From Gaussian beam to complex-source-point spherical wave

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It is shown that the paraxial Gaussian beam becomes the complex-source-point spherical wave when all-order corrections are made according to the method of Lax, Louisell, and McKnight. Apparent contradictions between previously published first-order corrections are also discussed.

I. INTRODUCTION

The paraxial wave equation[1] gives an accurate description of wave beams near the axis, when the characteristic transverse dimension, the beam waist, is larger than the wavelength. But for the important case of a strongly focused beam, this condition is not satisfied, and corrections to the paraxial beam must be made. Recently, Lax *et al.*[2] developed a consistent scheme to obtain corrections to the paraxial solution. The Gaussian beam (GB) appears as the first term of a series expansion of the powers of a small dimensionless parameter. Each successive correction term is related to the previous one through a differential recurrence relation. These authors did not calculate explicit expressions for the higher-order corrections; Davis[3] calculated a first-order correction using this scheme, and recently Agrawal and Pattanayak[4] also obtained a first-order correction by means of a completely different approach, but the corrections obtained by the two methods are different.

In this paper, we calculate a first-order correction which yields a result that also differs from the previous two. However, we indicate why the three results are different, and show that for the method of Lax *et al.* to yield a unique and self-consistent correction, a condition must be added.

All higher-order corrections are derived in Appendix A, and in Appendix B all the corrections are added to yield a closed-form expression for the corrected GB. We then show that this corrected wave is the so-called complex-source-point spherical wave.[5-6]

II. THE CORRECTION SCHEME

Consider the scalar Helmholtz equation

$$\nabla^2 A + k^2 A = 0 , \tag{1}$$

where A represents a Cartesian component of the vector potential. Substituting $A = \psi \exp(-ikz)$ into Eq. (1) yields

$$\nabla^2\psi - 2ik\frac{\partial\psi}{\partial z} = 0 . \tag{2}$$

Using a cylindrical coordinate system (r, z) with azimuthal symmetry, we define[3] the dimensionless variables as

$$\rho = r/W_0 , \qquad Q = (i + z/z_0)^{-1}, \tag{3}$$

where W_0 and z_0 are, respectively, the characteristic transverse and longitudinal dimensions of the beam. The parameters W_0 and z_0 may be linked with no loss of generality to the wave number k by setting $z_0 = \frac{1}{2}kW_0^2$.

We also define[3] the parameter f as

$$f = W_0/2z_0 . \tag{4}$$

Equation (2) is now written in these variables as

$$\frac{\partial^2\psi}{\partial\rho^2} + \left(\frac{1}{\rho}\right)\frac{\partial\psi}{\partial\rho} + 4iQ^2\frac{\partial\psi}{\partial Q} + 4f^2\frac{\partial}{\partial Q}\left(Q^2\frac{\partial\psi}{\partial Q}\right) = 0 . \tag{5}$$

Lax *et al.*,[2] using a perturbation method,[7] expressed this field as a power series

$$\psi = \sum_{n=0}^{\infty} f^{2n}\psi^{(2n)} , \tag{6}$$

and by grouping together equal powers of f, they obtained the following differential relations:

$$\mathcal{L}(\psi^{(0)}) = 0 , \tag{7}$$

$$\mathcal{L}(\psi^{(2n)}) = -\mathcal{D}(\psi^{(2n-2)}) \quad (n = 1, 2, 3, \ldots) \tag{8}$$

where $\mathcal{L}$ and $\mathcal{D}$ are the differential operators

$$\mathcal{L} = \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + 4iQ^2\frac{\partial}{\partial Q} ,$$

$$\mathcal{D} = 4Q^2\frac{\partial}{\partial Q}\left(Q^2\frac{\partial}{\partial Q}\right) .$$

The differential equation (7) is the well-known paraxial wave equation, which corresponds here to the lowest-order solution. The paraxial Gaussian beam (GB) which is to be corrected is taken as a solution, finite at the origin, of this differential equation. Following Siegman,[8] we write for cylindrical coordinates

$$\psi_m^{(0)} = Q^m L_m(iQ\rho^2)\psi_0^{(0)} , \tag{9}$$

where

$$\psi_0^{(0)} = Q\exp(-iQ\rho^2) , \tag{10}$$

and $L_m(x)$ is the Laguerre polynomial of order m. With this notation, $\psi_m^{(2n)}$ stands for the nth-order correction to the mth-order GB. In the following demonstration, for the higher-order GB we use the simpler form introduced by Siegman, instead of the usual one involving W.

