7} \qquad (9.21) \\
i_\beta N^{(8)}_{\bar{\mu}_1\ldots\bar{\mu}_7} &\rightarrow (-i_\beta\mathcal{N}^{(8)} + \mathcal{B}^{(6)}\wedge i_\beta\mathcal{C}^{(2)} - \mathcal{R}^{-2}i_\beta\mathcal{B}^{(6)}\wedge i_\beta\mathcal{C}^{(2)}\wedge\beta \\
&\quad + \frac{1}{3}i_\beta\mathcal{C}^{(4)}\wedge(\mathcal{C}^{(2)})^2 + \frac{2}{3}\mathcal{R}^{-2}i_\beta\mathcal{C}^{(4)}\wedge i_\beta\mathcal{C}^{(2)}\wedge\mathcal{C}^{(2)}\wedge\beta)_{\bar{\mu}_1\ldots\bar{\mu}_7} \qquad (9.22)
\end{aligned}
$$

where $\varphi^{(8)}$ is the 8-form magnetic dual of the dilaton and $\mathcal{N}^{(8)}$ is magnetically dual to a particular combination of the dilaton and axion [43, 85]. For completeness we state the inverse rules as well which are given by

$$
\begin{aligned}
\imath_\beta\varphi^{(8)}_{\bar{\mu}_1\ldots\bar{\mu}_7} &\rightarrow (i_\beta\phi^{(8)} - \frac{1}{2}N^{(7)} + \frac{1}{2}R^{-2}i_\beta N^{(7)}\wedge\beta \\
&\quad + \frac{1}{2}C^{(5)}\wedge i_\beta C^{(3)} - \frac{1}{2}R^{-2}i_\beta C^{(5)}\wedge i_\beta C^{(3)}\wedge\beta)_{\bar{\mu}_1\ldots\bar{\mu}_7} \qquad (9.23) \\
i_\beta\mathcal{N}^{(8)}_{\bar{\mu}_1\ldots\bar{\mu}_7} &\rightarrow \Big[-i_\beta N^{(8)} + i_\beta N^{(7)}\wedge(C^{(1)} - R^{-2}i_\beta C^{(1)}\beta) \\
&\quad - i_\beta C^{(3)}\wedge i_\beta C^{(5)}\wedge(C^{(1)} - R^{-2}i_\beta C^{(1)}\beta) \\
&\quad - \frac{1}{3}(i_\beta C^{(3)})^2\wedge(C^{(3)} - R^{-2}i_\beta C^{(3)}\wedge\beta)\Big]_{\bar{\mu}_1\ldots\bar{\mu}_7}. \qquad (9.24)
\end{aligned}
$$

The rules for the field $N^{(8)}$ have been given in [42] with different field definitions.

We then determine that the IIB field equations take the following form

$$
\begin{aligned}
\mathcal{H}^{(9)} &= d\varphi^{(8)} - l\mathcal{F}^{(9)} + \frac{1}{2}\mathcal{H}\wedge\mathcal{B}^{(6)} + \frac{1}{2}\mathcal{C}^{(2)}\wedge\mathcal{F}^{(7)} \qquad (9.25) \\
e^{-2\varphi}\mathcal{F}^{(9)} &= d\mathcal{N}^{(8)} + l^2\mathcal{F}^{(9)} + 2l\mathcal{H}^{(9)} - \frac{1}{2}\mathcal{F}^{(5)}\wedge(\mathcal{C}^{(2)})^2 \\
&\quad -(\mathcal{F}^{(3)} + l\mathcal{H})\wedge\mathcal{B}^{(6)} - \frac{1}{3}\mathcal{H}\wedge(\mathcal{C}^{(2)})^3. \qquad (9.26)
\end{aligned}
$$

Note that from the T-duality transformation we actually only produce the double dimensional reduction of the above field equations. However since neither field depends intrinsically on β, the full equations can be deduced trivially. This is not the case if we are dealing with fields that do depend on β in their definition for example $\mathcal{N}^{(7)}$; here the field equation for $\mathcal{N}^{(7)}_{\bar{\mu}_1\ldots\bar{\mu}_7}$ cannot be uniquely determined from the field equation of $i_\beta\mathcal{N}^{(7)}_{\bar{\mu}_1\ldots\bar{\mu}_6}$. A similar observation was made for $\hat{N}^{(8)}$ (8.18) in the $D = 11$ theory.

Given these definitions the Bianchi identities are calculated as

$$d\mathcal{H}^{(9)} = -\frac{1}{2}\mathcal{H}\wedge\mathcal{H}^{(7)} - \mathcal{F}^{(1)}\wedge\mathcal{F}^{(9)} + \frac{1}{2}\mathcal{F}^{(3)}\wedge\mathcal{F}^{(7)} \quad (9.27)$$

$$d(e^{-2\varphi}\mathcal{F}^{(9)}) = 2\mathcal{F}^{(1)}\wedge\mathcal{H}^{(9)} + \mathcal{F}^{(3)}\wedge\mathcal{H}^{(7)} \quad (9.28)$$

which confirms that (9.25) and (9.26) are gauge invariant.

In carrying out the T-dualities above, we find that the in order to account for the massive terms on the IIA side the IIB potentials are required to have the following Lie derivatives

$$\mathcal{L}_\beta\varphi^{(8)} = m(\mathcal{C}^{(8)} - \frac{1}{2}\mathcal{C}^{(6)}\wedge\mathcal{B}) \quad (9.29)$$

$$\mathcal{L}_\beta\mathcal{N}^{(8)} = m(-2\varphi^{(8)} - \mathcal{C}^{(6)}\wedge\mathcal{C}^{(2)} + \mathcal{C}^{(4)}\wedge\mathcal{C}^{(2)}\wedge\mathcal{B}) \quad (9.30)$$

$$\mathcal{L}_\beta\mathcal{N}^{(7)} = m(\frac{1}{2}i_\beta\mathcal{C}^{(8)} - \frac{1}{2}i_\beta\mathcal{C}^{(6)}\wedge\mathcal{B} + \frac{1}{2}\mathcal{C}^{(6)}\wedge i_\beta\mathcal{B} + \frac{1}{4}i_\beta\mathcal{C}^{(4)}\wedge(\mathcal{B})^2). \quad (9.31)$$

Combining these with (9.5)-(9.8) it is a simple matter to confirm that the definitions of the field strengths given above remain independent of the isometry direction. This is a non-trivial check of their structure.

As previously mentioned the fields $\varphi^{(8)}$ and $\mathcal{N}^{(8)}$ form an $SL(2,\mathbb{R})$ triplet along with the RR 8-form potential $\mathcal{C}^{(8)}$. We will re-express them in an $SL(2,\mathbb{R})$ covariant form in Chapter 12. Since they are magnetically dual to only two scalars, the axion and dilaton, they only have two independent degrees of freedom. This is usually expressed by a constraint on the three 9-form field strengths but is manifest in the field equations given above since they are only given in terms of the two independent 9-form field strengths, $\mathcal{F}^{(9)}$ and $\mathcal{H}^{(9)}$.

9.2.2 T-dualising from IIB to IIA

We now perform direct T-duality transformations on the IIB 8-form potentials to obtain 9-form potentials on the IIA side. The required T-duality rule for $e^{-2\varphi}\mathcal{F}^{(9)}$ can be inferred from the rule for $\mathcal{C}^{(8)}$ stated in Chapter 5 and is given by

$$e^{-2\varphi}\mathcal{F}^{(9)}_{\bar{\mu}_1\ldots\bar{\mu}_9} \rightarrow e^{-2\phi}R^2(-i_\beta F^{(10)} + F^{(8)}\wedge i_\beta B - R^{-2}i_\beta F^{(8)}\wedge i_\beta B\wedge\beta)_{\bar{\mu}_1\ldots\bar{\mu}_9}. \quad (9.32)$$

In the current scheme we have from (9.8) the relation $i_\beta d\varphi = 0$. This is because we are performing a Scherk-Schwarz reduction using only the subgroup of $SL(2,\mathbb{R})$ that involves shifts of the axion. Therefore strictly speaking when performing a direct T-duality transformation of $\mathcal{H}^{(9)}$ the IIA 10-form field strength that is produced will vanish. This is the field $H^{(10)}$ we have previously introduced. Since we will eventually encounter this field when we consider the non-covariant massive IIA theory as well as the more generalised T-duality involving the full $SL(2,\mathbb{R})$ group, we include it here for convenience.[2] The required T-duality rules follow analogously from those of $H^{(9)}$ in IIA and are given by

$$\mathcal{H}^{(9)}_{\bar{\mu}_1\ldots\bar{\mu}_9} \quad\rightarrow\quad \Big[-i_\beta H^{(10)} + (i_\beta H^{(9)} + \frac{1}{2}G^{(8)} + \frac{1}{2}R^{-2} i_\beta G^{(8)} \wedge \beta) \wedge i_\beta B\Big]_{\bar{\mu}_1\ldots\bar{\mu}_9}. \tag{9.33}$$

We then use these to T-dualise (9.25) and (9.26) and deduce the direct T-duality transformation rules for $\varphi^{(8)}$ and $\mathcal{N}^{(8)}$. We find that in order for the IIA field equations to be well formed we must have the following transformation rules for the 8-form potentials

$$\begin{aligned}
\varphi^{(8)}_{\bar{\mu}_1\ldots\bar{\mu}_8} \quad\rightarrow\quad & \Big[i_\beta B^{(9)} - \frac{1}{2} i_\beta C^{(3)} \wedge i_\beta C^{(7)} + (-i_\beta \phi^{(8)} + \frac{1}{2} N^{(7)} \\
& - \frac{1}{2} R^{-2} i_\beta N^{(7)} \wedge \beta - \frac{1}{2} i_\beta C^{(3)} \wedge C^{(5)} \\
& + \frac{1}{2} R^{-2} i_\beta C^{(3)} \wedge i_\beta C^{(5)} \wedge \beta) \wedge i_\beta B\Big]_{\bar{\mu}_1\ldots\bar{\mu}_8}
\end{aligned} \tag{9.34}$$

$$\begin{aligned}
\mathcal{N}^{(8)}_{\bar{\mu}_1\ldots\bar{\mu}_8} \quad\rightarrow\quad & \Big[-i_\beta D^{(9)} + \frac{1}{2} i_\beta C^{(5)} \wedge (i_\beta C^{(3)})^2 + (i_\beta N^{(8)} \\
& - i_\beta N^{(7)} \wedge (C^{(1)} - R^{-2} i_\beta C^{(1)} \beta) \\
& + i_\beta C^{(5)} \wedge i_\beta C^{(3)} \wedge (C^{(1)} - R^{-2} i_\beta C^{(1)} \beta) \\
& + \frac{1}{3} (i_\beta C^{(3)})^2 \wedge (C^{(3)} - R^{-2} i_\beta C^{(3)} \wedge \beta)) \wedge i_\beta B\Big]_{\bar{\mu}_1\ldots\bar{\mu}_8}.
\end{aligned} \tag{9.35}$$

Once again for the sake of completeness we also state the inverse rules which are

[2] The Hodge duality relation between $H^{(10)}$ and the mass parameters is given by (C.5). This can be compared to the generalised definition of $i_\beta d\varphi$ given by (D.2).

given by

$$
\begin{aligned}
i_\beta B^{(9)}_{\bar\mu_1\ldots\bar\mu_8} \;\;\rightarrow\;\; & \Big[\varphi^{(8)} + \mathcal{R}^{-2} i_\beta \varphi^{(8)} \wedge \beta + \frac{1}{2}\mathcal{C}^{(6)} \wedge \mathcal{C}^{(2)} \\
& + \frac{1}{2}\mathcal{R}^{-2} i_\beta \mathcal{C}^{(6)} \wedge \mathcal{C}^{(2)} \wedge \beta + \frac{1}{2}\mathcal{R}^{-2}\mathcal{C}^{(6)} \wedge i_\beta \mathcal{C}^{(2)} \wedge \beta\Big]_{\bar\mu_1\ldots\bar\mu_8} \qquad (9.36) \\
i_\beta D^{(9)}_{\bar\mu_1\ldots\bar\mu_8} \;\;\rightarrow\;\; & \Big[-\mathcal{N}^{(8)} - \mathcal{R}^{-2} i_\beta \mathcal{N}^{(8)} \wedge \beta + \frac{1}{2}\mathcal{C}^{(4)} \wedge (\mathcal{C}^{(2)})^2 \\
& + \frac{1}{2}\mathcal{R}^{-2} i_\beta \mathcal{C}^{(4)} \wedge (\mathcal{C}^{(2)})^2 + \mathcal{R}^{-2}\mathcal{C}^{(4)} \wedge i_\beta \mathcal{C}^{(2)} \wedge \mathcal{C}^{(2)}\Big]_{\bar\mu_1\ldots\bar\mu_8} . \qquad (9.37)
\end{aligned}
$$

We then determine that the IIA field equations take the following form

$$
\begin{aligned}
i_\beta H^{(10)} &= -d(i_\beta B^{(9)}) - i_\beta C^{(1)} i_\beta F^{(10)} - \frac{1}{2} i_\beta (H \wedge N^{(7)}) \\
& \quad + i_\beta \phi^{(8)} \wedge i_\beta H - \frac{1}{2} i_\beta (F^{(4)} + H \wedge C^{(1)}) \wedge i_\beta C^{(7)} \\
& \quad + m i_\beta C^{(9)} \wedge i_\beta B + \frac{1}{12} m i_\beta C^{(3)} \wedge i_\beta B \wedge (B)^3 \qquad (9.38) \\
e^{-2\phi} R^2 i_\beta F^{(10)} &= d(i_\beta D^{(9)}) + (i_\beta C^{(1)})^2 i_\beta F^{(10)} + 2 i_\beta C^{(1)} i_\beta H^{(10)} \\
& \quad - i_\beta N^{(8)} \wedge i_\beta H + i_\beta N^{(7)} \wedge i_\beta (F^{(4)} + H \wedge C^{(1)}) \\
& \quad + \frac{1}{3!} i_\beta (H \wedge C^{(3)}) \wedge (i_\beta C^{(3)})^2 - 2 m i_\beta B^{(9)} \wedge i_\beta B \\
& \quad - \frac{1}{4} m (i_\beta C^{(3)})^2 \wedge (B)^2 \wedge i_\beta B. \qquad (9.39)
\end{aligned}
$$

Given these we calculate the Bianchi identities as

$$
\begin{aligned}
d(i_\beta H^{(10)}) &= i_\beta F^{(2)} \wedge i_\beta F^{(10)} - \frac{1}{2} i_\beta F^{(4)} \wedge i_\beta F^{(8)} - i_\beta H \wedge i_\beta H^{(9)} \\
& \quad - \frac{1}{2} i_\beta (H \wedge G^{(10)}) \qquad (9.40) \\
d(e^{-2\phi} R^2 i_\beta F^{(10)}) &= -2 i_\beta F^{(2)} \wedge i_\beta H^{(10)} + i_\beta H \wedge i_\beta X^{(9)} \\
& \quad + i_\beta F^{(4)} \wedge i_\beta G^{(8)} \qquad (9.41)
\end{aligned}
$$

which confirms that the above field strength definitions are gauge invariant.

Equation (9.38) was also calculated from the dimensional reduction of $\hat{H}^{(11)}$ and given by (9.4). The agreement between both these methods is a non-trivial consistency check on (8.32) and (8.33) in the $D = 11$ theory. On the other hand, the potential $D^{(10)}$ did not arise in our treatment of the $D = 11$ fields. However uplifting (9.39) gives the following $D = 11$ field equation

$$
\begin{aligned}
(\hat{R}_{\hat{\beta}}^{2}\hat{R}_{\hat{\alpha}}^{-2} - 2\hat{R}_{\hat{\alpha}}^{-4}(i_{\hat{\alpha}}\hat{\beta})^{2})i_{\hat{\beta}\hat{\alpha}}\hat{F}^{(11)} &= -d(i_{\hat{\beta}\hat{\alpha}}\hat{D}^{(10)}) + i_{\hat{\alpha}}(i_{\hat{\beta}}\hat{T}^{(8)} \wedge i_{\hat{\beta}}\hat{F}) \\
&\quad -\frac{1}{4!}i_{\hat{\beta}\hat{\alpha}}(\hat{F} \wedge \hat{A} \wedge (i_{\hat{\beta}}\hat{A})^{2}) + 2\hat{m}i_{\hat{\beta}\hat{\alpha}}\hat{A} \wedge i_{\hat{\beta}\hat{\alpha}}\hat{B}^{(10)} \\
&\quad +\frac{1}{16}\hat{m}(i_{\hat{\beta}}\hat{A})^{2} \wedge (i_{\hat{\alpha}}\hat{A})^{2} \wedge i_{\hat{\beta}\hat{\alpha}}\hat{A}. \qquad (9.42)
\end{aligned}
$$

In the next chapter we will show that the new potential $\hat{D}^{(10)}$ forms an $SL(2,\mathbb{R})$ triplet along with $\hat{A}^{(10)}$ and $\hat{B}^{(10)}$.

Chapter 10

Generalised charges in $SL(2,\mathbb{R})$ covariant $D = 11$ supergravity

We now consider the $SL(2,\mathbb{R})$ covariant $D = 11$ supergravity which is an extension of the massive $D = 11$ theory considered previously in this thesis. This was first considered in [43] where it was interpreted as containing two M9-branes, and later generalised to the case of n M9-branes in [91]. Due to the M9-branes there are two compact isometry directions in the theory which we assume all the fields to be independent of. There are therefore two mutually commuting Killing vectors that define a T^2 manifold. Dimensional reduction over this manifold produces the triplet of $SL(2,\mathbb{R})$ $D = 9$ massive supergravities. These are also obtained by performing a Scherk-Schwarz reduction of the IIB theory containing 7-branes using the full $SL(2,\mathbb{R})$ symmetry group and in this way the M9-branes are mapped to the IIB 7-branes. Furthermore the $SL(2,\mathbb{R}) \subset GL(2,\mathbb{R})$ symmetry associated with the reduction of the $D = 11$ theory corresponds to the symmetry group of the IIB theory [70, 82–84]. To determine how the multiplets of IIB states map to the $D = 11$ theory it is therefore important to express this theory in an $SL(2,\mathbb{R})$ covariant manner. In this thesis our analysis of this theory simply amounts to the $SL(2,\mathbb{R})$ covariantisation of the equations we have so far presented for the massive $D = 11$ theory. To facilitate this we now adopt the more systematic notation $\hat{k}_a{}^{\hat{\mu}}$ to denote the Killing vectors of the theory. The index $a = 1, 2$ is an $SL(2,\mathbb{R})$ index.[1]

[1] We denote $SL(2,\mathbb{R})$ indices by lower case Roman characters.

Each field will contain a number of symmetrised $SL(2,\mathbb{R})$ indices depending on its representation. The Killing vectors therefore lie in the doublet representation. Instead of a single mass parameter we now have a 2×2 mass matrix $\hat{Q}^{mn}$ which we parameterise as

$$\hat{Q}^{mn} = \begin{pmatrix} \hat{m}_2+\hat{m}_3 & \hat{m}_1 \\ \hat{m}_1 & -(\hat{m}_2-\hat{m}_3) \end{pmatrix} = \begin{pmatrix} \hat{m}_+ & \hat{m}_1 \\ \hat{m}_1 & -\hat{m}_- \end{pmatrix}. \tag{10.1}$$

The massive $D = 11$ supergravity considered previously is obtained by making the following truncation

$$\hat{Q}^{mn} = \begin{pmatrix} \hat{m} & 0 \\ 0 & 0 \end{pmatrix} \tag{10.2}$$

and setting $\hat{k}_1 = \hat{\alpha}$ and $\hat{k}_2 = \hat{\beta}$. In this instance $\hat{\beta}$ is only considered as a standard Killing vector as opposed to a massive Killing vector due to the truncation of $\hat{Q}^{mn}$.

10.1 $SL(2,\mathbb{R})$ covariant $D = 11$ field equations

We will now present the field equations of this theory. The $SL(2,\mathbb{R})$ covariant field equation for the 3-form potential was given in [43] as

$$\hat{F} \;=\; d\hat{A} + \frac{1}{2}\hat{Q}^{mn} i_{\hat{k}_m}\hat{A}\wedge i_{\hat{k}_n}\hat{A} \tag{10.3}$$

and simply amounts to the straight forward covariantisation of (8.3). It is also a trivial task to deduce the covariant field equation for the 6-form potential from (8.4) as

$$\begin{aligned}\hat{F}^{(7)} \;&=\; d\hat{C}^{(6)} - \frac{1}{2}F\wedge\hat{A} + \hat{Q}^{mn}\Big[i_{\hat{k}_m}\hat{A}\wedge i_{\hat{k}_n}\hat{C} \\ &\quad + \frac{1}{12}\hat{A}\wedge i_{\hat{k}_m}\hat{A}\wedge i_{\hat{k}_n}\hat{A} + i_{\hat{k}_m}\hat{N}^{(8)}_n\Big]\end{aligned} \tag{10.4}$$

where $\hat{N}_a$ are the doublet of 8-form potentials which are considered below. Both these equations are easily deduced since the potentials $\hat{A}$ and $\hat{C}$ are $SL(2,\mathbb{R})$ scalars. The Bianchi identities are now given by

$$d\hat{F} \;=\; -\hat{Q}^{mn} i_{\hat{k}_m}\hat{F}\wedge i_{\hat{k}_n}\hat{A} \tag{10.5}$$

$$d\hat{F}^{(7)} \;=\; -\frac{1}{2}\hat{F}\wedge\hat{F} - \hat{Q}^{mn}\Big[i_{\hat{k}_m}\hat{G}^{(9)}_n + i_{\hat{k}_m}\hat{F}^{(7)}\wedge i_{\hat{k}_n}\hat{A}\Big] \tag{10.6}$$

where $\hat{G}_a^{(9)}$ are the 9-form field strengths of $\hat{N}_a$. The structure of these identities reflects the fact that the covariant gauge transformations of a general p-form $\hat{Y}^{(p)}$ are now given by the generalisation of (8.7) to

$$\delta\hat{Y}^{(p)} = \hat{Q}^{mn}\hat{\lambda}_m \wedge i_{\hat{k}_n}\hat{Y}^{(p)} \tag{10.7}$$

for $SL(2,\mathbb{R})$ scalars where $\hat{\lambda}_a = i_{\hat{k}_a}\hat{\chi}$. We re-emphasize the point that the gauge potentials generally exhibit more complicated massive gauge transformations than those described by the above rule and the generalisations thereof given below. The full gauge transformations of $\hat{A}$ are now given by

$$\delta\hat{A} = d\hat{\chi} + \hat{Q}^{mn}\hat{\lambda}_m \wedge i_{\hat{k}_n}\hat{A} \tag{10.8}$$

and therefore the torsion tensor (8.11) becomes

$$\hat{\tilde{T}}_{\hat{\mu}\hat{\nu}}{}^{\hat{\rho}} = -\hat{Q}^{mn}(i_{\hat{k}_m}\hat{A})_{\hat{\mu}\hat{\nu}}\hat{k}_n{}^{\hat{\rho}} \tag{10.9}$$

from which the contorsion tensor and total connection can be calculated.

Next we give the covariant field equations for the Killing vectors $\hat{k}_a$ which form $SL(2,\mathbb{R})$ doublets. There is not enough information in (8.13) to deduce the covariant form of the equation since loosely speaking it is not possible to distinguish the $SL(2,\mathbb{R})$ free indices from the dummy indices. This was why it was important for us to construct (8.22), from which we can deduce the following covariant field equation

$$\hat{G}_a^{(2)} = d\hat{k}_a + \hat{Q}^{mn}\Big[(\hat{k}_m \cdot \hat{k}_a)i_{\hat{k}_n}\hat{A} + i_{\hat{k}_m\hat{k}_a}\hat{A} \wedge \hat{k}_n\Big] \tag{10.10}$$

where we have $\hat{G}_1^{(2)} = \hat{G}^{(2)}$ and $\hat{G}_2^{(2)} = \hat{S}^{(2)}$. It is a simple matter to check that (10.10) gives both (8.13) and (8.22) when the truncation (10.2) is applied. Note that the last term on the RHS vanishes in (8.13). Due to $\hat{Q}^{mn}$ being symmetric there can be no other terms present in (10.10) since they would have had to show up in (8.22), hence (10.10) is uniquely determined.

The massive gauge transformations of $\hat{k}_a$ are given by the covariantisation of (8.21)

$$\delta\hat{k}_a = \hat{Q}^{mn}\Big[(\hat{k}_m \cdot \hat{k}_a)\hat{\lambda}_n + (\hat{k}_a \cdot \hat{\lambda}_m)\hat{k}_n\Big]. \tag{10.11}$$

This rule follows on from (10.7) which can be shown by considering the doublet $\hat{Y}_a^{(p)} = i_{\hat{k}_a}\hat{Y}^{(p+1)}$, where $\hat{Y}^{(p+1)}$ is a $(p+1)$-form $SL(2,\mathbb{R})$ scalar. Using (10.7) such fields are found to transform according to

$$\begin{aligned}\delta\hat{Y}_a^{(p)} &= \hat{Q}^{mn} i_{\hat{k}_a}\left[\hat{\lambda}_m \wedge i_{\hat{k}_n}\hat{Y}^{(p)}\right]\\ &= \hat{Q}^{mn}\left[\hat{\lambda}_m \wedge i_{\hat{k}_n}\hat{Y}_a^{(p)} + (\hat{k}_a\cdot\hat{\lambda}_m)\hat{Y}_n^{(p)}\right]. \qquad (10.12)\end{aligned}$$

We see that this has the same structure as (10.11) and conclude that it represents the general rule by which $SL(2,\mathbb{R})$ doublets transform. This explains why $\hat{\beta}$ did not transform according to (8.7) in Section 8.3. From now on we will use the term 'gauge covariant' to apply to fields transforming by this type of $SL(2,\mathbb{R})$ representation dependent rule. Applying (10.12) we find that both sides of (10.10) transform gauge covariantly.

Next we consider the 8-form potentials $\hat{N}_a^{(8)}$ which also form an $SL(2,\mathbb{R})$ doublet. In terms of the notation used in Chapter 8 we have $\hat{N}_1^{(8)} = \hat{N}^{(8)}$ and $\hat{N}_2^{(8)} = \hat{T}^{(8)}$, and also $\hat{G}_1^{(9)} = \hat{G}^{(9)}$ and $\hat{G}_2^{(9)} = \hat{S}^{(9)}$. In analogy to the case of the Killing vectors we find that (8.18) does not contain enough information to deduce the covariant equation whereas (8.32) does. The covariant equation is then uniquely determined to be

$$\begin{aligned}\hat{G}_a^{(9)} &= d\hat{N}_a^{(8)} + \frac{1}{3}\hat{F}\wedge i_{\hat{k}_a}\hat{C} - \frac{2}{3}i_{\hat{k}_a}\hat{F}\wedge\hat{C} + \frac{1}{3!}\hat{F}\wedge i_{\hat{k}_a}\hat{A}\wedge\hat{A}\\ &\quad +\hat{Q}^{mn}\left[\frac{1}{3}i_{\hat{k}_a}\hat{A}_{mn}^{(10)} - 2i_{\hat{k}_m}\hat{A}_{an}^{(10)} + i_{\hat{k}_m\hat{k}_a}\hat{A}\wedge\hat{N}_n^{(8)}\right.\\ &\quad \left. +i_{\hat{k}_m}\hat{A}\wedge i_{\hat{k}_n}\hat{N}_a^{(8)} - \frac{1}{4!}i_{\hat{k}_a}\hat{A}\wedge i_{\hat{k}_m}\hat{A}\wedge i_{\hat{k}_n}\hat{A}\wedge\hat{A}\right] \qquad (10.13)\end{aligned}$$

where $\hat{A}_{ab}^{(10)}$ are the triplet of 10-form potentials which we consider below. Applying the truncation (10.2) it is easy to see that (8.18) and (8.32) are recovered. The Bianchi identity is found by covariantising (8.28) and is given by

$$\begin{aligned}d\hat{G}_a^{(9)} &= -\frac{1}{3}\hat{F}^{(4)}\wedge i_{\hat{k}_a}\hat{F}^{(7)} + \frac{2}{3}i_{\hat{k}_a}\hat{F}^{(4)}\wedge\hat{F}^{(7)}\\ &\quad +\hat{Q}^{mn}\left[-\frac{1}{3}i_{\hat{k}_a}\hat{F}_{mn}^{(11)} + 2i_{\hat{k}_m}\hat{F}_{an}^{(11)} - i_{\hat{k}_m}\hat{A}\wedge i_{\hat{k}_n}\hat{G}_a^{(9)}\right.\\ &\quad \left. -i_{\hat{k}_m\hat{k}_a}\hat{A}\wedge\hat{G}_n^{(9)}\right]. \qquad (10.14)\end{aligned}$$

We can extend the rules (10.7) and (10.12) to find the covariant gauge transformation rule for $SL(2,\mathbb{R})$ triplets by considering objects such as $i_{\hat{k}_{(a}}\hat{Y}^{(p+1)}_{b)} = \hat{Y}^{(p)}_{ab}$, where $\hat{Y}^{(p+1)}_{a}$ is a $(p+1)$-form $SL(2,\mathbb{R})$ doublet. We then determine the gauge covariant rule for triplets as being

$$\begin{aligned}
\delta\hat{Y}^{(p)}_{ab} &= \hat{Q}^{mn} i_{\hat{k}_{(a}}\Big[\hat{\lambda}_{|m} \wedge i_{\hat{k}_{n|}}\hat{Y}^{(p)}_{b)} + (\hat{k}_{b)}\cdot\hat{\lambda}_m)\hat{Y}^{(p)}_{n}\Big] \\
&= \hat{Q}^{mn}\Big[\hat{\lambda}_m \wedge i_{\hat{k}_n}\hat{Y}^{(p)}_{ab} + 2 i_{\hat{k}_{(a}}\hat{\lambda}_{|m|}\hat{Y}^{(p)}_{b)n}\Big]. \qquad (10.15)
\end{aligned}$$

Using this rule together with (10.7) and (10.12) it is simple to show that both sides of (10.13) and (10.14) transform gauge covariantly.

