# $\Omega$-results for Beurling's zeta function and lower bounds for the generalised Dirichlet divisor problem[1]

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**Abstract**

In this paper we study generalised prime systems for which the integer counting function $N_{\mathcal{P}}(x)$ is asymptotically well-behaved, in the sense that $N_{\mathcal{P}}(x) = \rho x + O(x^\beta)$, where $\rho$ is a positive constant and $\beta < \frac{1}{2}$. For such systems, the associated zeta function $\zeta_{\mathcal{P}}(s)$ is holomorphic for $\sigma = \Re s > \beta$. We prove that for $\beta < \sigma < \frac{1}{2}$, $\int_0^T |\zeta_{\mathcal{P}}(\sigma+it)|^2 dt = \Omega(T^{2-2\sigma-\varepsilon})$ for any $\varepsilon > 0$, and also for $\varepsilon = 0$ for all such $\sigma$ except possibly one value.

The Dirichlet divisor problem for generalised integers concerns the size of the error term in $N_{k\mathcal{P}}(x) - \mathrm{Res}_{s=1}(\zeta_{\mathcal{P}}(s)^k x^s/s)$, which is $O(x^\theta)$ for some $\theta < 1$. Letting $\alpha_k$ denote the infimum of such $\theta$, we show that $\alpha_k \geq \frac{1}{2} - \frac{1}{2k}$.

*Keywords*: Beurling's generalised primes, Dirichlet divisor problem.
*AMS Mathematics subject classification* 2010: 11N80.

## 1. Introduction

A *generalised prime system* (or *g-prime system*) $\mathcal{P}$ is a sequence of positive reals $p_1, p_2, p_3, \ldots$ satisfying

$$1 < p_1 \leq p_2 \leq \cdots \leq p_n \leq \cdots$$

and for which $p_n \to \infty$ as $n \to \infty$. From these can be formed the system $\mathcal{N}$ of *generalised integers* or *Beurling integers*; that is, the numbers of the form

$$p_1^{a_1} p_2^{a_2} \ldots p_k^{a_k}$$

where $k \in \mathbb{N}$ and $a_1, \ldots, a_k \in \mathbb{N}_0$.[2] Such systems were first introduced by Beurling [2] and have been studied by many authors since then (see in particular [1]). Define the g-integer counting function $N_{\mathcal{P}}(x)$ and the associated Beurling zeta function, respectively, by

$$N_{\mathcal{P}}(x) = \sum_{n \in \mathcal{N}, n \leq x} 1, \quad \zeta_{\mathcal{P}}(s) = \sum_{n \in \mathcal{N}} \frac{1}{n^s}.$$

(Here, $\sum_{n \in \mathcal{N}}$ means a sum over all the g-integers, counting multiplicities.) In this paper, we shall be concerned with g-prime systems for which

$$N_{\mathcal{P}}(x) = \rho x + O(x^\beta), \tag{1.1}$$

for some $\beta < \frac{1}{2}$ and $\rho > 0$. Then $\zeta_{\mathcal{P}}(s)$ is defined and holomorphic for $\Re s > 1$, and has an analytic continuation to the half-plane $\Re s > \beta$ except for a simple pole at $s = 1$ with residue $\rho$. Furthermore, $\zeta_{\mathcal{P}}(s)$ has *finite order* for $\Re s > \beta$; i.e. $\zeta_{\mathcal{P}}(\sigma + it) = O(|t|^\lambda)$ for some $\lambda$ for $\sigma > \beta$. Let $\mu_{\mathcal{P}}(\sigma)$ denote the infimum of all such $\lambda$. It is well-known that $\mu_{\mathcal{P}}(\sigma)$ is non-negative,

[1] Journal of Number Theory **130** (2010) 707-715.
[2] Here, $\mathbb{N} = \{1, 2, 3, \ldots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and $\mathbb{P} = \{2, 3, 5, \ldots\}$ — the set of primes.

