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Rook Polynomials In Higher Dimensions

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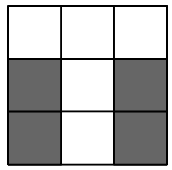
Abstract. A rook polynomial counts the number of placements of non-attacking rooks on a board. In this paper we describe generalizations of the definition and properties of rook polynomials to "boards" in three and higher dimensions. We also define generalizations of special two dimensional boards to three dimensions, including the triangle board and the board representing the probléme des rencontres. The number of rook placements on these three dimensional families of rook boards are shown to be related to famous number sequences, such as central factorial numbers, the number of Latin rectangles and the Genocchi numbers.

1. Introduction

The theory of rook polynomials provides a way of counting permutations with restricted positions. The theory was developed by Kaplansky and Riordan in [Kaplansky 1946]. The theory has been researched and studied quite extensively since then. Two rather comprehensive resources on the theory are [Riordan 1958] and [Stanley 1997]. The intent of research behind this paper was to generalize these properties and theorems into higher dimensions; this was done using a three dimensional board generalization first introduced by Benjamin Zindle. In the Appendix, we include a Maple program which calculates the rook numbers of a given three dimensional board using this generalization. In section 2 we describe rook polynomials in two dimensions, including the way in which rooks attack, and the boards that the rooks attack on. We also discuss many properties and theorems relating to rook polynomials in two dimensions. In sections 2.1, 2.2, and 2.3 we discuss specific two dimensional families of boards. Namely, the probléme des rencontres, the triangle board, and the type of boards known as the Ferrers board. Similarly, in section 3 we discuss rook polynomials in three dimensions. We describe the three dimensional board, as well as the way in which rooks attack in three dimensions. We also give the generalizations of the properties and theorems of two dimensional rook polynomials into three dimensions. In sections 3.1, and 3.2 we discuss the three dimensional counterparts of the probléme des rencontres and the triangle board. In section 3.3 we discuss an entirely new kind of board known as the Genocchi board. In the last section, section 3.4, we discuss future plans for the three dimensional version of the Ferrers board.

2. Rook Polynomials in Two Dimensions

A *rook polynomial* $R(x) = r_0 + r_1x + ... + r_kx^k + ...$ is a specific kind of generating function that represents all of the ways that one can place non-attacking rooks on a board. A *board* with m rows and n columns is a subset of $\{1, 2, ..., m\} \times \{1, 2, ..., n\}$. We call such a board an $m \times n$ board. If the board is the whole set $\{1, 2, ..., m\} \times \{1, 2, ..., n\}$, then we call the board the *full* $m \times n$ *board*. The coefficient of a rook polynomial, r_k represents the number of ways of placing k non-attacking rooks on the corresponding board. For a valid arrangement of non-attacking rooks on a board, no two rooks can lie in the same column, row, or restricted position. For an example, let us consider the board below with shaded regions representing restricted tiles on which rooks cannot be placed, and white regions representing unrestricted tiles.



$$R(x) = 1 + 5x + 4x^2$$

This picture is a visualization of the board which is a subset of $\{1, 2, 3\} \times \{1, 2, 3\}$, and consists of the tiles $(1, 1), (1, 2), (1, 3), (2, 2), (3, 2)$. There is only one way to place no rooks on the board, and this gives us our constant coefficient 1 in a rook polynomial for this or any other board. We can also see that there are five unrestricted tiles which correspond with five ways to place one rook on the board, as represented by the $5x$ in the rook polynomial. By exhaustion we can find that there are four ways to place two rooks on the board, and zero ways to place three or more rooks on the board. The four ways of placing two rooks gives us the $4x^2$ in the polynomial. Since there are zero ways to place three or more rooks on the board, our rook polynomial is of degree two. In general, the maximum number of non-attacking rooks that can be placed on a board cannot exceed either the number of rows or columns that compose the board, and hence the rook polynomial, as indicated by its name, is a polynomial with degree less than or equal to the minimum of the number of rows and columns.

In order to better understand the rook polynomials, we can observe the many properties. These allow us to more easily deduce information about a given board or polynomial. Let us now look at some properties of rook polynomials in two dimensions and their proofs.

The rook polynomial for a full $m \times n$ board can be found in a straightforward way as described in the next theorem.

Theorem. *The number of ways of placing k non-attacking rooks on the full $m \times n$ board is equal to* $\binom{m}{k}\binom{n}{k}k!$.

