ADMISSIBLE INFERENCE RULES IN TEMPORAL LINEAR LOGICS BASED AT INTEGER NUMBERS

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1 Transitive Temporal Logic of Integer Numbers

This research concerns rules admissible in temporal logics. The language of the temporal propositional logic (cf. [6]) consists of propositional letters, Boolean logical operations, and two modalities: $\Box_+$ and $\Box_-$. Formation rules for wff's are as usual and $\Box_+ A$ is read A *will always be true*, $\Box_- A$ - A *always was true*. Temporal Kripke frames can be represented as $\mathcal{F} := \langle W, R, R^{-1}\rangle$, where R^{-1} is the converse of R. Though we also can keep as notation the original one - $\mathcal{F} := \langle W, R\rangle$ - bearing in mind the presence of the converse to R relation. Formulas $\Diamond_+ A$ and $\Diamond_- A$ are abbreviations for $\neg\Box_+\neg A$ and $\neg\Box_-\neg A$. For a frame $\mathcal{F}$, $L(\mathcal{F})$ denotes the temporal logic generated by $\mathcal{F}$ (i.e. $L(\mathcal{F})$ is the set of all formulas which are true at $\mathcal{F}$ w.r.t. all valuations).

The logic **LDTL** (linear discrete temporal logic) is the set of all propositional temporal formulas valid in the frame $\mathcal{Z} := \langle Z, \leq, \geq\rangle$ consisting of all integer numbers with usual order-relations $\leq$ and $\geq$, i.e. **LDTL** $:= L(\mathcal{Z})$. The axiomatization for **LDTL** was proposed by K.Segerberg [5]. First we observe that linear temporal logics are quite different from linear temporal logics. In particular,

Theorem 1.1 *The logic* **LDTL** *has no finite model property.*

Also we show that the temporal logic of all natural numbers also does not have fmp, i.e. let $\mathcal{N} := \langle N, \leq, \geq\rangle$, that is $\mathcal{N}$ consists of all natural numbers with usual order relations $\leq$ and $\geq$ on N.

Theorem 1.2 *The logic $L(\mathcal{N})$ has no finite model property.*

For a collection of formulas $A_1(x_1,\ldots,x_n), \ldots, A_m(x_1,\ldots,x_n), B(x_1,\ldots,x_n)$ of formulas, the expression $inf := A_1(x_1,\ldots,x_n),\ldots,A_m(x_1,\ldots,x_n)/B(x_1,\ldots,x_n))$ is said to be an (structural) *inference rule.*

An inference rule $inf := A(x_1,\ldots,x_n), \ldots, A_m(x_1,\ldots,x_n)/B(x_1,\ldots,x_n)$ is *admissible* in a logic L if, for any formulas $C_1,\ldots,C_n$,$[A_1(C_1,\ldots,C_n) \in L\&\ldots\&A_m(C_1,\ldots,C_n) \in L] \Longrightarrow B(C_1,\ldots,C_n) \in L$. In another terms, an inference *inf* is admissible in L iff L is closed w.r.t. *inf.* The research devoted to finding algorithm recognizing admissible inference rules was initiated since H.Friedman [1], who directed this question to intuitionistic propositional logic IPC. Most progress since then is achieved for transitive modal and superintuitionistic logics (cf. [4, 2, 3]).

First we need is a special representation of all inference rules inf in a homogeneous form and with lowest temporal degree - degree 1. An inference rule inf has a *reduced normal form* if $inf = \bigvee_{1\leq j\leq m}(\bigwedge_{1\leq i\leq n}[x_i^{k(j,i,0)} \wedge (\Diamond_+ x_i)^{k(j,i,1)} \wedge (\Diamond_- x_i)^{k(j,i,2)}])/x_1$, where x_s are certain variables, $k(i,j,z) \in \{0,1\}$ and for any formula φ, $\varphi^0 := \varphi, \varphi^1 := \neg\varphi$.