decreasing, and convex (and hence continuous) (see, for example, [5]). For $\mathcal{P} = \mathbb{P}$ (so that $\mathcal{N} = \mathbb{N}$), the Lindelöf Hypothesis is the conjecture that $\mu_{\mathbb{P}}(\sigma) = \mu_0(\sigma)$ for all $\sigma$, where

$$\mu_0(\sigma) = \begin{cases} \frac{1}{2} - \sigma & \text{if } \sigma < \frac{1}{2} \\ 0 & \text{if } \sigma \geq \frac{1}{2} \end{cases}.$$

In [4], it was proven that for all g-prime systems satisfying (1.1), $\mu_{\mathcal{P}}(\sigma)$ must be *at least* as large as $\mu_0(\sigma)$: i.e. $\mu_{\mathcal{P}}(\sigma) \geq \frac{1}{2} - \sigma$ for $\sigma \in (\beta, \frac{1}{2})$. In this paper we prove a stronger result by considering the mean square behaviour of $\zeta_{\mathcal{P}}(\sigma + it)$. For $\sigma > \beta$, define $\nu_{\mathcal{P}}(\sigma)$ to be the infimum of numbers $\lambda$ such that

$$\int_1^T |\zeta_{\mathcal{P}}(\sigma + it)|^2 \, dt = O(T^{1+2\lambda}).$$

As in the case of $\mu_{\mathcal{P}}(\sigma)$, $\nu_{\mathcal{P}}(\sigma)$ is non-negative and convex decreasing (cf. [6], §7.8). Trivially, $\nu_{\mathcal{P}}(\sigma) \leq \mu_{\mathcal{P}}(\sigma)$. We show here that $\nu_{\mathcal{P}}(\sigma) \geq \mu_0(\sigma)$. In fact we prove slightly more.

**Theorem 1**
*Let $\mathcal{P}$ be a g-prime system for which (1.1) holds for some $\beta < \frac{1}{2}$ and $\rho > 0$. Then $\nu_{\mathcal{P}}(\sigma) \geq \mu_0(\sigma)$ for $\sigma \in (\beta, \frac{1}{2})$. Furthermore,*

$$\int_0^T |\zeta_{\mathcal{P}}(\sigma + it)|^2 \, dt = o(T^{2-2\sigma}) \tag{1.2}$$

*can hold for at most one value of $\sigma$ in this range. In this case $T^{2\sigma-2} \int_0^T |\zeta_{\mathcal{P}}(\sigma + it)|^2 dt$ is unbounded for all other values of $\sigma$.*

*Remark.* For $\mathcal{P} = \mathbb{P}$, we have $\nu_{\mathcal{P}}(\sigma) = \mu_0(\sigma)$, which shows the first part of Theorem 1 is best possible. However, in this case we have the asymptotic formula

$$\int_1^T |\zeta(\sigma + it)|^2 \, dt \sim \frac{\zeta(2 - 2\sigma)}{(2\pi)^{1-2\sigma}(2 - 2\sigma)} T^{2-2\sigma}$$

for $0 < \sigma < \frac{1}{2}$, showing that the exceptional value need not exist. In fact it seems unlikely an exceptional value exists and hence that $\int_0^T |\zeta_{\mathcal{P}}(\sigma + it)|^2 dt = \Omega(T^{2-2\sigma})$ for *all* $\sigma \in (\beta, \frac{1}{2})$, but we cannot quite show this. Furthermore it seems plausible that we should have $\int_0^T |\zeta_{\mathcal{P}}(\sigma + it)|^2 dt \geq C_\sigma T^{2-2\sigma}$ for some $C_\sigma > 0$.