From (10.10) explicit calculation reveals the identity

$$\epsilon^{ab} i_{\hat{k}_a}\hat{G}^{(2)}_{b} = 0 \qquad (10.16)$$

where ϵ^{ab} is the $SL(2,\mathbb{R})$ antisymmetric symbol and we use the convention

$$\epsilon^{12} = +1. \qquad (10.17)$$

This then leads to the following constraint on the 9-form field strengths

$$\epsilon^{ab}\epsilon^{cd} i_{\hat{k}_a\hat{k}_b}(\hat{G}^{(9)}_{c} \wedge \hat{k}_d) = 0 \qquad (10.18)$$

which maps to the constraint (12.33) on the 9-form field strengths in IIB.

Finally we consider the triplet of 10-form potentials $\hat{A}^{(10)}_{ab}$. In terms of the notation used in the earlier chapters we have $\hat{A}^{(10)}_{11} = \hat{A}^{(10)}$, $\hat{A}^{(10)}_{22} = \hat{D}^{(10)}$ and $\hat{A}^{(10)}_{12} = \hat{A}^{(10)}_{21} = \hat{B}^{(10)}$. Determining the 11-form covariant field equation here is less straight forward than the previous cases since it is not fully deducible from any of the non-covariant equations (8.12), (8.33) and (9.42) due to their overall contractions with the Killing vectors. However the structures of these equations do provide constraints on the full structure of the 11-form covariant equation. Further constraints are found by relating (10.13) and (10.14) but these still do not fully determine the 11-form covariant equation due to the contractions of the $SL(2,\mathbb{R})$ indices between the 10-form potentials and the mass matrix. However, by using the various constraints obtained after these considerations, neglecting the dimensionality of the spacetime and demanding gauge covariance it is possible to piece together the full structure of the 11-form covariant field equation which is found to be given by

$$\begin{aligned}\hat{F}^{(11)}_{ab} &= d\hat{A}^{(10)}_{ab} + \frac{3}{4} i_{\hat{k}_{(a}} \hat{F} \wedge \hat{N}^{(8)}_{b)} - \frac{1}{4} \hat{F} \wedge i_{\hat{k}_{(a}} \hat{N}^{(8)}_{b)} + \frac{1}{4!} \hat{F} \wedge i_{\hat{k}_a} \hat{A} \wedge i_{\hat{k}_b} \hat{A} \wedge \hat{A} \\ &\quad + \hat{Q}^{mn} \Big[-3 i_{\hat{k}_m} \hat{A}^{(12)}_{abn} + \frac{3}{4} i_{\hat{k}_{(a}} \hat{A}^{(12)}_{b)mn} + 2 i_{\hat{k}_m \hat{k}_{(a}} \hat{A} \wedge \hat{A}^{(10)}_{b)n} \\ &\quad + i_{\hat{k}_m} \hat{A} \wedge i_{\hat{k}_n} \hat{A}^{(10)}_{ab} - \frac{1}{80} i_{\hat{k}_a} \hat{A} \wedge i_{\hat{k}_b} \hat{A} \wedge i_{\hat{k}_m} \hat{A} \wedge i_{\hat{k}_n} \hat{A} \wedge \hat{A} \Big]. \end{aligned} \tag{10.19}$$

Note the presence of the quadruplet of 12-form potentials $\hat{A}^{(12)}_{abc}$, the inclusion of which are necessary for gauge covariance. Obviously these identically vanish in $D = 11$, however they appear as part of the full gauge algebra and in principal they would appear explicitly if the spacetime dimension could be extended beyond eleven. This observation is in agreement with [60] where Romans' IIA theory was considered and a 10-form potential was found to exist which contained a 10-form gauge parameter in its gauge transformations. This demonstrates that an 11-form potential must appear in its full field strength equation, the same conclusion is reached by performing a direct dimensional reduction on the (1,1) component of (10.19) after making the truncation (10.2).

Given the presence of the potentials $\hat{A}^{(12)}_{abc}$ in (10.19) it is important to determine whether their gauge transformations (which so far are only partly determined) are consistent overall by constructing a gauge covariant 13-form field equation. The situation here is essentially the same as that just encountered when determining the field equation (10.19). It is found that in order to construct the 13-form field equation it is necessary to introduce an $SL(2,\mathbb{R})$ quintuplet of 14-form potentials in an analogous fashion to the 12-form potentials previously, and so the situation repeats itself. This process then reveals the existence of an infinite tower of potentials whose field strengths can be written in the general form

$$
\begin{aligned}
\hat{F}^{(2j+7)}_{a_1\ldots a_j} \;=\;& d\hat{A}^{(2j+6)}_{a_1\ldots a_j} + \frac{j+1}{j+2} i_{k_{(a_1}} \hat{F}^{(4)} \wedge \hat{A}^{(2j+4)}_{a_2\ldots a_j)} - \frac{1}{j+2}\hat{F}^{(4)} \wedge i_{\hat{k}_{(a_1}} \hat{A}^{(2j+4)}_{a_2\ldots a_j)} \\
&+\frac{1}{(j+2)!}\hat{F}^{(4)} \wedge i_{\hat{k}_{(a_1}} \hat{A}^{(3)} \wedge \ldots \wedge i_{\hat{k}_{a_j)}} \hat{A}^{(3)} \wedge \hat{A}^{(3)} \\
&+\hat{Q}^{mn}\Big[-(j+1) i_{\hat{k}_m} \hat{A}^{(2j+8)}_{a_1\ldots a_j n} + \frac{j(j+1)}{2(j+2)} i_{\hat{k}_{(a_1}} \hat{A}^{(2j+8)}_{a_2\ldots a_j)mn} \\
&+j i_{\hat{k}_m \hat{k}_{(a_1}} \hat{A}^{(3)} \wedge \hat{A}^{(2j+6)}_{a_2\ldots a_j)n} + i_{\hat{k}_m}\hat{A}^{(3)} \wedge i_{\hat{k}_n}\hat{A}^{(2j+6)}_{a_1\ldots a_j} \\
&-\frac{(j+1)}{2(j+3)!} i_{\hat{k}_m}\hat{A}^{(3)} \wedge i_{\hat{k}_n}\hat{A}^{(3)} \wedge i_{\hat{k}_{(a_1}}\hat{A}^{(3)} \wedge \ldots \wedge i_{\hat{k}_{a_j)}}\hat{A}^{(3)} \wedge \hat{A}^{(3)}\Big]
\end{aligned}
\tag{10.20}
$$

where we have indicated the rank of the 4-form field strength and 3-form potential for convenience, and the a_j are $SL(2,\mathbb{R})$ indices. By equating $\hat{C} = -\hat{A}^{(6)}$ and $\hat{N}^{(8)}_a = \hat{A}^{(8)}_a$ we see that the field equations (10.4), (10.13) and (10.19) have this structure for the cases of $j = 0, 1, 2$ respectively. The Bianchi identity is found to be given by

$$
\begin{aligned}
d\hat{F}^{(2j+7)}_{a_1\ldots a_j} \;=\;& -\frac{j+1}{j+2} i_{\hat{k}_{(a_1}} \hat{F}^{(4)} \wedge \hat{F}^{(2j+5)}_{a_2\ldots a_j)} + \frac{1}{j+2}\hat{F}^{(4)} \wedge i_{\hat{k}_{(a_1}} \hat{F}^{(2j+5)}_{a_2\ldots a_j)} \\
&+\hat{Q}^{mn}\Big[(j+1) i_{\hat{k}_m}\hat{F}^{(2j+9)}_{a_1\ldots a_j n} - \frac{j(j+1)}{2(j+2)} i_{\hat{k}_{(a_1}}\hat{F}^{(2j+9)}_{a_2\ldots a_j)mn} \\
&-j i_{\hat{k}_m\hat{k}_{(a_1}}\hat{A}^{(3)} \wedge \hat{F}^{(2j+7)}_{a_2\ldots a_j)n} - i_{\hat{k}_m}\hat{A}^{(3)} \wedge i_{\hat{k}_n}\hat{F}^{(2j+7)}_{a_1\ldots a_j}\Big].
\end{aligned}
\tag{10.21}
$$

Note that this identity does not apply to the case $j = 0$ where instead we have (10.6). This is because the second and third terms on the RHS of (10.20) vanish for the case $j = 0$. By generalising the result (10.15) to a p-form in a general $SL(2,\mathbb{R})$ representation we obtain the general rule for gauge covariance

$$
\delta\hat{Y}^{(p)}_{a_1\ldots a_j} = \hat{Q}^{mn}\Big[\hat{\lambda}_m \wedge i_{\hat{k}_n}\hat{Y}^{(p)}_{a_1\ldots a_j} + j i_{\hat{k}_{(a_1}}\hat{\lambda}_{|m|}\hat{Y}^{(p)}_{a_2\ldots a_j)n}\Big]. \tag{10.22}
$$

From (10.21) we then see that the field strengths transform according to this rule.

The full gauge transformations of the potentials are then calculated as being

$$
\begin{aligned}
\delta\hat{A}^{(2j+6)}_{a_1\ldots a_j} &= \sum_{l=0}^{j}\frac{l+2}{(j+2)(j-l)!}i_{\hat{k}_{(a_1}}\hat{A}^{(3)}\wedge\ldots\wedge i_{\hat{k}_{a_{j-l}}}\hat{A}^{(3)}\wedge d\hat{\chi}^{(2l+5)}_{a_{j-l+1}\ldots a_j)}\\
&+\sum_{l=0}^{j-1}\frac{1}{(j+2)(j-l-1)!}i_{\hat{k}_{(a_1}}\hat{A}^{(3)}\wedge\ldots\wedge i_{\hat{k}_{a_{j-l-1}}}\hat{A}^{(3)}\wedge\hat{A}\wedge i_{\hat{k}_{a_{j-l}}}d\hat{\chi}^{(2l+5)}_{a_{j-l+1}\ldots a_j)}\\
&-\frac{(j+1)}{(j+2)!}i_{\hat{k}_{(a_1}}\hat{A}^{(3)}\wedge\ldots\wedge i_{\hat{k}_{a_j)}}\hat{A}^{(3)}\wedge\hat{A}\wedge d\hat{\chi}^{(2)}\\
&+\hat{Q}^{mn}\Bigg[-\sum_{l=1}^{j+1}\frac{l(l+1)}{(j+2)(j-l+1)!}i_{\hat{k}_{(a_1}}\hat{A}^{(3)}\wedge\ldots\wedge i_{\hat{k}_{a_{j-l+1|}}}\hat{A}^{(3)}\wedge i_{\hat{k}_m}\hat{\chi}^{(2l+5)}_{|a_{j-l+2}\ldots a_j)n}\\
&+\sum_{l=2}^{j+1}\frac{l(l-1)}{2(j+2)(j-l+1)!}i_{\hat{k}_{(a_1}}\hat{A}^{(3)}\wedge\ldots\wedge i_{\hat{k}_{a_{j-l+1}}}\hat{A}^{(3)}\wedge i_{\hat{k}_{a_{j-l+2}}}\hat{\chi}^{(2l+5)}_{a_{j-l+3}\ldots a_j)mn}\\
&-\sum_{l=1}^{j}\frac{l}{(j+2)(j-l)!}i_{\hat{k}_{(a_1}}\hat{A}^{(3)}\wedge\ldots\wedge i_{\hat{k}_{a_{j-l}}}\hat{A}^{(3)}\wedge\hat{A}^{(3)}\wedge i_{\hat{k}_{a_{j-l+1|}}}i_{\hat{k}_m}\hat{\chi}^{(2l+5)}_{|a_{j-l+2}\ldots a_j)n}\\
&+\hat{\lambda}_m\wedge i_{\hat{k}_n}\hat{A}^{(2j+6)}_{a_1\ldots a_j}+j\,i_{\hat{k}_{(a_1|}}\hat{\lambda}_m\wedge\hat{A}^{(2j+6)}_{|a_2\ldots a_j)n}\Bigg].
\end{aligned}
\tag{10.23}
$$

Returning to the 11-form field strengths, we find that they are related to the mass parameters by

$$
\hat{F}^{(11)}_{ab} = \hat{*}\hat{Q}^{mn}\Big[(\hat{k}_a\cdot\hat{k}_b)(\hat{k}_m\cdot\hat{k}_n) - 2(\hat{k}_a\cdot\hat{k}_m)(\hat{k}_b\cdot\hat{k}_n)\Big] \tag{10.24}
$$

which follows from the Hodge duality definitions of $\hat{F}^{(11)}$ and $\hat{H}^{(11)}$ as well as from the scalars that appear on the LHS of the field equation for $\hat{D}^{(10)}$ given by (9.42). This relation could also be derived from the action which contains a cosmological-type term of the form [91]

$$
\hat{Q}^{ab}\hat{Q}^{mn}\Big[\frac{1}{2}(\hat{k}_a\cdot\hat{k}_b)(\hat{k}_m\cdot\hat{k}_n) - (\hat{k}_a\cdot\hat{k}_m)(\hat{k}_b\cdot\hat{k}_n)\Big] \tag{10.25}
$$

by introducing the 10-forms as auxiliary fields along the lines of [61, 76].

Explicit calculation reveals that the three 11-form field strengths satisfy the following constraint

$$
\epsilon^{mp}\epsilon^{nq}(\hat{k}_m\cdot\hat{k}_n)\hat{F}^{(11)}_{pq} = 0. \tag{10.26}
$$

This constraint shows that there are in fact only two independent 11-forms and maps to the constraint (12.33) found for the IIB 9-form field strengths.

Although the equations given in this section were calculated for the $SL(2,\mathbb{R})$ case, the results should generalise to the case of n Killing vectors. In this case the theory would be $SL(n,\mathbb{R})$ covariant, where this group is a subgroup of the full U-duality group for n compact directions, and there would be a symmetric $n\times n$ mass matrix [91]. The exception to this are the constraints (10.18) and (10.26) which use the $SL(2,\mathbb{R})$ antisymmetric symbol and are therefore specific to the $n=2$ theory.

10.2 $SL(2,\mathbb{R})$ covariant $D=11$ generalised charges

We now extend the $D=11$ generalised charges already given to the $SL(2,\mathbb{R})$ covariant theory. As with the previous massive theories considered in this thesis, this exercise largely amounts to simply rechecking that the charges remain closed once the massive modifications to the theory have been included and generally requires a modification to the gauge conditions that the potentials must satisfy. However in this instance we find that the emphasis on the $SL(2,\mathbb{R})$ structure serves as a useful tool for exploring the possible states that can appear in theory.

Once again we begin by considering the modification to the Killing spinor equation (2.23). The supercovariant derivative operator (2.24) now becomes [92]

$$\begin{aligned}\hat{\hat{D}}_{\hat{\mu}} &= \hat{\nabla}_{\hat{\mu}} + \frac{1}{288}\left[\hat{\Gamma}_{\hat{\mu}}{}^{\hat{\nu}_1\ldots\hat{\nu}_4} - 8\hat{\delta}_{\hat{\mu}}^{\hat{\nu}_1}\hat{\Gamma}^{\hat{\nu}_2\hat{\nu}_3\hat{\nu}_4}\right]\hat{F}^{(4)}_{\hat{\nu}_1\ldots\hat{\nu}_4} \\ &\quad -\frac{1}{12}\hat{k}_{m\hat{\nu}}\hat{Q}^{mn}\hat{k}_n{}^{\hat{\nu}}\hat{\Gamma}_{\hat{\mu}} + \frac{1}{2}\hat{k}_{m\hat{\mu}}\hat{Q}^{mn}\hat{k}_{n\hat{\nu}}\hat{\Gamma}^{\hat{\nu}} \\ &\quad +\frac{1}{8}\hat{Q}^{mn}\left[2i_{\hat{k}_m}\hat{A}_{\hat{\mu}\hat{\nu}_1}\hat{k}_{n\hat{\nu}_2} - i_{\hat{k}_m}\hat{A}_{\hat{\nu}_1\hat{\nu}_2}\hat{k}_{n\hat{\mu}}\right]\hat{\Gamma}^{\hat{\nu}_1\hat{\nu}_2}. \end{aligned} \tag{10.27}$$

Note that the covariant derivative operator here is the massless one and we have explicitly written the massive terms involving the 3-form potential $\hat{A}$ which arise from the torsion tensor (10.9).

We briefly discuss the M2 and M5-brane cases. The generalisation of these charges to the covariant theory is trivial since both charges are $SL(2,\mathbb{R})$ scalars (note that all the bilinears are $SL(2,\mathbb{R})$ scalars). The relevant differential relations for the bilinears are calculated from (10.27) and are given by

$$
\begin{aligned}
d\hat{\omega} &= i_{\hat{K}}\hat{F}^{(4)} - \hat{Q}^{mn} i_{\hat{k}_m}\hat{A}\wedge i_{\hat{k}_n}\hat{\omega} &(10.28)\\
d\hat{\Sigma} &= i_{\hat{K}}\hat{F}^{(7)} - \hat{\omega}\wedge\hat{F}^{(4)} - \hat{Q}^{mn}\Big[(\hat{k}_m\cdot\hat{k}_n)\hat{\Lambda} + i_{\hat{k}_m}\hat{\Lambda}\wedge\hat{k}_n \\
&\quad + i_{\hat{k}_m}\hat{A}\wedge i_{\hat{k}_n}\hat{\Sigma}\Big] &(10.29)
\end{aligned}
$$

which amount to the covariantisation of (8.35) and (8.36) respectively, as we would expect. Using these relations together with (10.3) and (10.4) it is a simple matter to show that the M2-brane (2.43) and M5-brane (2.52) charges are closed providing the following gauge conditions are satisfied

$$
\begin{aligned}
\mathcal{L}_{\hat{K}}\hat{A} &= \hat{Q}^{mn} i_{\hat{k}_m}\hat{L}^{(2)}\wedge i_{\hat{k}_n}\hat{A} &(10.30)\\
\mathcal{L}_{\hat{K}}\hat{C} &= \hat{Q}^{mn}\Big[i_{\hat{k}_m}\hat{L}^{(2)}\wedge i_{\hat{k}_n}\hat{C} + \hat{L}^{(KK)}_{mn}\Big] &(10.31)
\end{aligned}
$$

where $\hat{L}^{(KK)}_{ab}$ are the triplet of KK-monopole charges which we discuss in the next subsection. Consistency of these conditions with the field equations (10.3) and (10.4) requires the following conditions to be true

$$
\begin{aligned}
\mathcal{L}_{\hat{K}}\hat{F} &= \hat{Q}^{mn} i_{\hat{k}_m}\hat{L}^{(2)}\wedge i_{\hat{k}_n}\hat{F} &(10.32)\\
\mathcal{L}_{\hat{K}}\hat{F}^{(7)} &= \hat{Q}^{mn} i_{\hat{k}_m}\hat{L}^{(2)}\wedge i_{\hat{k}_n}\hat{F}^{(7)} &(10.33)
\end{aligned}
$$

together with the condition on $\hat{N}^{(8)}_a$ given by (10.42). These can be determined independently by calculating the exterior derivatives of (10.28) and (10.29) and using the general definition of the Lie derivative (2.37). We observe that this type of consistency check on the conditions (10.30) and (10.31) is the same as that carried out to check the gauge covariance of the field equations. We therefore realise that the structure of the massive gauge transformations of both the field strengths and the potentials takes the same general form as the above Lie derivative conditions. This can be seen by comparing (10.30) with the massive terms in (10.8), (10.31) with the massive terms in (10.23) for $j = 0$ and making the identifications

$$
\hat{L}^{(2)} \sim \hat{\chi} \qquad \hat{L}^{(KK)}_{ab} \sim i_{\hat{k}_{(a}}\hat{\chi}^{(7)}_{b)}. \tag{10.34}
$$

From this correspondence we see that the presence of the charges $\hat{L}^{(KK)}_{mn}$ in (10.31) originates from the fact that $\hat{C}$ is a Stueckelberg field with $i_{\hat{k}_{(a}}\hat{N}^{(8)}_{b)}$ being the corresponding massive fields which is seen from (10.4).

10.2.1 $SL(2,\mathbb{R})$ covariant KK-monopole charges

We now consider the KK-monopole charge (6.15) in the $SL(2,\mathbb{R})$ covariant theory. Using the isometry structure of (6.15) it is trivial to covariantise to give the following triplet of charges

$$
\begin{aligned}
\hat{L}^{(KK)}_{ab} \;=\;& (\hat{k}_a\cdot\hat{k}_b)\hat{\Lambda} + i_{\hat{k}_{(a}}\hat{\Lambda}\wedge\hat{k}_{b)} - i_{\hat{K}}(i_{\hat{k}_{(a}}\hat{N}^{(8)}_{b)} - \frac{1}{3!}\hat{A}\wedge i_{\hat{k}_a}\hat{A}\wedge i_{\hat{k}_b}\hat{A}) \\
& - i_{\hat{k}_{(a}}\hat{\omega}\wedge(i_{\hat{k}_{b)}}\hat{C} + \frac{1}{2}\hat{A}\wedge i_{\hat{k}_{b)}}\hat{A}) + i_{\hat{k}_{(a}}\hat{L}^{(5)}\wedge i_{\hat{k}_{b)}}\hat{A} \\
& - \frac{1}{2}\hat{L}^{(2)}\wedge i_{\hat{k}_a}\hat{A}\wedge i_{\hat{k}_b}\hat{A}. \qquad (10.35)
\end{aligned}
$$

In order to confirm that these charges are closed we must determine the differential relation for $(\hat{k}_a\cdot\hat{k}_b)\hat{\Lambda}$. Following the usual procedure we find that here we have two algebraic Killing spinor equations formed by taking the projection of (10.27) along each of the isometry directions. Projecting along $\hat{k}_1$ gives

$$
\begin{aligned}
\hat{\tilde{D}}_x\hat{\epsilon} \;=\;& -\frac{1}{8}d(\hat{k}_1)_{AB}\hat{\Gamma}^{AB}\hat{\epsilon} - \frac{1}{4}|\hat{k}_1|^{-1}\partial_A(|\hat{k}_1|^2)\hat{\Gamma}^{A\underline{x}}\hat{\epsilon} \\
& + \frac{1}{288}|\hat{k}_1|\hat{\Gamma}_{\underline{x}}{}^{ABCD}\hat{F}_{ABCD}\hat{\epsilon} - \frac{1}{36}|\hat{k}_1|\hat{\Gamma}^{ABC}\hat{F}_{\underline{x}ABC}\hat{\epsilon} \\
& + \hat{Q}^{mn}\Big[-\frac{1}{12}|\hat{k}_1|(\hat{k}_m\cdot\hat{k}_n)\hat{\Gamma}^{\underline{x}} + \frac{1}{2}(\hat{k}_1\cdot\hat{k}_m)\hat{k}_{nA}\hat{\Gamma}^A \\
& + \frac{1}{2}|\hat{k}_1|^{-1}(\hat{k}_1\cdot\hat{k}_m)(\hat{k}_1\cdot\hat{k}_n)\hat{\Gamma}^{\underline{x}}\Big]\hat{\epsilon} + \frac{1}{8}\hat{Q}^{mn}\Big[2i_{\hat{k}_1\hat{k}_m}\hat{A}_A\hat{k}_{nB} \\
& - (\hat{k}_1\cdot\hat{k}_m)i_{\hat{k}_n}\hat{A}_{AB}\Big]\hat{\Gamma}^{AB}\hat{\epsilon} + \frac{1}{2}\hat{Q}^{mn}|\hat{k}_1|^{-1}(\hat{k}_1\cdot\hat{k}_m)i_{\hat{k}_1\hat{k}_n}\hat{A}_A\hat{\Gamma}^{A\underline{x}}\hat{\epsilon} \\
& = 0. \qquad (10.36)
\end{aligned}
$$

where we work in a basis adapted to $\hat{k}_1$ analogous to that described in Section 6.1 where x parametrises the $\hat{k}_1$ direction. We could produce a similar relation to (10.36) by working with $\hat{k}_2$. These can then be used to calculate $d[(\hat{k}_1\cdot\hat{k}_1)\hat{\Lambda}]$ and $d[(\hat{k}_2\cdot\hat{k}_2)\hat{\Lambda}]$ respectively following the procedure used in Chapter 6. However the calculation for $d[(\hat{k}_1\cdot\hat{k}_2)\hat{\Lambda}]$ is more complicated since we must now rewrite the two algebraic Killing spinor equations in a basis that is adapted to both isometries simultaneously. Fortunately, each of these differential relations must take the same general form so we need only calculate one of these and this will uniquely determine the covariant form of the relation.

The process of calculating $d[(\hat{k}_1 \cdot \hat{k}_1)\hat{\Lambda}]$ follows precisely the same lines as that described in Section 8.4.1 when calculating $d(\hat{R}^2_\beta\hat{\Lambda})$. We can therefore simply covariantise (8.48) and determine

$$\begin{aligned} d[(\hat{k}_a \cdot \hat{k}_b)\hat{\Lambda} + i_{\hat{k}_{(a}}\hat{\Lambda} \wedge \hat{k}_{b)}] \;=\; & i_{\hat{K}\hat{k}_{(a}}\hat{G}^{(9)}_{b)} + i_{\hat{k}_{(a}}\hat{\omega} \wedge i_{\hat{k}_{b)}}\hat{F}^{(7)} + i_{\hat{k}_{(a}}\hat{\Sigma} \wedge i_{\hat{k}_{b)}}\hat{F}^{(4)} \\ & +\hat{Q}^{mn}\Big[2(\hat{k}_m \cdot \hat{k}_{(a})i_{\hat{k}_{b)}\hat{k}_n}\hat{A} \wedge \hat{\Lambda} - i_{\hat{k}_m\hat{k}_{(a}}\hat{A} \wedge i_{\hat{k}_{b)}}\hat{\Lambda} \wedge \hat{k}_n \\ & +(\hat{k}_m \cdot \hat{k}_{(a})i_{\hat{k}_{b)}}\hat{\Lambda} \wedge i_{\hat{k}_n}\hat{A} + i_{\hat{k}_m\hat{k}_{(a}}\hat{A} \wedge \hat{k}_{b)} \wedge i_{\hat{k}_n}\hat{\Lambda} \\ & -i_{\hat{k}_m}\hat{A} \wedge i_{\hat{k}_n\hat{k}_{(a}}\hat{\Lambda} \wedge \hat{k}_{b)} - (\hat{k}_a \cdot \hat{k}_b)i_{\hat{k}_m}\hat{A} \wedge i_{\hat{k}_n}\hat{\Lambda} \\ & +2(\hat{k}_m \cdot \hat{k}_{(a})i_{\hat{k}_{b)}\hat{k}_n}\hat{\Pi}\Big]. \end{aligned} \tag{10.37}$$

Using this relation it is then a straight forward task to check that (10.35) is closed. We discuss the gauge condition on $i_{\hat{k}_{(a}}\hat{N}^{(8)}_{b)}$ below in equation (10.42).

From the $SL(2,\mathbb{R})$ index structure of (10.35) we see that the KK-monopole charges form a triplet and we will show in the following chapters that these map to the triplet of 7-branes charges in IIB. In the current set-up we have a Taub-NUT isometry from the KK-monopole as well as the two massive isometries from the M9-branes. The charges $\hat{L}^{(KK)}_{11}$ and $\hat{L}^{(KK)}_{22}$ are each associated with the cases where the Taub-NUT isometry coincides with one of the massive isometries and so are in this sense equivalent. However if we make the truncation (10.2) then this equivalence disappears. In this scenario the charge $\hat{L}^{(KK)}_{11}$ represents the case where the Taub-NUT isometry and the remaining massive isometry coincide, whereas $\hat{L}^{(KK)}_{22}$ represents the case where they are distinct.

The third charge in the triplet is $\hat{L}^{(KK)}_{12}$, and we will refer to the state that this corresponds to as the $D = 11$ $r7$-brane. From looking at the charge we see that it explicitly involves both Killing vectors, which suggests that the brane itself should have a 7-dimensional worldvolume with two isometries in its transverse space. From the leading term of this charge we deduce that the tension of this brane should scale as $\hat{k}_1 \cdot \hat{k}_2$.