Proof: Since we wish to place k rooks, on an $m \times n$ board, we first choose k of the m rows, and k of the n columns on which the rooks will be placed. This can be done in $\binom{m}{k}\binom{n}{k}$ ways. Once we have selected which rows and columns that we wish to place the k rooks on, we can consider the number of ways that we can arrange k rooks among the selected rows and columns. Let us start at the top of the selected rows and work our way down. For the first row we have k columns we can place a rook in. If we then place a rook somewhere in the first row, when we move down to the second row we will have eliminated one possibility for a column, so we now have $k-1$ columns to place the second rook in. Continuing this method we have $k-2$ ways to place a rook in the third row, and so on. Hence there are $k!$ ways to place k rooks on the selected rows and columns. Since we have $\binom{m}{k}\binom{n}{k}$ ways to choose the columns, we have $\binom{m}{k}\binom{n}{k}k!$ ways to place k non-attacking rooks on a full $m \times n$ board. □

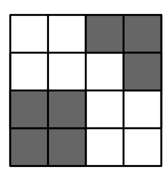
One of the elementary properties of the rook polynomials provides an important tool in calculating the rook polynomial recursively. The rook polynomial of a board consisting of two disjoint boards, meaning that the boards share no rows or columns, can be calculated in terms of the rook polynomials of the disjoint boards as follows:

Theorem. *Let A and B be boards that share no rows or columns. The rook polynomial for the board $A \cup B$ consisting of the union of the tiles in A and B is $R_{A\cup B}(x) = R_A(x) \times R_B(x)$.*

Proof: Let $R_A(x) = \sum_{i=0}^{\infty} r_k x^k$ and $R_B(x) = \sum_{i=0}^{\infty} r'_k x^k$ be the rook polynomials for A and B, respectively. Let us consider the number of ways to place k rooks on $A \cup B$. We can place k rooks on $A \cup B$ in the same number of ways as placing k rooks on A and 0 rooks on B, which is equal to $r_k r_0{}'$, plus the number of ways of placing $k-1$ rooks on A and 1 rook on B, which is $r_{k-1}r'_1$, and so on. Hence, we find that the number of ways to place k rooks on $A \cup B$ is $\sum_{i=0}^{k} r_{k-i}r'_i$ which is the coefficient of x^k in $A(x) \times B(x)$. Therefore $R_{A\cup B}(x) = R_A(x) \times R_B(x)$. □

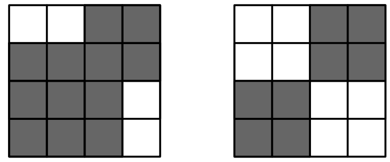
Sometimes, a given board may not be immediately decomposed into disjoint boards. We can then swap rows and columns to obtain disjoint boards. Doing this will not change the rook polynomial of the board. We are also able to rotate a board or take its reflection without changing the polynomial.

Much like decomposing into disjoint boards, cell decomposition is another method of expressing the rook polynomial of a board in terms of smaller boards. Consider the board B shown below.



B

This board cannot be decomposed into two disjoint boards, and swapping rows and columns does not fix this problem. The cell decomposition method allows us to find the rook polynomial for this board by breaking rook placements down into cases. We will consider two cases: when there is a rook in the $(2,3)$ position, and when there is no rook in the $(2,3)$ position. If there is a rook in the $(2,3)$ position we cannot have another rook in row two or column three. By deleting row two and column three we create a new board B' on which the rest of the rooks can be placed. In the case that no rook is placed on $(2,3)$ we will create a board B'' by adding a restriction to $(2,3)$.



B' B''

In order to find the number of ways to place k rooks on B, we must simply exhaust all cases of k rooks being placed. If we let $k = 3$ then we must consider putting 2 more rooks on B' and add this number to the remaining case of placing 3 rooks on B'' where no rooks are to be placed on $(3,3)$. We can see that there are disjoint boards within both B' and B'', so we can compute the rook polynomials of B' and B'' by multiplying the rook polynomials for the disjoint sub-boards that lie within the larger boards. There are 6 ways to arrange 2 additional rooks on B'. By multiplying the rook polynomials for the two disjoint boards found in B'' we can see that the coefficient of x^3 is 16, so we have 16 ways to place 3 rooks on B''. Adding these two up gives us a total of 24 ways to place 3 rooks on B. This idea, generalized to an arbitrary number of rooks, proves the following theorem.

Theorem. *Let B be a board, B' be the board obtained by removing the row and column corresponding to a cell from B, and B'' be the board obtained by restricting the same cell on B. Then $R_B(x) = xR_{B'}(x) + R_{B''}(x)$.*

Another property of the rook polynomials, that will prove useful when dealing with the probléme des rencontres, is the property which relates the rook polynomials of two complementary boards. We say that given a board A, the *complement* of A, denoted $\bar{A}$ is defined as the collection of all tiles/cells so that the union of A and $\bar{A}$ is a full board and A and $\bar{A}$ share no tiles/cells.