## 2. Dirichlet divisor problems for g-primes

For a g-prime system satisfying (1.1) (with $\beta < 1$), we can study the equivalent of the Dirichlet divisor problem concerning the error term in the asymptotic formula for the average of the 'generalised divisor' function. For $k \in \mathbb{N}$, let $k\mathcal{P}$ denote the g-prime system obtained from $\mathcal{P}$ by letting every g-prime from $\mathcal{P}$ be counted $k$ times. (If an original g-prime has multiplicity $m$, then in the new system it will have multiplicity $km$.) The Beurling zeta function of $k\mathcal{P}$ is

$$\zeta_{k\mathcal{P}}(s) = \zeta_{\mathcal{P}}(s)^k.$$

By standard methods using Perron's formula,

$$N_{k\mathcal{P}}(x) = \mathrm{Res}_{s=1}\Big\{\frac{\zeta_{\mathcal{P}}(s)^k}{s} x^s\Big\} + \Delta_{\mathcal{P},k}(x) = xP_{k-1}(\log x) + \Delta_{\mathcal{P},k}(x),$$

where $P_{k-1}(\cdot)$ is a polynomial of degree $k-1$ and $\Delta_{\mathcal{P},k}(x) = O(x^\theta)$ for some $\theta < 1$, depending on $k$. Let $\alpha_k$ denote the infimum of such $\theta$. The *generalised Dirichlet divisor problem* is the problem of determining $\alpha_k$. Also let $\beta_k$ denote the infimum of $\phi$ for which

$$\int_0^x \Delta_{\mathcal{P},k}(y)^2\, dy = O(x^{1+2\phi}).$$

Trivially, $\beta_k \le \alpha_k$.

For $\mathbb{P}$, it is known that

$$\alpha_k \ge \beta_k \ge \frac{1}{2} - \frac{1}{2k} \tag{2.1}$$

and it is conjectured that there is equality throughout (actually $\beta_k = \frac{1}{2} - \frac{1}{2k}$ for all $k$ is equivalent to the Lindelöf Hypothesis — see [6], Theorem 13.4). We use Theorem 1 to show that (2.1) remains true for $\mathcal{P}$ satisfying (1.1). In fact we have the following two corollaries:

**Corollary 2**
*Let $\mathcal{P}$ satisfy* (1.1) *for some $\beta < \frac{1}{2}$. Then for $\sigma \in (\beta, \frac{1}{2} - \frac{1}{2k})$,*

$$\int_{-\infty}^{\infty} \frac{|\zeta_{\mathcal{P}}(\sigma+it)|^{2k}}{|\sigma+it|^2}\, dt \tag{2.2}$$

*diverges. Further, if $\frac{1}{2} - \frac{1}{2k}$ is not the exceptional value in* (1.2)*, then the integral also diverges for $\sigma = \frac{1}{2} - \frac{1}{2k}$.*

**Corollary 3**
*Let $\mathcal{P}$ satisfy* (1.1) *for some $\beta < \frac{1}{2}$. With $\alpha_k$ and $\beta_k$ as above, $\alpha_k \ge \beta_k \ge \max\{\beta, \frac{1}{2} - \frac{1}{2k}\}$.*

## 3. Proofs

*Proof of Theorem 1.* If $\nu_{\mathcal{P}}(\sigma') < \frac{1}{2} - \sigma'$ for some $\sigma' \in (\beta, \frac{1}{2})$ then, by continuity of $\nu_{\mathcal{P}}(\cdot)$, $\nu_{\mathcal{P}}(\sigma) < \frac{1}{2} - \sigma$ throughout some interval around $\sigma'$ and (1.2) holds for all such $\sigma$; in particular for two such values. We shall show that this is impossible.

Suppose, for a contradiction, that (1.2) holds for $\sigma = \sigma_0, \sigma_1$ where $\beta < \sigma_0 < \sigma_1 < \frac{1}{2}$.

