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# Hereditariness, Strongness and Relationship between Brown-McCoy and Behrens Radicals

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**Abstract.** In this paper we explore the properties of being hereditary and being strong among the radicals of associative rings, and prove certain results such as a relationship between Brown-McCoy and Behrens radicals.

MSC 2000: 16N80

## I.

In this paper rings are all associative, but not necessarily with a unit element. As usual,  $I \triangleleft A$  and  $L \triangleleft_l A$  ( $R \triangleleft_r A$ ) denote that  $I$  is an ideal and  $L$  is a left ideal ( $R$  is a right ideal) in  $A$ , respectively.  $A^\circ$  will stand for the ring on the additive group  $(A, +)$  with multiplication  $xy = 0$ , for all  $x, y \in A$ .

Let us recall that a (Kurosh-Amitsur) *radical*  $\gamma$  is a class of rings which is closed under homomorphisms, extensions ( $I$  and  $A/I$  in  $\gamma$  imply  $A$  in  $\gamma$ ), and has the inductive property (if  $I_1 \subseteq \dots \subseteq I_\lambda \subseteq \dots$  is a chain of ideals,  $A = \cup I_\lambda$ , and each  $I_\lambda$  is in  $\gamma$ , then  $A$  is in  $\gamma$ ).

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The unique largest  $\gamma$ -ideal  $\gamma(A)$  of  $A$  is then the  $\gamma$ -radical of  $A$ . A hereditary radical containing all nilpotent rings is called a *supernilpotent radical*. Let  $\mathcal{M}$  be a class of rings. Put

$$\overline{\mathcal{M}} = \{A \mid \text{every ideal of } A \text{ is in } \mathcal{M}\}.$$

A radical  $\gamma$  is said to be *principally left (right) hereditary* if  $a \in A \in \gamma$  implies  $Aa \in \gamma$  ( $aA \in \gamma$ , respectively). A radical  $\gamma$  is said to be *left (right) strong* if  $L \triangleleft_l A$  ( $R \triangleleft_r A$ ) and  $L \in \gamma$  ( $R \in \gamma$ ) imply  $L \subseteq \gamma(A)$  ( $R \subseteq \gamma(A)$ , respectively). A radical  $\gamma$  is *normal* if  $\gamma$  is left strong and principally left hereditary. We shall make use of the following condition a left ideal  $L$  of a ring  $A$  may satisfy with respect to a class  $\mathcal{M}$  of rings:

$$(*) \quad L \triangleleft_l A \text{ and } Lz \in \mathcal{M} \text{ for all } z \in L \cup \{1\}.$$

A radical  $\gamma$  is said to be *principally left strong* if  $L \subseteq \gamma(A)$  whenever the left ideal  $L$  of a ring  $A$  satisfies condition  $(*)$  with respect to the class  $\gamma(= \mathcal{M})$ . Principally right strongness is defined analogously.

We will focus on two conditions that a class  $\mathcal{M}$  can satisfy.

$$(H) \quad \text{If } A^\circ \in \mathcal{M} \text{ then } S \in \mathcal{M} \text{ for every subring } S \subseteq A^\circ.$$

$$(Z) \quad \text{If } A \in \mathcal{M} \text{ then } A^\circ \in \mathcal{M}.$$

A class  $\mathcal{M}$  of rings is said to be *regular* if every nonzero ideal of a ring in  $\mathcal{M}$  has a nonzero homomorphic image in  $\mathcal{M}$ . Starting from a regular (in particular, hereditary) class  $\mathcal{M}$  of rings the *upper radical operator*  $\mathcal{U}$  yields a radical class

$$\mathcal{U}\mathcal{M} = \{A \mid A \text{ has no nonzero homomorphic image in } \mathcal{M}\}.$$

Recall that the *Baer radical*  $\beta$  is the upper radical determined by all prime rings, the *Brown-McCoy radical*  $\mathcal{G}$  is the upper radical determined by all simple rings with unity element, and the *Behrens radical*  $\mathcal{B}$  is the upper radical of all subdirectly irreducible rings having a nonzero idempotent in their hearts.

The lower principally left strong radical construction  $\mathcal{L}_{ps}(\mathcal{M})$  is similar to the lower (left) strong radical construction  $\mathcal{L}_s(\mathcal{M})$  (see [1]).