Theorem. *Let A be an $m \times n$ board with restrictions, $\bar{A}$ be the complement of A, and $R_{\bar{A}}(x) = r_0 + r_1x + r_2x^2 + ... + r_kx^k...$ be the rook polynomial for $\bar{A}$. Then the number of ways to place k non-attacking rooks on A is*

$$\sum_{i=0}^{k} (-1)^i \binom{m-i}{k-i}\binom{n-i}{k-i}(k-i)!r_i$$

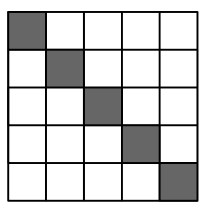
.

Proof: We know from from earlier that we can place k rooks on the full $m \times n$ board in $\binom{m}{k}\binom{n}{k}k!$ ways. We wish to find the number of ways to place k non-attacking rooks on a board A which has restrictions. In order to do this we will consider the total number of placements of k non-attacking rooks on an unrestricted $m \times n$ board and subtract the cases where one or more rooks are placed on the restricted positions using the Inclusion-Exclusion Principle. We will first consider the total number of ways to place k rooks on an unrestricted $m \times n$ board, that is $\binom{m}{k}\binom{n}{k}k!$. Let us now consider placing one rook on a restricted tile, and placing the $k-1$ rooks on the remaining$(m-1)\times(n-1)$ tiles. This can be done in $\binom{m}{k-1}\binom{n}{k-1}(k-1)!r_1$ ways. We will subtract $\binom{m-1}{k-1}\binom{n-1}{k-1}(k-1)!r_1$ from $\binom{m}{k}\binom{n}{k}k!$. We now wish to consider placing two rooks on restrictions and $k-2$ rooks the the remaining $(m-2)\times(n-2)$ tiles. When subtracting the number of placements where one rook was placed on a restriction, we also double counted number of ways to place two rooks on restrictions. So we add $\binom{m}{k-2}\binom{n}{k-2}(k-2)!r_2$ to $\binom{m}{k}\binom{n}{k}k! - \binom{m}{k-1}\binom{n}{k-1}(k-1)!r_1$. Continuing this way we get the number of ways to place k rooks on A is

$$\sum_{i=0}^{k}(-1)^i\binom{m-i}{k-i}\binom{n-i}{k-i}(k-i)!r_i$$

□

2.1. **Probléme des rencontres.** We now consider a special family of boards which correspond to the famous Probléme des rencontres, or equivalently to derangements. An example of such a problem is as follows. Suppose that five people enter a restaurant, each person with their own unique hat. We are interested in finding the number of ways that everyone can leave the restaurant without their own hat, ignoring the order in which they leave. We will use this problem to demonstrate how to find the rook polynomial of a board with restricted positions. In the case of the probléme des rencontres we are given an $m \times m$ board with restrictions placed along the main diagonal. For this case we will let $m = 5$ and place restrictions on (i,i) for $i = 1,..,5$. We will find the number of ways to place 5 rooks on the board B. This corresponds with the number of permutations of five elements where no element is in its original position.



All of the restrictions are disjoint single tiles, and the rook polynomial for a single tile is $1+x$. Using the theorem on rook polynomials of disjoint boards, the rook polynomial for the restrictions is $(1+x)^5$, which equals $1+5x+10x^2+10x^3+5x^4+x^5$. Now, using the theorem on rook polynomials of completementary boards, we find that the number of ways to place 5 rooks on B is equal to

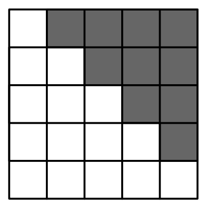
$$\begin{aligned}\binom{5}{5}\binom{5}{5}5!\cdot 1 - \binom{4}{4}\binom{4}{4}4!\cdot 5 + \binom{3}{3}\binom{3}{3}3!\cdot 10 - \binom{2}{2}\binom{2}{2}2!\cdot 10 \\ +\binom{1}{1}\binom{1}{1}1!\cdot 5 - \binom{0}{0}\binom{0}{0}0!\cdot 1 &= 44\end{aligned}$$

In general, r_k of the rook polynomial of a $m \times m$ probléme des rencontres board is

$$\sum_{i=0}^{k}(-1)^i \binom{m-i}{k-i}^2 (k-i)! \binom{m}{k}$$

2.2. **Triangle Board.** Another famous family of 2-dimensional boards is the triangle boards. We say that a *triangle board* of size m consists of all of the tiles of the form (i, j) where $j \leq i$ and $1 \leq i, j \leq m$. The rook numbers of this family of boards correspond with the Stirling numbers of the second kind. Recall that the Stirling numbers of the second kind, $S(n, k)$, count the number of ways to partition a set of size n into k non-empty sets, and can be defined recursively by $S(n, k) = S(n-1, k-1) + kS(n-1, k)$ with $S(n, n) = 1$ and $S(n, 1) = 1$.