We are not aware of this state having been directly investigated in the literature. However the triplet of 7-branes in IIB has been discussed [42,43,68,78–80] and many of the results should map to the triplet of states here, in particular the charge matrix q^{ab} (see for example [68]) which describes the particular combination of branes that make up a given state. The charge of any state is then given by

$$\hat{L} = q^{ab}\hat{L}^{(KK)}_{ab}. \tag{10.38}$$

Due to the constraint on the field strengths (10.18) there will be restrictions on the combinations of charges that can occur.

There exist three conjugacy classes of states defined by whether $\det(q)$ is positive, negative or zero. Each conjugacy class has a family of solutions that are related by $SL(2,\mathbb{R})$ transformations, with the single KK-monopole belonging to the $\det(q) = 0$ class. However, since these transformations preserve $\det(q)$, they do not map between conjugacy classes. It was shown in [79, 80] that globally well defined IIB solutions can be constructed for the cases of $\det(q)$ zero or positive, but not negative. Such a restriction does not necessarily apply here due to the different co-dimension of the branes, but it would be interesting to explore this. Furthermore the worldvolume action for the $\det(q) = 0$ class was constructed in [93] with the more general case being given in [80] to second order in the Born-Infeld field strength. In Appendix B we construct the $SL(2,\mathbb{R})$ covariant worldvolume kinetic term for the $\det(q) = 0$ branes and discuss a first step to generalising this for arbitrary $\det(q)$.

We now offer another interpretation of this triplet of charges. For this we require the relation for $\hat{\Lambda}\wedge d\hat{k}_a$ which can be calculated by hitting (10.36) from the left with $\hat{\epsilon}\hat{\Gamma}_{A_1\ldots A_8}$. We then find

$$\begin{aligned}(d\hat{k}_1\wedge\hat{\Lambda})_{A_1\ldots A_8} \;=\;& (i_{\hat{K}}\hat{G}^{(9)}_1 - \frac{1}{3}i_{\hat{k}_1}\hat{\Sigma}\wedge\hat{F} - \frac{2}{3}\hat{\Sigma}\wedge i_{\hat{k}_1}\hat{F} - \frac{1}{3}\hat{\omega}\wedge i_{\hat{k}_1}\hat{F}^{(7)} \\ &+\frac{2}{3}i_{\hat{k}_1}\hat{\omega}\wedge\hat{F}^{(7)})_{A_1\ldots A_8} + \hat{Q}^{mn}\Big[-(\hat{k}_1\cdot\hat{k}_m)i_{\hat{k}_n}\hat{A}\wedge\hat{\Lambda} \\ &+i_{\hat{k}_1\hat{k}_m}\hat{A}\wedge\hat{\Lambda}\wedge\hat{k}_n + \frac{1}{3}(\hat{k}_m\cdot\hat{k}_n)i_{\hat{k}_1}\hat{\Pi} - 2(\hat{k}_1\cdot\hat{k}_m)i_{\hat{k}_n}\hat{\Pi}\Big]_{A_1\ldots A_8}.\end{aligned} \tag{10.39}$$

Next we convert this expression into the co-ordinate basis using (6.12). The additional terms that result from this process can be substituted using (10.37) and we end up with the $a = 1$ component of

$$\begin{aligned} d(\hat{\Lambda}\wedge\hat{k}_a) &= i_{\hat{K}}\hat{G}^{(9)}_a - \frac{1}{3}i_{\hat{k}_a}\hat{\Sigma}\wedge\hat{F} - \frac{2}{3}\hat{\Sigma}\wedge i_{\hat{k}_a}\hat{F} - \frac{1}{3}\hat{\omega}\wedge i_{\hat{k}_a}\hat{F}^{(7)} \\ &\quad + \frac{2}{3}i_{\hat{k}_a}\hat{\omega}\wedge\hat{F}^{(7)} + \hat{Q}^{mn}\Big[-(\hat{k}_a\cdot\hat{k}_m)i_{\hat{k}_n}\hat{A}\wedge\hat{\Lambda} + i_{\hat{k}_a\hat{k}_m}\hat{A}\wedge\hat{\Lambda}\wedge\hat{k}_n \\ &\quad - i_{\hat{k}_m}\hat{A}\wedge i_{\hat{k}_n}\hat{\Lambda}\wedge\hat{k}_a + \frac{1}{3}(\hat{k}_m\cdot\hat{k}_n)i_{\hat{k}_a}\hat{\Pi} - 2(\hat{k}_a\cdot\hat{k}_m)i_{\hat{k}_n}\hat{\Pi}\Big]. \end{aligned} \tag{10.40}$$

The $a = 2$ component of this relation is calculated similarly. Note that (10.37) is merely the contraction of the above expression with a Killing vector followed by the symmetrisation of the $SL(2,\mathbb{R})$ indices.

Given (10.40) we then follow the usual procedure and produce the following charge-like expression

$$\begin{aligned} \hat{L}^{(7)}_a &= \hat{\Lambda}\wedge\hat{k}_a + i_{\hat{K}}\hat{N}^{(8)}_a + \frac{1}{3}i_{\hat{k}_a}\hat{L}^{(5)}\wedge\hat{A} + \frac{2}{3}\hat{L}^{(5)}\wedge i_{\hat{k}_a}\hat{A} \\ &\quad - \frac{1}{2}\hat{L}^{(2)}\wedge i_{\hat{k}_a}\hat{A}\wedge\hat{A} - \frac{1}{3}\hat{\omega}\wedge i_{\hat{k}_a}\hat{C} + \frac{2}{3}i_{\hat{k}_a}\hat{\omega}\wedge\hat{C} \\ &\quad + \frac{1}{6}i_{\hat{K}}\hat{A}\wedge i_{\hat{k}_a}\hat{A}\wedge\hat{A}. \end{aligned} \tag{10.41}$$

For this expression to be closed we require the gauge conditions (10.30) and (10.31) to be satisfied as well as

$$\begin{aligned} \mathcal{L}_{\hat{K}}\hat{N}^{(8)}_a &= \hat{Q}^{mn}\Big[i_{\hat{k}_m}\hat{L}^{(2)}\wedge i_{\hat{k}_n}\hat{N}^{(8)}_a + i_{\hat{k}_a\hat{k}_m}\hat{L}^{(2)}\wedge\hat{N}^{(8)}_n \\ &\quad - \frac{2}{3}\hat{L}^{(KK)}_{mn}\wedge i_{\hat{k}_a}\hat{A} + \frac{1}{3}i_{\hat{k}_a}\hat{L}^{(KK)}_{mn}\wedge\hat{A} \\ &\quad - \frac{1}{3}i_{\hat{k}_a}\hat{L}^{(9)}_{mn} + 2i_{\hat{k}_m}\hat{L}^{(9)}_{na}\Big] \end{aligned} \tag{10.42}$$

where $\hat{L}^{(9)}_{ab}$ are related to the M9-brane charges which are discussed in the following subsection. Their presence here is a consequence of the fact that $\hat{N}^{(8)}_a$ are Stueckelberg fields with the corresponding massive fields being certain components of $\hat{A}^{(10)}_{ab}$, which is seen from (10.13). Comparing the above gauge condition with the massive gauge transformations in (10.23) for the $j = 1$ case we see that they have the same structure if we make the identifications

$$\hat{L}^{(9)}_{ab} \sim -\hat{\chi}^{(9)}_{ab} \tag{10.43}$$

along with (10.34). Consistency with the field strength equation (10.13) then requires the condition (10.53) given in the next subsection along with the following condition on $\hat{G}^{(9)}_a$

$$\mathcal{L}_{\hat{K}}\hat{G}^{(9)} = \hat{Q}^{mn}\left[i_{\hat{k}_m}\hat{L}^{(2)} \wedge i_{\hat{k}_n}\hat{G}^{(9)}_a + i_{\hat{k}_a\hat{k}_m}\hat{L}^{(2)} \wedge \hat{G}^{(9)}_n\right]. \tag{10.44}$$

This can be determined independently by calculating the exterior derivative of (10.40) and using the general definition of the Lie derivative (2.37).

Note that the leading term of $\hat{L}^{(7)}_a$ has a different structure to that usually found in the generalised charges. In all the other examples this term had the simple structure of a bilinear and a multiplying factor which reflected the brane's tension and allowed the charge to be interpreted as a generalised calibration. In this instance though such an interpretation is not obvious. However, we observe the following relation with the KK-monopole charges

$$\hat{L}^{(KK)}_{ab} = i_{\hat{k}_{(a}}\hat{L}^{(7)}_{b)} \tag{10.45}$$

which corresponds precisely to the (spacetime) components of $\hat{L}^{(7)}_a$ which contain a leading bilinear term of the standard form. This does not offer a spacetime interpretation of the other components, which naively would seem to correspond to some sort of generalised KK-monopole state. We suspect however that the existence of $\hat{L}^{(7)}$ is a necessary requirement in order to construct the KK-monopole charges due to the fact that they contain Killing vectors, and does not therefore have a more general spacetime interpretation. We find a similar example of this for the M9-branes in the next subsection. It would be interesting to understand precisely why this expression does occur. Expressing the KK-monopole charges in terms of $\hat{L}^{(7)}$ is however useful since it makes identities such as $i_{\hat{k}_1}\hat{L}^{(KK)}_{11} = 0$ explicit which is important in determining the types of multiplets that exist which we now briefly discuss.

The index structure of the charges is obviously related to the type of $SL(2,\mathbb{R})$ multiplet which they form. In addition to this, the index structure also determines how the charges should be mapped to IIB up to a discrete S-duality transformation (or alternatively to the $D = 9$ theory), i.e. whether a direct or double dimensional reduction should be performed etc. Therefore, as for the branes themselves, a given charge can belong to different multiplets depending on how it is related to IIB. An example of this was given above by (10.45) where the doublet $\hat{L}^{(7)}_a$ was used to form the triplet of KK-monopole charges. The form of this triplet then contains the information of how the KK-monopoles are mapped to IIB to form the triplet of charges of the 7-branes. The KK-monopoles can also be mapped to IIB to form the doublet of states that consist of the D5 and NS5-branes. In terms of the charges this is expressed by

$$i_{\hat{k}_b \hat{k}_c} \hat{L}^{(7)}_a = \pm i_{\hat{k}_1 \hat{k}_2} \hat{L}^{(7)}_a. \tag{10.46}$$

This charge is interpreted as a doublet since the indices on the Killing vectors are fixed by their antisymmetry. Note that this is only true for the $SL(2,\mathbb{R})$ covariant theory. No other charge multiplets can be formed from $\hat{L}^{(7)}_a$. The quadruplet $i_{\hat{k}_{(a}\hat{k}_b} \hat{L}^{(7)}_{c)}$ is identically zero, the doublet $\epsilon^{bc} i_{\hat{k}_a \hat{k}_b} \hat{L}^{(7)}_c$ merely reduces to (10.46), and the scalar $\epsilon^{ab} i_{\hat{k}_a} \hat{L}^{(7)}_b$ has the same problem as $\hat{L}^{(7)}_a$ in that it does not have the correct structure of a charge. We therefore find that the only multiplets we can construct correspond precisely to those found in the IIB theory as we would expect.

10.2.2 $SL(2,\mathbb{R})$ covariant M9-brane charges

We now consider how the M9-brane charge generalises to the $SL(2,\mathbb{R})$ covariant theory. The charge (8.56) already constructed was interpreted as that of a single M9-brane. It is a trivial task to covariantise this to determine the structure of a quadruplet of charges which map to the charges of the quadruplet of 9-branes in IIB. However, we expect there also to exist a doublet and triplet of charges that map to the charges of the doublet of 9-branes and triplet of 7-branes in IIB. These multiplets are not directly attainable from (8.56).

Following the example of the KK-monopole we find that the multiplet structures of the M9-branes is more naturally expressed in terms of a 9-form expression $\hat{L}^{(9)}_{ab}$, which we refer to as the principal M9-brane charge and which plays an analogous role to $\hat{L}^{(7)}_a$. We therefore must determine the structure of $\hat{L}^{(9)}_{ab}$. From looking at the leading bilinear term of (8.56) we expect that the leading term of $\hat{L}^{(9)}_{ab}$ should be[2] $(\hat{k}_a \cdot \hat{k}_b)\hat{\Pi}$. We therefore need to determine the exterior derivative of this term

[2]Note that at this point we cannot rule out the possibility that the leading term of $\hat{L}^{(9)}_{ab}$ takes the form $i_{\hat{k}_{(a}} \hat{\Pi} \wedge \hat{k}_{b)}$. However, the exterior derivative of this expression is calculated similarly to

which we now do by first calculating the (1,1) component and then covariantising.

From the usual method we find that the expression for $d\hat{\Pi}$ (2.35) is now modified to

$$d\hat{\Pi} = -\frac{1}{3}\hat{F}\wedge\hat{\Lambda} + \hat{Q}^{mn}\left[-\frac{5}{3}(\hat{k}_m\cdot\hat{k}_n)\hat{\Upsilon} - i_{\hat{k}_m}\hat{\Upsilon}\wedge\hat{k}_n - i_{\hat{k}_m}\hat{\Pi}\wedge i_{\hat{k}_n}\hat{A}\right]. \quad (10.47)$$

It will also be necessary to determine the relation for $d(|\hat{k}_1|^2 i_{\hat{k}_1}\hat{\Pi})$. This is achieved by obtaining the algebraic relation that results, working in the usual basis adapted to $\hat{k}_1$, from hitting (10.36) from the left with $\hat{\Gamma}_{A_1\ldots A_9}$. After multiplication by $4|\hat{k}_1|^2$ this is given by

$$\begin{aligned} 0 = &\left[i_{\hat{k}_1}\hat{\omega}\wedge i_{\hat{k}_1}\hat{G}_1^{(9)} - d(|\hat{k}_1|^2)\wedge i_{\hat{k}_1}\hat{\Pi} - \frac{1}{3}|\hat{k}_1|^2 i_{\hat{k}_1}\hat{\Lambda}\wedge\hat{F} + \frac{2}{3}|\hat{k}_1|^2\hat{\Lambda}\wedge i_{\hat{k}_1}\hat{F}\right]_{A_1\ldots A_9} \\ &+\hat{Q}^{mn}\left[\frac{1}{3}|\hat{k}_1|^2(\hat{k}_m\cdot\hat{k}_n) i_{\hat{k}_1}\hat{\Upsilon} - 2|\hat{k}_1|^2(\hat{k}_1\cdot\hat{k}_m) i_{\hat{k}_n}\hat{\Upsilon}\right. \\ &\left.+2(\hat{k}_1\cdot\hat{k}_m) i_{\hat{k}_1\hat{k}_n}\hat{A}\wedge i_{\hat{k}_1}\hat{\Pi}\right]_{A_1\ldots A_9}. \end{aligned} \quad (10.48)$$

Combining this with (10.47) contracted with $\hat{k}_1$ and multiplied by $|\hat{k}_1|^2$, and then converting to the co-ordinate basis using the $\hat{k}_1$ analogue of (6.12), yields

$$\begin{aligned} d(|\hat{k}_1|^2 i_{\hat{k}_1}\hat{\Pi}) = &\ |\hat{k}_1|^2\hat{\Lambda}\wedge i_{\hat{k}_1}\hat{F} + i_{\hat{k}_1}\hat{\Lambda}\wedge\hat{k}_1\wedge i_{\hat{k}_1}\hat{F} + i_{\hat{k}_1}\hat{\omega}\wedge i_{\hat{k}_1}\hat{G}^{(9)} - i_{\hat{K}\hat{k}_1}\hat{F}_{11}^{(11)} \\ &+\hat{Q}^{mn}\Big[|\hat{k}_1|^2(\hat{k}_m\cdot\hat{k}_n) i_{\hat{k}_1}\hat{\Upsilon} + |\hat{k}_1|^2 i_{\hat{k}_1}(i_{\hat{k}_m}\hat{\Upsilon}\wedge\hat{k}_n) \\ &+2(\hat{k}_1\cdot\hat{k}_m)(\hat{k}_1\cdot\hat{k}_n) i_{\hat{k}_1}\hat{\Upsilon} + 2(\hat{k}_1\cdot\hat{k}_m) i_{\hat{k}_1\hat{k}_n}\hat{\Upsilon}\wedge\hat{k}_1 \\ &-2|\hat{k}_1|^2(\hat{k}_1\cdot\hat{k}_m) i_{\hat{k}_n}\hat{\Upsilon} + |\hat{k}_1|^2 i_{\hat{k}_1}(i_{\hat{k}_n}\hat{A}\wedge i_{\hat{k}_m}\hat{\Pi}) \\ &+2(\hat{k}_1\cdot\hat{k}_m) i_{\hat{k}_1\hat{k}_n}\hat{A}\wedge i_{\hat{k}_1}\hat{\Pi}\Big] \end{aligned} \quad (10.49)$$

which is trivially satisfied for the (spacetime) components parallel to $\hat{k}_1$ and is therefore fully tensorial.

Next we require the algebraic relation obtained by hitting (10.36) from the left with $\hat{\epsilon}\hat{\Gamma}_{A_1\ldots A_{10}\underline{x}}$ which, after multiplying by $|\hat{k}_1|$, is given by

$$\begin{aligned} 0 = &\left[-d(|\hat{k}_1|^2)\wedge\hat{\Pi} - \frac{1}{4}\hat{\omega}\wedge i_{\hat{k}_1}\hat{G}_1^{(9)} + \frac{3}{4} i_{\hat{k}_1}\hat{\omega}\wedge\hat{G}_1^{(9)} + \frac{1}{12}|\hat{k}_1|^2\hat{\Lambda}\wedge\hat{F}\right]_{A_1\ldots A_{10}} \\ &+\hat{Q}^{mn}\left[-\frac{1}{12}|\hat{k}_1|^2(\hat{k}_m\cdot\hat{k}_n)\hat{\Upsilon} + \frac{1}{2}(\hat{k}_1\cdot\hat{k}_m) i_{\hat{k}_1}(\hat{\Upsilon}\wedge\hat{k}_n)\right. \\ &\left.+2(\hat{k}_1\cdot\hat{k}_m) i_{\hat{k}_1\hat{k}_n}\hat{A}\wedge\hat{\Pi}\right]_{A_1\ldots A_{10}}. \end{aligned} \quad (10.50)$$

(10.50) and it turns out that a charge cannot be constructed from such a relation.

Note that due to the dual interpretations of the terms involving $\hat{\omega}$ and $\hat{\Pi}$ in the first line that arises from the identity (2.39) there exists some freedom in the specific combination of these terms which appear. The particular combination we have stated is the most natural for constructing the generalised charge given below.

The differential relation for $d[(\hat{k}_1\cdot\hat{k}_1)\hat{\Pi}]_{\mu_1\ldots\mu_{10}}$ is obtained by combining (10.50) with (10.47) multiplied by $|\hat{k}_1|^2$, then converting to the co-ordinate basis and substituting in (10.49) for the additional terms that arise. The remaining spacetime components $d[(\hat{k}_1\cdot\hat{k}_1)\hat{\Pi}]_{\mu_1\ldots\mu_9 x}$ of this expression match (10.49) and are therefore also seen to be satisfied, allowing us to write a fully tensorial expression. After covariantising, this expression is given by

$$\begin{aligned}
d[(\hat{k}_a\cdot\hat{k}_b)\hat{\Pi}] \;=\;& -\frac{1}{4}\hat{\omega}\wedge i_{\hat{k}_{(a}}\hat{G}^{(9)}_{b)}+\frac{3}{4}i_{\hat{k}_{(a}}\hat{\omega}\wedge\hat{G}^{(9)}_{b)}-\frac{1}{4}(\hat{k}_a\cdot\hat{k}_b)\hat{\Lambda}\wedge\hat{F}\\
&-\frac{1}{4}i_{\hat{k}_{(a}}\hat{\Lambda}\wedge\hat{F}\wedge\hat{k}_{b)}+\frac{3}{4}\hat{\Lambda}\wedge i_{\hat{k}_{(a}}\hat{F}\wedge\hat{k}_{b)}-i_{\hat{K}}\hat{F}^{(11)}_{ab}\\
&+\hat{Q}^{mn}\Big[-\frac{3}{4}(\hat{k}_a\cdot\hat{k}_b)(\hat{k}_m\cdot\hat{k}_n)\hat{\Upsilon}-\frac{3}{2}(\hat{k}_a\cdot\hat{k}_m)(\hat{k}_b\cdot\hat{k}_n)\hat{\Upsilon}\\
&-(\hat{k}_a\cdot\hat{k}_b)i_{\hat{k}_m}\hat{\Upsilon}\wedge\hat{k}_n+\frac{1}{4}(\hat{k}_m\cdot\hat{k}_n)i_{\hat{k}_{(a}}\hat{\Upsilon}\wedge\hat{k}_{b)}\\
&+\frac{1}{2}(\hat{k}_m\cdot\hat{k}_{(a})i_{\hat{k}_{b)}}\hat{\Upsilon}\wedge\hat{k}_n-2(\hat{k}_m\cdot\hat{k}_{(a})i_{\hat{k}_{|n|}}\hat{\Upsilon}\wedge\hat{k}_{b)}\\
&+2(\hat{k}_m\cdot\hat{k}_{(a})i_{\hat{k}_{b)}\hat{k}_n}\hat{A}\wedge\hat{\Pi}-(\hat{k}_a\cdot\hat{k}_b)i_{\hat{k}_m}\hat{A}\wedge i_{\hat{k}_n}\hat{\Pi}\Big].
\end{aligned}\tag{10.51}$$

From this relation we can determine the structure of $\hat{L}^{(9)}_{ab}$ using the usual method. The result is found to be

$$\begin{aligned}
\hat{L}^{(9)}_{ab} \;=\;& (\hat{k}_a\cdot\hat{k}_b)\hat{\Pi}-i_{\hat{K}}\hat{A}^{(10)}_{ab}+\frac{3}{4}i_{\hat{k}_{(a}}\hat{\omega}\wedge\hat{N}^{(8)}_{b)}-\frac{1}{4}\hat{\omega}\wedge i_{\hat{k}_{(a}}\hat{N}^{(8)}_{b)}\\
&+\frac{3}{4}\hat{L}^{(7)}_{(a}\wedge i_{\hat{k}_{b)}}\hat{A}+\frac{1}{4}i_{\hat{k}_{(a}}\hat{L}^{(7)}_{b)}\wedge\hat{A}-\frac{1}{4}\hat{L}^{(5)}\wedge i_{\hat{k}_a}\hat{A}\wedge i_{\hat{k}_b}\hat{A}\\
&-\frac{1}{4}i_{\hat{k}_{(a}}\hat{L}^{(5)}\wedge i_{\hat{k}_{b)}}\hat{A}\wedge\hat{A}+\frac{1}{3!}\hat{L}^{(2)}\wedge\hat{A}\wedge i_{\hat{k}_a}\hat{A}\wedge i_{\hat{k}_b}\hat{A}\\
&-\frac{1}{4!}i_{\hat{K}}\hat{A}\wedge\hat{A}\wedge i_{\hat{k}_a}\hat{A}\wedge i_{\hat{k}_b}\hat{A}.
\end{aligned}\tag{10.52}$$

The fact that this expression matches that found in (10.42) acts as a consistency check on its structure. In order for it to be closed we must satisfy the gauge conditions (10.30), (10.31) and (10.42) together with

$$\begin{aligned}\mathcal{L}_{\hat{K}}\hat{A}^{(10)}_{ab} &= \hat{Q}^{mn}\Big[i_{\hat{k}_m}\hat{L}^{(2)}\wedge i_{\hat{k}_n}\hat{A}^{(10)}_{ab} - 2i_{\hat{k}_m\hat{k}_{(a}}\hat{L}^{(2)}\wedge\hat{A}^{(10)}_{b)n}\\ &-\frac{1}{4}\hat{L}^{(KK)}_{mn}\wedge i_{\hat{k}_{(a}}\hat{A}\wedge i_{\hat{k}_{b)}}\hat{A} + \frac{1}{4}i_{\hat{k}_{(a}}\hat{L}^{(KK)}_{|mn|}\wedge i_{\hat{k}_{b)}}\hat{A}\wedge\hat{A}\\ &-\frac{1}{4}i_{\hat{k}_{(a}}\hat{L}^{(9)}_{|mn|}\wedge i_{\hat{k}_{b)}}\hat{A} + \frac{3}{2}i_{\hat{k}_m}\hat{L}^{(9)}_{n(a}\wedge i_{\hat{k}_{b)}}\hat{A} + \frac{1}{2}i_{\hat{k}_m\hat{k}_{(a}}\hat{L}^{(9)}_{b)n}\wedge\hat{A}\\ &-3i_{\hat{k}_m}\hat{L}^{(11)}_{abn} + \frac{3}{4}i_{\hat{k}_{(a}}\hat{L}^{(11)}_{b)mn}\Big]. \end{aligned} \tag{10.53}$$

Note the presence of the quadruplet of terms $\hat{L}^{(11)}_{abc}$ in the above expression. These arise since $\hat{A}^{(10)}_{ab}$ are Stueckelberg fields with the corresponding massive fields being certain components of $\hat{A}^{(12)}_{abc}$ which can be seen from (10.19). The situation is analogous to that found for $\hat{C}$ and $\hat{N}^{(8)}_a$.

The structure of (10.53) matches the massive terms in (10.23) for $j=2$ if we make the identifications (10.34) and (10.43) together with

$$\hat{L}^{(11)}_{abc} \sim \hat{\chi}^{(11)}_{abc} \tag{10.54}$$

which demonstrates consistency with the gauge algebra.

The structure of the $\hat{L}^{(11)}_{abc}$ terms can be uniquely inferred from (10.53) and are given by

$$\begin{aligned}\hat{L}^{(11)}_{abc} &= (\hat{k}_{(a}\cdot\hat{k}_b)\hat{\Upsilon}\wedge\hat{k}_{c)} + i_{\hat{K}}\hat{A}^{(12)}_{abc} + \frac{1}{5}\hat{\omega}\wedge i_{\hat{k}_{(a}}\hat{A}^{(10)}_{bc)} - \frac{4}{5}i_{\hat{k}_{(a}}\hat{\omega}\wedge\hat{A}^{(10)}_{bc)}\\ &+\frac{4}{5}\hat{L}^{(9)}_{(ab}\wedge i_{\hat{k}_{c)}}\hat{A} + \frac{1}{5}i_{\hat{k}_{(a}}\hat{L}^{(9)}_{bc)}\wedge\hat{A} - \frac{3}{10}\hat{L}^{(7)}_{(a}\wedge i_{\hat{k}_b}\hat{A}\wedge i_{\hat{k}_{c)}}\hat{A}\\ &-\frac{1}{5}i_{\hat{k}_{(a}}\hat{L}^{(7)}_{b}\wedge i_{\hat{k}_{c)}}\hat{A}\wedge\hat{A} + \frac{1}{15}\hat{L}^{(5)}\wedge i_{\hat{k}_a}\hat{A}\wedge i_{\hat{k}_b}\hat{A}\wedge i_{\hat{k}_c}\hat{A}\\ &+\frac{1}{10}i_{\hat{k}_{(a}}\hat{L}^{(5)}\wedge i_{\hat{k}_b}\hat{A}\wedge i_{\hat{k}_{c)}}\hat{A}\wedge\hat{A} - \frac{1}{4!}\hat{L}^{(2)}\wedge i_{\hat{k}_a}\hat{A}\wedge i_{\hat{k}_b}\hat{A}\wedge i_{\hat{k}_c}\hat{A}\wedge\hat{A}\\ &+\frac{1}{5!}i_{\hat{K}}\hat{A}\wedge i_{\hat{k}_a}\hat{A}\wedge i_{\hat{k}_b}\hat{A}\wedge i_{\hat{k}_c}\hat{A}\wedge\hat{A}.\end{aligned} \tag{10.55}$$

They have the form of a quadruplet of 11-form charges and in principal would correspond to some type of 11-branes. Obviously spacetime solutions of these branes do not exist but the fact that such charge structures do exist seems to be related to the fact that the gauge algebra can be treated independently of the dimensionality of the background spacetime. Note that double dimensional reduction of the $(1,1,1)$ component, using

$$i_{\hat{\alpha}}\hat{A}^{(12)}_{111} \;\rightarrow\; -C^{(11)} + \frac{1}{5!}C^{(3)}\wedge(B)^4 \tag{10.56}$$

where $C^{(11)}$ is an 11-form RR potential, gives an expression in IIA with the structure of a D10-brane charge i.e. a charge of the form (3.33) for $n = 5$. Furthermore, we believe the quadruplet $i_{\hat{k}_1\hat{k}_2}\hat{L}^{(11)}_{abc}$ should map to the quadruplet of 9-brane charges in IIB (12.44), although we do not explicitly investigate this in this thesis.

We now discuss the expression $\hat{L}^{(9)}_{ab}$. The situation here is analogous to $\hat{L}^{(7)}_a$ although in this instance we observe that the leading bilinear term of $\hat{L}^{(9)}_{ab}$ is of the form found in the other charges. However, like for $\hat{L}^{(7)}_a$ we suspect that not all the spacetime components correspond to supersymmetric spacetime solutions. Certain combinations do however seem to map to the charges of known states in IIB and so we propose that these correspond to states in the $D = 11$ $SL(2,\mathbb{R})$ covariant theory as well. These consist of the following triplet and quadruplet

$$\begin{aligned} \text{Triplet} &= i_{\hat{k}_1\hat{k}_2}\hat{L}^{(9)}_{ab} \\ \text{Quadruplet} &= i_{\hat{k}_{(a}}\hat{L}^{(9)}_{bc)}. \end{aligned}$$

The (1,1,1) component of the triplet was given by (8.56). In the following chapters we will show that these multiplets map to the charges of the triplet of 7-branes and quadruplet of 9-branes in IIB respectively.