We shall construct the lower principally left strong radical (see also [7]) in the following way. Let  $\mathcal{M}$  be a homomorphically closed class of rings and define  $\mathcal{M} = \mathcal{M}_1$ ,

$$\mathcal{M}_{\alpha+1} = \left\{ A \mid \begin{array}{l} \text{every nonzero homomorphic image of } A \text{ has a} \\ \text{nonzero left ideal with } (*) \text{ in } \mathcal{M}_\alpha \text{ or a nonzero} \\ \text{ideal } I \in \mathcal{M}_\alpha \end{array} \right\}$$

for ordinals  $\alpha \geq 1$  and  $\mathcal{M}_\lambda = \bigcup_{\alpha < \lambda} \mathcal{M}_\alpha$  for limit ordinals  $\lambda$ . In particular,

$$\mathcal{M}_2 = \left\{ A \mid \begin{array}{l} \text{every nonzero homomorphic image of } A \text{ has a} \\ \text{nonzero left ideal with } (*) \text{ in } \mathcal{M} \text{ or a nonzero ideal} \\ I \in \mathcal{M} \end{array} \right\}.$$

The class  $\mathcal{L}_{ps}(\mathcal{M}) = \bigcup_{\alpha} \mathcal{M}_\alpha$  is called the lower principally left strong radical class. As shown in [6]  $\mathcal{L}_{ps}(\mathcal{M})$  is the smallest principally left strong radical containing  $\mathcal{M}$  and

$$\mathcal{M} \subseteq \mathcal{L}(\mathcal{M}) \subseteq \mathcal{L}_{ps}(\mathcal{M}) \subseteq \mathcal{L}_s(\mathcal{M}).$$

For any class  $\mathcal{M}$  let us define  $\mathcal{M}^\circ = \{A \mid A^\circ \in \mathcal{M}\}$ . It is easy to see that if  $\mathcal{M}$  is a radical then so is  $\mathcal{M}^\circ$ . Let

$$\gamma_l = \{A \in \gamma \mid \text{every left ideal of } A \text{ is in } \gamma\}$$

and

$$\gamma_r = \{A \in \gamma \mid \text{every right ideal of } A \text{ is in } \gamma\}.$$

Next, we recall some results which will be used later on.

**Proposition 1.** [2, Lemma 1] *Let  $\gamma$  be a radical. If  $S$  is a subring of a ring  $A$  such that  $S^\circ \in \gamma$ , then also  $(S^*)^\circ \in \gamma$  where  $S^*$  denotes the ideal of  $A$  generated by  $S$ .*

**Proposition 2.** [5, Lemma 2.4] *Let  $\gamma$  be a radical. If  $(\beta(A))^\circ \in \gamma$ , then  $\beta(A) \in \gamma$ .*

**Proposition 3.** [2, Corollary 1] *If  $\mathcal{M} \subseteq \mathcal{M}^\circ$  then  $\mathcal{L}(\mathcal{M}) \subseteq (\mathcal{L}(\mathcal{M}))^\circ$  and  $\mathcal{L}_s(\mathcal{M}) \subseteq (\mathcal{L}_s(\mathcal{M}))^\circ$ .*

**Proposition 4.** [4, Theorem 4] *If a radical  $\gamma$  is left strong and principally left hereditary, then  $\gamma$  is normal.*

**Proposition 5.** [2, Lemma 2] *For any element  $a$  of a ring  $A$ ,  $I = r(a)a$ , where  $r(a) = \{x \in A \mid ax = 0\}$  is an ideal of  $Aa$  and  $I^2 = 0$ . In addition  $Aa/I$  is a homomorphic image of  $aA$ .*

**Proposition 6.** [5, Corollary 4.2] *A radical  $\gamma$  is hereditary and normal if and only if  $\gamma$  is principally left strong, principally left hereditary and satisfies condition (H).*

**Proposition 7.** [7, Theorem 6] *A radical  $\gamma$  is normal if and only if  $\gamma$  is principally left or right hereditary and principally left or right strong.*

**Proposition 8.** [6, Theorem 3.3] *Let  $\mathcal{M}$  be a homomorphically closed class of rings satisfying:*

- 1)  $\mathcal{M}$  contains all zero rings;
- 2)  $\mathcal{M}$  is hereditary;
- 3) if  $I \triangleleft A$ ,  $I^2 = 0$  and  $A/I \in \mathcal{M}$  then  $A \in \mathcal{M}$ .

Then  $\mathcal{L}_{ps}(\mathcal{M}) = \mathcal{M}_2$ .