Theorem. *The number of ways to place k non-attacking rooks on a triangle board of size m is equal to $S(m+1, m+1-k)$, where $1 \leq k \leq m$ and $S(n, k)$ are the Stirling numbers of the second kind.*



Proof: We will prove this theorem by induction on m.

The rook polynomial for the board B with a single cell, the triangle board for $m = 1$, is $R_B(x) = 1+x$. Since $S(2, 1) = S(2, 2) = 1$ the theorem checks in this case.

Assume now the theorem is true for some m, i.e. that the number of ways of placing k rooks on a size m triangle board is equal to $S(m+1, m+1-k)$ for $0 \leq k \leq m$. We will show that the number of ways of placing k rooks on an $(m+1)\times(m+1)$ board is equal to $S((m+1)+1, (m+1)+1-k) = S(m+2, m+2-k)$ for $0 \leq k \leq m+1$.

For $k = m+1$, there is only 1 way to place k non-attacking rooks on a size $m+1$ triangle board, this is when all rooks lie on the diagonal. This corresponds to $S(m+2, m+2-k) = S(m+2, 1)$ which by definition is equal to one. For $k = 0$, placing k rooks on the board can be done in only one way which corresponds with $S(m+2, m+2-k) = S(m+2, m+2)$. Therefore the rook numbers and the Stirling numbers agree for $k = 0$ and $k = m+1$.

Let us now show that these numbers agree for $0 < k < m+1$. In order to get from a size m triangle board to a size $m+1$ triangle board, let us add an additional row on the bottom. When finding the number of ways to place k rooks on the size $m+1$ triangle board, we can consider two cases. First consider the case where we place all k rooks on the top m rows, which form a size m triangle board. This gives us $S(m+1, m+1-k)$ possibilities by our inductive hypothesis. Second consider the case where a rook lies in the bottom row. If this is the case then we must place $k-1$ rooks on the size m board board and we then have $m+1-(k-1)$ tiles in the last row to place our last rook, giving us $(m+2-k)S(m+1, m+2-k)$ ways to place k rooks on the board with one rook lying in the last row. So there are $S(m+1, m+1-k) + (m+2-k)S(m+1, m+2-k)$ ways to place k rooks on a size $m+1$ triangle board. Using the recursive definition of the Stirling numbers, this sum corresponds to $S(m+2, m+2-k)$.

Therefore, by induction, the rook number r_k for size n triangle board is given by $S(n+1, n+1-k)$ for any n. □

2.3. **Ferrers Boards.** Both triangle boards and rectangular boards are special cases of a more general group of boards known as Ferrers Boards. A *Ferrers board* is a board whose tiles can be arranged in such a way that all of the columns have a common base, are contiguous and have non-decreasing heights from left to right. Algebraically, such a board corresponds to a sequence of integers $0 \leq b_1 \leq ... \leq b_n$ where the cellsof the board are (i, j) with $1 \leq i \leq n$ and $1 \leq j \leq b_i$. We then refer to this board as the Ferrers board of shape $(b_1, ..., b_n)$. If we consider any full rectangular board we know that as the column number increases the number of tiles per column remain equal. In addition, the number of tiles per column is equal to the column height. Since the number of tiles per column is non-decreasing as we move in an increasing fashion among the columns, we know that rectangular boards are Ferrers boars. With triangle boards the number of tiles per column increases by exactly one as we move from left to right. It is also the case that for any column with k unrestricted tiles, the tiles occupy the bottom k rows.

The next theorem gives an explicit formula for the rook coefficients of a Ferrers board in terms of the heights of the columns in the board.

Theorem. *Let $\sum r_k x^k$ be the rook polynomial for a Ferrers board B with shape $(b_1, b_2, ..., b_m)$ If we then let $s_i = b_i - i + 1$ then we have*

$$\sum r_k P(x, (m-k)) = \prod_{1}^{m} (x + s_i)$$

A proof of this theorem can be found in [Stanley 1997].