Assuming that $\psi^{(2n-2)}$ has been calculated, the differential recurrence relation (8) is in fact an inhomogeneous differential equation. The solution of this equation is therefore the sum of a particular solution $\psi_{\rm part}^{(2n)}$ and any linear combination of the solutions $\psi_m^{(0)}$ of the corresponding homogeneous equation (7).

Now the solution

$$\psi_0^{(2n)} = \psi_{\rm part}^{(2n)} + \sum_{m=0}^{\infty} a_m \psi_m^{(0)} \tag{11}$$

can be inserted in the inhomogeneous term of Eq. (8), defining an inhomogeneous equation for $\psi^{(2n+2)}$. The solution is again the sum of a particular solution [which depends on the choice of the coefficients a_m in Eq. (11)] and any linear combination of the $\psi_m^{(0)}$ solutions, the combination possibly being different from that of Eq. (11). Such a degree of arbitrariness for each order of correction may lead to inconsistencies, unless a way is found to uniquely specify the linear combination that may be added to each order. This may be achieved by means of a boundary condition.

A physically reasonable condition is that the solution $\psi_m^{(0)}$ be the *paraxial* solution of the problem. This means that the correction terms $\psi_m^{(2n)}$ must be zero along the axis ($\rho = 0$) for all n different from zero. It is easy to see that no linear combination of the $\psi_m^{(0)}$ can be zero along the axis, so they must be eliminated from all orders of correction.

The elimination of all the GB's also means that in the correction terms of a given GB, for instance $\psi_0^{(0)}$, no other order GB $\psi_m^{(0)}$, $m = 1, 2, \ldots,$ should be present; these terms should only be present when correcting the mth-order beam. Obviously, $\psi_0^{(0)}$ itself should not be added to any order of correction, because it is already present in the correction series as the zero-order term.

III. FIRST-ORDER CORRECTION

As an example consider the first-order correction to the lowest-order GB $\psi_0^{(0)}$. By inserting expression (10) for $\psi_0^{(0)}$ in the differential relation (8) with n equal to unity, it can be shown that a particular solution of the resulting inhomogeneous differential equation is

$$\psi_0^{(2)} = [2(iQ\rho^2) - (iQ\rho^2)](iQ)\psi_0^{(0)} . \tag{12}$$

By inspecting the arrangement of the variables Q and $(iQ\rho^2)$, it is easy to verify that this particular solution does not contain any GB $\psi_m^{(0)}$ of any order. For the same problem, Davis[3] obtained for the first-order correction to the lowest-order GB the expression

$$(\psi_0^{(2)})_{\rm Davis} = [2 - (iQ\rho^2)^2](iQ)\psi_0^{(0)} . \tag{13}$$

Note that this correction is not zero along the axis. Using our notation, it may be written

$$(\psi_0^{(2)})_{\rm Davis} = \psi_0^{(2)} + \psi_1^{(0)} , \tag{14}$$

where $\psi_0^{(2)}$ is our correction term of Eq. (12). Thus Davis's correction is equal to our correction plus the first-order GB $\psi_1^{(0)}$. Of course this is still a mathematical solution of Eq. (8), because $\psi_1^{(0)}$ is a solution of the homogeneous equation.

Davis was led to this particular solution by imposing, as it is always the case when assuming sources of a finite extent, that the corrected wave behaves paraxially like a spherical wave for large z, and by making the judicious change of the variable z to the complex one found in the paraxial GB, namely, $z + iz_0$. However, in the expansion of the spherical wave, he took into account only the phase of the spherical wave. Following the same reasoning as Davis, but taking into account in addition the photometric term $1/R$ of the spherical wave, we were able to obtain the result given in Eq. (12). The validity of this process and its ability to generate the correct first-order, as well as any higher-order correction, will become obvious when considering our final result [Eq. (21)].