**Proposition 9.** [5, Theorem 5.1] *The Behrens radical class  $\mathcal{B}$  is the largest principally left hereditary subclass of the Brown-McCoy radical class  $\mathcal{G}$ , in fact*

$$\mathcal{B} = \mathcal{M}\mathcal{G},$$

where

$$\mathcal{M}\mathcal{G} = \{A \mid Aa \in \mathcal{G} \text{ for all } a \in A\}.$$

A ring  $A$  is said to be (right) strongly prime if every non-zero ideal  $I$  of  $A$  contains a finite subset  $F$  such that  $r_A(F) = 0$ , where  $r_A(F) = \{x \in A \mid Fx = 0\}$ .

The (right) strongly prime radical  $S$  is defined as the upper radical determined by the class of all strongly prime rings, i.e. for any ring  $A$ ,

$$S(A) = \cap \{I \triangleleft A \mid A/I \text{ is strongly prime}\}.$$

It is known that the radical  $S$  is special: so, in particular,  $S$  is hereditary and contains the prime radical  $\beta$ .

**Proposition 10.** [3, Corollary 1] *The (right) strongly prime radical  $S$  is right strong.*

II.

**Proposition 11.** *Let  $\gamma$  be a principally left strong radical satisfying the conditions (H) and (Z). Then the largest hereditary subclass  $\bar{\gamma}$  of  $\gamma$  will be principally left strong.*

*Proof.* Let  $L \triangleleft_l A$  be such that  $L \in \bar{\gamma}$  and  $Lz \in \bar{\gamma}$  for every  $z \in L$ . Let  $L^*$  be the ideal in  $A$  generated by  $L$ ,  $L^* = L + LA$  and suppose  $I \triangleleft L^*$ . Then  $IL \triangleleft L$ ,  $IL \triangleleft_l I$  and  $ILz \triangleleft Lz \in \bar{\gamma}$  for all  $z \in L$ . Since  $\gamma$  satisfies condition (H),  $\bar{\gamma}$  is hereditary, and so  $ILz \in \bar{\gamma}$  for all  $z \in IL$ . Since  $\gamma$  is principally left strong  $IL \subseteq \gamma(I)$ . We have

$$I(L^*)^2 = I(L + LA)L^* = (IL + ILA)L^* \subseteq ILL^* \subseteq \gamma(I)L^* \subseteq \gamma(I).$$

So  $I^3 \subseteq I(L^*)^2 \subseteq \gamma(I)$  and therefore  $I/\gamma(I)$  is nilpotent, implying  $I/\gamma(I) \in \beta$ . We claim that  $I^\circ \in \gamma$ . Since  $L \in \bar{\gamma} \subseteq \gamma$ , by (Z) we conclude that  $L^\circ \in \gamma$ . Now Proposition 1 implies that  $(L^*)^\circ \in \gamma$  and so by (H) it follows  $I^\circ \in \gamma$ . Hence  $(I/\gamma(I))^\circ \in \gamma \cap \beta$  and applying Proposition 2 and taking into consideration that  $I/\gamma(I)$  is nilpotent, we get

$$I/\beta(I) = \beta(I/\gamma(A)) \in \gamma.$$

Thus  $I \in \gamma$  and so  $\bar{\gamma}$  is principally left strong. □

**Corollary 12.** *If a class  $\mathcal{M}$  is hereditary and satisfies (Z) then  $\mathcal{L}_{ps}(\mathcal{M})$  is hereditary.*

*Proof.* By Proposition 3, we have  $\mathcal{L}_{ps}(\mathcal{M}) \subseteq \mathcal{L}_s(\mathcal{M}) \subseteq (\mathcal{L}_s(\mathcal{M}))^\circ$ . Let  $A \in \mathcal{L}_{ps}(\mathcal{M})$  then we get  $A^\circ \in \mathcal{L}_s(\mathcal{M})$  and so  $A^\circ \in \mathcal{L}(\mathcal{M})$ . Since  $\mathcal{L}(\mathcal{M})$  is hereditary, we conclude that  $A^\circ \in \overline{\mathcal{L}(\mathcal{M})}$  and so  $A^\circ \in \overline{\mathcal{L}_{ps}(\mathcal{M})}$ . This means that  $\mathcal{L}_{ps}(\mathcal{M})$  satisfies the conditions (Z) and (H). By Proposition 11,  $\overline{\mathcal{L}_{ps}(\mathcal{M})}$  is principally left strong and  $\mathcal{M} \subseteq \overline{\mathcal{L}_{ps}(\mathcal{M})} \subseteq \mathcal{L}_{ps}(\mathcal{M})$  and this implies  $\overline{\mathcal{L}_{ps}(\mathcal{M})} = \mathcal{L}_{ps}(\mathcal{M})$ . □

**Proposition 13.** *Let  $\gamma$  be a principally left strong radical satisfying the conditions (H) and (Z). Then  $\gamma_r$  is left strong.*