3. Rook Polynomials in Three and Higher Dimensions

The theory of rook polynomials in two dimensions as described above can be generalized to three and higher dimensions. The theory for the three dimensions is introduced in [Zindle] and the theory we describe in this paper is a more generalized version of this theory.

In three dimensions, our boards will be subsets of $\{1, 2, \ldots, m\} \times \{1, 2, \ldots, n\} \times \{1, 2, \ldots, p\}$. We refer to such a board as an $m \times n \times p$ board. More generally, a board in d dimensions is a subset of $\{1, 2, \ldots, m_1\} \times \{1, 2, \ldots, m_2\} \times \ldots \times \{1, 2, \ldots, m_d\}$. A *full board* is again a board with no restrictions.

In three and higher dimensions, we use the word *cell* to refer to positions where rooks can be placed. In particular, in three dimensions, a cell is a 3-tuple (i, j, k) with $1 \leq i \leq m, 1 \leq j \leq n, 1 \leq k \leq p$.

In two dimensions rows correspond to tiles with the same second coordinate, and the columns correspond with the same first coordinate. In three dimensions, we can still consider the cells with the same fixed coordinate, but we modify the terminology. All cells with the same third coordinate are said to lie in the same *layer*, all cells with the same first coordinate are in the same *slab*, and all cells with the same second coordinate are in the same *wall*. We will still have rows and columns within a layer. We also have *towers* which correspond with the cells with the same first and second coordinate.

We generalize the rook theory to three dimensions so that a rook in three dimensions will attack along walls, slabs and layers. In higher dimensions, rooks attack along hyperplanes, which correspond to layers of cells with one fixed coordinate. In two dimensions, when we place a rook on a tile, we are no longer able to place a rook on any tile in the same row or column. In three dimensions, when we place a rook in a cell, we can no longer place another rook in the same wall, slab, or layer. In higher dimensions, when we place a rook in a cell, we cannot place another rook in the hyperplanes that this cell belongs to.

Another potential generalization of rook polynomials to three and higher dimensions involves a different method to generalize how rooks attack. Instead of a rook attacking over hyperplanes, we can assume that the rooks attack along lines, just as the two-dimensional case. This approach seems to have possible applications as well, however, we will not pursue this generalization in this report.

Our first theorem on the generalized rook theory deals with sweeping a two dimensional board in the z-direction. Sweeping a two dimensional board will give us a three dimensional version of the board which will be identical with respect to x and y. Since our new three dimensional board is based off of the two dimensional board, it makes sense that there is a relationship between the rook polynomials for each board.

Theorem. *Let A be an $m \times n$ board and B be a three dimensional sweep of A of magnitude p. The number of ways of placing k rooks on B is equal to $P(p, k)$ times the the number of ways of placing k rooks on A.*

Proof: Let $r_{A,k}$ represent the number of ways of placing k non-attacking rooks on A, and let p be the magnitude of our sweep. Let us consider the number of ways of placing k rooks on B, or $r_{B,k}$. Since each rook must be non-attacking with all other rooks we know that no two rooks can lie on the same layer, wall, or slab. We know that there are $r_{A,k}$ ways to place k non-attacking rooks with respect to walls and slabs, or placing k non-attacking rooks ignoring the layer. Given such a placement, we must disperse the k rooks among p layers. This is equivalent to k permutations of p numbers, which corresponds with $P(p, k)$. We should note that we are not using $\binom{p}{k}$ since each rook is in a unique tower, so each permutation is unique. So we have $r_{A,k}P(p,k)$ ways to place k rooks on B. It follows that for values of k that are greater than p or values of k greater than m yield zero ways of placing k rooks. □

In two dimensions we are able to re-order the rows and columns so that we may obtain disjoint boards. Similarly, in three dimensions we are able to re-order the layers, walls, and slabs so that we can create disjoint boards, once again this will not change to rook polynomial of the board. We are also able to rotate or reflect a board in three dimensions without changing the polynomial. This allows us to look at a board from numerous perspectives so that we can better understand a board. Much like in two dimensions, we are able to use the method of cell decomposition when dealing with three dimensional boards.

Theorem. *Let B be a board, B' be the board obtained by removing the wall, slab, and column corresponding to a cell from B, and B'' be the board obtained by restricting the same cell on B. Then $R_B(x) = xR_{B'}(x) + R_{B''}(x)$.*

More generally:

Theorem. *Let B be a board, B' be the board obtained by removing the hyperplanes that correspond to a cell from B, and B'' be the board obtained by restricting the same cell on B. Then $R_B(x) = xR_{B'}(x) + R_{B''}(x)$.*

Similar to two dimensions, when dealing with two disjoint boards, the number of ways to place k rooks on the union of two three dimensional boards is the coefficient of x^k when we multiply the rook polynomials of both boards together.