Considering the triplet first, the charges $i_{\hat{k}_1\hat{k}_2}\hat{L}^{(9)}_{11}$ and $i_{\hat{k}_1\hat{k}_2}\hat{L}^{(9)}_{22}$ merely correspond to the usual M9-branes with an extra isometry parallel to the worldvolume. The extra isometry in each case is not however intrinsic to the charge and therefore neither to the brane itself. Their presence is purely a result of the type of mapping required to produce the IIB triplet. On the other hand the charge $i_{\hat{k}_1\hat{k}_2}\hat{L}^{(9)}_{12}$ does contain two Killing vectors that are intrinsic to the charge and suggest that the corresponding state, which we will refer to as the $D = 11$ r9-brane, contains two isometries parallel to its worldvolume. This therefore corresponds to a different brane which maps to the IIB r7-brane and which we are not aware of having been discussed in the literature. As for the triplet of monopoles, much of the discussion for the IIB 7-branes should be applicable to the states here.

Similarly the charges $i_{\hat{k}_1}\hat{L}^{(9)}_{11}$ and $i_{\hat{k}_2}\hat{L}^{(9)}_{22}$ of the quadruplet simply correspond to the usual M9-brane charges, this time however with no extra isometry parallel to the worldvolume. The other two charges $i_{\hat{k}_{(1}}\hat{L}^{(9)}_{12)}$ and $i_{\hat{k}_{(1}}\hat{L}^{(9)}_{22)}$ involve combinations of the M9-brane and r9-brane charges, but with different contractions with the Killing vectors. In each case, when compared to the triplet, the difference involves the contraction of a Killing vector which is intrinsic to the main charge expression. This might therefore mean that the corresponding states are inherently different to those associated with the triplet, despite the charges all being derived from the principal M9-brane charge. One would have to investigate the spacetime solutions to resolve these issues. In [68] it was shown that, unlike for the triplet of 7-branes, there is only a single conjugacy class for the quadruplet of 9-branes, and also that there are two constraints on the combinations of states that correspond to supersymmetric spacetime solutions, reducing the independent degrees of freedom to two. One would expect these observations to apply here as well, but it might have an obvious geometrical interpretation in terms of the spacetime solutions.

The remaining task is to determine the doublet of states and charges that map to the doublet of 9-branes in IIB. A natural candidate would be the following doublet

$$\epsilon^{bc} i_{\hat{k}_b} \hat{L}^{(9)}_{ca}. \tag{10.57}$$

However, on mapping this expression to IIB it is found that it produces a doublet of charges that each intrinsically depend on a Killing vector. This suggests that the corresponding IIB states are some sort of KK9-monopole. Although we do not fully analyse these charges in this thesis we give a brief discussion in Appendix C where we consider how the $D = 11$ 10-form potentials map to IIB. We propose that the KK9-monopoles might source a doublet of mass parameters in a non-covariant (in the spacetime sense) massive deformation of the IIB theory. Furthermore, it seems that this doublet might be the IIB origin of the doublet of mass parameters $(m_4, \tilde{m}_4)$ discussed in [90] which arise by performing a Scherk-Schwarz dimensional reduction of the massless IIA theory using the global $SO(1,1)$ symmetry.

10.2.3 $SL(2,\mathbb{R})$ covariant charge doublet

Since no more charge multiplets can be constructed from $\hat{L}^{(9)}_{ab}$ we must look elsewhere to find the $D = 11$ origin of the charges of the IIB 9-brane doublet. In order to construct an appropriate charge doublet it seems necessary to introduce a doublet of 11-form gauge potentials $\hat{A}_a^{(11)}$. Neglecting mass terms, the field equation of this doublet is given simply by

$$d\hat{A}_a^{(11)} \;=\; \hat{F}_a^{(12)}. \tag{10.58}$$

Taking the $a = 1$ component and performing a double dimensional reduction one obtains the field equation given in [60] for a 10-form potential in IIA. Furthermore, it is a simple task to show that this doublet maps to the field equation for the 10-form doublet in IIB given in [67], which we show in the following chapters. The gauge algebra of this doublet seems to be independent of the other potentials already considered. A related observation was made in [93] where the worldvolume action of the doublet of 9-branes in IIB also seems to follow a different structure from the other branes in the theory. We do however suspect that (10.58) is incomplete and should in fact include massive terms. The full structure could in principal be determined by investigating the closure of the SUSY transformations as done in [67, 93] but we do not investigate this in this thesis.

We then propose the following structure for a doublet of 10-form 'charges'

$$\hat{L}_a^{(10)} = \hat{\Pi} \wedge \hat{k}_a + i_{\hat{K}}\hat{A}_a^{(11)}. \tag{10.59}$$

It is problematic to show that (10.59) is closed generally without making use of the dimensionality of the background spacetime. This is because the differential relation for the leading bilinear term is calculated using the Killing spinor equation (10.27) which is derived from supersymmetry transformations and therefore must take into account the spacetime dimension. We can however calculate the exterior derivative of say $i_{\hat{k}_1}(\hat{\Pi} \wedge \hat{k}_1)$ by using (10.47) together with the algebraic relation obtained by hitting (10.36) from the left with $\hat{\bar{\epsilon}}\hat{\Gamma}_{A_1 \ldots A_{10}\underline{x}}$. Doing this we find that it vanishes if we neglect the massive terms. We therefore conjecture the following formal relation

$$d(\hat{\Pi} \wedge \hat{k}_a) \;=\; i_{\hat{K}}\hat{F}_a^{(12)} + \text{massive terms} \tag{10.60}$$

which we interpret as an analytic extension of the usual relations derived from the Killing spinor equations.

The expression (10.59) is then easily shown to be closed in the massless case. The massive case is more complicated and requires us to take account of the massive terms that would appear in (10.60) and (10.58) and also in the condition on $\mathcal{L}_{\hat{K}}\hat{A}_a^{(11)}$ which would appear to include higher rank charge structures just as for the other potentials already considered. Due to the difficulties in obtaining the full algebraic structures of the differential relations of the high rank bilinears, we do not explore the massive case in this thesis.

In the following chapters we will show that the doublet

$$i_{\hat{k}_1\hat{k}_2}\hat{L}_a^{(10)} \tag{10.61}$$

maps to the charges of the 9-brane doublet in IIB supporting the proposed structure (10.59). From this we would expect the charge doublet (10.61) to represent a doublet of 9-branes each with a single isometry direction transverse to the worldvolume (since each charge contains only one intrinsic Killing vector) and tensions which scale as $|\hat{k}_1|^2$ and $|\hat{k}_2|^2$. The situation here is essentially just a higher dimensional version of $i_{\hat{k}_1\hat{k}_2}\hat{L}_a^{(7)}$.

Chapter 11

Generalised charges in non-covariant IIA supergravity

We now give the IIA charges produced from dimensionally reducing the $D = 11$ charges of the previous chapter. These charges are guaranteed to be closed due to their $D = 11$ origin. Therefore, for the sake of brevity we do not give the full field strength equations, bilinear differential relations or the gauge conditions on the potentials, but these are easily calculated. The reduction is performed over $\hat{k}_1 = \hat{\alpha}$ and the reduction rules for the fields and bilinears can be found in Appendix A.

11.1 Dimensionally reducing KK-monopole charges

We begin by considering the KK-monopole charges. Reducing $\hat{L}_{11}^{(KK)}$ gives the D6-brane charge, (3.33) for $n = 3$, so we do not need to restate this here. Reducing $\hat{L}_{22}^{(KK)}$ on the other hand produces the charge for the KK6-monopole which is given by

$$\begin{aligned} M^{(KK6)} &= \left[e^{-3\phi}R^2 + e^{-\phi}(i_\beta C^{(1)})^2\right] \Psi^{(6)} + e^{-3\phi} i_\beta \Psi^{(6)} \wedge \beta - e^{-2\phi} i_\beta C^{(1)}\,\tilde{\Sigma} \wedge \beta \\ &\quad + (e^{-2\phi}R^2\,\tilde{\Sigma} - e^{-2\phi} i_\beta \tilde{\Sigma} \wedge \beta + e^{-\phi} i_\beta C^{(1)}\, i_\beta \Psi^{(6)}) \wedge C^{(1)} \\ &\quad - i_{K\beta} N^{(8)} + i_\beta M^{(NS5)} \wedge i_\beta C^{(3)} - i_\beta(e^{-\phi}\Psi^{(2)} + \tilde{K} \wedge C^{(1)}) \wedge i_\beta B^{(6)} \\ &\quad - \frac{1}{2} i_\beta \left[(e^{-\phi}\Psi^{(2)} + \tilde{K} \wedge C^{(1)}) \wedge C^{(3)}\right] \wedge i_\beta C^{(3)} - M^{(0)} i_\beta N^{(7)} \\ &\quad - \frac{1}{3} i_\beta (i_K C^{(3)} \wedge C^{(3)}) \wedge i_\beta C^{(3)}. \end{aligned} \qquad (11.1)$$

From this charge we see that the KK6-monopole tension scales as

$$e^{-3\phi}R^2 + e^{-\phi}(i_\beta C^{(1)})^2$$

and also that it minimally couples to $i_\beta N^{(8)}$. This is in agreement with [42].

Next we consider $\hat{L}_{12}^{(KK)}$ which reduces to the charge of what we will refer to as the IIA r6-brane. This is found to be given by

$$\begin{aligned} M^{(r6)} &= e^{-\phi} i_\beta C^{(1)} \Psi^{(6)} - i_{K\beta}\phi^{(8)} + \frac{1}{2}\Big(i_K N^{(7)} - e^{-2\phi}\tilde{\Sigma}\wedge\beta + i_\beta M^{(NS5)}\wedge B \\ &\quad - M^{(0)} i_\beta C^{(7)} + (e^{-\phi}\Psi^{(4)} - \frac{1}{2}\tilde{K}\wedge C^{(3)} - i_K C^{(5)})\wedge i_\beta C^{(3)} \\ &\quad - i_\beta(e^{-\phi}\Psi^{(2)} + \tilde{K}\wedge C^{(1)})\wedge C^{(5)} + e^{-\phi} i_\beta \Psi^{(6)}\wedge C^{(1)} \\ &\quad + \tilde{K}\wedge i_\beta B^{(6)}\Big). \end{aligned} \tag{11.2}$$

We see that the tension here scales as $e^{-\phi} i_\beta C^{(1)}$ and the brane minimally couples to the combination of potentials $i_\beta \phi^{(8)} - \frac{1}{2}N^{(7)}$. In [60] the potential $\phi^{(8)}$ was considered and it was found that it does not minimally couple to any supersymmetric state on its own. This does not contradict the result here since in that reference states containing isometries were not considered, nor was the potential $N^{(7)}$.

11.2 Dimensionally reducing M9-brane triplet

We now consider the triplet of M9-brane charges. The charge $i_{\hat{\alpha}\hat{\beta}}\hat{L}_{11}^{(9)}$ reduces to the D8-brane charge, (3.33) for $n = 4$, with an overall contraction with β so we do not need to restate this here. The charge $i_{\hat{\alpha}\hat{\beta}}\hat{L}_{22}^{(9)}$ reduces to the KK8-monopole charge which is given by

$$\begin{aligned} i_\beta M^{(KK8)} &= (e^{-3\phi}R^2 + e^{-\phi}(i_\beta C^{(1)})^2) i_\beta \Psi^{(8)} + i_{K\beta} D^{(9)} + i_\beta \tilde{K} i_\beta N^{(8)} \\ &\quad - i_\beta(e^{-\phi}\Psi^{(2)} + \tilde{K}\wedge C^{(1)})\wedge i_\beta N^{(7)} + \frac{1}{3} i_\beta(\tilde{K}\wedge C^{(3)})\wedge(i_\beta C^{(3)})^2 \\ &\quad - \frac{1}{2} i_\beta(e^{-\phi}\Psi^{(4)} - i_K C^{(5)})\wedge(i_\beta C^{(3)})^2 + M^{(KK6)}\wedge i_\beta B \\ &\quad + M^{(KK)}\wedge i_\beta C^{(3)} - i_\beta M^{(NS5)}\wedge i_\beta C^{(3)}\wedge i_\beta B. \end{aligned} \tag{11.3}$$

The KK8-monopole was considered for example in [42] and is known to have a tension that scales as

$$e^{-3\phi}R^3 + e^{-\phi}R(i_\beta C^{(1)})^2.$$

This is in agreement with the leading bilinear terms in the above charge, remembering that the contraction of β with $\Psi^{(8)}$ creates an additional factor of R. Furthermore we see that the KK8-monopole minimally couples to the potential $i_\beta D^{(9)}$.

Next we reduce the charge $i_{\hat{\alpha}\hat{\beta}}\hat{L}^{(9)}_{12}$. We will denote the corresponding brane as the IIA r8-brane and find its charge to be given by

$$\begin{aligned}
i_\beta M^{(r8)} &= e^{-\phi} i_\beta C^{(1)} i_\beta \Psi^{(8)} + i_{K\beta} B^{(9)} + i_\beta \tilde{K} i_\beta \phi^{(8)} + M^{(r6)} \wedge i_\beta B \\
&+ \frac{1}{2}\Big[i_\beta (e^{-\phi}\Psi^{(6)} + \tilde{K} \wedge C^{(5)} + i_K C^{(7)}) \wedge i_\beta C^{(3)} \\
&- i_\beta (e^{-\phi}\Psi^{(2)} + \tilde{K} \wedge C^{(1)}) \wedge i_\beta C^{(7)} - i_\beta (\tilde{K} \wedge N^{(7)}) \\
&+ M^{(KK)} \wedge B - i_\beta M^{(NS5)} \wedge B \wedge i_\beta B \Big]. \qquad (11.4)
\end{aligned}$$

The r8-brane is related by T-duality to the r7-brane in IIB. Its tension can be seen to scale as $e^{-\phi} R i_\beta C^{(1)}$ and it minimally couples to $i_\beta B^{(9)}$.

11.3 Dimensionally reducing M9-brane quadruplet

Next we consider the dimensional reduction of the quadruplet of M9-brane charges. The charge $i_{\hat{\alpha}}\hat{L}_{11}$ merely reduces to the D8-brane charge, (3.33) for $n = 4$, and so we do not need to restate it here. The next charge is then $\frac{2}{3} i_{\hat{\alpha}} \hat{L}^{(9)}_{12} + \frac{1}{3} i_{\hat{\beta}} \hat{L}^{(9)}_{11}$, which reduces to

$$\begin{aligned}
i_\beta M^{(r9)} &= \frac{1}{3} e^{-2\phi} i_\beta \Pi + e^{-\phi} i_\beta C^{(1)} \Psi^{(8)} + \frac{1}{3} e^{-\phi} i_\beta \Psi^{(8)} \wedge C^{(1)} + \frac{1}{3} i_{K\beta} A^{(10)} \\
&- \frac{2}{3} i_K B^{(9)} + \tilde{K} \wedge (\frac{2}{3} i_\beta \phi^{(8)} - \frac{1}{3} N^{(7)}) + \frac{1}{3} i_\beta (e^{-\phi}\Psi^{(2)} + \tilde{K} \wedge C^{(1)}) \wedge C^{(7)} \\
&+ \frac{1}{3} (e^{-\phi}\Psi^{(6)} + \tilde{K} \wedge C^{(5)} + i_K C^{(7)}) \wedge i_\beta C^{(3)} + \frac{2}{3} M^{(r6)} \wedge B \\
&- \frac{1}{6} i_\beta M^{(NS5)} \wedge (B)^2 + \frac{1}{3} M^{(0)} i_\beta C^{(9)}. \qquad (11.5)
\end{aligned}$$

We will refer to the state to which this charge corresponds as the IIA r9-brane. It is seen to have a tension which scales as

$$e^{-\phi}\sqrt{\frac{1}{9} e^{-2\phi} R^2 + (i_\beta C^{(1)})^2}$$

and minimally couples to the combination of potentials $\frac{1}{3} i_\beta A^{(10)} - \frac{2}{3} B^{(9)}$. In [60] the potential $A^{(10)}$ was considered and it was found that it does not minimally couple to any supersymmetric state on its own. This does not contradict the result here since in that reference states containing isometries were not considered, nor was the potential $B^{(9)}$.

Next we reduce the charge $\frac{2}{3}i_{\hat{\beta}}\hat{L}^{(9)}_{12} + \frac{1}{3}i_{\hat{\alpha}}\hat{L}^{(9)}_{22}$. In IIA this gives

$$\begin{aligned} i_\beta M^{(s9)} &= \frac{2}{3}e^{-2\phi}i_\beta C^{(1)} i_\beta \Pi + \left[\frac{1}{3}e^{-3\phi}R^2 + e^{-\phi}(i_\beta C^{(1)})^2\right]\Psi^{(8)} \\ &\quad + \frac{2}{3}e^{-\phi}i_\beta C^{(1)} i_\beta \Psi^{(8)} \wedge C^{(1)} + \frac{2}{3}i_{K\beta}B^{(10)} - \frac{1}{3}i_K D^{(9)} + \frac{1}{3}\tilde{K}\wedge i_\beta N^{(8)} \\ &\quad - \frac{1}{6}(e^{-\phi}\Psi^{(4)} - i_K C^{(5)} - \frac{2}{3}\tilde{K}\wedge C^{(3)}) \wedge (i_\beta C^{(3)})^2 \\ &\quad + i_\beta(e^{-\phi}\Psi^{(2)} + \tilde{K}\wedge C^{(1)}) \wedge (-\frac{2}{3}\phi^{(8)} + \frac{1}{3}N^{(7)}) + \frac{1}{3}M^{(KK6)}\wedge B \\ &\quad + \frac{2}{3}M^{(r6)}\wedge i_\beta C^{(3)} - \frac{1}{3}i_\beta M^{(NS5)} \wedge i_\beta C^{(3)} \wedge B + \frac{2}{3}M^{(0)} i_\beta B^{(9)}. \end{aligned} \tag{11.6}$$

We will refer to the state to which this charge corresponds as the IIA s9-brane. We see that it has a tension which scales as

$$e^{-\phi}\sqrt{\frac{4}{9}e^{-2\phi}R^2(i_\beta C^{(1)})^2 + (\frac{1}{3}e^{-2\phi}R^2 + (i_\beta C^{(1)})^2)^2}$$

and minimally couples to the combination of potentials $\frac{2}{3}i_\beta B^{(10)} - \frac{1}{3}D^{(9)}$.

The last charge in this multiplet is $i_{\hat{\beta}}\hat{L}^{(9)}_{22}$. It reduces to

$$\begin{aligned} i_\beta M^{(q9)} &= \left[e^{-4\phi}R^2 + e^{-2\phi}(i_\beta C^{(1)})^2\right] i_\beta \Pi + \left[e^{-3\phi}R^2 + e^{-\phi}(i_\beta C^{(1)})^2\right] i_\beta(\Psi^{(8)}\wedge C^{(1)}) \\ &\quad + i_{K\beta}D^{(10)} + i_\beta(e^{-\phi}\Psi^{(2)} + \tilde{K}\wedge C^{(1)}) \wedge (-i_\beta N^{(8)} + \frac{1}{6}(i_\beta C^{(3)})^2) \\ &\quad + \frac{1}{8}i_K C^{(3)} \wedge C^{(3)} \wedge (i_\beta C^{(3)})^2 + M^{(KK6)} \wedge i_\beta C^{(3)} \\ &\quad - \frac{1}{2}i_\beta M^{(NS5)} \wedge (i_\beta C^{(3)})^2 + M^{(0)} i_\beta D^{(9)}. \end{aligned} \tag{11.7}$$

We will refer to the state to which this charge corresponds as the IIA q9-brane. We see that it has a tension which scales as

$$e^{-\phi}(e^{-2\phi}R^2 + (i_\beta C^{(1)})^2)^{3/2}$$

and minimally couples to $i_\beta D^{(10)}$.

11.4 Dimensionally reducing 9-brane doublet

We now dimensionally reduce the 9-brane doublet (10.61). We first consider the charge $i_{\hat{\alpha}\hat{\beta}}\hat{L}_1^{(10)}$ which produces

$$M^{(t9)} = e^{-2\phi}\Pi + i_K\overline{A}^{(10)}. \tag{11.8}$$

In this instance we may remove the overall contraction of β since this Killing vector is not intrinsically present in the charge. We will refer to the brane that this charge corresponds to as the IIA t9-brane. This brane was discussed in [67] where it was shown to have a tension which scales with $e^{-2\phi}$ which can also be read from the above charge.

Finally we dimensionally reduce the charge $i_{\hat{\alpha}\hat{\beta}}\hat{L}_2^{(10)}$ which produces

$$i_\beta M^{(u9)} = e^{-2\phi} i_\beta C^{(1)} i_\beta \Pi - e^{-3\phi}(R^2\Psi^{(8)} + i_\beta\Psi^{(8)} \wedge \beta) - i_{K\beta}\overline{B}^{(10)}. \tag{11.9}$$

We will refer to the brane that this charge corresponds to as the IIA u9-brane. We see that it has a tension that scales as

$$e^{-2\phi}R\sqrt{e^{-2\phi}R^2 + (i_\beta C^{(1)})^2} \tag{11.10}$$

and minimally couples to $i_\beta\overline{B}^{(10)}$.

Chapter 12

$SL(2,\mathbb{R})$ covariant generalised charges in IIB supergravity

We now revisit the IIB supergravity theory. We start by T-dualising the IIA charges given in the previous chapter to produce a set of IIB charges that were not considered in Chapter 4. These charges correspond to the remaining branes required to fill the $SL(2,\mathbb{R})$ multiplets in the IIB theory. These multiplets consist of a doublet of 1-branes, a 3-brane singlet, a doublet of 5-branes, a triplet of 7-branes, a quadruplet of 9-branes and a doublet of 9-branes. We also include the KK-monopole which forms a singlet. We present all the charges in an $SL(2,\mathbb{R})$ covariant fashion in Section 12.2 and show that they form the expected $SL(2,\mathbb{R})$ multiplets, in contrast to the flatspace SUSY charges.

When performing the T-duality transformations we adopt the second of the schemes discussed at the beginning of Chapter 5 which involves the massless T-duality rules and assumes that the IIB potentials are *independent* of the T-duality isometry direction. We present the pseudo-reformulation of IIB that accounts for the mass parameters in this scheme in Section 12.3 and discuss the nature of the relation between the IIB and IIA charges when the non-covariant massive IIA theory is being considered. We delay giving the $SL(2,\mathbb{R})$ covariant differential relations for the bilinears until this section.

12.1 T-dualising from IIA

12.1.1 Triplet of 7-brane charges

We start by considering the triplet of IIB 7-branes. For recent discussions on the spacetime solutions of these branes see [78–80]. We have already considered the D7-brane charge, (4.23) with $n = 3$, so we do not need to repeat it here. We have not yet however considered the NS7-brane charge which is obtained from IIA by either performing a direct T-duality transformation on the KK6-monopole charge (11.1) or a double T-duality transformation on the KK8-monopole charge (11.3). The result is found to be

$$\begin{aligned} N^{(NS7)} &= (e^{-3\varphi} + l^2 e^{-\varphi})\Phi^{(7)} + i_{K^+}\mathcal{N}^{(8)} + (e^{-\varphi}\Phi^{(1)} - lK^-)\wedge\mathcal{B}^{(6)} \\ &+ \Big(e^{-\varphi} l\Phi^{(5)} + e^{-2\varphi}\Sigma^- - i_{K^+}\mathcal{B}^{(6)} + \frac{1}{2}e^{-\varphi}\Phi^{(3)}\wedge\mathcal{C}^{(2)} \\ &+ \frac{1}{3!}K^-\wedge(\mathcal{C}^{(2)})^2\Big)\wedge\mathcal{C}^{(2)}. \end{aligned} \tag{12.1}$$

The tension of the NS7-brane was shown to scale as $e^{-3\varphi} + l^2 e^{-\varphi}$ in [68], which can also be trivially read off from the above charge. Furthermore this brane minimally couples to $\mathcal{N}^{(8)}$.

We refer to the third brane in this triplet as the IIB r7-brane. Its charge can be found by either performing a direct T-duality transformation of the IIA r6-brane charge (11.2) or a double T-duality transformation of the IIA r8-brane charge (11.4). It is found to be given by

$$\begin{aligned} N^{(r7)} &= e^{-\varphi} l\Phi^{(7)} - i_{K^+}\varphi^{(8)} + \frac{1}{2}\Big(e^{-\varphi}\Phi^{(5)}\wedge\mathcal{C}^{(2)} - K^-\wedge\mathcal{B}^{(6)} \\ &+ N^{(NS5)}\wedge\mathcal{B} - N^{(1)}\wedge\mathcal{C}^{(6)}\Big). \end{aligned} \tag{12.2}$$

The tension for this brane was also given in [68] and was shown to scale as $le^{-\varphi}$, which is in agreement with the above charge. Furthermore, it minimally couples to $\varphi^{(8)}$.

12.1.2 Quadruplet of 9-brane charges

We now consider the charges for the quadruplet of 9-branes. These are obtained by T-dualising the IIA charges which are produced from dimensionally reducing

(10.57). In order to do this we must first determine the T-duality rules for the potentials which minimally couple to these branes. We present the details of this task in Appendix C and simply use the results in this section.

The D9-brane belongs to this quadruplet. However since we have already given its charge, (4.23) with $n = 4$, we do not need to repeat it here. We will refer to the next brane as the IIB s9-brane. The (double dimensional reduction of the) charge of this brane can be calculated by T-dualising the IIA s9-brane charge (11.6) along β. Doing this yields

$$\begin{aligned} N^{(s9)} &= (\frac{1}{3}e^{-3\varphi} + l^2 e^{-\varphi})\Phi^{(9)} + \frac{2}{3}le^{-2\varphi}\Omega^- + \frac{1}{3}(e^{-3\varphi} + l^2 e^{-\varphi})\Phi^{(7)} \wedge \mathcal{B} \\ &+\frac{2}{3}le^{-\varphi}\Phi^{(7)} \wedge \mathcal{C}^{(2)} + \frac{1}{6}e^{-\varphi}\Phi^{(5)} \wedge (\mathcal{C}^{(2)})^2 + \frac{1}{3}(e^{-2\varphi}\Sigma^- + le^{-\varphi}\Phi^{(5)}) \wedge \mathcal{C}^{(2)} \wedge \mathcal{B} \\ &+\frac{1}{6}e^{-\varphi}\Phi^{(3)} \wedge (\mathcal{C}^{(2)})^2 \wedge \mathcal{B} + (e^{-\varphi}\Phi^{(1)} - lK^-) \wedge (\frac{2}{3}\varphi^{(8)} + \frac{1}{3}\mathcal{B}^{(6)} \wedge \mathcal{B}) \\ &+K^- \wedge (\frac{1}{3}\mathcal{N}^{(8)} - \frac{1}{3}\mathcal{B}^{(6)} \wedge \mathcal{C}^{(2)} + \frac{1}{18}(\mathcal{C}^{(2)})^3 \wedge \mathcal{B}) + i_{K^+}\mathcal{A}^{(10)} \\ &+\frac{1}{3}i_{K^+}\mathcal{N}^{(8)} \wedge \mathcal{B} + \frac{2}{3}\varphi^{(8)} \wedge i_{K^+}\mathcal{C}^{(2)} - \frac{1}{3}i_{K^+}\mathcal{B}^{(6)} \wedge \mathcal{C}^{(2)} \wedge \mathcal{B}. \end{aligned} \tag{12.3}$$

From this charge we see that the tension of the IIB s9-brane scales as

$$e^{-\varphi}\sqrt{\frac{4}{9}l^2 e^{-2\varphi} + (\frac{1}{3}e^{-2\varphi} + l^2)^2}$$

which is in agreement with [68], and minimally couple to the potential $\mathcal{A}^{(10)}$.