Theorem 1.3 *There exists an algorithm which, for any given inference rule inf, constructs its normal reduced form $rf(inf)$.*

For a temporal logic L and a model $\mathcal{M}$ with a valuation defined for a set of propositional letters p_1, ... , p_k, $\mathcal{M}$ is said to be *k-characterizing* for L if the following holds. For any formula $A(p_1,\ldots,p_k)$ built using letters $p_1,\ldots,p_k$, $A(p_1,\ldots,p_k) \in L$ iff $\mathcal{M} \Vdash A(p_1,\ldots,p_k)$. We say a model $\mathcal{M} := \langle M, R, V\rangle$ refutes an inference $\varphi_1, ..., \varphi_n/\psi$ if $\forall i, \forall a \in M((\mathcal{M},a) \Vdash_V \varphi_i)$ & $\exists b \in M((\mathcal{M},b) \nVdash_V \psi)$. We need the following simple fact (cf., for instance, [4], p. 297).

Lemma 1.4 *A consecution* **cs** *is not admissible in a logic $\mathcal{L}$ iff, for any sequence of k-characterizing models, there are a number n and n-*characterizing model *$Ch_{\mathcal{L}}(n)$ from this sequence such that the frame of $Ch_{\mathcal{L}}(n)$ refutes* **cs** *by a certain definable in $Ch_{\mathcal{L}}(n)$ valuation.*

To construct k-characterizing models for modal and superintuitionistic logics, the finite model property usually has been used (cf. [4]). However, $LDTL$ does not have fmp. Therefore we construct these models using infinite linear frames. Let $\mathcal{M}_k$ be the disjoint union of all models $\langle \mathcal{Z}, V\rangle$ based on $\mathcal{Z}$, where V are all possible valuations with $Dom(V) = \{p_1,\ldots,p_k\}$, and of all models based on the single reflexive element with all possible valuations of letters $p_1,\ldots,p_k$. The base sets of these models are evidently uncountable, $||\mathcal{M}|| = 2^\omega$.

Theorem 1.5 *The model $\mathcal{M}_k$ is k-*characterizing *for $LDTL$.*

Based on this fact and Theorem 1.3 we can prove

Theorem 1.6 *The logic* **LDTL** *is decidable w.r.t. admissible inference rules.*

Using same approach, we can prove

Theorem 1.7 *The temporal logic of natural numbers $\mathcal{L}(\mathcal{N})$ is decidable w.r.t. admissible consecutions.*

In particular, as a consequence, we immediately obtain that the logics **LDTL** and $\mathcal{L}(\mathcal{N})$ are decidable (w.r.t. theorems), though neither possesses the finite model property.

2 Intransitive Temporal Logic of Integer Numbers

Next temporal logic, which we will study from viewpoint of inference rules, is the intransitive linear logic of natural numbers. we define this logic as follows. The temporal frame $\mathcal{T}_n := \langle \{1, 2, \ldots, n\}, Next, Prev \rangle$ has the base set $[1, n]$ and the accessibility relations $Next$ and $Prev$. The relation $Next$ is the binary relation *next natural number*, i.e. $Next(n, x) = false$ for all $x \in \mathcal{T}_n$ and $Next(k, m)$ is true iff $k < n$ and $m = k + 1$. Similarly, $Prev$ is the binary relation *previous natural number*, i.e. $Prev(1, x) = false$ for all $x \in \mathcal{T}_n$ and $Prev(k, m)$ is true iff $k > 1$ and $m = k - 1$. We can also understand $Next$ as the one-to-one partial function where $Next(n) := n + 1$, the same regarding $Prev$, with $Prev(n) := n - 1$.

For any $x \in \mathcal{T}_k$, $Next_0(x) := Next(x)$ if $x \neq k$ otherwise $Next_0(k) := k$, and $Prev_0(x) := Prev(x)$ if $x \neq 1$, otherwise $Prev_0(1) := 1$.

The temporal Tomorrow/Yesterday logic **TYL** *is the set of all formulas which are valid in any frame* $\mathcal{T}_n$*, i.e.* $\mathbf{TYL} := L(\{\mathcal{T}_n \mid n \geq 1\})$.

The following statement would be quite trivial if we would consider infinite intervals of numbers, like $\mathcal{Z}$, instead finite ones. But because our intervals are finite we need a double induction on size of formulas and on distances worlds from initial and terminal points.