Theorem. *Let A and B be boards that share no walls, slabs, or layers. The rook polynomial for the board $A \cup B$ consisting of the union of the tiles in A and B is $R_{A\cup B}(x) = R_A(x) \times R_B(x)$.*

More generally:

Theorem. *Let A and B be boards that share no hyperplanes. The rook polynomial for the board $A \cup B$ consisting of the union of the tiles in A and B is $R_{A\cup B}(x) = R_A(x) \times R_B(x)$.*

As in the two-dimensional case, the rook polynomial of the full boards can be calculated in a straightforward way in three and higher dimensions.

Theorem. *There are $\binom{m_1}{k}\binom{m_2}{k}\dots\binom{m_d}{k}(k!)^{d-1}$ ways to place k non attacking rooks in a full $m_1 \times m_2 \times \dots \times m_d$ board in d dimensions.*

Proof: For simplicity, let us consider placing k rooks in a full $m \times n \times p$ board in three dimensions. Since we are placing k rooks on m slabs, n walls, and p layers, we have $\binom{m}{k}$ ways to choose k slabs to place the rooks on, $\binom{n}{k}$ ways to choose k walls to place the rooks on, and $\binom{p}{k}$ ways to choose k layers to place the rooks on. This would give us $\binom{m}{k}\binom{n}{k}\binom{p}{k}$ total selections of walls, slabs, and layers. For any given selection of layers, slabs, and walls, we will first assign the rooks to an individual layer. Since we have k rooks and k layers, there will be exactly one rook on each layer. When assigning them to slabs and walls, the rook on the first layer has k walls and k slabs to be placed on. After placing the first rook, the second layer rook will have $k-1$ slabs and $k-1$ walls as options. Continuing this way, we find that we have $k \cdot k \cdot (k-1) \cdot (k-1) \cdot (k-2) \cdot (k-2) \cdot ... \cdot 2 \cdot 2 \cdot 1 \cdot 1$ which is the same as $((k)(k-1)(k-2)...(2)(1))^2$ which equals $(k!)^2$. So there are $\binom{m}{k}\binom{n}{k}\binom{r}{k}(k!)^2$ ways to place k non attacking rooks on an unrestricted $m \times n \times r$ board so long as k is less than or equal to the smallest of m,n, and r.

This method generalizes to d dimensions so that for an $m_1 \times m_2 \times m_3 \times ... \times m_d$ board, we can place k rooks on the board in $\binom{m_1}{k}\binom{m_2}{k}\binom{m_3}{k}...\binom{m_d}{k}(k!)^{d-1}$ ways. □

Applying the same idea in higher dimensions produces the following more general result:

Theorem. *There are $\binom{m_1}{k}\binom{m_2}{k}\dots\binom{m_d}{k}(k!)^{d-1}$ ways to place k non attacking rooks in a full $m_1 \times m_2 \times \dots \times m_d$ board in d dimensions.*

The theorem on complementary boards can be generalized to three dimensions as follows:

Theorem. *Let B be a restricted board with m slabs, n walls, and p layers. Let $R_{B'}(x) = r_0 + r_1 x + r_2 x^2 + ...$ be the rook polynomial of the restricted positions of B. Then the number of ways that we can place k non-attacking rooks on B with respect to the restrictions is equal to*

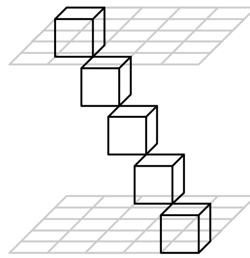
$$\sum_{i=0}^{min(m,n,p)} \binom{m}{k-i}\binom{n}{k-i}\binom{p}{k-i} r_i$$

Proof: In order to find the number of ways to place k rooks with respect to the restrictions, we can take the total number of ways of placing k non-attacking rooks in an $m \times n \times p$ board and subtract the number of arrangements when some of these rooks are on restrictions. As in the two dimensional case, using the inclusion exclusion principle we find that the number of ways to place k non-attacking rooks on B with respect to the restrictions is

$$r_k \sum_{i=0}^{min(m,n,p)} \binom{m}{k-i}\binom{n}{k-i}\binom{p}{k-i} r_i$$

□

3.1. **Probléme des Rencontres in Three Dimensions.** Recall the probléme des rencontres from earlier. The probléme des rencontres dealt with a board with restrictions along the main diagonal. When creating a three dimensional version of the probléme des rencontres board, we will again place restrictions along the diagonal. In two dimensions we explained the probléme des rencontres by considering five people leaving a restaurant without their own hat. For this type of problem to make sense in three dimensions we will have to alter the scenario. We will once again consider five people entering a restaurant. We will also introduce another dimensions to the story. Let each person now have their own unique hat and coat. We are now interested in the number of ways that the five people can leave the restaurant without both of their original items. Let B be an $5\times5\times5$ board with restricted positions (i,i,i) for $i=1,...,5$; we will consider placing 5 rooks on B. The board representing the restrictions of B is shown below.