On the other hand, Agrawal *et al.*[4] (Ag) developed a very different method to obtain corrections to the GB: they specified a different boundary condition, that the field be equal to the GB in the plane $z = 0$, and they used the angular spectrum representation to calculate the field distribution at an arbitrary plane z. The result was then expanded in powers of the parameter f^2, which were considered as higher-order corrections to the GB. With our notation, their first-order correction is

$$(\psi_0^{(2)})_{\rm Ag} = \psi_0^{(2)} - 2(i\psi_1^{(0)} + \psi_2^{(0)}) . \tag{15}$$

This result is again a solution of Eq. (8), because the two GB's of orders 1 and 2 in the parentheses are solutions of the homogeneous equation. The difference between their result and ours may come either from the different approximations made in their analysis, or from the fact that their method of correction, with their boundary condition, cannot avoid mixing together contributions from different GB's in addition to the one being corrected.

IV. HIGHER-ORDER CORRECTIONS

We may now calculate the next order of correction $\psi_0^{(4)}$ by means of the method described above, and so on. In Appendix A the reduced variables Q and $(iQ\rho^2)$ are used. The differential relation (8) may then be separated to obtain a second-order differential relation for the variable $iQ\rho^2$. From this differential relation we also obtain a power-series solution whose zero-order term is the lowest-order GB. The nth term reads

$$\psi_0^{(2n)}=[(iQ\rho^2)^n L_n^n(iQ\rho^2)](iQ)^n\psi_0^{(0)} , \qquad (16)$$

where $L_n^n(x)$ is the associated Laguerre polynomial

$$L_n^n(x)=(2n)! \sum_{m=0}^{n} \frac{(-x)^m}{m!(n-m)!(n+m)!} . \qquad (17)$$

Furthermore, the higher-order correction ($n\neq 0$) satisfies the boundary condition along the axis. When all the corrections are added according to Eq. (6), the total corrected field is

$$\psi_0=\psi_0^{(0)} \sum_{n=0}^{\infty} (-1)^n (Q\rho f)^{2n} L_n^n(iQ\rho^2) . \qquad (18)$$

In Appendix B, a generating function for the spherical Hankel function is used to show that ψ_0 can be written in the closed form

$$\psi_0=\frac{C}{2f^2}\left[\exp\left(\frac{i}{2Qf^2}\right)\right] \times\left(\frac{\exp\{(-i/2Qf^2)[1+(2Q\rho f)^2]^{1/2}\}}{(-i/2Qf^2)[1+(2Q\rho f)^2]^{1/2}}\right), \qquad (19)$$

which, written in terms of the real coordinates of the problem, is

$$\psi_0=ce^{ikz}\frac{e^{-ikR_c}}{R_c} , \qquad (20)$$

where

$$R_c=[r^2+(z+iz_0)^2]^{1/2} ,$$

and where c is a constant. Thus the final expression for the vector potential is

$$A_0=\frac{e^{-ikR_c}}{R_c} . \qquad (21)$$

Clearly, Eq. (21) is a solution of the Helmholtz equation (1). It was first observed by Deschamps[5] that such a complex-source-point spherical wave could be expanded near the z axis in a power series whose first term was the GB. This wave has been extensively analyzed by Felsen.[6]

V. CORRECTIONS TO HIGHER-ORDER GAUSSIAN BEAMS

The same analysis can be applied to calculate corrections to higher-order GB's. It appears reasonable to suppose that those corrections could be summed, and the result expressed in terms of higher-order complex-source-point spherical waves

$$A_{mn}=h_m^{(2)}(kR_c)P_m^n(\cos\theta_c)e^{\pm in\phi}, \qquad (22)$$

where the complex polar angle θ_c is defined by

$$\cos\theta_c=\frac{z+iz_0}{R_c} . \qquad (23)$$

Preliminary studies have shown that, at least for the azimuthally symmetrical Siegman-type Gaussian beams, the sum of all corrections may be expressed as simple linear combinations of complex-source-point spherical waves. It is probable that for nonsymmetrical higher-order beams, such a correspondence can also be made. This is also suggested by a result obtained by Shin and Felsen.[9] These authors showed that multipole fields generated by the complex-source-point spherical wave behave, near the axis, like the Hermite-Gaussian beams introduced by Siegman.