The next brane we refer to as the IIB r9-brane. The (double dimensional reduction of the) charge of this brane can be calculated by T-dualising the IIA r9-brane charge (11.5) along β. Doing this yields

$$\begin{aligned} N^{(r9)} &= le^{-\varphi}\Phi^{(9)} + \frac{1}{3}e^{-2\varphi}\Omega^- + \frac{1}{3}e^{-\varphi}\Phi^{(7)} \wedge \mathcal{C}^{(2)} + \frac{2}{3}le^{-\varphi}\Phi^{(7)} \wedge \mathcal{B} \\ &+\frac{1}{6}(e^{-2\varphi}\Sigma^- + le^{-\varphi}\Phi^{(5)}) \wedge (\mathcal{B})^2 + \frac{1}{3}e^{-\varphi}\Phi^{(5)} \wedge \mathcal{C}^{(2)} \wedge \mathcal{B} \\ &+\frac{1}{6}e^{-\varphi}\Phi^{(3)} \wedge \mathcal{C}^{(2)} \wedge (\mathcal{B})^2 \\ &+(e^{-\varphi}\Phi^{(1)} - lK^-) \wedge (\frac{1}{3}\mathcal{C}^{(8)} - \frac{1}{3}\mathcal{C}^{(6)} \wedge \mathcal{B} + \frac{1}{6}\mathcal{C}^{(4)} \wedge (\mathcal{B})^2) \\ &+K^- \wedge (-\frac{2}{3}\varphi^{(8)} - \frac{1}{3}\mathcal{B}^{(6)} \wedge \mathcal{B} + \frac{1}{12}(\mathcal{C}^{(2)})^2 \wedge (\mathcal{B})^2) + i_{K^+}\mathcal{B}^{(10)} \\ &-\frac{2}{3}i_{K^+}\varphi^{(8)} \wedge \mathcal{B} + \frac{1}{3}\mathcal{C}^{(8)} \wedge i_{K^+}\mathcal{C}^{(2)} - \frac{1}{6}i_{K^+}\mathcal{B}^{(6)} \wedge (\mathcal{B})^2 \\ &-\frac{1}{3}\mathcal{C}^{(6)} \wedge i_{K^+}\mathcal{C}^{(2)} \wedge \mathcal{B} + \frac{1}{6}\mathcal{C}^{(4)} \wedge i_{K^+}\mathcal{C}^{(2)} \wedge (\mathcal{B})^2. \end{aligned} \tag{12.4}$$

From this charge we see that the tension of the IIB r9-brane scales as

$$e^{-\varphi}\sqrt{\frac{1}{9}e^{-2\varphi}+l^2}$$

which is in agreement with [68], and minimally couple to the potential $\mathcal{B}^{(10)}$.

We refer to the final brane in the quadruplet as the IIB q9-brane. The (double dimensional reduction of the) charge of this brane can be calculated by T-dualising the IIA q9-brane charge (11.7) along β. Doing this yields

$$\begin{aligned} N^{(q9)} \;=\;& (e^{-4\varphi}+l^2e^{-2\varphi})\Omega^- + (le^{-3\varphi}+l^3e^{-\varphi})\Phi^{(9)} + (e^{-3\varphi}+l^2e^{-\varphi})\Phi^{(7)}\wedge\mathcal{C}^{(2)} \\ &+\frac{1}{2}(e^{-2\varphi}\Sigma^- + le^{-\varphi}\Phi^{(5)})\wedge(\mathcal{C}^{(2)})^2 + \frac{1}{6}e^{-\varphi}\Phi^{(3)}\wedge(\mathcal{C}^{(2)})^3 \\ &+(e^{-\varphi}\Phi^{(1)} - lK^-)\wedge(-\mathcal{N}^{(8)}+\mathcal{B}^{(6)}\wedge\mathcal{C}^{(2)}) + \frac{1}{24}K^-\wedge(\mathcal{C}^{(2)})^4 \\ &-i_{K^+}\mathcal{D}^{(10)} + i_{K^+}\mathcal{N}^{(8)}\wedge\mathcal{C}^{(2)} - \frac{1}{2}i_{K^+}\mathcal{B}^{(6)}\wedge(\mathcal{C}^{(2)})^2. \end{aligned} \tag{12.5}$$

From this charge we see that the tension of the IIB q9-brane scales as

$$e^{-\varphi}(e^{-2\varphi}+l^2)^{3/2}$$

which is in agreement with [68], and minimally couple to the potential $\mathcal{D}^{(10)}$.

In [68] it was shown that there were two constraints which restrict which combinations of these branes can be supersymmetrically coupled to the IIB supergravity action. Such constraints are not deducible from the structural form of the above charges. Furthermore, when dealing with spacetime filling branes the total charge must vanish for consistency which involves some $\mathcal{N}=1$ truncation. It is interesting to note that no such truncation is required in order for the 9-brane charges to be closed. This is presumably related to the fact that the gauge algebra, and therefore the charge structures, seem to be independent of the dimensionality of the background spacetime, as already noticed due to the existence of $\hat{L}^{(11)}_{abc}$ in $D=11$.

12.1.3 Doublet of 9-brane charges

We now consider the charges for the doublet of 9-branes. These are obtained by T-dualising the IIA charges which are produced from the dimensional reduction of (10.61). We first perform a double T-duality transformation on the IIA t9-brane charge (11.8) using the following rule for the IIA potential

$$i_\beta \overline{A}^{(10)}_{\bar{\mu}_1\ldots\bar{\mu}_9} \to i_\beta \overline{\mathcal{A}}^{(10)}_{\bar{\mu}_1\ldots\bar{\mu}_9} \tag{12.6}$$

where we have split the co-ordinates in the same fashion as done in Chapter 5. The result is then (the double dimensional reduction of)

$$N^{(t9)} = e^{-2\varphi}\Omega^- - i_{K^+}\overline{\mathcal{A}}^{(10)}. \tag{12.7}$$

We see that the brane that this charge corresponds to, which we will refer to as the IIB t9-brane, has a tension that scales as $e^{-2\varphi}$. This is in agreement with the observations of [68].

Next we T-dualise the IIA u9-brane charge (11.9) along β using the following rule for the IIA potential

$$i_\beta \overline{B}^{(10)}_{\bar{\mu}_1\ldots\bar{\mu}_9} \to i_\beta \overline{\mathcal{B}}^{(10)}_{\bar{\mu}_1\ldots\bar{\mu}_9}. \tag{12.8}$$

The result is

$$N^{(u9)} = e^{-3\varphi}\Phi^{(9)} - le^{-2\varphi}\Omega^- + i_{K^+}\overline{\mathcal{B}}^{(10)}. \tag{12.9}$$

We see that the brane that this charge corresponds to, which we will refer to as the IIB u9-brane, has a tension that scales as

$$e^{-2\varphi}\sqrt{e^{-2\varphi}+l^2}$$

in agreement with [68].

12.2 $SL(2,\mathbb{R})$ covariant IIB charges

We now consider the $SL(2,\mathbb{R})$ covariant version of IIB and show that the charges transform in the same multiplets as the states they correspond to. This involves re-expressing the fields, bilinears and charges given so far in this thesis in an $SL(2,\mathbb{R})$ covariant fashion. The 1, 3 and 5-form field equations have been given previously in [43] and the 7, 9 and 11-form ones have been given in an $SU(1,1)$ covariant form in [67, 85].

So far in this thesis we have been working with the string frame metric, however the $SL(2,\mathbb{R})$ covariant theory is more naturally expressed in the Einstein frame. For the generalised charges the only effect this has is a scaling of the bilinears due to their construction from Γ matrices. Since this offers little simplification we will remain working in the string frame to avoid excessive changes of notation, and simply write the bilinear multiplets in terms of the string frame bilinears.

The bilinear multiplets can be determined by calculating their transformations under the discrete S-duality transformation. This is done by mapping a given bilinear to $D = 11$, performing the transformation $\hat{k}_1 \to -\hat{k}_2$, $\hat{k}_2 \to \hat{k}_1$, then mapping back to IIB. We find that the bilinears fall into groups that transform according to essentially the same rule. To demonstrate we temporarily denote the three bilinears K^-, Σ^- and Ω^- by the symbol $Y^{(p)}$ for $p = 1, 5, 9$ respectively. We then find that for $p = 1, 5, 9$ we have

$$\begin{aligned} \Phi^{(p)} &\to |\lambda|^{\frac{p-1}{2}}(-e^{-\varphi}Y^{(p)} - l\Phi^{(p)}) \\ Y^{(p)} &\to |\lambda|^{\frac{p-1}{2}}(e^{-\varphi}\Phi^{(p)} - lY^{(p)}) \end{aligned} \tag{12.10}$$

whereas for $p = 3, 7$ we get

$$\Phi^{(p)} \to |\lambda|^{\frac{p+1}{2}}\Phi^{(p)} \tag{12.11}$$

where $\lambda = l + ie^{-\varphi}$ is the axion-dilaton. The only other bilinear transformation we need is for K^+ with its spacetime index up, which appears in the charges contracted with various gauge potentials. This can be shown to transform as a scalar. Note that also this object is invariant when going between the Einstein and string frame since the scaling of both K^+_μ and the inverse metric cancel.

The scalar fields φ and l parametrise the coset $SL(2,\mathbb{R})/SO(2)$ and transform as the following matrix [43]

$$\mathcal{M}_{ab} = e^{\varphi}\begin{pmatrix} |\lambda|^2 & l \\ l & 1 \end{pmatrix}. \tag{12.12}$$

From this it can be determined that under the discrete S-duality transformations the scalars obey

$$e^{-\varphi} \to \frac{e^{-\varphi}}{|\lambda|^2} \qquad l \to \frac{-l}{|\lambda|^2}. \tag{12.13}$$

12.2.1 Doublet of 1-brane charges

This multiplet consists of the D1-brane and the F-string, the non-covariant generalised charges of which are given by (4.23) for $n = 0$ and (4.26) respectively. The leading bilinear terms are $e^{-\varphi}\Phi^{(1)} - lK^{-}$ and K^{-} respectively. From (12.10) it can be shown that these transform as a doublet, specifically we define

$$K_a = \begin{pmatrix} e^{-\varphi}\Phi^{(1)} - lK^{-} \\ -K^{-} \end{pmatrix}. \tag{12.14}$$

The two 3-form field strengths and 2-form potentials form the doublets

$$\mathcal{H}_a = \begin{pmatrix} \mathcal{F}^{(3)} + l\mathcal{H} \\ \mathcal{H} \end{pmatrix} \qquad \mathcal{B}_a = \begin{pmatrix} \mathcal{C}^{(2)} \\ \mathcal{B} \end{pmatrix} \tag{12.15}$$

and the field equations become simply

$$\mathcal{H}_a = d\mathcal{B}_a. \tag{12.16}$$

We then find that the 1-brane charges can be written as the following doublet

$$N_a^{(1)} = K_a + i_{K^+}\mathcal{B}_a = \begin{pmatrix} N^{(1)} \\ -N^{(F1)} \end{pmatrix}. \tag{12.17}$$

It is trivial to check that this expression is generally closed, however we delay giving the required covariant form of the bilinear differential relations until Section 12.3 when we discuss how the presence of the mass parameters on the IIA side are dealt with in IIB.

12.2.2 D3-brane charge singlet

The D3-brane transforms as a singlet under $SL(2,\mathbb{R})$, and the non-covariant charge is given by (4.23) for $n = 1$. The leading bilinear term here is $e^{-\varphi}\Phi^{(3)}$ which from (12.11) can be seen to transform as a scalar. We therefore define $\tilde{\Phi}^{(3)} = e^{-\varphi}\Phi^{(3)}$ which is actually equivalent to the Einstein frame definition of the bilinear.

The 5-form field equation is also an $SL(2,\mathbb{R})$ scalar which can be seen by making the following redefinition of the RR 4-form

$$\mathcal{C}^{(4)} \to \tilde{\mathcal{C}}^{(4)} + \frac{1}{2}\mathcal{C}^{(2)} \wedge \mathcal{B}. \tag{12.18}$$

We then get the following field equation

$$\mathcal{F}^{(5)} = d\tilde{\mathcal{C}}^{(4)} - \frac{1}{2}\epsilon^{ab}\mathcal{B}_a \wedge d\mathcal{B}_b \tag{12.19}$$

where $\epsilon^{12} = +1$. Given these definitions we can write the D3-brane charge as the following $SL(2,\mathbb{R})$ scalar

$$N^{(3)} = \tilde{\Phi}^{(3)} - i_{K^+}\tilde{\mathcal{C}}^{(4)} + \epsilon^{ab}K_a \wedge \mathcal{B}_b + \frac{1}{2}\epsilon^{ab}i_{K^+}\mathcal{B}_a \wedge \mathcal{B}_b. \tag{12.20}$$

12.2.3 Doublet of 5-brane charges

Next we consider the D5-brane and NS5-brane which form an $SL(2,\mathbb{R})$ doublet. The non-covariant forms of the charges are given by (4.23) for $n = 2$ and (4.30) respectively. The respective leading bilinear terms in these instances are $e^{-\varphi}\Phi^{(5)}$ and $e^{-2\varphi}\Sigma^{-} + e^{-\varphi}l\Phi^{(5)}$. We then define the doublet

$$\Sigma_a = \begin{pmatrix} e^{-2\varphi}\Sigma^{-} + e^{-\varphi}l\Phi^{(5)} \\ e^{-\varphi}\Phi^{(5)} \end{pmatrix} \tag{12.21}$$

which can be checked to transform correctly using (12.10). Note that this doublet has a qualitatively different structure to (12.14). The two types of doublet structure can however be mapped to one another using ${\mathcal{M}_a}^b$.

The two 7-form field strengths and 6-form potentials can also be written as a doublet. To show this we need to make the following field redefinitions

$$\mathcal{C}^{(6)} \quad \rightarrow \quad -\tilde{\mathcal{C}}^{(6)} + \frac{1}{6}\mathcal{C}^{(2)} \wedge (\mathcal{B})^2 \tag{12.22}$$

$$\mathcal{B}^{(6)} \quad \rightarrow \quad \tilde{\mathcal{B}}^{(6)} + \tilde{\mathcal{C}}^{(4)} \wedge \mathcal{C}^{(2)} + \frac{1}{6}\mathcal{B} \wedge (\mathcal{C}^{(2)})^2. \tag{12.23}$$

The doublets are then given by

$$\mathcal{H}_a^{(7)} = \begin{pmatrix} \mathcal{H}^{(7)} - l\mathcal{F}^{(7)} \\ -\mathcal{F}^{(7)} \end{pmatrix} \qquad \mathcal{B}_a^{(6)} = \begin{pmatrix} \tilde{\mathcal{B}}^{(6)} \\ \tilde{\mathcal{C}}^{(6)} \end{pmatrix} \tag{12.24}$$

from which we can write the field equations as

$$\mathcal{H}_a^{(7)} = d\mathcal{B}_a^{(6)} + \tilde{\mathcal{C}}^{(4)} \wedge d\mathcal{B}_a + \frac{1}{6}\epsilon^{bc}\mathcal{B}_a \wedge \mathcal{B}_b \wedge d\mathcal{B}_c. \tag{12.25}$$

Using these definitions the D5-brane charge and the NS5-brane charge can be written as the following doublet

$$\begin{aligned} N_a^{(5)} &= \Sigma_a + \tilde{\Phi}^{(3)} \wedge \mathcal{B}_a + K_a \wedge \tilde{\mathcal{C}}^{(4)} + \frac{1}{2}\epsilon^{bc} K_b \wedge \mathcal{B}_c \wedge \mathcal{B}_a - i_{K^+}\mathcal{B}_a^{(6)} \\ &\quad - i_{K^+}\tilde{\mathcal{C}}^{(4)} \wedge \mathcal{B}_a + \frac{1}{6}\epsilon^{bc} i_{K^+}\mathcal{B}_b \wedge \mathcal{B}_c \wedge \mathcal{B}_a \\ &= \begin{pmatrix} N^{(NS5)} \\ N^{(5)} \end{pmatrix}. \end{aligned} \tag{12.26}$$

12.2.4 Triplet of 7-brane charges

We next consider the triplet of 7-branes which is comprised of the D7-brane, NS7-brane and r7-brane. As we have already mentioned it is often stated that these branes are an example of a discrepancy between the charges appearing in the SUSY algebra and the spectrum of BPS states that couple to the theory. While this is true for the flatspace SUSY algebra where the 7-form charge transforms as a singlet, we now show that the generalised charges transform as a triplet.

The D7-brane generalised charge is given by (4.23) for $n = 3$ and NS7-brane and r7-brane charges are given by (12.1) and (12.2) respectively. The leading bilinear terms in all cases involve the 7-form bilinear $\Phi^{(7)}$ and some scalar factor consisting of the axion and dilaton. From (12.11) we see that the combination $\tilde{\Phi}^{(7)} = e^{-2\varphi}\Phi^{(7)}$ is an $SL(2,\mathbb{R})$ scalar. To form a term that transforms as a triplet we must use $\mathcal{M}_{ab}$. It is the presence of this term, which is neglected for flat spacetimes, which causes the discrepancy between the flatspace SUSY algebra and the brane multiplet.

The three 8-form potentials $\mathcal{N}^{(8)}$, $\varphi^{(8)}$ and the RR potential $\mathcal{C}^{(8)}$ form a triplet. The field strength equations are given by (9.26), (9.25) along with the standard RR field strength. In order to express these as a triplet we need to make the following redefinitions

$$\varphi^{(8)} \to \tilde{\varphi}^{(8)} + \frac{1}{2}\mathcal{C}^{(2)} \wedge \tilde{\mathcal{C}}^{(6)} - \frac{1}{48}(\mathcal{C}^{(2)})^2 \wedge (\mathcal{B})^2 \tag{12.27}$$

$$\mathcal{C}^{(8)} \to \tilde{\mathcal{C}}^{(8)} + \frac{1}{24}\mathcal{C}^{(2)} \wedge (\mathcal{B})^3 \tag{12.28}$$

$$\mathcal{N}^{(8)} \to -\tilde{\mathcal{N}}^{(8)} + \frac{1}{2}\tilde{\mathcal{C}}^{(4)} \wedge (\mathcal{C}^{(2)})^2 + \frac{1}{8}(\mathcal{C}^{(2)})^3 \wedge \mathcal{B}. \tag{12.29}$$

We then define the following triplet structures

$$\mathcal{F}^{(9)}_{ab} = \begin{pmatrix} 2l\mathcal{H}^{(9)} + (l^2 - e^{-2\varphi})\mathcal{F}^{(9)} & \mathcal{H}^{(9)} + l\mathcal{F}^{(9)} \\ \mathcal{H}^{(9)} + l\mathcal{F}^{(9)} & \mathcal{F}^{(9)} \end{pmatrix} \tag{12.30}$$

$$\mathcal{C}^{(8)}_{ab} = \begin{pmatrix} \tilde{\mathcal{N}}^{(8)} & \tilde{\varphi}^{(8)} \\ \tilde{\varphi}^{(8)} & \tilde{\mathcal{C}}^{(8)} \end{pmatrix}. \tag{12.31}$$

The 9-form field equations can then be written as

$$\mathcal{F}^{(9)}_{ab} = d\mathcal{C}^{(8)}_{ab} + \mathcal{B}^{(6)}_{(a} \wedge d\mathcal{B}_{b)} - \frac{1}{4!}\epsilon^{cd}\mathcal{B}_a \wedge \mathcal{B}_b \wedge \mathcal{B}_c \wedge d\mathcal{B}_d \tag{12.32}$$

with the constraint

$$\mathcal{M}^{ab}\mathcal{F}^{(9)}_{ab} = 0 \tag{12.33}$$

which follows from (12.30) and demonstrates that there are only two independent 9-form field strengths. We use the following convention to raise $SL(2,\mathbb{R})$ indices

$$\epsilon^{ac}\mathcal{M}_{cb} = \mathcal{M}^a{}_b. \tag{12.34}$$

The 7-form charges (12.1) and (12.2) together with the D7-brane charge can then be seen to form the following triplet

$$\begin{aligned} N^{(7)}_{ab} &= \mathcal{M}_{ab}\tilde{\Phi}^{(7)} + \Sigma_{(a} \wedge \mathcal{B}_{b)} + \frac{1}{2}\tilde{\Phi}^{(3)} \wedge \mathcal{B}_a \wedge \mathcal{B}_b + \frac{1}{6}\epsilon^{cd}K_c \wedge \mathcal{B}_d \wedge \mathcal{B}_a \wedge \mathcal{B}_b \\ &\quad + K_{(a} \wedge \mathcal{B}^{(6)}_{b)} + K_{(a} \wedge \tilde{C}^{(4)} \wedge \mathcal{B}_{b)} - i_{K^+}\mathcal{B}^{(6)}_{(a} \wedge \mathcal{B}_{b)} - i_{K^+}\mathcal{C}^{(8)}_{ab} \\ &\quad - \frac{1}{2}i_{K^+}\tilde{\mathcal{C}}^{(4)} \wedge \mathcal{B}_a \wedge \mathcal{B}_b + \frac{1}{4!}\epsilon^{cd}i_{K^+}\mathcal{B}_c \wedge \mathcal{B}_d \wedge \mathcal{B}_a \wedge \mathcal{B}_b \\ &= \begin{pmatrix} N^{(NS7)} & N^{(r7)} \\ N^{(r7)} & N^{(7)} \end{pmatrix}. \end{aligned} \tag{12.35}$$

12.2.5 Quadruplet of 9-brane charges

We now discuss the quadruplet of 9-branes which was discussed in [68]. It consists of the D9-brane along with other branes that we refer to as the r9, s9 and q9-branes. The non-covariant form of the D9-brane charge is given by (4.23) for $n = 4$, and the other three in Section 12.1.2. In each case the leading bilinear terms consist of combinations of $\Phi^{(9)}$ and Ω^- with factors made up form the dilaton and axion.

Using (12.11) we define the following bilinear doublet

$$\Omega_a = \begin{pmatrix} e^{-3\varphi}\Omega^- + e^{-2\varphi} l \Phi^{(9)} \\ e^{-2\varphi}\Phi^{(9)} \end{pmatrix} . \tag{12.36}$$

We then redefine the 10-form potentials according to

$$\mathcal{A}^{(10)} \;\rightarrow\; \tilde{\mathcal{A}}^{(10)} - \frac{2}{3}\tilde{\varphi}^{(8)} \wedge \mathcal{C}^{(2)} - \frac{1}{6}\tilde{\mathcal{C}}^{(6)} \wedge (\mathcal{C}^{(2)})^2 + \frac{1}{360}(\mathcal{C}^{(2)})^3 \wedge (\mathcal{B})^2 \tag{12.37}$$

$$\mathcal{B}^{(10)} \;\rightarrow\; \tilde{\mathcal{B}}^{(10)} - \frac{1}{3}\tilde{\mathcal{C}}^{(8)} \wedge \mathcal{C}^{(2)} - \frac{1}{360}(\mathcal{C}^{(2)})^2 \wedge (\mathcal{B})^3 \tag{12.38}$$

$$\mathcal{C}^{(10)} \;\rightarrow\; \tilde{\mathcal{C}}^{(10)} + \frac{1}{5!}\mathcal{C} \wedge (\mathcal{B})^4 \tag{12.39}$$

$$\mathcal{D}^{(10)} \;\rightarrow\; -\tilde{\mathcal{D}}^{(10)} + \frac{1}{6}\tilde{\mathcal{C}}^{(4)} \wedge (\mathcal{C}^{(2)})^3 + \frac{1}{20}\mathcal{B} \wedge (\mathcal{C}^{(2)})^4 . \tag{12.40}$$

Then using (C.14)-(C.17) we can write the quadruplet of 11-form field strength equations as

$$\mathcal{F}^{(11)}_{abc} = d\mathcal{C}^{(10)}_{abc} - \mathcal{C}^{(8)}_{(ab} \wedge d\mathcal{B}_{c)} - \frac{1}{5!}\epsilon^{de}\mathcal{B}_a \wedge \mathcal{B}_b \wedge \mathcal{B}_c \wedge \mathcal{B}_d \wedge d\mathcal{B}_e \tag{12.41}$$

where we have defined the components of the quadruplet of field strengths $\mathcal{F}^{(11)}_{abc}$ as

$$\begin{aligned} \mathcal{F}^{(11)}_{111} &= l^3\mathcal{F}^{(11)} + 2l^2\mathcal{H}^{(11)} + 2l\mathcal{X}^{(11)} - \mathcal{G}^{(11)} \\ \mathcal{F}^{(11)}_{112} &= l^2\mathcal{F}^{(11)} + \frac{4}{3}l\mathcal{H}^{(11)} + \frac{2}{3}\mathcal{X}^{(11)} \\ \mathcal{F}^{(11)}_{122} &= l\mathcal{F}^{(11)} + \frac{2}{3}\mathcal{H}^{(11)} \\ \mathcal{F}^{(11)}_{222} &= \mathcal{F}^{(11)} \end{aligned} \tag{12.42}$$

and potentials $\mathcal{C}^{(10)}_{abc}$ as

$$\begin{aligned} \mathcal{C}^{(10)}_{111} &= \tilde{\mathcal{D}}^{(10)} \qquad & \mathcal{C}^{(10)}_{112} &= \tilde{\mathcal{A}}^{(10)} \\ \mathcal{C}^{(10)}_{122} &= \tilde{\mathcal{B}}^{(10)} \qquad & \mathcal{C}^{(10)}_{222} &= \tilde{\mathcal{C}}^{(10)} . \end{aligned} \tag{12.43}$$

The quadruplet of 9-form charges is then given by

$$\begin{aligned} N^{(9)}_{abc} = \;& \mathcal{M}_{(ab}\Omega_{c)} + \mathcal{M}_{(ab}\tilde{\Phi}^{(7)} \wedge \mathcal{B}_{c)} + \frac{1}{2}\Sigma_{(a} \wedge \mathcal{B}_b \wedge \mathcal{B}_{c)} \\ & + \frac{1}{3!}\tilde{\Phi}^{(3)} \wedge \mathcal{B}_{(a} \wedge \mathcal{B}_b \wedge \mathcal{B}_{c)} + \frac{1}{4!}\epsilon^{de}K_d \wedge \mathcal{B}_e \wedge \mathcal{B}_a \wedge \mathcal{B}_b \wedge \mathcal{B}_c \\ & + K_{(a} \wedge \mathcal{C}^{(8)}_{bc)} + K_{(a} \wedge \mathcal{B}^{(6)}_b \wedge \mathcal{B}_{c)} + \frac{1}{2}K_{(a} \wedge \tilde{\mathcal{C}}^{(4)} \wedge \mathcal{B}_b \wedge \mathcal{B}_{c)} \\ & + i_{K^+}\mathcal{C}^{(10)}_{abc} - i_{K^+}\mathcal{C}^{(8)}_{(ab} \wedge \mathcal{B}_{c)} - \frac{1}{2}i_{K^+}\mathcal{B}^{(6)}_{(a} \wedge \mathcal{B}_b \wedge \mathcal{B}_{c)} \\ & - \frac{1}{3!}i_{K^+}\tilde{\mathcal{C}}^{(4)} \wedge \mathcal{B}_{(a} \wedge \mathcal{B}_b \wedge \mathcal{B}_{c)} + \frac{1}{5!}\epsilon^{ab}i_{K^+}\mathcal{B}_d \wedge \mathcal{B}_e \wedge \mathcal{B}_{(a} \wedge \mathcal{B}_b \wedge \mathcal{B}_{c)} \end{aligned} \tag{12.44}$$

where we have

$$\begin{aligned} N^{(9)}_{111} &= N^{(q9)} & N^{(9)}_{112} &= N^{(s9)} \\ N^{(9)}_{122} &= N^{(r9)} & N^{(9)}_{222} &= N^{(9)}. \end{aligned} \tag{12.45}$$

12.2.6 Doublet of 9-brane charges

It is a simple task to show that the doublet of 9-branes (12.7) and (12.9) can be written as a doublet. The 10-form potentials transform as a doublet as they are. We therefore define

$$\mathcal{C}^{(10)}_a = \begin{pmatrix} \overline{\mathcal{B}}^{(10)} \\ \overline{\mathcal{A}}^{(10)} \end{pmatrix} \tag{12.46}$$

which gives rise to the field equation

$$\mathcal{F}^{(11)}_a = d\mathcal{C}^{(10)}_a \tag{12.47}$$

which agrees with the field equations implicitly given in [67]. Then using (12.36) we can write the charges as

$$N^{(9)}_a = \mathcal{M}_a{}^b \Omega_b - i_{K^+}\mathcal{C}^{(10)}_a = \begin{pmatrix} -N^{(u9)} \\ N^{(t9)} \end{pmatrix}. \tag{12.48}$$

12.2.7 KK-monopole singlet

Finally we consider the KK-monopole which transforms as a singlet and which has a charge given by (6.27). In this case the $SL(2,\mathbb{R})$ transformation of the leading bilinear terms does not follow from (12.10) or (12.11). However explicit calculation reveals that

$$\tilde{\Sigma}^+ = e^{-2\varphi}\mathcal{R}^2\Sigma^+ - e^{-2\varphi} i_\beta \Sigma^+ \wedge \beta \tag{12.49}$$

transforms as a singlet.