For $N \ge 1$ let $\zeta_{N,\mathcal{P}}(s) = \sum_{n \le N} n^{-s}$, where the sum ranges over $n \in \mathcal{N}$. As was stated in [4] (and shown in [3]), for $\sigma < \frac{1}{2}$ there exist constants $c_1, c_2 > 0$ such that for $R \ge c_1 N$,

$$\sum_{r=1}^{R} \int_0^{2r-1} |\zeta_{N,\mathcal{P}}(\sigma+it)|^2\, dt \ge c_2 R^2 N^{1-2\sigma}. \tag{3.1}$$

Also, writing $s = \sigma + it$, and following the arguments in [3], we have

$$\zeta_{N,\mathcal{P}}(s) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{\zeta_{\mathcal{P}}(s+w)N^w}{w}\, dw + O\Big(\frac{N^c}{T(c+\sigma-1)}\Big) + O\Big(\frac{N^{1-\sigma}}{T} \sum_{\substack{\frac{N}{2} < n < 2N \\ n \in \mathcal{N}}} \frac{1}{|n-N|}\Big), \tag{3.2}$$

for $|t| < T$, $c > 1 - \sigma$ and $N \notin \mathcal{N}$. We shall put $c = 1 - \sigma + \frac{1}{\log N}$ and choose $N$ in such a way that $(N-\alpha, N+\alpha) \cap \mathcal{N} = \emptyset$. (As was shown in [4], this is possible for arbitrarily large $N$ if

$0 < \alpha < \frac{1}{4\rho}$.) With this choice of $N$, the final sum in (3.2) was shown to be $O(\sqrt{N})$. As such (3.2) becomes

$$\zeta_{N,\mathcal{P}}(s) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{\zeta_{\mathcal{P}}(s+w)N^w}{w}\,dw + O\Big(\frac{N^{\frac{3}{2}-\sigma}}{T}\Big). \tag{3.3}$$

Now put $\sigma = \sigma_1$ and push the contour in the integral to the left as far as $\Re w = \sigma_0 - \sigma_1 < 0$, picking up the residues at $w = 0$ and $w = 1 - s$ (since $|t| < T$).

The contribution along the horizontal line $[\sigma_0 - \sigma_1 + iT, c + iT]$ is, in modulus, less than

$$\frac{1}{2\pi T}\int_{\sigma_0-\sigma_1}^{c} N^y|\zeta_{\mathcal{P}}(\sigma_1 + y + i(t+T))|\,dy.$$

Using the uniform bound $|\zeta_{\mathcal{P}}(\sigma + it)| = O(t^{\frac{1-\sigma}{1-\beta}+\varepsilon})$, this is at most a constant times

$$\frac{1}{T}\int_{\sigma_0-\sigma_1}^{1-\sigma_1} T^{\frac{1-\sigma_1-y}{1-\beta}+\varepsilon}N^y\,dy + \frac{1}{T}\int_{1-\sigma_1}^{1-\sigma_1+\frac{1}{\log N}} T^{\varepsilon}N^y\,dy = O(T^{\frac{\beta-\sigma_0}{1-\beta}+\varepsilon}N^{\sigma_0-\sigma_1}) + O(T^{\varepsilon-1}N^{1-\sigma_1}). \tag{3.4}$$

Similarly on $[\sigma_0 - \sigma_1 - iT, c - iT]$.

The integral along $\Re w = \sigma_0 - \sigma_1$ is at most

$$\begin{aligned}\frac{N^{\sigma_0-\sigma_1}}{2\pi}\int_{-T}^{T}\frac{|\zeta_{\mathcal{P}}(\sigma_0 + i(t+y))|}{\sqrt{(\sigma_1-\sigma_0)^2+y^2}}\,dy &= O\Big(N^{\sigma_0-\sigma_1}\int_1^{2T}\frac{|\zeta_{\mathcal{P}}(\sigma_0+iy)|}{y}\,dy\Big)\\ &= o(N^{\sigma_0-\sigma_1}T^{\frac{1}{2}-\sigma_0}),\end{aligned} \tag{3.5}$$

using[3] the hypothetical bound $\int_0^T|\zeta_{\mathcal{P}}(\sigma_0+it)|^2dt = o(T^{2-2\sigma_0})$.