VI. CONCLUSION

By means of the perturbation method used by Lax *et al.*,[2] with the additional condition that the corrections be zero along the axis, it is possible to calculate in a consistent manner all the corrections to the lowest-order paraxial Gaussian beam. We have shown that the sum of all the corrections transforms the GB into the complex-source-point spherical wave. Because the complex-source-point spherical wave has a very simple mathematical form, it should be used more widely to deal with very strong focusing of laser beams.

APPENDIX A: SOLUTION OF THE RECURRENCE RELATION FOR $\psi_0^{(2n)}$

The method of separation of variables may be used to solve the following recurrence relation (5):

$$\frac{\partial^2\psi^{(2n)}}{\partial\rho^2}+\left(\frac{1}{\rho}\right)\frac{\partial\psi^{(2n)}}{\partial\rho}+4iQ^2\frac{\partial\psi^{(2n)}}{\partial Q} =-4Q^2\frac{\partial}{\partial Q}\left(Q^2\frac{\partial\psi^{(2n-2)}}{\partial Q}\right). \qquad (A1)$$

As stated in the text, solutions for $\psi^{(0)}$ are obtained by setting the right-hand side of this recurrence relation to zero. The solution of the resulting differential equation is then obtained by a separation of the variables Q and $iQ\rho^2$. Here we seek a solution for Eq. (A1) with the limiting form $\psi^{(0)}$ when n goes to zero. Let

$$u = iQ\rho^2 \ . \tag{A2}$$

The recurrence relation for the variables u and Q becomes

$$u\frac{\partial^2\psi^{(2n)}}{\partial u^2} + (1+u)\frac{\partial\psi^{(2n)}}{\partial u} + Q\frac{\partial\psi^{(2n)}}{\partial Q} = iQ\left[\frac{\partial}{\partial u}\left(u^2\frac{\partial\psi^{(2n-2)}}{\partial Q}\right) + 2uQ\frac{\partial^2\psi^{(2n-2)}}{\partial u\partial Q} + \frac{\partial}{\partial Q}\left(Q^2\frac{\partial\psi^{(2n-2)}}{\partial Q}\right)\right] . \tag{A3}$$

When n is equal to zero, that is, when the right-hand side of the recurrence relation is equal to zero, the result is

$$\psi_m^{(0)} = [e^{-u}L_m(u)](iQ)^{m+1} \quad (m=0,1,2,\dots) \tag{A4}$$

which is the product of a function of u multiplied by a power of Q. The integer m is associated with higher-order GB's and will be set equal to zero here, in order to find corrections for the lowest-order GB. This result for n equal to zero suggests a separation of the same form, namely, a function of u multiplied by a power of Q for other values of n. A close examination of Eq. (A3) shows that a separation of the variables is achieved for

$$\psi_0^{(2n)} = [U^{(2n)}(u)](iQ)^{n+1} , \tag{A5}$$

where $U^{(2n)}(u)$ is a solution of the differential relation

$$u\frac{d^2U^{(2n)}}{du^2} + (1+u)\frac{dU^{(2n)}}{du} + (n+1)U^{(2n)} = u^2\frac{d^2U^{(2n-2)}}{du^2} + 2(n+1)u\frac{dU^{(2n-2)}}{du} + n(n+1)U^{(2n-2)} \ . \tag{A6}$$

A solution which is finite at the origin may be obtained by means of the power-series method. We have found that the form

$$U^{(2n)}(u) = \sum_{m=0}^{\infty} B_m^{(n)}u^{m+n} \tag{A7}$$

leads to a simple recurrence relation for $B_m^{(n)}$:

$$(m+n)B_m^{(n)} + (m+2n)B_{m-1}^{(n)} = (m+2n)(m+2n-1)B_m^{(n-1)} \tag{A8}$$

for $m=0,1,2,\dots,$ and $B_{-1}^{(n)} \equiv 0$. We were able to show that the coefficient $B_m^{(n)}$

$$B_m^{(n)} = \frac{C(-1)^m\Gamma(m+2n+1)}{m!\,n!\,\Gamma(m+n+1)} , \tag{A9}$$

where C is a constant, satisfying the recurrence relation (A8).