The potential that the KK-monopole minimally couples to, $i_\beta \mathcal{N}^{(7)}$, can be shown to transform as a scalar. However for a more complete treatment we consider the full potential $\mathcal{N}^{(7)}$. In this case we see from (9.15) and (12.32) that this potential only transforms as a singlet when combined with $i_\beta\tilde{\varphi}^{(8)}$. After making the redefinition

$$\begin{aligned}\mathcal{N}^{(7)} \quad \rightarrow \quad & \tilde{\mathcal{N}}^{(7)} + \frac{1}{2}i_\beta\tilde{\mathcal{C}}^{(6)}\wedge\mathcal{C}^{(2)} - \frac{1}{2}\tilde{\mathcal{C}}^{(6)}\wedge i_\beta\mathcal{C}^{(2)} + \frac{1}{4}i_\beta\tilde{\mathcal{C}}^{(4)}\wedge\mathcal{C}^{(2)}\wedge\mathcal{B} \\ & +\frac{1}{16}i_\beta\mathcal{C}^{(2)}\wedge\mathcal{C}^{(2)}\wedge(\mathcal{B})^2 + \frac{1}{48}i_\beta\mathcal{B}\wedge\mathcal{B}\wedge(\mathcal{C}^{(2)})^2 \end{aligned} \tag{12.50}$$

we find that the combination $\tilde{\mathcal{N}}^{(7)} - \frac{1}{2}i_\beta\tilde{\varphi}^{(8)}$ forms a singlet. We can then write

$$\begin{aligned}\tilde{\mathcal{G}}^{(8)} \quad = \quad & d(\tilde{\mathcal{N}}^{(7)} - \frac{1}{2}i_\beta\tilde{\varphi}^{(8)}) - \frac{3}{4}\epsilon^{ab}\mathcal{B}^{(6)}_a\wedge i_\beta d\mathcal{B}_b + \frac{1}{4}\epsilon^{ab}i_\beta\mathcal{B}^{(6)}_a\wedge d\mathcal{B}_b + \frac{1}{2}i_\beta d\tilde{\mathcal{C}}^{(4)}\wedge\tilde{\mathcal{C}}^{(4)} \\ & +\frac{1}{4}\epsilon^{ab}i_\beta\tilde{\mathcal{C}}^{(4)}\wedge\mathcal{B}_a\wedge d\mathcal{B}_b - \frac{1}{4}\epsilon^{ab}\tilde{\mathcal{C}}^{(4)}\wedge i_\beta(\mathcal{B}_a\wedge d\mathcal{B}_b) \\ & -\frac{1}{4!}\epsilon^{ab}\epsilon^{cd}i_\beta\mathcal{B}_a\wedge\mathcal{B}_b\wedge\mathcal{B}_c\wedge d\mathcal{B}_d \end{aligned} \tag{12.51}$$

where we have defined the field strength singlet

$$\tilde{\mathcal{G}}^{(8)} = \mathcal{G}^{(8)} + \frac{1}{2}i_\beta\mathcal{H}^{(9)}. \tag{12.52}$$

It is interesting to note that $i_\beta\tilde{\varphi}^{(8)}$ appears both here as part of a singlet and also in (12.31) as part of a triplet.

We find that the KK-monopole charge can then be expressed as the following singlet

$$\begin{aligned} N^{(KK)} \quad = \quad & \tilde{\Sigma}^{+} - i_{K+\beta}\tilde{\mathcal{N}}^{(7)} - \epsilon^{ab}i_\beta\Sigma_a\wedge i_\beta\mathcal{B}_b - \frac{1}{2}\epsilon^{ab}i_\beta(\tilde{\Phi}^{(3)}\wedge\mathcal{B}_a)\wedge i_\beta\mathcal{B}_b \\ & +i_\beta\tilde{\Phi}^{(3)}\wedge i_\beta\tilde{\mathcal{C}}^{(4)} - \epsilon^{ab}i_\beta K_a\wedge i_\beta\mathcal{B}^{(6)}_b - \epsilon^{ab}i_\beta(K_a\wedge\tilde{\mathcal{C}}^{(4)})\wedge i_\beta\mathcal{B}_b \\ & -\frac{1}{3!}\epsilon^{ab}\epsilon^{cd}i_\beta(K_a\wedge\mathcal{B}_b\wedge\mathcal{B}_c)\wedge i_\beta\mathcal{B}_d - \epsilon^{ab}i_{K+\beta}\mathcal{B}^{(6)}_a\wedge i_\beta\mathcal{B}_b \\ & +\frac{1}{2}i_{K+\beta}\tilde{\mathcal{C}}^{(4)}\wedge i_\beta\tilde{\mathcal{C}}^{(4)} + \frac{1}{2}\epsilon^{ab}i_\beta(i_{K^+}\tilde{\mathcal{C}}^{(4)}\wedge\mathcal{B}_a)\wedge i_\beta\mathcal{B}_b \\ & -\frac{1}{4!}\epsilon^{ab}\epsilon^{cd}i_\beta(i_{K^+}\mathcal{B}_a\wedge\mathcal{B}_b\wedge\mathcal{B}_c)\wedge i_\beta\mathcal{B}_d. \end{aligned} \tag{12.53}$$

12.3 $SL(2,\mathbb{R})$ T-duality and 'massive' IIB

We now discuss how the IIB theory is related to the non-covariant massive IIA theory, and hence also to the $SL(2,\mathbb{R})$ covariant $D=11$ theory, through T-duality. Our first motivation for doing this is to justify the use of the massless T-duality rules to map between the IIA and IIB charges as done in the previous chapters. A second reason is to show how $D = 11$ field equations are related to those given in IIB which provides a check on the massive terms.

As discussed at the beginning of Chapter 5 one may incorporate the mass parameters into the T-duality via the Scherk-Schwarz mechanism [69], see for example [43]. In this scheme the IIB potentials are multi-valued along the compact T-duality isometry direction with the different values being related by a global $SL(2,\mathbb{R})$ transformation where the mass parameters correspond to the generators of this transformation. Here however we follow a different approach, also discussed in [43], where the IIB potentials are taken to be *independent* of the T-duality direction and the information corresponding to the IIA masses is instead encoded via a modification to the derivative operator. This modification can be interpreted as a gauging of the $SL(2,\mathbb{R})$ symmetry using a gauge field that takes values of the mass parameters in one direction only. For the most part this corresponds to making the following substitution for the derivative operator acting on an arbitrary p-form $Y^{(p)}_{a_1\ldots a_n}$

$$dY^{(p)}_{a_1\ldots a_n} \to DY^{(p)}_{a_1\ldots a_n} = dY^{(p)}_{a_1\ldots a_n} + nm_{(a_1}{}^{b}dy \wedge Y^{(p)}_{a_2\ldots a_n)b} \tag{12.54}$$

where

$$m_a{}^b = \begin{pmatrix} m_1 & m_+ \\ m_- & -m_1 \end{pmatrix} \tag{12.55}$$

is the mass matrix, which also acts as the generator of an $SL(2,\mathbb{R})$ transformation. Here y is the co-ordinate that parameterises the T-duality isometry direction. An advantage of this scheme is that the mass parameters are made explicit through this modification.

Making this substitution effectively modifies the field equations, the new ones being given by

$$
\begin{aligned}
\check{\mathcal{H}}_a &= \mathcal{H}_a + m_a{}^b \mathcal{B}_b \wedge dy && (12.56)\\
\check{\mathcal{F}}^{(5)} &= \mathcal{F}^{(5)} - \frac{1}{2} m^{ab} \mathcal{B}_a \wedge \mathcal{B}_b \wedge dy && (12.57)\\
\check{\mathcal{H}}_a^{(7)} &= \mathcal{H}_a^{(7)} + m_a{}^b (\mathcal{B}_b^{(6)} + \tilde{\mathcal{C}}^{(4)} \wedge \mathcal{B}_b) \wedge dy \\
&\quad + \frac{1}{3!} m^{bc} \mathcal{B}_b \wedge \mathcal{B}_c \wedge \mathcal{B}_a \wedge dy && (12.58)\\
\check{\mathcal{F}}_{ab}^{(9)} &= \mathcal{F}_{ab}^{(9)} + m_{(a}{}^c (2\mathcal{C}_{b)c}^{(8)} + \mathcal{B}_{b)}^{(6)} \wedge \mathcal{B}_c) \wedge dy \\
&\quad - \frac{1}{4!} m^{cd} \mathcal{B}_c \wedge \mathcal{B}_d \wedge \mathcal{B}_a \wedge \mathcal{B}_b \wedge dy && (12.59)\\
\check{\mathcal{F}}_{abc}^{(11)} &= \mathcal{F}_{abc}^{(11)} + m_{(a}{}^d (3\mathcal{C}_{bc)d}^{(10)} - \mathcal{C}_{bc)}^{(8)} \wedge \mathcal{B}_d) \wedge dy \\
&\quad - \frac{1}{5!} m^{de} \mathcal{B}_d \wedge \mathcal{B}_e \wedge \mathcal{B}_a \wedge \mathcal{B}_b \wedge \mathcal{B}_c \wedge dy && (12.60)\\
\check{\mathcal{F}}_a^{(11)} &= \mathcal{F}_a^{(11)} + m_a{}^b \mathcal{C}_b^{(10)} \wedge dy. && (12.61)
\end{aligned}
$$

Since all the fields are now assumed to be independent of the isometry direction one uses the massless rules when T-dualising the potentials. The information of the massive terms on the IIA side is now represented by the 'massive' terms produced by making the substitution (12.54), and it is in this way that the T-duality rules are modified implicitly. In order to T-dualise the field strength equations it is more convenient to work with the non-covariant fields so we give these in Appendix D. Then it is a straight forward task to check that the IIB equations map to the equations given in $SL(2,\mathbb{R})$ covariant $D = 11$ supergravity. This acts as a non-trivial check on those equations.

In order to check that the IIB charges given in the previous subsection are still closed under this scheme we now give the 'massive' $SL(2,\mathbb{R})$ covariant differential relations of the bilinears. The majority of these relations have already been given in the usual form in Chapter 4 and the remaining ones are calculated using the same technique. The massive modifications are obtained by making the above field modifications as well as the substitution (12.54). The results are then given by

$$\begin{aligned}
dK_a &= i_{K^+}\check{\mathcal{H}}_a + m_a{}^b K_b \wedge dy && (12.62)\\
d\tilde{\Phi}^{(3)} &= \epsilon^{ab} K_a \wedge \check{\mathcal{H}}_b - i_{K^+}\check{\mathcal{F}}^{(5)} && (12.63)\\
d\Sigma_a &= \tilde{\Phi}^{(3)} \wedge \check{\mathcal{H}}_a + K_a \wedge \check{\mathcal{F}}^{(5)} - i_{K^+}\check{\mathcal{F}}^{(7)}_a + m_a{}^b \Sigma_b \wedge dy && (12.64)\\
d(\mathcal{M}_{ab}\tilde{\Phi}^{(7)}) &= \Sigma_{(a} \wedge \check{\mathcal{H}}_{b)} + K_{(a} \wedge \check{\mathcal{F}}^{(7)}_{b)} - i_{K^+}\check{\mathcal{F}}^{(9)}_{ab} \\
&\quad + 2m_{(a}{}^c \mathcal{M}_{b)c}\tilde{\Phi}^{(7)} \wedge dy && (12.65)\\
d(\mathcal{M}_{(ab}\Omega_{c)}) &= \mathcal{M}_{(ab}\tilde{\Phi}^{(7)} \wedge \check{\mathcal{H}}_{c)} + K_{(a} \wedge \check{\mathcal{F}}^{(9)}_{bc)} + i_{K^+}\check{\mathcal{F}}^{(11)}_{abc} \\
&\quad + 2m_{(a}{}^d \mathcal{M}_{|d|b}\Omega_{c)} \wedge dy + m_{(a}{}^d \mathcal{M}_{bc)}\Omega_d \wedge dy && (12.66)\\
d(\mathcal{M}_a{}^b\Omega_b) &= -i_{K^+}\check{\mathcal{F}}^{(11)}_a + m_a{}^b \mathcal{M}_b{}^c \Omega_c \wedge dy. && (12.67)
\end{aligned}$$

Here the inclusion of the 11-form field strengths in the last two differential relation are conjectures along the lines of (10.60).

Finally in order for the charges to be closed we require the potentials to satisfy the following gauge conditions

$$\begin{aligned}
\mathcal{L}_{K^+}\mathcal{B}_a &= -K^{+y} m_a{}^b \mathcal{B}_b - m_a{}^b N^{(1)}_b \wedge dy && (12.68)\\
\mathcal{L}_{K^+}\tilde{\mathcal{C}}^{(4)} &= \frac{1}{2} m^{ab} N^{(1)}_a \wedge \mathcal{B}_b \wedge dy && (12.69)\\
\mathcal{L}_{K^+}\mathcal{B}^{(6)}_a &= -K^{+y} m_a{}^b \mathcal{B}^{(6)}_b + m_a{}^b N^{(5)}_b \wedge dy - \frac{1}{3} m^{bc} N^{(1)}_b \wedge \mathcal{B}_c \wedge \mathcal{B}_a \wedge dy \\
&&& (12.70)\\
\mathcal{L}_{K^+}\mathcal{C}^{(8)}_{ab} &= -2K^{+y} m_{(a}{}^c \mathcal{C}^{(8)}_{b)c} + 2m_{(a}{}^c N^{(7)}_{b)c} \wedge dy - m_{(a}{}^c N^{(5)}_{|c|} \wedge \mathcal{B}_{b)} \wedge dy \\
&\quad + \frac{1}{8} m^{cd} N^{(1)}_c \wedge \mathcal{B}_d \wedge \mathcal{B}_a \wedge \mathcal{B}_b \wedge dy && (12.71)\\
\mathcal{L}_{K^+}\mathcal{C}^{(10)}_{abc} &= -3K^{+y} m_{(a}{}^d \mathcal{C}^{(10)}_{bc)d} - 3m_{(a}{}^d N^{(9)}_{bc)d} \wedge dy + 2m_{(a}{}^d N^{(7)}_{|d|b} \wedge \mathcal{B}_{c)} \wedge dy \\
&\quad - \frac{1}{2} m_{(a}{}^d N^{(5)}_{|d|} \wedge \mathcal{B}_b \wedge \mathcal{B}_{c)} \wedge dy + \frac{1}{30} m^{de} N^{(1)}_d \wedge \mathcal{B}_e \wedge \mathcal{B}_a \wedge \mathcal{B}_b \wedge \mathcal{B}_c \\
&&& (12.72)\\
\mathcal{L}_{K^+}\mathcal{C}^{(10)}_a &= -K^{+y} m_a{}^b \mathcal{C}^{(10)}_b + m_a{}^b N^{(9)}_b \wedge dy. && (12.73)
\end{aligned}$$

These are merely the T-duals of the conditions imposed on the IIA potentials. Using these relations one can then check that the IIB charges are closed in this 'massive' theory and it is in this sense they can be said to be related through T-duality to the charges in non-covariant massive IIA. Note that the scalar K^{+y} must be closed due to the relation (5.24).

A complication arises for the KK-monopole charge in this scheme due to its dependence on the isometry. To demonstrate this we consider the potentials contracted with β. Applying the rule (12.54) in these instances creates ambiguities since different results are obtained depending on whether the contraction with β is applied before or after the substitution (12.54) is made. From T-duality we find that the correct order is to contract with β *after* the substitution (12.54) is made. Specifically we may consider the potentials $i_\beta\mathcal{B}_a$ as an example, in which case we have

$$d(i_\beta\mathcal{B}_a) = -i_\beta d\mathcal{B}_a \rightarrow -i_\beta(d\mathcal{B}_a + m_a{}^b dy \wedge \mathcal{B}_b). \tag{12.74}$$

For the KK-monopole we must consider the potential $i_\beta\tilde{\mathcal{N}}^{(7)}$. The isometry is present here not just through the contraction with β but is also intrinsic to $\tilde{\mathcal{N}}^{(7)}$. We cannot therefore remove this isometry dependence before applying the rule (12.54). It seems that for this reason the rule (12.54) is different in this instance and we find that the modified field strength is given by

$$\begin{aligned} i_\beta\check{\mathcal{G}}^{(8)} &= i_\beta\mathcal{G}^{(8)} + m^{ab}\Big[-i_\beta\mathcal{C}^{(8)}_{ab} - i_\beta\mathcal{B}^{(6)}_a \wedge i_\beta(\mathcal{B}_b \wedge dy) \\ &\quad -\frac{1}{2}i_\beta\tilde{\mathcal{C}}^{(4)} \wedge i_\beta(\mathcal{B}_a \wedge \mathcal{B}_b \wedge dy) + \frac{1}{4!}\epsilon^{cd} i_\beta\mathcal{B}_c \wedge i_\beta(\mathcal{B}_d \wedge \mathcal{B}_a \wedge \mathcal{B}_b \wedge dy)\Big] \end{aligned}$$

which is determined through T-duality (compare with (6.29)). Note the presence of the potentials $\mathcal{C}^{(8)}_{ab}$ which do not arise from making the substitution (12.54). We find a similar a situation for the differential relation of Σ^+. Specifically we have

$$\begin{aligned} d\tilde{\Sigma}^+ &= i_{K+\beta}\check{\mathcal{G}}^{(8)} - \epsilon^{ab} i_\beta K_a \wedge i_\beta\check{\mathcal{H}}^{(7)}_b - \epsilon^{ab} i_\beta\Sigma_a \wedge i_\beta\check{\mathcal{H}}_b + i_\beta\tilde{\Phi}^{(3)} \wedge i_\beta\check{\mathcal{F}}^{(5)} \\ &\quad -m^{ab}\mathcal{M}_{ab} i_\beta\tilde{\Phi}^{(7)} \end{aligned} \tag{12.75}$$

which is again determined from T-duality. Note the presence of the term involving $\tilde{\Phi}^{(7)}$ (compare with (6.28)).

Using these relations we can confirm that the KK-monopole charge, given by (6.27), is closed in this 'massive' IIB scheme. The gauge condition on $i_\beta\tilde{\mathcal{N}}^{(7)}$ is

given by

$$\begin{aligned}
\mathcal{L}_{K^+}(i_\beta \mathcal{N}^{(7)}) \;&=\; -m^{ab} i_\beta N^{(7)}_{ab} + m^{ab} i_\beta (N^{(5)}_a \wedge dy) \wedge i_\beta \mathcal{B}_b \\
&\quad + \frac{1}{4} m^{ab} i_\beta (N^{(1)}_a \wedge \mathcal{B}_b \wedge dy) \wedge i_\beta \tilde{\mathcal{C}}^{(4)} \\
&\quad + \frac{1}{8} m^{ab} \epsilon^{cd} i_\beta (N^{(1)}_a \wedge \mathcal{B}_b \wedge \mathcal{B}_c \wedge dy) \wedge i_\beta \mathcal{B}_d
\end{aligned} \tag{12.76}$$

which can be shown to T-dualise to the IIA conditions. Note that we have

$$\mathcal{L}_{K^+}(i_\beta \mathcal{N}^{(7)}) = i_\beta (\mathcal{L}_{K^+} \mathcal{N}^{(7)}) \tag{12.77}$$

since we assume independence of the isometry direction.

Chapter 13

Discussion

In this thesis we have, following the original work of [1], successfully formulated the generalised charges for the more commonly encountered BPS branes in the $D = 11$ and IIA and IIB supergravity theories. These generalise the usual topological charges found in the flatspace SUSY algebras to curved supersymmetric backgrounds, and have the alternative interpretation of generalised calibrations [57]. We found that several of the shortcomings usually attributed to the flatspace charges did not apply to the generalised charges, most notably the breakdown of the one-to-one correspondence with the branes. Additionally it was found that the structure of the generalised charges contains information about any isometry that occurs in the spacetime solutions of the branes as well as the dependence of the brane tensions on any scalars in the theory.

In order to properly investigate the M9-brane and D8-brane charges we considered massive versions of the $D = 11$ and IIA theories where it was found that the inclusion of the mass parameters has only an implicit effect on the charges by virtue of altering the gauge conditions satisfied by the potentials. Of particular interest was the $D = 11$ theory where we considered a double M9-brane background. The $SL(2, \mathbb{R})$ symmetry that arises from the pair of compact Killing isometries that are present in these backgrounds maps to the $SL(2, \mathbb{R})$ symmetry in the IIB theory through an $SL(2, \mathbb{R})$ triplet of massive $D = 9$ theories [43]. The geometric nature of this symmetry in the $D = 11$ theory allows for a systematic analysis of the types of charge multiplets that occur. The expressions $\hat{L}^{(7)}_a$ and $\hat{L}^{(9)}_{ab}$ play a key role in this process although they do not seem to correspond to the charges of any (known) branes by themselves. By explicitly mapping these charge multiplets to IIB it was shown that they exhibit the same multiplet structure as the IIB branes themselves, a trait that is not fully satisfied by the flatspace charges. In order to complete this correspondence an additional $D = 11$ doublet $\hat{L}_a^{(10)}$ was also conjectured, whose structure is unrelated to that of the other charges, and which maps to the doublet of 9-brane charges in IIB.

En route to determining the structure of these charges we first investigated the $D = 11$ gauge fields and determined their field strength equations. This included the 8-form potentials that are the magnetic duals of the Killing vectors, as well as the 10-form potentials which can be introduced into the action as auxiliary fields and whose 11-form field strengths are related to the mass parameters through Hodge duality. Once again the $SL(2, \mathbb{R})$ symmetry played a key role in this analysis and it was found that the full gauge algebra consists of an infinite tower of fields of increasing rank which in reality is truncated after the 10-form potentials due to the dimensionality of the background spacetime.

Although our analysis focused on the $SL(2, \mathbb{R})$ case, with $n = 2$ Killing vectors, the field equations should generalise to arbitrary n, with the resulting $SL(n, \mathbb{R})$ symmetry being a subgroup of the full U-duality group for n compact directions. A natural extension to the work presented in this thesis would then be to extend the analysis to other values of n and determine the charge multiplets that occur. One could then map these to the lower dimensional supergravity theories to obtain their brane spectra and then compare with previous results in the literature.

An interesting observation was that the potential $i_\beta C^{(1)}$ in IIA was treated as an independent scalar field with an 8-form magnetic dual potential. We might therefore expect new scalar fields to emerge in $D = 11$ for $n = 3, 6$ and 8. Specifically we would have $i_{\hat{k}_1\hat{k}_2\hat{k}_3}\hat{A}$, $i_{\hat{k}_1\ldots\hat{k}_6}\hat{C}$ and $i_{\hat{k}_1\ldots\hat{k}_8}\hat{N}^{(8)}$. These would then each have a 9-form dual potential given by $\hat{A}^{(9)}$, $\hat{C}^{(9)}$ and $\hat{N}^{(9)}$ respectively, which would suggest the existence of new families of branes emerging for each of these values of n. Schematically the generalised charges for these could have leading bilinear and potential terms given by

$$
\begin{aligned}
i_{\hat{k}_1\hat{k}_2\hat{k}_3}(\hat{\Sigma}\wedge\hat{k}_1\wedge\hat{k}_2\wedge\hat{k}_3)+i_{\hat{K}\hat{k}_1\hat{k}_2\hat{k}_3}\hat{A}^{(9)} \\
i_{\hat{k}_1\ldots\hat{k}_6}(\hat{\omega}\wedge\hat{k}_1\wedge\ldots\wedge\hat{k}_6)+i_{\hat{K}\hat{k}_1\ldots\hat{k}_6}\hat{C}^{(9)} \\
i_{\hat{k}_1\ldots\hat{k}_8}(i_{\hat{k}_1}\hat{K}\wedge\hat{k}_1\wedge\ldots\hat{k}_8)+i_{\hat{K}\hat{k}_1\ldots\hat{k}_8}\hat{N}^{(9)}.
\end{aligned}
\tag{13.1}
$$

The expressions for the tensions would include the terms $|\hat{k}_1|^2|\hat{k}_2|^2|\hat{k}_3|^2$, $|\hat{k}_1|^2\ldots|\hat{k}_6|^2$ and $|\hat{k}_1|^3|\hat{k}_2|^2\ldots|\hat{k}_8|^2$ respectively which is in agreement with previous results from considering the full U-duality group [94, 95]. However from the approach here we would also expect there to be terms in the tensions involving the contraction of different Killing vectors which are not found using the U-duality approach. A similar observation was made in [41] where the spacetime solutions of a series of new families of branes were calculated which share the same main features as the branes proposed here. It would be interesting to explore these states further and construct the full generalised charge expressions.

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Appendix A

IIA from $D = 11$

Here we present the dimensional reduction rules of the $D = 11$ supergravity theory to IIA. To begin with we use the non-covariant notation of Chapter 8 where $\hat{\alpha}$ is the Killing vector that generates the isometry over which the reduction is being performed. We work in a co-ordinate system adapted to $\hat{\alpha}$ and split the co-ordinates into $\{\mu_i, z\}$ where z parametrises the isometry direction and μ_i represent the other 10 directions.

The metric reduces via a scaled Kaluza Klein reduction given by

$$\hat{g}_{\hat{\mu}\hat{\nu}} \rightarrow \begin{pmatrix} e^{-\frac{2}{3}\phi} g_{\mu\nu} + e^{\frac{4}{3}\phi} C^{(1)}_\mu C^{(1)}_\nu & e^{\frac{4}{3}\phi} C^{(1)}_\mu \\ e^{\frac{4}{3}\phi} C^{(1)}_\nu & e^{\frac{4}{3}\phi} \end{pmatrix}. \tag{A.1}$$

The inverse metric then reduces according to

$$\hat{g}^{\hat{\mu}\hat{\nu}} \rightarrow \begin{pmatrix} e^{\frac{2}{3}\phi} g^{\mu\nu} & -e^{\frac{2}{3}\phi} C^{(1)\mu} \\ -e^{\frac{2}{3}\phi} C^{(1)\nu} & e^{\frac{2}{3}\phi} |C^{(1)}|^2 + e^{-\frac{4}{3}\phi} \end{pmatrix}. \tag{A.2}$$

The $D = 11$ gauge potentials and Killing vectors (with indices down) reduce according to

$$\begin{aligned}
\hat{\alpha} &\rightarrow e^{\frac{4}{3}\phi}C^{(1)} + e^{\frac{4}{3}\phi}dz && \text{(A.3)}\\
\hat{\beta} &\rightarrow e^{-\frac{2}{3}\phi}\beta + e^{\frac{4}{3}\phi}i_\beta C^{(1)}C^{(1)} + e^{\frac{4}{3}\phi}i_\beta C^{(1)}dz && \text{(A.4)}\\
\hat{A} &\rightarrow C^{(3)} + B \wedge dz && \text{(A.5)}\\
\hat{C} &\rightarrow B^{(6)} - (C^{(5)} - \frac{1}{2}C^{(3)} \wedge B) \wedge dz && \text{(A.6)}\\
\hat{N}^{(8)} &\rightarrow \frac{4}{3}\phi^{(8)} + (C^{(7)} - \frac{1}{3!}C \wedge B^2) \wedge dz && \text{(A.7)}\\
\hat{T}^{(8)} &\rightarrow N^{(8)} + (N^{(7)} - \frac{2}{3}i_\beta\phi^{(8)} - \frac{1}{3}C \wedge i_\beta C \wedge B) \wedge dz && \text{(A.8)}\\
\hat{A}^{(10)} &\rightarrow A^{(10)} + (C^{(9)} - \frac{1}{4!}C^{(3)} \wedge (B)^3) \wedge dz && \text{(A.9)}\\
\hat{B}^{(10)} &\rightarrow B^{(10)} + (B^{(9)} - \frac{1}{16}C^{(3)} \wedge i_\beta C^{(3)} \wedge (B)^2) \wedge dz && \text{(A.10)}\\
\hat{D}^{(10)} &\rightarrow D^{(10)} + (D^{(9)} - \frac{1}{8}C^{(3)} \wedge (i_\beta C^{(3)})^2 \wedge B) \wedge dz. && \text{(A.11)}
\end{aligned}$$

In addition to these we define the double dimensional reduction rules for the doublet of 11-forms as (in covariant notation)

$$\begin{aligned}
i_{\hat{\alpha}}\hat{A}_1^{(11)} &\rightarrow \overline{A}^{(10)} && \text{(A.12)}\\
i_{\hat{\alpha}}\hat{A}_2^{(11)} &\rightarrow \overline{B}^{(10)}. && \text{(A.13)}
\end{aligned}$$

The spinors reduce according to

$$\hat{\epsilon} \rightarrow e^{-\frac{1}{6}\phi}\epsilon. \tag{A.14}$$

The reduction rules for the $\hat{\Gamma}$ matrices can be inferred from their defining property (2.11) as well as the reduction rule for the metric (A.1). We then find

$$\hat{\Gamma} \rightarrow e^{-\frac{1}{3}\phi}\Gamma + e^{\frac{2}{3}\phi}\Gamma_{11} \wedge (C^{(1)} + dz). \tag{A.15}$$

From the above these we can then determine the reduction rules for the bilinear forms. These are given by

$$\hat{K} \rightarrow e^{-\frac{2}{3}\phi} K + e^{\frac{1}{3}\phi} \Psi^{(0)} \wedge (C^{(1)} + dz) \tag{A.16}$$

$$\hat{\omega} \rightarrow e^{-\phi} \Psi^{(2)} + \tilde{K} \wedge (C^{(1)} + dz) \tag{A.17}$$

$$\hat{\Sigma} \rightarrow e^{-2\phi} \Sigma + e^{-\phi} \Psi^{(4)} \wedge (C^{(1)} + dz) \tag{A.18}$$

$$\hat{\Lambda} \rightarrow e^{-\frac{7}{3}\phi} \Psi^{(6)} + e^{-\frac{4}{3}\phi} \tilde{\Sigma} \wedge (C^{(1)} + dz) \tag{A.19}$$

$$\hat{\Pi} \rightarrow e^{-\frac{10}{3}\phi} \Pi + e^{-\frac{7}{3}\phi} \Psi^{(8)} \wedge (C^{(1)} + dz) \tag{A.20}$$

$$\hat{\Upsilon} \rightarrow e^{-\frac{11}{3}\phi} \Psi^{(10)} + e^{-\frac{8}{3}\phi} \tilde{\Pi} \wedge (C^{(1)} + dz). \tag{A.21}$$

To consider the reduction rules for the generalised charges we also require the rule for $\hat{K}^{\hat{\mu}}$. This can be determined from the rule for the inverse metric (A.2) and (A.16). We then find

$$\begin{aligned} \hat{K}^{\mu} &= \hat{g}^{\mu\nu} \hat{K}_{\nu} + \hat{g}^{\mu z} \hat{K}_z \\ &\rightarrow e^{\frac{2}{3}\phi} g^{\mu\nu} (e^{-\frac{2}{3}\phi} K_{\nu} + e^{\frac{1}{3}\phi} \Psi^{(0)} C^{(1)}_{\nu}) - e^{\frac{2}{3}\phi} C^{(1)\mu} e^{\frac{1}{3}\phi} \Psi^{(0)} \\ &= K^{\mu} \end{aligned} \tag{A.22}$$

and

$$\begin{aligned} \hat{K}^{z} &= \hat{g}^{z\nu} \hat{K}_{\nu} + \hat{g}^{zz} \hat{K}_z \\ &\rightarrow -e^{\frac{2}{3}\phi} C^{(1)\nu} (e^{-\frac{2}{3}\phi} K_{\nu} + e^{\frac{1}{3}\phi} \Psi^{(0)} C^{(1)}_{\nu}) + (e^{\frac{2}{3}\phi} |C^{(1)}|^2 + e^{-\frac{4}{3}\phi}) e^{\frac{1}{3}\phi} \Psi^{(0)} \\ &= M^{(0)}. \end{aligned} \tag{A.23}$$

Applying these rules we then determine that the M2-brane (2.43) and M5-brane (2.52) charges reduce according to

$$\hat{L}^{(2)} \rightarrow M^{(2)} + M^{(F1)} \wedge dz \tag{A.24}$$

$$\hat{L}^{(5)} \rightarrow M^{(NS5)} + M^{(4)} \wedge dz \tag{A.25}$$

which is in agreement with the relationship found between the corresponding branes as we would expect. For the remaining charges we work in the $SL(2,\mathbb{R})$ covariant notation of Chapter 10 and it is to be understood that the reduction always occurs along the direction described by $\hat{k}_1$. We then have the following collection of relations

$$i_{\hat{k}_1\hat{k}_2} L_2^{(7)} \rightarrow -M^{(KK)} \tag{A.26}$$

$$i_{\hat{k}_1} \hat{L}_1^{(7)} \rightarrow M^{(6)} \tag{A.27}$$

$$i_{\hat{k}_{(1}} \hat{L}_{2)}^{(7)} \rightarrow M^{(r6)} \tag{A.28}$$

$$i_{\hat{k}_2} \hat{L}_2^{(7)} \rightarrow M^{(KK6)} \tag{A.29}$$

$$i_{\hat{k}_1} \hat{L}_{11}^{(9)} \rightarrow M^{(8)} \tag{A.30}$$

$$i_{\hat{k}_{(1}} \hat{L}_{12)}^{(9)} \rightarrow i_\beta M^{(r9)} \tag{A.31}$$

$$i_{\hat{k}_{(1}} \hat{L}_{22)}^{(9)} \rightarrow i_\beta M^{(s9)} \tag{A.32}$$

$$i_{\hat{k}_2} \hat{L}_{22}^{(9)} \rightarrow i_\beta M^{(q9)} \tag{A.33}$$

$$i_{\hat{k}_1\hat{k}_2} \hat{L}_{12}^{(9)} \rightarrow -i_\beta M^{(r8)} \tag{A.34}$$

$$i_{\hat{k}_1\hat{k}_2} \hat{L}_{22}^{(9)} \rightarrow -M^{(KK8)} \tag{A.35}$$

$$i_{\hat{k}_1} \hat{L}_1^{(10)} \rightarrow -M^{(t9)} \tag{A.36}$$

$$i_{\hat{k}_1\hat{k}_2} \hat{L}_2^{(10)} \rightarrow i_\beta M^{(u9)}. \tag{A.37}$$

Appendix B

$SL(2,\mathbb{R})$ covariant $D = 11$ KK-monopole worldvolume action

Here we construct the kinetic term of the worldvolume action of the KK-monopoles in the $SL(2,\mathbb{R})$ covariant $D = 11$ massive supergravity. The purpose here is not to study the worldvolume actions in any great detail but rather just to show that an action can be constructed with two gauged isometries which is invariant under the massive gauge transformations (10.11). We begin by first reviewing the case of a single gauged isometry to demonstrate the general mechanisms at work, before constructing the double isometry case.