The residues at $w = 0$ and $w = 1-s$ are, respectively, $\zeta_{\mathcal{P}}(s)$ and $\rho N^{1-s}/(1-s) = O(\frac{N^{1-\sigma_1}}{|t|+1})$. Putting (3.3), (3.4), and (3.5) together gives

$$\zeta_{N,\mathcal{P}}(\sigma_1+it) = \zeta_{\mathcal{P}}(\sigma_1+it) + O\Big(\frac{N^{1-\sigma_1}}{|t|+1}\Big) + O(N^{1-\sigma_1}T^{\varepsilon-1}) + o(N^{\sigma_0-\sigma_1}T^{\frac{1}{2}-\sigma_0}) + O\Big(\frac{N^{\frac{3}{2}-\sigma_1}}{T}\Big),$$

for $|t| < T$. (Note that the first $O$-term in (3.4) is superfluous since $\frac{\beta-\sigma_0}{1-\beta} < \frac{1}{2} - \sigma_0$.) Hence, using $(a+b+c+d+e)^2 \le 5(a^2+b^2+c^2+d^2+e^2)$, we have

$$|\zeta_{N,\mathcal{P}}(\sigma_1+it)|^2 \le 5|\zeta_{\mathcal{P}}(\sigma_1+it)|^2 + O\Big(\frac{N^{2-2\sigma_1}}{t^2+1}\Big) + O(N^{2-2\sigma_1}T^{2\varepsilon-2}) + o(N^{2\sigma_0-2\sigma_1}T^{1-2\sigma_0}) + O\Big(\frac{N^{3-2\sigma_1}}{T^2}\Big).$$

Now apply $\sum_{r=1}^R\int_0^{2r-1}\ldots dt$ to both sides to give (for $2R-1 < T$)

$$\begin{aligned}\sum_{r=1}^R\int_0^{2r-1}|\zeta_{N,\mathcal{P}}(\sigma_1+it)|^2\,dt &= O\Big(\sum_{r=1}^R\int_0^{2r-1}|\zeta_{\mathcal{P}}(\sigma_1+it)|^2\,dt\Big) + O\Big(\sum_{r=1}^R\int_0^{2r-1}\frac{N^{2-2\sigma_1}}{(t+1)^2}\,dt\Big)\\ &\quad + O(R^2N^{2-2\sigma_1}T^{2\varepsilon-2}) + O\Big(\frac{R^2N^{3-2\sigma_1}}{T^2}\Big) + o(R^2N^{2(\sigma_0-\sigma_1)}T^{1-2\sigma_0})\\ &= o(R^{3-2\sigma_1}) + O(RN^{2-2\sigma_1}) + O(R^2N^{2-2\sigma_1}T^{2\varepsilon-2}) + O\Big(\frac{R^2N^{3-2\sigma_1}}{T^2}\Big) + o(R^2N^{2(\sigma_0-\sigma_1)}T^{1-2\sigma_0})\end{aligned}$$

[3] If $f \ge 0$ and $\int_0^T f^2 = o(T^\lambda)$ (some $\lambda > 1$), then $\int_{T/2}^T\frac{f(y)}{y}dy \le \frac{2}{T}\int_0^T f \le \frac{2}{T}\sqrt{T\int_0^T f^2} = o(T^{\frac{\lambda-1}{2}})$, and $\int_1^T\frac{f(y)}{y}dy = o(T^{\frac{\lambda-1}{2}})$ follows.

using (1.2) for $\sigma_1$. Let $T = 2R$. The left-hand side above is at least $c_2 R^2 N^{1-2\sigma_1}$ by (3.1) if $R \geq c_1 N$. Dividing both sides through by $R^2 N^{1-2\sigma_1}$ gives

$$c_2 \leq o\Big(\Big(\frac{R}{N}\Big)^{1-2\sigma_1}\Big) + O\Big(\frac{N}{R}\Big) + O(NR^{2\varepsilon-2}) + O\Big(\frac{N^2}{R^2}\Big) + o\Big(\Big(\frac{R}{N}\Big)^{1-2\sigma_0}\Big). \tag{3.6}$$

Put $R = KN$ where $K \geq c_1$ is a fixed, but arbitrary, constant. Letting $N \to \infty$, the $o$-terms both tend to zero as does the middle $O$-term. Hence

$$c_2 \leq \frac{A}{K} + \frac{B}{K^2}$$

for some absolute constants $A, B$. But $K$ can be made arbitrarily large, so this gives a contradiction.