*Proof.* Let  $L \triangleleft_l A$  and  $L \in \gamma_r$  and let  $K$  be a left ideal of  $L^* = L + LA$ . Since  $L \in \gamma_r$ ,  $kL \in \gamma$  for every  $k \in K$ . Let  $R \triangleleft_r kL$ . Then it is easy to see that  $RkL \in \gamma$ , and by conditions (Z) and (H),  $R/RkL \in \gamma$  and so  $R \in \gamma$ . Hence  $kL \in \gamma_r$  for every  $k \in K$ . An argument similar to the proof of Proposition 5 will show that  $(Lk + r(k)k)/r(k)k$  is a homomorphic image of  $kL$ , where  $r(k) = \{x \in L^*/kx = 0\}$ . Hence  $(Lk + r(k)k)/r(k)k \in \gamma$ . By (H) and (Z) we have  $r(k)k \in \gamma$  and so  $Lk \in \gamma$  for every  $k \in K$ . Therefore  $Lk \subseteq \gamma(K)$  and  $LK \subseteq \gamma(K)$ . Clearly

$$K^3 \subseteq (L^*K)K \subseteq (LA^1K)K \subseteq LL^*K \subseteq LK \subseteq \gamma(K)$$

hence  $K \in \gamma$  by Proposition 2. □

The next result is a generalization of [2, Corollary 4].

**Corollary 14.** *If  $\mathcal{M}$  is a right hereditary class with (Z), then  $\mathcal{L}_{ps}(\mathcal{M})$  is one-sided hereditary and  $\mathcal{L}_{ps}(\mathcal{M}) = \mathcal{L}_s(\mathcal{M})$  (i.e.  $\mathcal{L}_{ps}(\mathcal{M})$  is left and right hereditary).*

*Proof.* By Corollary 12,  $\mathcal{L}_{ps}(\mathcal{M})$  satisfies condition (H). Let  $A \in \mathcal{L}_{ps}(\mathcal{M})$ . Then it is easy to see that  $A^\circ \in \mathcal{L}_{ps}(\mathcal{M})$ . Hence  $\mathcal{L}_{ps}(\mathcal{M})$  satisfies condition (Z). Hence  $\mathcal{L}_{ps}(\mathcal{M})_r$  is a radical. By Proposition 13,  $\mathcal{L}_{ps}(\mathcal{M})_r$  is left strong. Since  $\mathcal{M} \subseteq \mathcal{L}_{ps}(\mathcal{M})_r$  we get  $\mathcal{M} \subseteq \mathcal{L}_{ps}(\mathcal{M})_r \subseteq \mathcal{L}_{ps}(\mathcal{M}) \subseteq \mathcal{L}_s(\mathcal{M})$  and  $\mathcal{L}_{ps}(\mathcal{M})_r = \mathcal{L}_s(\mathcal{M})$ . Hence  $\mathcal{L}_{ps}(\mathcal{M}) = \mathcal{L}_s(\mathcal{M})$ . Since  $\mathcal{L}_{ps}(\mathcal{M})_r$  is right hereditary and left strong, we have that  $\mathcal{L}_{ps}(\mathcal{M})$  is one-sided hereditary.  $\square$

**Theorem 15.** *Let  $\gamma \neq 0$  be a principally left strong radical with (Z) and (H). Then  $\gamma_r$  is contained in  $\gamma$  as a largest nonzero hereditary and normal subradical. Furthermore,  $\bar{\gamma}$  is contained in  $\gamma$  as a largest non-zero hereditary principally left strong subradical.*

*Proof.* Let  $0 \neq A \in \gamma$ . By (Z),  $A^\circ \in \gamma$  and by (H),  $A^\circ \in \gamma_r$ . All zero-rings of  $\gamma$  are in  $\gamma_r$  and so  $\gamma_r \neq 0$ . Hence  $\gamma_r$  satisfies conditions (Z) and (H). By Propositions 13, 6 and 4,  $\gamma$  is normal and hereditary.

The second part of the theorem follows from Proposition 11.  $\square$

**Corollary 16.** *The largest left hereditary subclass  $S_l$  of strongly prime radical  $S$  is the largest normal radical contained in  $S$ .*

**Theorem 17.** *The following statements are equivalent for a radical  $\gamma$ .*

- 1)  $\gamma$  is hereditary and normal.
- 2)  $\gamma$  is left or right principally hereditary, principally left or right strong and satisfies condition (H).
- 3) There exists a principally left (right, respectively) strong radical  $\delta$  such that  $\delta_r = \gamma$  ( $\delta_l = \gamma$ , respectively) and satisfies conditions (Z) and (H).
- 4) There exists a right (left, respectively) hereditary class  $\mathcal{M}$  of rings satisfying (Z) such that  $\gamma = \mathcal{L}_{ps}(\mathcal{M})$  ( $\gamma = \mathcal{L}'_{ps}(\mathcal{M})$ , respectively), where  $\mathcal{L}'_{ps}(\mathcal{M})$  is principally right strong radical generated by  $\mathcal{M}$ .