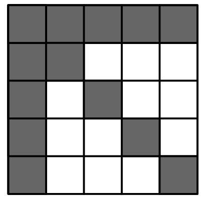


For this board we let each layer represent a person and we let the walls and slabs represent the corresponding coats and hats. These restrictions correspond with no person leaving with both their hat and coat. We can see that our 5 restrictions are disjoint, and since the rook polynomial for each restriction is $1+x$, like in the two dimensions case we get $(1+x)^5 = 1+5x+10x^2+10x^3+5x^4+x^5$ as the rook polynomial for the restricted positions. Using the previous theorem we find that the number of ways to place 5 rooks on B is

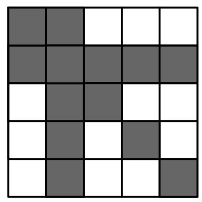
$$\binom{5}{5}\binom{5}{5}\binom{5}{5}(5!)^2 - \binom{4}{4}\binom{4}{4}\binom{4}{4}((4!)^2(5) + \binom{3}{3}\binom{3}{3}\binom{3}{3}(3!)^2(10) -$$
$$\binom{2}{2}\binom{2}{2}\binom{2}{2}(2!)^2(10) + \binom{1}{1}\binom{1}{1}\binom{1}{1}(1!)^2(5) - \binom{0}{0}\binom{0}{0}\binom{0}{0}(0!)^2(1) \quad = \quad 11844$$

More generally, the number of ways that we can place k rooks on an $m\times m\times m$ probléme des rencontres board of this kind is $\sum_{j=0}^{k}\binom{m-j}{k-j}^3(k-j)!^2\binom{k}{j}$

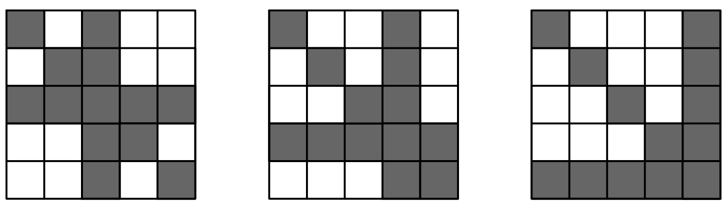
Another generalization of the probléme des rencontres is to restrict the rows, columns, and towers that pass through a position of the form (i,i,j), (i,j,i) and (j,i,i). This second generalization to three dimensions corresponds with the number of ways that the five people can leave the restaurant without their coat, hat, or any proper pairing of a coat and hat. This means that each person must leave the restaurant with a hat that isn't theirs, and a coat that is neither theirs nor the owner of the hat.. The rook board for this problem is a bit more difficult to visualize so we will first discuss how to construct it. For this problem we will let the layers represent a person, and the slabs and walls represent the coats and hats. For the n^{th} layer of an $m\times m\times m$ probléme des rencontres board, we will have restrictions on cells of the form (i,i,j), (i,j,i) and (j,i,i). Since our story problem deals with five people we will demonstrate the construction of a $5\times5\times5$ probléme des rencontres board. For bottom layer we will restrict $(1,1,k)$ and $(k,1,1)$ for k from one to 5. This restricts the row and column corresponding with person one not leaving with their coat or hat. We will also restrict all cells along the main diagonal, meaning $(k, k, 1)$ for k from one to m, this corresponds with person one not leaving with a proper pairing of a coat and hat. The first layer of the board will then appear as follows.



For the second layer from the bottom we will restrict $(2, 2, k)$ and $(k, 2, 2)$ for k from one to m. This will restrict the row and column associated person two leaving with their own coat and hat. We will also restrict $(k, k, 2)$ for k from one to m. This corresponds with person two not leaving with their own coat, and person two not leaving with their own hat. This layer will appear as follows:



Continuing this method for the final three layers we get:



The computation of the rook polynomial for this board is a bit cumbersome so we will use a computer program for this task. Using Maple we find that the rook polynomial for this board is 552. The rook numbers of the three dimensional probléme des rencontres boards for sizes up to 7 are given in the following table:

$m\backslash k$	0	1	2	3	4	5	6	7
0								
1								
2								
3	1	6	6	2				
4	1	24	132	176	24			
5	1	60	960	4580	5040	552		
6	1	120	4260	52960	213000	206592	21280	
7	1	210	14070	368830	3762360	13109712	11404960	1073160

From the table we see that the numbers for $k = m$ correspond to the number of $3 \times m$ Latin rectangles.