For the final part, suppose (1.2) holds for $\sigma = \sigma_0$ say. If $\int_0^T |\zeta_{\mathcal{P}}(\sigma' + it)|^2 dt = O(T^{2-2\sigma'})$ for some $\sigma' \in (\beta, \frac{1}{2})$ with $\sigma' \neq \sigma_0$, then (1.2) actually holds for all $\sigma$ between $\sigma_0$ and $\sigma'$. (This follows from the Phragmen-Lindelöf Theorem for a strip (see [6], §7.8, with $\varepsilon$ in the place of $C$)). This was shown to be impossible, and hence $T^{2\sigma-2}\int_0^T |\zeta_{\mathcal{P}}(\sigma' + it)|^2 dt$ must be unbounded for all $\sigma \neq \sigma_0$.

□

Now we apply Theorem 1 to find lower bounds in the Dirichlet divisor problem. Note that Theorem 1 actually shows that given $\varepsilon > 0$,

$$\int_{T/2}^{T} |\zeta_{\mathcal{P}}(\sigma + it)|^2\, dt = \Omega(T^{2-2\sigma-\varepsilon}),$$

for if it was $o(T^{2-2\sigma-\varepsilon})$, then by telescoping it would follow that $\int_0^T |\zeta_{\mathcal{P}}(\sigma+it)|^2\, dt = o(T^{2-2\sigma-\varepsilon})$ which is false.

*Proofs of Corollaries 2 and 3.* By Hölder's inequality,

$$\int_{T/2}^{T} |\zeta_{\mathcal{P}}(\sigma + it)|^{2k}\, dt \geq \frac{2^{k-1}}{T^{k-1}}\Big(\int_{T/2}^{T} |\zeta_{\mathcal{P}}(\sigma + it)|^2\, dt\Big)^k,$$

for every $k \in \mathbb{N}$. By Theorem 1, given $\varepsilon > 0$, $\int_{T/2}^{T} |\zeta_{\mathcal{P}}(\sigma + it)|^2 dt \geq aT^{2-2\sigma-\varepsilon}$ for some $a > 0$ and some arbitrarily large $T$. Hence for such $T$,

$$\int_{T/2}^{T} |\zeta_{\mathcal{P}}(\sigma + it)|^{2k}\, dt \geq a^k T^{k(1-2\sigma)+1-\varepsilon k}.$$

It follows that

$$\int_{T/2}^{T} \frac{|\zeta_{\mathcal{P}}(\sigma + it)|^{2k}}{|\sigma + it|^2}\, dt \geq a' T^{k(1-2\sigma)-1-\varepsilon k}$$

for some $a' > 0$. But for $\sigma < \frac{1}{2} - \frac{1}{2k}$, we have $k(1-2\sigma) - 1 > 0$. Hence for $\varepsilon$ sufficiently small, $k(1-2\sigma) - 1 - \varepsilon k > 0$ also, and so $\int_{T/2}^{T} \frac{|\zeta_{\mathcal{P}}(\sigma+it)|^{2k}}{|\sigma+it|^2} dt \not\to 0$ as $T \to \infty$, and Corollary 2 follows. Of course, if $\frac{1}{2} - \frac{1}{2k}$ is not the exceptional value in Theorem 1, then we can take $\varepsilon = 0$ in the above and the result also holds for $\sigma = \frac{1}{2} - \frac{1}{2k}$.