*Proof.* 2)  $\implies$  1): By Proposition 7,  $\gamma$  is normal and by Proposition 6,  $\gamma$  is hereditary.

1)  $\implies$  3): We claim that  $\gamma$  is one-sided hereditary. So let  $L \triangleleft_l A \in \gamma$ . Since  $\gamma$  is normal,  $\gamma$  is principally left hereditary, so  $Aa \in \gamma$ , for all  $a \in L$ . Therefore  $Aa \cdot z \in \gamma$  for every  $z \in Aa$ . Hence  $Aa \subseteq \gamma(L)$  for all  $a \in L$ , and this gives  $L^2 \subseteq \gamma(L)$ . Again, since  $\gamma$  is normal and satisfies condition (Z),  $A^\circ \in \gamma$  and by hereditariness  $L^\circ \in \gamma$ . Therefore  $L \in \gamma$ . Right hereditariness is proved analogously. Now we choose  $\delta$  to be  $\gamma$ ,  $\delta = \gamma$  and we have  $\gamma = \delta = \delta_l = \delta_r$ .

3)  $\implies$  4): We choose  $\mathcal{M} = \delta_r$  ( $\mathcal{M} = \delta_l$ , respectively). Then  $\delta_r = \mathcal{L}_{ps}(\delta_r) = \mathcal{L}_{ps}(\mathcal{M})$  ( $\delta_l = \mathcal{L}'_{ps}(\delta_l) = \mathcal{L}'_{ps}(\mathcal{M})$ , respectively) by Proposition 13 and clearly  $\delta_r$  satisfies (Z).

4)  $\implies$  2): By Corollary 14,  $\gamma = \mathcal{L}_{ps}(\mathcal{M})$  ( $\gamma = \mathcal{L}'_{ps}(\mathcal{M})$ ) is one-sided hereditary and left strong. Hence by Proposition 4 it is normal. It is easy to see that  $\gamma$  satisfies 2).  $\square$

**Proposition 18.** *Let  $\gamma$  be a supernilpotent radical and let us assume that  $\gamma_l = \gamma_r$  is the largest principally left hereditary subclass of  $\gamma$  which we will denote by  $\delta$ . Then*

$$\mathcal{L}_{ps}(\gamma) = \mathcal{L}_{ps}(\delta) \vee \gamma$$

where  $\vee$  denotes the union in the lattice of all radicals (i.e. the lower radical determined by the union of the components).

*Proof.* Clearly  $\mathcal{L}_{ps}(\delta) \vee \gamma \subseteq \mathcal{L}_{ps}(\gamma)$ . Conversely, let  $A \in \mathcal{L}_{ps}(\gamma)$ . Under our hypothesis, we can apply Proposition 8 and so  $\mathcal{L}_{ps}(\gamma) = \gamma_2$ . Thus any non-zero homomorphic image  $A'$  of  $A$  has a non-zero  $\gamma$ -ideal or a nonzero left ideal  $L$  such that  $La \in \gamma$  for all  $a \in L \cup \{1\}$ . Using our hypothesis again, we conclude that  $L \in \delta$  and therefore the  $\mathcal{L}_{ps}(\delta)$ -radical of  $A'$  is nonzero. Hence  $A'$  has a nonzero ideal in  $\mathcal{L}_{ps}(\delta) \cup \gamma$  and so  $A \in \mathcal{L}_{ps}(\delta) \vee \gamma$ .  $\square$

**Corollary 19.**  $\mathcal{L}_{ps}(\mathcal{G}) = \mathcal{L}_{ps}(\mathcal{B}) \vee \mathcal{G}$  and  $\mathcal{G}_2 = \mathcal{B}_2 \vee \mathcal{G}$ .

*Proof.* By Proposition 9, the Brown-McCoy radical satisfies the assumption of Proposition 18, in fact,  $\mathcal{M}\mathcal{G} = \mathcal{G}_l = \mathcal{G}_r = \mathcal{B}$ .  $\square$

**Remark.** This corollary can also be obtained as an application of Proposition 8 to the radicals  $\mathcal{G}$  and  $\mathcal{B}$ .

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