B.1 Single gauged isometry case

Our discussion of the single gauged isometry case is based on [27] which is an extension of the earlier work [26]. Here the action for the KK-monopole in a massive background where the Taub-NUT and massive isometries coincided was considered. In this case we therefore have just a single massive isometry ($\hat{\alpha}$) which is gauged in the worldvolume action by replacing the partial derivatives with covariant derivatives whenever the pullback of a spacetime field to the worldvolume is performed. The action is then that of a gauged sigma model [96] where one of the embedding co-ordinates is eliminated through the gauging. Specially we make the substitution

$$\partial_i \hat{X}^{\hat{\mu}} \to D_i \hat{X}^{\hat{\mu}} = \partial_i \hat{X}^{\hat{\mu}} - \hat{C}_i \hat{\alpha}^{\hat{\mu}} \tag{B.1}$$

where

$$\hat{C}_i = \hat{\alpha}^{-2}\partial_i \hat{X}^{\hat{\mu}}\hat{\alpha}_{\hat{\mu}}. \tag{B.2}$$

The kinetic term of the action is then given by[1]

$$\hat{S}_{kin} = -\hat{T}_{KK}\int d^7\xi\ |\hat{\alpha}|^2\sqrt{|\det(D_i\hat{X}^{\hat{\mu}}D_j\hat{X}^{\hat{\nu}}\hat{g}_{\hat{\mu}\hat{\nu}} + \hat{\alpha}^{-1}\hat{\mathcal{F}}_{ij})|} \tag{B.3}$$

where ξ^i are the worldvolume co-ordinates, $\hat{T}_{KK}$ is the monopole tension and the field strength $\hat{\mathcal{F}}$ is given by

$$\hat{\mathcal{F}}_{ij} = d\hat{\omega}^{(1)}_{ij} + i_{\hat{\alpha}}\hat{A}_{ij} \tag{B.4}$$

where $\hat{\omega}^{(1)}_i$ is the Born-Infeld 1-form. This field describes intersections of the KK-monopole with an M2-brane where the M2-brane wraps the compact isometry direction.

The action (B.3) is invariant under both local isometry transformations with parameter $\hat{\sigma}^{(0)}(\xi^i)$ and massive gauge transformations which we will now give. Under local isometry transformations the embedding co-ordinates transform as

$$\delta\hat{X}^{\hat{\mu}} \quad = \quad \hat{\sigma}^{(0)}\hat{\alpha}^{\hat{\mu}} \tag{B.5}$$

whereas a spacetime field $\hat{T}^{\hat{\mu}_1\ldots\hat{\mu}_i}{}_{\hat{\nu}_1\ldots\hat{\nu}_j}$ varies according to the general rule

$$\delta_{\hat{\sigma}}\hat{T}^{\hat{\mu}_1\ldots\hat{\mu}_i}{}_{\hat{\nu}_1\ldots\hat{\nu}_j} = \hat{\sigma}^{(0)}\hat{\alpha}^{\hat{\lambda}}\partial_{\hat{\lambda}}\hat{T}^{\hat{\mu}_1\ldots\hat{\mu}_i}{}_{\hat{\nu}_1\ldots\hat{\nu}_j}. \tag{B.6}$$

Then, using (8.7) for $\hat{\alpha}$, one determines that $\hat{C}_i$ transforms as follows

$$\delta\hat{C}_i \quad = \quad d\hat{\sigma}^{(0)}_i + \hat{m}\hat{\lambda}_i \tag{B.7}$$

from which it is a simple matter to calculate

$$\delta D_i\hat{X}^{\hat{\mu}} \quad = \quad \hat{\sigma}^{(0)}D_i\hat{X}^{\hat{\nu}}\partial_{\hat{\nu}}\hat{\alpha}^{\hat{\mu}} - \hat{m}\hat{\lambda}_i\hat{\alpha}^{\hat{\mu}}. \tag{B.8}$$

We also have the following transformation rules for the Born-Infeld vector and spacetime metric

$$\delta\hat{\omega}^{(1)}_i \quad = \quad d\hat{\rho}^{(0)}_i + \hat{\lambda}_i \tag{B.9}$$

$$\delta\hat{g}_{\hat{\mu}\hat{\nu}} \quad = \quad 2\hat{m}\hat{\lambda}_{(\hat{\mu}}\hat{\alpha}_{\hat{\nu})} + \hat{\sigma}^{(0)}\hat{\alpha}^{\hat{\rho}}\partial_{\hat{\rho}}\hat{g}_{\hat{\mu}\hat{\nu}}. \tag{B.10}$$

[1]We set $2\pi\alpha' = 1$.

Finally we need the transformation rule for $\hat{A}_{\hat{\mu}_1\hat{\mu}_2\hat{\mu}_3}$ which is given by (8.8) together with (B.6).

Using this collection of transformation rules together with the fact that $\hat{\alpha}$ is a Killing isometry it is a straight forward task to show that the action (B.3) is invariant under both local isometry transformations and the massive gauge transformations. This concludes our discussion of the single gauged isometry action.

B.2 Double gauged isometry case

We now extend this method to the case where two isometries are gauged in the action. The results here apply to both massive and non-massive isometries. So for example, they can be applied to the case of a KK-monopole in a single M9-brane background where the Taub-NUT and massive isometries do not coincide, as considered in [28]. Alternatively, they can also be applied to the case of a KK-monopole in a double M9-brane background where the Taub-NUT isometry does coincide with one of the massive isometries. Which case is described depends on whether the truncation of the mass matrix $\hat{Q}$ (10.2) or the general mass matrix (10.1) is used respectively.

In order to write the action in an $SL(2,\mathbb{R})$ covariant manner, we need to make use of the charge matrix q^{ab} as done in [93]. The entries of this matrix determines which combination of states within the KK-monopole triplet the action is describing. However, as outlined below we find that we are also required to include the charge vector q^a. The only way that this can be meaningfully related to the combination of states being considered is if it is derived from the charge matrix according to $q^{ab} = q^a q^b$, which adds a restriction to q^{ab}, in particular we must have $\det(q^{ab}) = 0$. The same restriction was found in [93] for the triplet of 7-branes where the Wess-Zumino term of the action was also considered. Mapping this restriction from IIB to $D = 11$ also gives us a restriction on the mass parameters

$$\det(\hat{Q}^{ab}) = 0 \tag{B.11}$$

which is equivalent to $\hat{m}_1^2 + \hat{m}_- \hat{m}_+ = 0$. We therefore might also expect this restriction to apply to the action here, however we will show below that this does not seem to be required for the action (or at least the kinetic term) to be invariant under the massive gauge transformations.

We start constructing the action by generalising the field $\hat{C}_i$ of the previous section. In the present case we have two isometries and so expect to have two such fields. Treating these symmetrically, we propose that each should be defined analogously to $\hat{C}_i$ but each with a 'naive' gauging using the other isometry. Explicitly we propose that we have two such fields defined as

$$\begin{aligned}\hat{B}_i &= \hat{\beta}^{-2}(\partial_i \hat{X}^{\hat{\mu}} - \hat{C}_i \hat{\alpha}^{\hat{\mu}})\hat{\beta}_{\hat{\mu}} \\ \hat{C}_i &= \hat{\alpha}^{-2}(\partial_i \hat{X}^{\hat{\mu}} - \hat{B}_i \hat{\beta}^{\hat{\mu}})\hat{\alpha}_{\hat{\mu}} \end{aligned} \tag{B.12}$$

where we have used a non-covariant notation with $\hat{\beta}$ denoting the second isometry. These rearrange to give

$$\hat{B}_i = \frac{\hat{\alpha}^2}{\hat{\gamma}}\partial_i \hat{X}^{\hat{\mu}}\hat{\beta}_{\hat{\mu}} - \frac{\hat{\alpha}\cdot\hat{\beta}}{\hat{\gamma}}\partial_i \hat{X}^{\hat{\mu}}\hat{\alpha}_{\hat{\mu}} \tag{B.13}$$

$$\hat{C}_i = \frac{\hat{\beta}^2}{\hat{\gamma}}\partial_i \hat{X}^{\hat{\mu}}\hat{\alpha}_{\hat{\mu}} - \frac{\hat{\alpha}\cdot\hat{\beta}}{\hat{\gamma}}\partial_i \hat{X}^{\hat{\mu}}\hat{\beta}_{\hat{\mu}} \tag{B.14}$$

where we have defined

$$\hat{\gamma} = \hat{\alpha}^2\hat{\beta}^2 - (\hat{\alpha}\cdot\hat{\beta})^2. \tag{B.15}$$

These expressions can be written in covariant notation as

$$\hat{C}_{ia} = \frac{2\epsilon^{bc}(\hat{k}_a\cdot\hat{k}_b)\partial_i \hat{X}^{\hat{\mu}}\hat{k}_{\hat{\mu}c}}{\epsilon^{bc}\epsilon^{de}(\hat{k}_b\cdot\hat{k}_d)(\hat{k}_c\cdot\hat{k}_e)} \tag{B.16}$$

with

$$\hat{C}_{ia} = \begin{pmatrix} \hat{B}_i \\ -\hat{C}_i \end{pmatrix}. \tag{B.17}$$

The covariant derivative in this case is then defined as

$$D_i \hat{X}^{\hat{\mu}} = \partial_i \hat{X}^{\hat{\mu}} - \epsilon^{ab}\hat{C}_{ia}\hat{k}_b^{\hat{\mu}}. \tag{B.18}$$

Note that this is only well defined in the instance where the two isometries are not parallel.

In order to construct an action that is invariant under the various transformations we consider each subterm individually beginning with the metric term. The embedding co-ordinates now change under the two local isometry transformations, with parameters $\hat{\sigma}_a^{(0)}(\xi^i)$, according to

$$\delta \hat{X}^{\hat{\mu}} = \epsilon^{ab} \hat{\sigma}_a^{(0)} \hat{k}_b^{\hat{\mu}} \tag{B.19}$$

whereas a spacetime field $\hat{T}^{\hat{\mu}_1 \ldots \hat{\mu}_i}{}_{\hat{\nu}_1 \ldots \hat{\nu}_j}$ transforms by the general rule

$$\delta_{\hat{\sigma}} \hat{T}^{\hat{\mu}_1 \ldots \hat{\mu}_i}{}_{\hat{\nu}_1 \ldots \hat{\nu}_j} = \epsilon^{ab} \hat{\sigma}_a^{(0)} \hat{k}_b^{\hat{\lambda}} \partial_{\hat{\lambda}} \hat{T}^{\hat{\mu}_1 \ldots \hat{\mu}_i}{}_{\hat{\nu}_1 \ldots \hat{\nu}_j}. \tag{B.20}$$

Then, using (10.11) one can show through explicit calculation that

$$\delta \hat{C}_{ia} = d\hat{\sigma}_{ia}^{(0)} - \hat{Q}_a{}^b D_i \hat{X}^{\hat{\mu}} \hat{\lambda}_{\hat{\mu} b} \tag{B.21}$$

and therefore

$$\delta D_i \hat{X}^{\hat{\mu}} = \epsilon^{ab} \hat{\sigma}_a^{(0)} D_i \hat{X}^{\hat{\nu}} \partial_{\hat{\nu}} \hat{k}_b^{\hat{\mu}} - \hat{Q}^{ab} D_i \hat{X}^{\hat{\nu}} \hat{\lambda}_{\hat{\nu} a} \hat{k}_b^{\hat{\mu}}. \tag{B.22}$$

The rule for the metric (B.10) now becomes

$$\delta \hat{g}_{\hat{\mu}\hat{\nu}} = \hat{Q}^{ab} (\hat{\lambda}_{\hat{\mu} a} \hat{k}_{\hat{\nu} b} + \hat{\lambda}_{\hat{\nu} a} \hat{k}_{\hat{\mu} b}) + \epsilon^{ab} \hat{\sigma}_a^{(0)} \hat{k}_b^{\hat{\rho}} \partial_{\hat{\rho}} \hat{g}_{\hat{\mu}\hat{\nu}}. \tag{B.23}$$

It is then a straight forward task to show that the term

$$D_i \hat{X}^{\hat{\mu}} D_j \hat{X}^{\hat{\nu}} \hat{g}_{\hat{\mu}\hat{\nu}} \tag{B.24}$$

is invariant under these covariant transformations using the fact that $\hat{k}_a$ define Killing isometries.

Next we consider the Born-Infeld field strength. Since the Born-Infeld field describes the intersection of the KK-monopole with an M2-brane where the M2-brane wraps a compact isometry direction, we get a doublet of Born-Infeld fields $\hat{\omega}_{ia}^{(1)}$ in the covariant theory since we now have two compact isometry directions. We begin determining the structure of the field strength $\hat{\mathcal{F}}_{ija}$ by generalising the transformation rule (B.9) to

$$\delta \hat{\omega}_{ia}^{(1)} = d\hat{\rho}_{ia}^{(0)} + D_i \hat{X}^{\hat{\mu}} \hat{\lambda}_{\hat{\mu} a} \tag{B.25}$$

from which we calculate

$$\delta (d\hat{\omega}_a^{(1)}) = d\hat{\lambda}_a - i_{\hat{k}_2} \hat{\lambda}_1 d\hat{C}_a - d(i_{\hat{k}_2} \hat{\lambda}_1) \wedge \hat{C}_a. \tag{B.26}$$

From (B.4) we would also expect there to be a term $D_i\hat{X}^{\hat{\mu}}D_j\hat{X}^{\hat{\nu}}i_{\hat{k}_a}\hat{A}_{\hat{\mu}\hat{\nu}}$ which, using (10.8), we find transforms as

$$\begin{aligned}\delta(D_i\hat{X}^{\hat{\mu}}D_j\hat{X}^{\hat{\nu}}i_{\hat{k}_a}\hat{A}_{\hat{\mu}\hat{\nu}}) &= -D_i\hat{X}^{\hat{\mu}}D_j\hat{X}^{\hat{\nu}}d\hat{\lambda}_{\hat{\mu}\hat{\nu}a} \\ &\quad +i_{\hat{k}_2}\hat{\lambda}_1\hat{Q}_a{}^b D_i\hat{X}^{\hat{\mu}}D_j\hat{X}^{\hat{\nu}}i_{\hat{k}_b}\hat{A}_{\hat{\mu}\hat{\nu}}. \end{aligned} \tag{B.27}$$

Combining these results we see that the naive generalisation of (B.4) is not invariant under the massive gauge transformations. Following [28] we introduce the worldvolume scalar $\hat{\omega}^{(0)}$ which has the following gauge transformation

$$\delta\hat{\omega}^{(0)} = \frac{1}{2}\epsilon^{ab}i_{\hat{k}_a}\hat{\lambda}_b = -i_{\hat{k}_2}\hat{\lambda}_1. \tag{B.28}$$

We then consider the term $\hat{\omega}^{(0)}d\hat{C}_{ija}$ which transforms as

$$\delta(\hat{\omega}^{(0)}d\hat{C}_a) = -i_{\hat{k}_2}\hat{\lambda}_1 d\hat{C}_a - \hat{\omega}^{(0)}\hat{Q}_a{}^b(d\hat{\lambda}_b - i_{\hat{k}_2}\hat{\lambda}_1 d\hat{C}_a - d(i_{\hat{k}_2}\hat{\lambda}_1)\wedge\hat{C}_a). \tag{B.29}$$

Putting these together we see that the following field definition is invariant up to quadratic order of the mass parameters

$$\begin{aligned}\hat{\mathcal{F}}_{ija} &= d\hat{\omega}^{(1)}_{ija} + (\mathbb{1}_a{}^b + \hat{\omega}^{(0)}\hat{Q}_a{}^b)D_i\hat{X}^{\hat{\mu}}D_j\hat{X}^{\hat{\nu}}i_{\hat{k}_b}\hat{A}_{\hat{\mu}\hat{\nu}} \\ &\quad -\hat{\omega}^{(0)}(\mathbb{1}_a{}^b + \frac{1}{2}\hat{\omega}^{(0)}\hat{Q}_a{}^b)d\hat{C}_{ijb}. \end{aligned} \tag{B.30}$$

Note that we have the identity $\hat{Q}_a{}^b\hat{Q}_b{}^c = -\det(\hat{Q})\mathbb{1}_a{}^c$. We can therefore neglect terms that are quadratic or higher order in the mass matrix if we impose the constraint (B.11).

Finally we must consider how terms such as $\hat{k}_a\cdot\hat{k}_b$ transform. We find, using (10.11),

$$\delta(\hat{k}_a\cdot\hat{k}_b) = 2\hat{Q}_{(a}{}^c(\hat{k}_{b)}\cdot\hat{k}_c)i_{\hat{k}_2}\hat{\lambda}_1. \tag{B.31}$$

We then construct the invariant expression

$$\hat{k}_{ab} = (\mathbb{1}_{(a}{}^c + 2\hat{\omega}^{(0)}\hat{Q}_{(a}{}^c)(\hat{k}_{b)}\cdot\hat{k}_c) \tag{B.32}$$

where once again we restrict ourselves to the case where the terms quadratic in the mass parameters vanish.

At this point we realise that there are no appropriate fields with which we can contract the $SL(2,\mathbb{R})$ free index of the Born-Infeld field strength. The Killing vectors $\hat{k}_a$ are not suitable since this would leave a free spacetime index. We are therefore forced to include the charge vector q^a discussed at the start of this section. Putting these expressions we find that an $SL(2,\mathbb{R})$ covariant action with two gauged isometries can be written as

$$\hat{S}_{kin} \quad = \quad -\hat{T}_{KK}\int d^7\xi\; q^{ab}\hat{k}_{ab}\sqrt{|\det(D_i\hat{X}^{\hat{\mu}}D_j\hat{X}^{\hat{\nu}}\hat{g}_{\hat{\mu}\hat{\nu}} + \frac{q^a\hat{\mathcal{F}}_{ija}}{(q^{ab}\hat{k}_{ab})^{\frac{1}{2}}})|} \tag{B.33}$$

which is invariant under both the massive gauge transformations and local isometry transformations. Note that after performing the truncation (10.2) there are differences between the action presented here and the action given in [28]. This originates from the differences between the massive gauge transformation rules (10.11) and the symmetric manner in which we gauged both isometries (B.12).

In [93] the IIB 7-brane action was constructed for the $\det(q) = 0$ case and the result has the same general structure as (B.33). This action was then generalised to positive $\det(q)$ in [80] to second order in the Born-Infeld term. In the current situation, while it is not obvious how to generalise the action fully, we note that the restriction (B.11) can be lifted without spoiling the gauge invariance if we define the Born-Infeld field strength as

$$\begin{aligned}\hat{\mathcal{F}}_{ija} \quad = \quad & d\hat{\omega}^{(1)}_{ija} + \exp(\hat{\omega}^{(0)}\hat{Q})_a{}^b D_i\hat{X}^{\hat{\mu}}D_j\hat{X}^{\hat{\nu}} i_{\hat{k}_b}\hat{A}_{\hat{\mu}\hat{\nu}} \\ & -(\hat{Q}^{-1})_a{}^b(\exp(\hat{\omega}^{(0)}\hat{Q})_b{}^c - \mathbb{1}_b{}^c)d\hat{C}_{ijc}\end{aligned} \tag{B.34}$$

and the Killing vector term as

$$\hat{k}_{ab} = \exp(2\hat{\omega}^{(0)}\hat{Q})_{(a}{}^c(\hat{k}_{b)}\cdot\hat{k}_c). \tag{B.35}$$

The action is then exponentially dependent on the mass parameters. This demonstrates a first step in generalising this action for $\det(q) \neq 0$. Note that the inverse mass matrix only appears in (B.34) if the powers of the masses are collected in terms of an exponential function, but cancels if these are expressed as an explicit power series.

Appendix C

Mapping $D = 11$ 10-forms to IIB

We now map the $D = 11$ 10-form potentials to IIB and determine the resulting quadruplet of field equations for the IIB 10-form potentials. These equations were calculated in [67] in an $SU(1,1)$ covariant form. The reason we re-derive them here is to determine the T-duality rules for the IIA potentials which map to these potentials, which are required in order to T-dualise the IIA 8/9-form charges. Furthermore, we reveal the existence of a pair of IIB 9-form potentials which form part of a doublet which we conjecture should couple to a doublet of KK-type 9-branes. For the purposes of this calculation we must neglect the dimensionality of the spacetime in order to determine the full structure of the field equations.

We begin by considering (10.26) which shows that there are only in fact two independent 11-form field strengths in the $D = 11$ theory. We make this fact explicit upon performing the dimensional reduction of the triplet of $D = 11$ 11-form field strengths to IIA. Specifically we have (reducing along z)

$$
\begin{aligned}
\hat{F}^{(11)}_{11} &\rightarrow G^{(11)} + F^{(10)} \wedge (C^{(1)} + dz) && \text{(C.1)}\\
\hat{F}^{(11)}_{12} &\rightarrow i_\beta C^{(1)} G^{(11)} + H^{(11)} + (i_\beta C^{(1)} F^{(10)} + H^{(10)}) \wedge (C^{(1)} + dz) && \text{(C.2)}\\
\hat{F}^{(11)}_{22} &\rightarrow ((i_\beta C^{(1)})^2 - e^{-2\phi} R^2) G^{(11)} + 2 i_\beta C^{(1)} H^{(11)} \\
&\quad + \Big[((i_\beta C^{(1)})^2 - e^{-2\phi} R^2) F^{(10)} + 2 i_\beta C^{(1)} H^{(10)} \Big] \wedge (C^{(1)} + dz). && \text{(C.3)}
\end{aligned}
$$

The combinations of the IIA 10-forms here are determined from the Hodge dual relation (10.24). The RR 10-form $F^{(10)}$ is then related to the mass parameters via

$$\begin{aligned} F^{(10)} &= *(-m_+ + ((i_\beta C^{(1)})^2 - e^{-2\phi}R^2)m_- - 2i_\beta C^{(1)}m_1) \\ &= - * F^{(0)} \end{aligned} \tag{C.4}$$

which is a generalisation of the usual relation found in Romans' theory. $H^{(10)}$ is a second 10-form related to the mass parameters by

$$\begin{aligned} H^{(10)} &= e^{-2\phi}R^2 * (2i_\beta C^{(1)}m_- - 2m_1) \\ &= e^{-2\phi} * H^{(0)}. \end{aligned} \tag{C.5}$$

Obviously the two 11-form fields $G^{(11)}$ and $H^{(11)}$ have no Hodge dual interpretation and in reality are identically zero due to the dimensionality of the spacetime. Because of this the combinations of them appearing in the reduction rule above is solely motivated by considering the mappings of their associated field strength equations to IIB. We find that the following set of T-duality rules then produces a well formed set of IIB field strength equations

$$\begin{aligned} e^{-2\phi}R_\beta^2 F^{(10)}_{\bar\mu_1\ldots\bar\mu_{10}} &\to (\frac{3}{4}\mathcal{S}^{(10)} + \frac{3}{4}\mathcal{R}^{-2}i_\beta\mathcal{S}^{(10)}\wedge\beta \\ &\quad + e^{-2\varphi}(\mathcal{F}^{(9)} - \mathcal{R}^{-2}i_\beta\mathcal{F}^{(9)}\wedge\beta)\wedge i_\beta\mathcal{B})_{\bar\mu_1\ldots\bar\mu_{10}} && \text{(C.6)} \\ H^{(10)}_{\bar\mu_1\ldots\bar\mu_{10}} &\to (i_\beta\mathcal{H}^{(11)} - \frac{1}{2}\mathcal{G}^{(10)} - \frac{1}{2}\mathcal{R}^{-2}i_\beta\mathcal{G}^{(10)}\wedge\beta \\ &\quad + (\mathcal{H}^{(9)} - \mathcal{R}^{-2}i_\beta\mathcal{H}^{(9)}\wedge\beta)\wedge i_\beta\mathcal{B})_{\bar\mu_1\ldots\bar\mu_{10}} && \text{(C.7)} \\ e^{-2\phi}R_\beta^2 i_\beta G^{(11)}_{\bar\mu_1\ldots\bar\mu_{10}} &\to i_\beta\mathcal{G}^{(11)}_{\bar\mu_1\ldots\bar\mu_{10}} && \text{(C.8)} \\ i_\beta G^{(11)}_{\bar\mu_1\ldots\bar\mu_{10}} &\to (\mathcal{G}^{(10)} + \mathcal{R}^{-2}i_\beta\mathcal{G}^{(10)}\wedge\beta)_{\bar\mu_1\ldots\bar\mu_{10}} && \text{(C.9)} \\ i_\beta H^{(11)}_{\bar\mu_1\ldots\bar\mu_{10}} &\to (i_\beta\mathcal{X}^{(11)} + \frac{3}{8}\mathcal{S}^{(10)} + \frac{3}{8}\mathcal{R}^{-2}i_\beta\mathcal{S}^{(10)}\wedge\beta)_{\bar\mu_1\ldots\bar\mu_{10}} && \text{(C.10)} \end{aligned}$$

along with the usual rule for the RR 10-form. Note that the co-ordinates $\bar\mu_i$ correspond to directions transverse to the T-duality isometry direction and that these number more than 9 since we are neglecting the dimensionality of the spacetime. It may seem strange that $F^{(10)}$ and $i_\beta G^{(11)}$ obey different rules depending on whether a factor of $e^{-2\phi}R$ is present or not, however essentially the same situation occurs when T-dualising $i_\beta\mathcal{F}^{(9)}$ from IIB to IIA which maps to either $F^{(8)}$ or $i_\beta X^{(9)}$ (9.20). In that case the reason for it can be traced back to the fact that both the IIA field strengths are related to $F^{(2)}$ through Hodge duality. Obviously there is no such interpretation in the current case.