Let $\gamma_k$ be the infimum of $\sigma$ (with $\sigma > \beta$) for which $\int_{-\infty}^{\infty} \frac{|\zeta_{\mathcal{P}}(\sigma+it)|^{2k}}{|\sigma+it|^2} dt$ converges. By Corollary 2, $\gamma_k \geq \frac{1}{2} - \frac{1}{2k}$. An identical argument as in the $\mathcal{P} = \mathbb{P}$ case (see [6], Theorem 12.5) shows that $\gamma_k = \beta_k$. (The argument is simply based upon Parseval's formula for Mellin transforms, which in this case is the identity

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|\zeta_{\mathcal{P}}(\sigma+it)|^{2k}}{|\sigma+it|^2} dt = \int_0^{\infty} \frac{\Delta_{\mathcal{P},k}(x)^2}{x^{1+2\sigma}} dx$$

for $\sigma$ in some interval $(\theta, 1)$ with $\theta < 1$.) Hence $\beta_k \geq \frac{1}{2} - \frac{1}{2k}$.

□

## 4. On the line $\sigma = \frac{1}{2}$

In this article, we have considered the mean-value along vertical lines $\Re s = \sigma$ with $\sigma < \frac{1}{2}$. This raises the question of what happens on the line $\sigma = \frac{1}{2}$. For $\mathcal{P} = \mathbb{P}$, we have $\int_0^T |\zeta(\frac{1}{2}+it)|^2 dt \sim T \log T$, so do we have $\int_0^T |\zeta_{\mathcal{P}}(\frac{1}{2}+it)|^2 dt = \Omega(T \log T)$ in general? As in the $\sigma < \frac{1}{2}$ case, we relate the behaviour of the mean-square value at $\sigma = \frac{1}{2}$ to the behaviour of the mean-square for some $\sigma = \sigma_0 < \frac{1}{2}$.

**Theorem 4**
*Let $\mathcal{P}$ be a g-prime system for which (1.1) holds. If $\int_1^T \frac{|\zeta_{\mathcal{P}}(\sigma+it)|}{t} dt = o((T \log T)^{\frac{1}{2}-\sigma})$ for some $\sigma \in (\beta, \frac{1}{2})$, then $\int_0^T |\zeta_{\mathcal{P}}(\frac{1}{2}+it)|^2 dt = \Omega(T \log T)$.*

Note that the assumption is implied by $\int_1^T |\zeta_{\mathcal{P}}(\sigma+it)|^2 dt = o(T^{2-2\sigma}(\log T)^{1-2\sigma})$.

*Sketch of Proof.* We follow the proof of Theorem 1 as much as possible, this time taking $\sigma_1 = \frac{1}{2}$.

Using the argument in [3] for $\sigma = \frac{1}{2}$, (3.1) becomes: *there exist constants* $c_1, c_2 > 0$ *such that for* $R \geq c_1 N / \log N$,

$$\sum_{r=1}^{R} \int_0^{2r-1} \left| \zeta_{N,\mathcal{P}}\Big(\frac{1}{2}+it\Big) \right|^2 dt \geq c_2 R^2 \log N. \tag{4.1}$$

To see this, note that we have

$$\int_0^T \left| \zeta_{N,\mathcal{P}}\Big(\frac{1}{2}+it\Big) \right|^2 dt = T \sum_{n \leq N}^{*} \frac{1}{n} + 2 \sum_{n \leq N} \frac{1}{\sqrt{n}} \sum_{m<n} \frac{S_{m,n}(T)}{\sqrt{m}},$$

where $S_{m,n}(T) = \frac{\sin(T \log(n/m))}{\log(n/m)}$. (Here $m, n \in \mathcal{N}$ and the $*$ indicates that any multiplicities must be squared.) In any case, we have $\sum_{n \leq N}^{*} \frac{1}{n} \geq \sum_{n \leq N} \frac{1}{n} \geq k_1 \log N$ for some $k_1 > 0$.[4] For $m \leq \frac{n}{2}$, $|S_{m,n}(T)| \leq 1/\log 2$, so this part of the double sum is $O(\sum_{n \leq N} \frac{1}{\sqrt{n}} \sum_{m \leq n/2} \frac{1}{\sqrt{m}}) = O(N)$. Thus, for some positive constants $k_1, k_2$, independent of $T$ and $N$,

$$\int_0^T \left| \zeta_{N,\mathcal{P}}\Big(\frac{1}{2}+it\Big) \right|^2 dt \geq k_1 T \log N + 2 \sum_{n \leq N} \frac{1}{\sqrt{n}} \sum_{\frac{n}{2}<m<n} \frac{S_{m,n}(T)}{\sqrt{m}} - k_2 N.$$