Theorem 2.1 Small Models Theorem. For any formula A, if $A \notin \mathbf{TYL}$, then there is a frame $\mathcal{T}_n$ of size linear in the length of A where $\mathcal{T}_n \not\Vdash A$.

Corollary 2.2 The temporal logic **TYL** is decidable.

Definition 2.3 Given a model $\mathcal{M} := \langle \mathcal{F}, V \rangle$ based upon the frame $\mathcal{F}$ and a new valuation V_1 in $\mathcal{F}$ of a set of propositional letters q_i, V_1 is definable in $\mathcal{M}$ if, for any q_i, $V_1(q_i) = V(\phi_i)$ for some formula ϕ_i.

Lemma 2.4 (cf., for instance, [4]) A rule **cs** is not admissible in a logic L iff, for any sequence of k-characterizing models, there are a number n and an *n-characterizing model* $Ch_L(n)$ from this sequence such that the frame of $Ch_L(n)$ refutes **cs** by a certain definable in $Ch_L(n)$ valuation.

The construction of n-characterizing models for **TYL**, comparing with similar ones for modal logics, is surprisingly simple (though we will need to pay a cost for this simplicity). Indeed, consider any temporal frame $\mathcal{T}_n$ and any valuation V of letters p_1, ... , p_k in $\mathcal{T}_n$. Take the disjoint union $\bigsqcup \mathcal{T}_n$ of all such non-isomorphic models. It is a constructive countable model which we denote by $Ch_k(\mathbf{TYL})$.

Lemma 2.5 The model $Ch_k(\mathbf{TYL})$ is k-characterizing for **TYL**.

A model $\mathcal{M}$ is definable if, for any element a of $\mathcal{M}$, there is a formula φ_a which is true in $\mathcal{M}$ only at a.

Lemma 2.6 The model $Ch_k(\mathbf{TYL})$ is definable.

For any inference rule $\mathbf{c_{nf}}$ in normal reduced form, $Pr(c_{nf}) = \{\varphi_i \mid i \in I\}$ is the set of all disjunctive members of the premise of $\mathbf{c_{nf}}$. $Sub(c_{nf})$ is the set of all subformulas of $\mathbf{c_{nf}}$.

Lemma 2.7 If a rule $\mathbf{c_{nf}}$ in the normal reduced form is not admissible in **TYL** then

(i) For any $\mathcal{T}_m$ $(m \geq 1)$, there is a valuation S for variables of $\mathbf{c_{nf}}$ in $\mathcal{T}_m$ such that $\mathcal{T}_m \Vdash_S \bigvee Pr(c_{nf})$.

(ii) For some $k \in N$, linearly computable in the size of $\mathbf{c_{nf}}$, there exists a valuation S for variables of $\mathbf{c_{nf}}$ in $\mathcal{T}_k$, where

(1) $\mathcal{T}_k \Vdash_S \bigvee Pr(c_{nf})$;

(2) There are $\varphi_i \in Pr(c_{nf})$ and $j \in \mathcal{T}_k$, where

$$(\mathcal{T}_k, j) \Vdash_S \varphi_i,\ (\mathcal{T}_k, j+1) \Vdash_S \varphi_i,\ (\mathcal{T}_k, j+2) \Vdash_S \varphi_i.$$

Lemma 2.8 If a rule $\mathbf{c_{nf}}$ in normal reduced form is not admissible in **LTY** then there is a valuation S for $\mathbf{c_{nf}}$ in the frame $\mathcal{T}_n$ for some $n \geq 1$ refuting $\mathbf{c_{nf}}$, where the size of $\mathcal{T}_n$ is linear in the size of $\mathbf{c_{nf}}$.

Lemma 2.9 If a rule $\mathbf{c_{nf}}$ in normal reduced form satisfies the conclusions of Lemmas 2.7 and 2.8 then $\mathbf{c_{nf}}$ is not admissible in **TYL**.

Since the conditions of Lemmas 2.7 and 2.8 have to be verified only for frames $\mathcal{T}_n$ with sizes linearly bounded in the size of the rule (normal reduced form of the rule), from Theorem 1.3, Lemmas 2.7, 2.8, and Lemma 2.9 we immediately derive

Theorem 2.10 *The logic* **TYL** *is decidable w.r.t. admissible inference rules.*

References

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