We discuss the field strengths $\mathcal{S}^{(10)}$ and $\mathcal{G}^{(10)}$ further below but note that due to a cancellation they do not appear in the field equations for the quadruplet of 10-forms which is obtained by mapping $i_{\hat{k}_{(a}}\hat{F}^{(10)}_{bc)}$ from $D = 11$, which we now consider. We find that in order to produce well-formed IIB field strength equations the relevant IIA potentials T-dualise as

$$\begin{aligned}
B^{(9)}_{\bar{\mu}_1\ldots\bar{\mu}_9} \quad\rightarrow\quad & \Big[-\frac{3}{2}i_\beta\mathcal{B}^{(10)} - \frac{1}{2}\mathcal{N}^{(9)} + \frac{1}{2}\mathcal{R}^{-2}i_\beta\mathcal{N}^{(9)}\wedge\beta - \frac{3}{8}i_\beta\mathcal{C}^{(8)}\wedge\mathcal{C}^{(2)} \\
& -\frac{3}{8}\mathcal{R}^{-2}i_\beta\mathcal{C}^{(8)}\wedge i_\beta\mathcal{C}^{(2)}\wedge\beta + (\varphi^{(8)} + \mathcal{R}^{-2}i_\beta\varphi^{(8)}\wedge\beta + \frac{1}{2}\mathcal{C}^{(6)}\wedge\mathcal{C}^{(2)} \\
& +\frac{1}{2}\mathcal{R}^{-2}i_\beta\mathcal{C}^{(6)}\wedge\mathcal{C}^{(2)}\wedge\beta + \frac{1}{2}\mathcal{R}^{-2}\mathcal{C}^{(6)}\wedge i_\beta\mathcal{C}^{(2)}\wedge\beta)\wedge i_\beta\mathcal{B}\Big]_{\bar{\mu}_1\ldots\bar{\mu}_9} \qquad \text{(C.11)}
\end{aligned}$$

$$\begin{aligned}
i_\beta\mathcal{B}^{(10)}_{\bar{\mu}_1\ldots\bar{\mu}_9} \quad\rightarrow\quad & \Big[\frac{3}{2}i_\beta\mathcal{A}^{(10)} - \frac{3}{8}\mathcal{T}^{(9)} + \frac{3}{8}\mathcal{R}^{-2}i_\beta\mathcal{T}^{(9)}\wedge\beta + \frac{3}{4}i_\beta\varphi^{(8)}\wedge\mathcal{C}^{(2)} \\
& +\frac{3}{4}\mathcal{R}^{-2}i_\beta\varphi^{(8)}\wedge i_\beta\mathcal{C}^{(2)}\wedge\beta - \frac{1}{4}\mathcal{N}^{(7)}\wedge\mathcal{C}^{(2)} + \frac{1}{4}\mathcal{R}^{-2}i_\beta\mathcal{N}^{(7)}\wedge\mathcal{C}^{(2)}\wedge\beta \\
& -\frac{1}{4}\mathcal{R}^{-2}\mathcal{N}^{(7)}\wedge i_\beta\mathcal{C}^{(2)}\wedge\beta + \frac{1}{8}i_\beta\mathcal{C}^{(6)}\wedge(\mathcal{C}^{(2)})^2 \\
& +\frac{1}{4}\mathcal{R}^{-2}i_\beta\mathcal{C}^{(6)}\wedge i_\beta\mathcal{C}^{(2)}\wedge\mathcal{C}^{(2)}\wedge\beta\Big]_{\bar{\mu}_1\ldots\bar{\mu}_9} \qquad \text{(C.12)}
\end{aligned}$$

$$\begin{aligned}
i_\beta D^{(10)}_{\bar{\mu}_1\ldots\bar{\mu}_9} \quad\rightarrow\quad & \Big[-i_\beta\mathcal{D}^{(10)} + \mathcal{N}^{(8)}\wedge i_\beta\mathcal{C}^{(2)} - \mathcal{R}^{-2}i_\beta\mathcal{N}^{(8)}\wedge i_\beta\mathcal{C}^{(2)}\wedge\beta \\
& +\frac{1}{8}i_\beta\mathcal{C}^{(4)}\wedge(\mathcal{C}^{(2)})^3 + \frac{3}{8}\mathcal{R}^{-2}i_\beta\mathcal{C}^{(4)}\wedge i_\beta\mathcal{C}^{(2)}\wedge(\mathcal{C}^{(2)})^2\wedge\beta\Big]_{\bar{\mu}_1\ldots\bar{\mu}_9} \qquad \text{(C.13)}
\end{aligned}$$

along with the standard rule for the RR potential $C^{(9)}$.

We then find the field equations to be given by

$$\begin{aligned}
\frac{2}{3}\mathcal{X}^{(11)} &= d\mathcal{A}^{(10)} - l^2\mathcal{F}^{(11)} - \frac{4}{3}l\mathcal{H}^{(11)} + \frac{2}{3}\mathcal{H}^{(9)}\wedge\mathcal{C}^{(2)} + \frac{2}{3}l\mathcal{F}^{(9)}\wedge\mathcal{C}^{(2)} \\
&\quad +\frac{1}{3}\mathcal{H}\wedge\mathcal{N}^{(8)} - \frac{1}{6}\mathcal{F}^{(7)}\wedge(\mathcal{C}^{(2)})^2 - \frac{1}{3}\mathcal{H}\wedge\mathcal{B}^{(6)}\wedge\mathcal{C}^{(2)} \qquad \text{(C.14)} \\
\frac{2}{3}\mathcal{H}^{(11)} &= d\mathcal{B}^{(10)} - l\mathcal{F}^{(11)} + \frac{1}{3}\mathcal{F}^{(9)}\wedge\mathcal{C}^{(2)} - \frac{2}{3}\mathcal{H}\wedge\varphi^{(8)} \qquad \text{(C.15)} \\
\mathcal{F}^{(11)} &= d\mathcal{C}^{(10)} - \mathcal{C}^{(8)}\wedge\mathcal{H} \qquad \text{(C.16)} \\
\mathcal{G}^{(11)} &= d\mathcal{D}^{(10)} + l^3\mathcal{F}^{(11)} + 2l^2\mathcal{H}^{(11)} + 2l\mathcal{X}^{(11)} \\
&\quad -\mathcal{N}^{(8)}\wedge d\mathcal{C}^{(2)} - \frac{1}{6}d\mathcal{C}^{(4)}\wedge(\mathcal{C}^{(2)})^3 + \frac{1}{24}\mathcal{H}\wedge(\mathcal{C}^{(2)})^4 \qquad \text{(C.17)}
\end{aligned}$$

where we have included the RR field equation which takes the usual form. These can be written as a quadruplet which is shown in Section 12.2.5. The Bianchi identities for these equations are calculated to be

$$\frac{2}{3}d\mathcal{X}^{(11)} = -\frac{4}{3}\mathcal{F}^{(1)}\wedge\mathcal{H}^{(11)}+\frac{2}{3}\mathcal{F}^{(3)}\wedge\mathcal{H}^{(9)}+\frac{1}{3}e^{-2\varphi}\mathcal{F}^{(9)}\wedge\mathcal{H} \tag{C.18}$$

$$\frac{2}{3}d\mathcal{H}^{(11)} = -\mathcal{F}^{(1)}\wedge\mathcal{F}^{(11)}+\frac{1}{3}\mathcal{F}^{(3)}\wedge\mathcal{F}^{(9)}+\frac{2}{3}\mathcal{H}\wedge\mathcal{H}^{(9)} \tag{C.19}$$

$$d\mathcal{G}^{(11)} = 2\mathcal{F}^{(1)}\wedge\mathcal{X}^{(11)}-e^{-2\varphi}\mathcal{F}^{(9)}\wedge\mathcal{F}^{(3)} \tag{C.20}$$

$$d\mathcal{F}^{(11)} = \mathcal{H}\wedge\mathcal{F}^{(9)} \tag{C.21}$$

which confirms that the above definitions are gauge invariant.

The field equations for $\mathcal{G}^{(10)}$ and $\mathcal{S}^{(10)}$ can be calculated using the following T-duality rules

$$\begin{aligned} i_\beta A^{(10)}_{\bar\mu_1\ldots\bar\mu_9} \rightarrow\ & \Big[-\mathcal{N}^{(9)}+\mathcal{R}^{-2}i_\beta\mathcal{N}^{(9)}\wedge\beta+\frac{1}{4}i_\beta\mathcal{C}^{(8)}\wedge\mathcal{C}^{(2)} \\ & +\frac{1}{4}\mathcal{R}^{-2}i_\beta\mathcal{C}^{(8)}\wedge i_\beta\mathcal{C}^{(2)}\wedge\beta\Big]_{\bar\mu_1\ldots\bar\mu_9} \end{aligned} \tag{C.22}$$

$$\begin{aligned} D^{(9)}_{\bar\mu_1\ldots\bar\mu_9} \rightarrow\ & \Big[-\frac{3}{4}\mathcal{T}^{(9)}+\frac{3}{4}\mathcal{R}^{-2}i_\beta\mathcal{T}^{(9)}\wedge\beta-\frac{1}{2}i_\beta\varphi^{(8)}\wedge\mathcal{C}^{(2)} \\ & -\frac{1}{2}\mathcal{R}^{-2}i_\beta\varphi^{(8)}\wedge i_\beta\mathcal{C}^{(2)}\wedge\beta-\frac{1}{2}\mathcal{N}^{(7)}\wedge\mathcal{C}^{(2)}+\frac{1}{2}\mathcal{R}^{-2}i_\beta\mathcal{N}^{(7)}\wedge\mathcal{C}^{(2)}\wedge\beta \\ & -\frac{1}{2}\mathcal{R}^{-2}\mathcal{N}^{(7)}\wedge i_\beta\mathcal{C}^{(2)}\wedge\beta-\frac{1}{4}i_\beta\mathcal{C}^{(6)}\wedge(\mathcal{C}^{(2)})^2 \\ & -\frac{1}{2}\mathcal{R}^{-2}i_\beta\mathcal{C}^{(6)}\wedge i_\beta\mathcal{C}^{(2)}\wedge\mathcal{C}^{(2)}\wedge\beta+(-\mathcal{N}^{(8)}-\mathcal{R}^{-2}i_\beta\mathcal{N}^{(8)}\wedge\beta \\ & +\frac{1}{2}\mathcal{C}^{(4)}\wedge(\mathcal{C}^{(2)})^2+\frac{1}{2}\mathcal{R}^{-2}i_\beta\mathcal{C}^{(4)}\wedge(\mathcal{C}^{(2)})^2\wedge\beta \\ & +\mathcal{R}^{-2}\mathcal{C}^{(4)}\wedge i_\beta\mathcal{C}^{(2)}\wedge\mathcal{C}^{(2)}\wedge\beta)\wedge i_\beta\mathcal{B}\Big]_{\bar\mu_1\ldots\bar\mu_9} \end{aligned} \tag{C.23}$$

and are found to be given by

$$\begin{aligned} \mathcal{G}^{(10)} =\ & d\mathcal{N}^{(9)}-li_\beta\mathcal{F}^{(11)}-\mathcal{F}^{(9)}\wedge i_\beta\mathcal{C}^{(2)}+\mathcal{H}\wedge i_\beta\varphi^{(8)}-\frac{1}{2}\mathcal{H}\wedge\mathcal{N}^{(7)} \\ & +\frac{1}{4}i_\beta d\mathcal{C}^{(4)}\wedge\mathcal{C}^{(6)}+\frac{1}{4}i_\beta\mathcal{F}^{(9)}\wedge\mathcal{C}^{(2)} \end{aligned} \tag{C.24}$$

$$\begin{aligned} \mathcal{S}^{(10)} =\ & d\mathcal{T}^{(9)}+\frac{8}{3}li_\beta\mathcal{H}^{(11)}+\frac{4}{3}l^2i_\beta\mathcal{F}^{(11)}-\frac{4}{3}l\mathcal{G}^{(10)}-\frac{4}{3}\mathcal{N}^{(8)}\wedge i_\beta\mathcal{H}-\frac{1}{3}i_\beta\mathcal{N}^{(8)}\wedge\mathcal{H} \\ & +\frac{2}{3}d\mathcal{N}^{(7)}\wedge\mathcal{C}^{(2)}-\frac{2}{3}i_\beta d\varphi^{(8)}\wedge\mathcal{C}^{(2)}-\frac{1}{3}i_\beta\mathcal{F}^{(7)}\wedge(\mathcal{C}^{(2)})^2-\frac{1}{3}i_\beta\mathcal{F}^{(5)}\wedge\mathcal{B}^{(6)} \\ & +\frac{1}{3}i_\beta d\mathcal{C}^{(4)}\wedge\mathcal{C}^{(4)}\wedge\mathcal{C}^{(2)}. \end{aligned} \tag{C.25}$$

Confirmation that such equations are gauge invariant is found by calculating the

Bianchi identities

$$\begin{aligned}
d\mathcal{G}^{(10)} &= -\mathcal{F}^{(1)} \wedge i_\beta \mathcal{F}^{(11)} - i_\beta \mathcal{F}^{(3)} \wedge \mathcal{F}^{(9)} + \frac{1}{4}\mathcal{F}^{(3)} \wedge i_\beta \mathcal{F}^{(9)} + \frac{1}{4} i_\beta \mathcal{F}^{(5)} \wedge \mathcal{F}^{(7)} \\
&\quad + \mathcal{H} \wedge i_\beta \mathcal{H}^{(9)} + \frac{1}{2}\mathcal{H} \wedge \mathcal{G}^{(8)} \qquad \text{(C.26)} \\
d\mathcal{S}^{(10)} &= \frac{8}{3}\mathcal{F}^{(1)} \wedge i_\beta \mathcal{H}^{(11)} - \frac{4}{3}\mathcal{F}^{(1)} \wedge \mathcal{G}^{(10)} - \frac{4}{3} e^{-2\varphi} i_\beta \mathcal{F}^{(9)} \wedge i_\beta \mathcal{H} \\
&\quad + \frac{1}{3} e^{-2\varphi} i_\beta \mathcal{F}^{(9)} \wedge \mathcal{H} + \frac{2}{3}\mathcal{F}^{(3)} \wedge \mathcal{G}^{(8)} - \frac{2}{3}\mathcal{F}^{(3)} \wedge i_\beta \mathcal{H}^{(9)} \\
&\quad - \frac{1}{3} i_\beta \mathcal{F}^{(5)} \wedge \mathcal{H}^{(7)}. \qquad \text{(C.27)}
\end{aligned}$$

The field strengths $\mathcal{G}^{(10)}$ and $\mathcal{S}^{(10)}$ are also related via T-duality to the IIA field strengths $H^{(9)}$ and $X^{(9)}$ respectively. Mapping from IIA to IIB we have the following rules

$$\begin{aligned}
H^{(9)}_{\bar\mu_1 \ldots \bar\mu_9} &\to \left[-i_\beta \mathcal{G}^{(10)} + (i_\beta \mathcal{H}^{(9)} + \frac{1}{2}\mathcal{G}^{(8)} + \frac{1}{2}\mathcal{R}^{-2} i_\beta \mathcal{G}^{(8)} \wedge \beta) \wedge i_\beta \mathcal{B} \right]_{\bar\mu_1 \ldots \bar\mu_9} \qquad \text{(C.28)} \\
X^{(9)}_{\bar\mu_1 \ldots \bar\mu_9} &\to (-i_\beta \mathcal{S}^{(10)} - e^{-2\varphi} i_\beta \mathcal{F}^{(9)} \wedge i_\beta \mathcal{B})_{\bar\mu_1 \ldots \bar\mu_9}. \qquad \text{(C.29)}
\end{aligned}$$

The field equations (C.24) and (C.25) contracted with β can then be shown to be obtained from the IIA field equations for $X^{(9)}$ and $H^{(9)}$ using the following mappings[1]

$$\begin{aligned}
\phi^{(8)}_{\bar\mu_1 \ldots \bar\mu_8} &\to \Big[i_\beta \mathcal{N}^{(9)} - (i_\beta \varphi^{(8)} - \frac{1}{2}\mathcal{N}^{(7)} + \frac{1}{2}\mathcal{R}^{-2} i_\beta \mathcal{N}^{(7)} \wedge \beta + \frac{1}{4} i_\beta \mathcal{C}^{(6)} \wedge \mathcal{C}^{(2)} \\
&\quad + \frac{1}{4}\mathcal{R}^{-2} i_\beta \mathcal{C}^{(6)} \wedge i_\beta \mathcal{C}^{(2)} \wedge \beta) \wedge i_\beta \mathcal{B} \Big]_{\bar\mu_1 \ldots \bar\mu_8} \qquad \text{(C.30)} \\
N^{(8)}_{\bar\mu_1 \ldots \bar\mu_8} &\to \Big[i_\beta \mathcal{T}^{(9)} + \frac{2}{3} i_\beta \mathcal{N}^{(7)} \wedge \mathcal{C}^{(2)} + \frac{2}{3}\mathcal{R}^{-2} i_\beta \mathcal{N}^{(7)} \wedge i_\beta \mathcal{C}^{(2)} \wedge \beta \\
&\quad + (i_\beta \mathcal{N}^{(8)} - \mathcal{B}^{(6)} \wedge i_\beta \mathcal{C}^{(2)} + \mathcal{R}^{-2} i_\beta \mathcal{B}^{(6)} \wedge i_\beta \mathcal{C}^{(2)} \wedge \beta \\
&\quad - \frac{1}{3} i_\beta \mathcal{C}^{(4)} \wedge (\mathcal{C}^{(2)})^2 - \frac{2}{3}\mathcal{R}^{-2} i_\beta \mathcal{C}^{(4)} \wedge i_\beta \mathcal{C}^{(2)} \wedge \mathcal{C}^{(2)} \wedge \beta) \wedge i_\beta \mathcal{B} \Big]_{\bar\mu_1 \ldots \bar\mu_8} \qquad \text{(C.31)}
\end{aligned}$$

Since the IIA potentials $\phi^{(8)}$ and $N^{(8)}$ are both the magnetic duals of scalars, the above T-duality relations imply that $\mathcal{G}^{(10)}$ and $\mathcal{S}^{(10)}$ vanish under a normal T-duality transformation. However, they are non-zero if one performs the T-duality via a Scherk-Schwarz reduction on the IIA side using the global $SO(1,1)$ symmetry [78, 90]. We therefore conclude that $\mathcal{G}^{(10)}$ and $\mathcal{S}^{(10)}$ exist in a non-covariant (in the

[1] We have not explicitly given the full field equation for $X^{(9)}$ in this thesis. However it can be determined via a direct dimensional reduction of (8.32).

spacetime sense) massive deformation of IIB. Furthermore these fields form part of a doublet which can be seen by mapping the $D = 11$ doublet $\epsilon^{bc} i_{\hat{k}_b} \hat{F}^{(11)}_{ca}$ to IIB. The IIB doublet structure is then found to be

$$\mathcal{G}^{(10)}_a = \begin{pmatrix} -i_\beta \mathcal{X}^{(11)} - \frac{9}{8}\mathcal{S}^{(10)} + l(i_\beta \mathcal{H}^{(11)} - \frac{3}{2}\mathcal{G}^{(10)}) \\ i_\beta \mathcal{H}^{(11)} - \frac{3}{2}\mathcal{G}^{(10)} \end{pmatrix}. \tag{C.32}$$

The way that the 10-form and 11-form fields combine here is analogous to that seen for the 8-form singlet (12.51). It is possible that this doublet is related to the doublet of masses $(m_4, \tilde{m}_4)$ discussed in [90] through Hodge duality.

The potentials $\mathcal{N}^{(9)}$ and $\mathcal{T}^{(9)}$ appear in a doublet of IIB charges that map to the $D = 11$ doublet (10.57). These charges appear to correspond to a doublet of 9-branes in IIB with leading bilinear terms given by ${\mathcal{M}_a}^b i_\beta \Omega_b$ and which minimally couple to the combination of potentials $\frac{3}{2} i_\beta \mathcal{B}^{(10)} + \frac{3}{2}\mathcal{N}^{(9)}$ and $\frac{3}{2} i_\beta \mathcal{A}^{(10)} - \frac{9}{8}\mathcal{T}^{(9)}$ which, after the appropriate redefinitions, form a doublet. Due to the form of the leading bilinear terms we determine that the 9-branes each contain an isometry direction parallel to the worldvolume. Therefore even though they are spacetime filling, due to the presence of the isometry directions and the fact that they minimally couple to the potentials $\mathcal{T}^{(9)}$ and $\mathcal{N}^{(9)}$, these branes should act as the sources of a doublet of masses. These branes therefore play an analogous role to the M9-branes, and would appear to be the IIB origin of the mass doublet $(m_4, \tilde{m}_4)$ discussed in [90].

A proper treatment of these branes should therefore take into account the role of these mass parameters which we do not investigate in this thesis. However we note that the differential relation for the leading bilinear doublet must be of the form

$$d({\mathcal{M}_a}^b i_\beta \Omega_b) \sim i_{K^+} \mathcal{G}^{(10)}_a \tag{C.33}$$

in order for the charges to be closed. Comparing this with the conjectured relation (12.67) it would seem that the 11-forms $\mathcal{H}^{(11)}$ and $\mathcal{X}^{(11)}$ are related to the 11-form doublet $\mathcal{F}^{(11)}_a$, although this is difficult to show directly since there is no Hodge Dual interpretation of these fields.

Finally we comment on the role of the isometry in these charges. Since the charges are 8-forms we denote them by $N^{(m8)}_a$. Following the examples of the KK-monopoles and M9-brane we might expect them to take the form $N^{(m8)}_a = i_\beta N^{(m9)}_a$.

However this is incorrect since we find that $i_\beta N_a^{(m8)} \neq 0$ which is apparent due to the appearance of the 9-form potentials in $N_a^{(m8)}$. In fact it is not clear how to interpret $i_\beta N_a^{(m8)}$ since the leading bilinear terms do vanish. It would therefore seem to be the opposite of the situation encountered with $\hat{L}_a^{(7)}$ and $\hat{L}_{ab}^{(9)}$ in $D = 11$. In those instances the interpretation in terms of generalised calibrations is only obvious after a contraction with a Killing vector, whereas here it is only obvious if such a contraction does not occur.

Appendix D

Non-$SL(2,\mathbb{R})$ covariant 'massive' IIB fields

Here we give the 'massive' IIB field strength equations obtained after making the substitution (12.54) in a non-$SL(2,\mathbb{R})$ covariant fashion. These can then be T-dualised using the rules given throughout this thesis to obtain the field equations in the non-covariant massive IIA theory. They are given by

$$\check{\mathcal{F}}^{(1)} = \mathcal{F}^{(1)} + \left[m_+ + m_-(e^{-2\varphi} - l^2) + 2m_1 l\right] \wedge dy \tag{D.1}$$

$$(\check{d\varphi}) = d\varphi + \left[2m_- l - 2m_1\right] \wedge dy \tag{D.2}$$

$$\check{\mathcal{F}}^{(3)} = \mathcal{F}^{(3)} + \left[m_+\mathcal{B} - m_- l\mathcal{C}^{(2)} + m_1(\mathcal{C}^{(2)} + l\mathcal{B})\right] \wedge dy \tag{D.3}$$

$$\check{\mathcal{H}} = \mathcal{H} + \left[m_-\mathcal{C}^{(2)} - m_1\mathcal{B}\right] \wedge dy \tag{D.4}$$

$$\check{\mathcal{F}}^{(5)} = \mathcal{F}^{(5)} + \left[\frac{1}{2}m_+(\mathcal{B})^2 - \frac{1}{2}m_-(\mathcal{C}^{(2)})^2 + m_1\mathcal{B}\wedge\mathcal{C}^{(2)}\right] \wedge dy \tag{D.5}$$

$$\begin{aligned}\check{\mathcal{F}}^{(7)} &= \mathcal{F}^{(7)} + \Big[\frac{1}{3!}m_+(\mathcal{B})^3 - m_-\mathcal{B}^{(6)} \\ &\quad + m_1(-\mathcal{C}^{(6)} + \mathcal{C}^{(4)}\wedge\mathcal{B})\Big] \wedge dy\end{aligned} \tag{D.6}$$

$$\begin{aligned}\check{\mathcal{H}}^{(7)} &= \mathcal{H}^{(7)} + m_+\Big[-\mathcal{C}^{(6)} + \mathcal{C}^{(4)}\wedge\mathcal{B} - \frac{1}{2}\mathcal{C}^{(2)}\wedge(\mathcal{B})^2 + \frac{1}{3!}l(\mathcal{B})^3\Big] \wedge dy \\ &\quad + m_-\Big[\frac{1}{3!}(\mathcal{C}^{(2)})^3 - l(\mathcal{B})^3\Big] \wedge dy + m_1\Big[\mathcal{B}^{(6)} - \frac{1}{2}(\mathcal{C}^{(2)})^2\wedge\mathcal{B} \\ &\quad - l\mathcal{C}^{(6)} + l\mathcal{C}^{(4)}\wedge\mathcal{B}\Big] \wedge dy\end{aligned} \tag{D.7}$$

$$\check{\mathcal{F}}^{(9)} = \mathcal{F}^{(9)} + \left[\frac{1}{4!}m_+(\mathcal{B})^4 + 2m_-\varphi^{(8)} + m_1(-2\mathcal{C}^{(8)} + \mathcal{C}^{(6)}\wedge\mathcal{B})\right] \wedge dy \tag{D.8}$$

$$\begin{aligned}\check{\mathcal{H}}^{(9)} &= \mathcal{H}^{(9)} + m_+\Big[\mathcal{C}^{(8)} - \frac{1}{2}\mathcal{C}^{(6)}\wedge\mathcal{B} + \frac{1}{12}\mathcal{C}^{(2)}\wedge(\mathcal{B})^3 - \frac{1}{4!}l(\mathcal{B})^4\Big] \wedge dy \\ &\quad m_-\Big[-\mathcal{N}^{(8)} + \frac{1}{2}\mathcal{B}^{(6)}\wedge\mathcal{C}^{(2)} - 2l\varphi^{(8)}\Big] \wedge dy + m_1\Big[-\frac{1}{2}\mathcal{C}^{(6)}\wedge\mathcal{C}^{(2)} \\ &\quad -\frac{1}{2}\mathcal{B}^{(6)}\wedge\mathcal{B} + \frac{1}{2}\mathcal{C}^{(4)}\wedge\mathcal{C}^{(2)}\wedge\mathcal{B} + 2l\mathcal{C}^{(8)} - l\mathcal{C}^{(6)}\wedge\mathcal{B}\Big] \wedge dy\end{aligned} \tag{D.9}$$

$$\begin{aligned}(e^{-2\varphi}\check{\mathcal{F}}^{(9)}) &= e^{-2\varphi}\mathcal{F}^{(9)} + m_+\Big[-2\varphi^{(8)} - \mathcal{B}^{(6)}\wedge\mathcal{B} - \mathcal{C}^{(6)}\wedge\mathcal{C}^{(2)} + \mathcal{C}^{(4)}\wedge\mathcal{C}^{(2)}\wedge\mathcal{B} \\ &\quad -\frac{1}{4}(\mathcal{C}^{(2)})^2\wedge(\mathcal{B})^2 + 2l\mathcal{C}^{(8)} - l\mathcal{C}^{(6)}\wedge\mathcal{B} + \frac{1}{3!}l\mathcal{C}^{(2)}\wedge(\mathcal{B})^3 \\ &\quad -\frac{1}{4!}l^2(\mathcal{B})^4\Big] \wedge dy + m_-\Big[\frac{1}{4!}(\mathcal{C}^{(2)})^4 - 2l\mathcal{N}^{(8)} + l\mathcal{B}^{(6)}\wedge\mathcal{C}^{(2)} \\ &\quad -2l^2\varphi^{(8)}\Big] \wedge dy + m_1\Big[2\mathcal{N}^{(8)} - \mathcal{B}^{(6)}\wedge\mathcal{C}^{(2)} - \frac{1}{3!}(\mathcal{C}^{(2)})^3\wedge\mathcal{B} + 2l^2\mathcal{C}^{(8)} \\ &\quad -l^2\mathcal{C}^{(6)}\wedge\mathcal{B} - l\mathcal{C}^{(6)}\wedge\mathcal{C}^{(2)} - l\mathcal{B}^{(6)}\wedge\mathcal{B} \\ &\quad + l\mathcal{C}^{(4)}\wedge\mathcal{C}^{(2)}\wedge\mathcal{B}\Big] \wedge dy.\end{aligned} \tag{D.10}$$