Using the hypergeometric notation, $U^{(2n)}(u)$ may be written

$$U^{(2n)}(u) = \frac{C(2n)!}{(n!)^2}u^n\,{}_1F_1\left(\begin{matrix}2n+1\\ n+1\end{matrix}\middle| -u\right), \tag{A10}$$

where ${}_1F_1$ is a confluent hypergeometric function. It can be shown by means of a Kummer transformation of this function[10] that

$$U^{(2n)}(u) = Ce^{-u}u^nL_n^n(u) , \tag{A11}$$

where $L_n^n(u)$ is the associated Laguerre polynomial.

Finally, our solution for $\psi_0^{(2n)}$ is

$$\psi_0^{(2n)} = C(-Q^2\rho^2)^nL_n^n(iQ\rho^2)\psi_0^{(0)} , \tag{A12}$$

where $\psi_0^{(0)}$ is the lowest-order GB

$$\psi_0^{(0)} = (iQ)\exp(-iQ\rho^2) . \tag{A13}$$

A second solution to the recurrence relation for $U^{(2n)}$ of Eq. (A6) may readily be obtained by inspection:

$$U^{(2n)} = C_2 n! ,$$

where C_2 is a constant. This solution is linearly independent of the previous one. The corresponding form for $\psi_0^{(2n)}$ is

$$\psi_0^{(2n)} = C_2 n!(iQ)^{n+1} ,$$

which differs from the expression for the paraxial Gaussian beam when $n=0$. This solution is therefore rejected as unrelated to our problem.

APPENDIX B: SUMMATION OF THE $\psi_0^{(2n)}$

In this appendix the $\psi_0^{(2n)}$ found in Appendix A are added according to Eq. (6),

$$\psi_0 = \sum_{n=0}^{\infty} f^{2n}\psi_0^{(2n)} \tag{B1}$$

taking for $\psi_0^{(2n)}$ its power-series expansion obtained in Appendix A, Eqs. (A7) and (A9). Equation (B1) may then be expressed as the double summation

$$\psi_0 = C(iQ)\sum_{n,m=0}^{\infty}\frac{(-Q^2\rho^2f^2)^n(-iQ\rho^2)^m\Gamma(m+2n+1)}{n!\,m!\,\Gamma(m+n+1)} . \tag{B2}$$

Changing the order of summation permits us to write

$$\psi_0 = C(iQ)\sum_{m=0}^{\infty}\frac{(-iQ\rho^2)^m}{m!}\sum_{n=0}^{m}\frac{\Gamma(m+n+1)}{n!\,\Gamma(m-n+1)}(-iQf^2)^n . \tag{B3}$$

The last summation represents a polynomial of order m which may be related to the spherical Hankel function $h_m^{(2)}$,[10] yielding

$$\psi_0 = \frac{C}{2f^2} \exp\left(\frac{i}{2Qf^2}\right) \sum_{m=0}^{\infty} \frac{(-Q\rho^2)^m}{m!} h_m^{(2)}\left(\frac{1}{2Qf^2}\right) . \tag{B4}$$

Using the generating functions for the spherical Bessel functions y_n and j_n (Ref. 10), it can be shown that a generating function for $h_n^{(2)}$ is

$$\left(\frac{1}{Z}\right)\exp[-i(Z^2+2Zt)]^{1/2} = \sum_{n=0}^{\infty} \frac{(-t)^n h_{n-1}^{(2)}(Z)}{n!} , \tag{B5}$$

where $2|t|<|Z|$.

By differentiating this generating function relative to t, one obtains

$$\frac{\exp[-i(Z^2+2Zt)^{1/2}]}{-i(Z^2+2Zt)^{1/2}} = \sum_{n=0}^{\infty} \frac{(-t)^n h_n^{(2)}(Z)}{n!} . \tag{B6}$$

This last generating function leads to a closed-form expression for ψ_0,

$$\psi_0 = \frac{C}{2f^2}\left[\exp\left(\frac{i}{2Qf^2}\right)\right] \times \left(\frac{\exp\{(-i/2Qf^2)[1+(2Q\rho f)^2]^{1/2}\}}{(-i/2Qf^2)[1+(2Q\rho f)^2]^{1/2}}\right) . \tag{B7}$$

The convergence condition $2|t|<|Z|$ now reads $|Q\rho f|<1$. By analytical continuation this condition can be relaxed, ensuring the validity of Eq. (B7) for all the possible values of the arguments, except at the two points of discontinuity: $Q\rho f = \pm i$.

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