Putting $T = 2r-1$ for $r = 1, 2, \ldots, R$, and summing both sides gives, on noticing that $\sum_{r=1}^{R} \sin((2r-1)\log\frac{n}{m}) = \frac{\sin^2(R \log n/m)}{\sin(\log n/m)} \geq 0$ since $0 < \log n/m < \log 2$,

$$\sum_{r=1}^{R} \int_0^{2r-1} \left| \zeta_{N,\mathcal{P}}\Big(\frac{1}{2}+it\Big) \right|^2 dt \geq k_1 R^2 \log N - k_2 R N,$$

[4] This follows readily from $N_{\mathcal{P}}(x) \sim \rho x$.

and (4.1) follows.

In (3.2), we need a better estimate for the final sum. Let $M \in \mathbb{N}$. Then, with $N$ such that $(N-\alpha, N+\alpha) \cap \mathcal{N} = \emptyset$,

$$\sum_{\substack{\frac{N}{2} < n < 2N \\ n \in \mathcal{N}}} \frac{1}{|n-N|} = \sum_{m=1}^{M} \sum_{\alpha N^{\frac{m-1}{M}} \le |n-N| < \alpha N^{\frac{m}{M}}} \frac{1}{|n-N|} + O(1)$$

$$\le \frac{1}{\alpha} \sum_{m=1}^{M} \frac{1}{N^{\frac{m-1}{M}}} \Big( N(N+\alpha N^{m/M}) - N(N-\alpha N^{m/M}) \Big) + O(1)$$

$$= O(N^{1/M}) + O(N^{\beta}),$$

using (1.1). Since $M$ is arbitrary, this is $O(N^{\beta+\varepsilon})$ for every $\varepsilon > 0$ in any case. Thus (3.3) becomes

$$\zeta_{N,\mathcal{P}}(s) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{\zeta_{\mathcal{P}}(s+w)N^w}{w}\, dw + O\Big(\frac{N^{\frac{1}{2}+\beta+\varepsilon}}{T}\Big).$$

The analysis up to (3.5) remains the same (with $\sigma_0 = \sigma$ and $\sigma_1 = \frac{1}{2}$) but in (3.5) we use the bound assumed in the statement to give $o(N^{\sigma-\frac{1}{2}}(T\log T)^{\frac{1}{2}-\sigma})$. The arguments following (3.5) remain valid and we put $T = 2R$ again, but this time we divide through by $R^2 \log N$. On assuming $\int_0^T |\zeta_{\mathcal{P}}(\frac{1}{2}+it)|^2 dt = o(T\log T)$, (3.6) now becomes

$$c_2 \le o\Big(\frac{\log R}{\log N}\Big) + O\Big(\frac{N}{R\log N}\Big) + O\Big(\frac{NR^{2\varepsilon-2}}{\log N}\Big) + O\Big(\frac{N^{1+2\beta+2\varepsilon}}{R^2}\Big) + o\bigg(\Big(\frac{R\log R}{N}\Big)^{1-2\sigma}\frac{1}{\log N}\bigg).$$

Put $R = KN/\log N$ where $K \ge c_1$ is a fixed, but arbitrary, constant. Letting $N \to \infty$, all the terms tend to zero except the first $O$-term. Hence

$$c_2 \le \frac{A}{K}$$

for some absolute constant $A$. As $K$ can be made arbitrarily large, this gives a contradiction. Hence $\int_0^T |\zeta_{\mathcal{P}}(\frac{1}{2}+it)|^2 dt = \Omega(T\log T)$.

□

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