Theorem. *The number of ways to place m rooks on an $m \times m \times m$ probléme des rencontres board is equal to the number of $3 \times m$ Latin rectangles.*

Proof: A $3 \times m$ Latin rectangle is an array where on each of the three rows is a permutation of the elements $1, 2, 3, ..., m$ and where each of the m columns contains each element at most one time. We can create an ordered triple to represent each column by taking the the entry in rows one, two and three as (r_1, r_2, r_3). We can then take the m ordered triples and place rooks in the corresponding positions of an $m \times m \times m$ board. Because each element appears in each row from the Latin rectangle exactly one time, we will have only one rook per slab, one rook per wall, and one rook per tower. So the ordered triple associated with any $3 \times m$ Latin rectangle corresponds with a valid placement of m rooks.

Since each of the entries in a Latin rectangle column must be unique, we know that for any ordered triple the entries are different, and thus we will not have any ordered triples of the form (i, i, j), (i, j, i) or (j, i, i). This matches exactly with the restrictions for an $m \times m \times m$ probléme des rencontres board. So we know that any ordered triple from a $3 \times m$ Latin rectangle will correspond with a valid placement of m rooks on an $m \times m \times m$ probléme des rencontres board.

Let us consider an arbitrary placement of m rooks on an $m \times m \times m$ probléme des rencontres board. The position of each of the m rooks can be represented as an ordered triple (i, j, k) in the same way that we can represent the entries of the m columns of a $3 \times m$ Latin rectangle as ordered triples. We know that for any ordered triple from an $m \times m \times m$ probléme des rencontres board we will not have any ordered triples of the form (k, k, n), (k, n, n) and (n, k, n) for some n and k from 1 to m. So every placement of m rooks on an $m \times m \times m$ probléme des rencontres board creates a valid $3 \times m$ Latin rectangle.

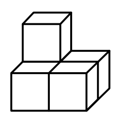
Since we know every placement of m rooks on an $m \times m \times m$ probléme des rencontres board corresponds to a valid $3 \times m$ Latin rectangle, and every $3 \times m$ Latin rectangle corresponds to a valid placement of m rooks on an $m \times m \times m$ probléme des rencontres board, we know that the total number of placements of m rooks on an $m \times m \times m$ probléme des rencontres board is equal to the total number of $3 \times m$ latin rectanglesL

This second generalization of the probléme des rencontres board can be generalized to dimensions higher than 3 as follows: a size m probléme des rencontres board in d dimensions is a subset of the set $\{1, 2, \ldots m\}^d$ and the cells with two equal coordinates are restricted. With this generalization, using a method similar to the proof of the above theorem, we obtain the following theorem:

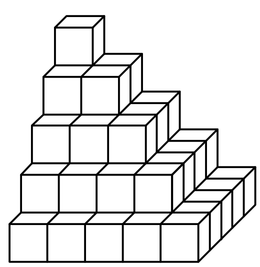
Theorem. *The number of ways to place m rooks on a size m probléme des rencontres board in d dimensions is equal to the number of $d \times m$ Latin rectangles.*

Another generalization of the probléme des rencontres board to three dimensions involves sweeping an $m \times m$ probléme des rencontres board by m units. We would then have an $m \times m \times m$ board with restrictions on all cells of the form (i, i, k). In relation to the white elephant gift exchange, this would correspond to the case where nobody is allowed to draw any name two times. □

3.2. **Triangle Board in Three Dimensions.** In two dimensions a triangle board was constructed by restricting all tiles above the (i, i) position for $i = 1, .., n$ on an $n \times n$ board. We noticed that the only way to place n rooks on an $n \times n$ triangle board was to place the rooks on the main diagonal. Another property of the triangle board was removing both the row and column corresponding with a rook placed on the diagonal of a size n triangle board resulted in a size $n-1$ triangle board. We want to replicate these aspects of a triangle board in three dimensions, and this is how our three dimensional triangle board evolved. In three dimensions a size 1 triangle board is simply a cell. The size 2 triangle board is obtained by placing a $2 \times 2 \times 1$ unrestricted board below the size 1 triangle board as follows.



For any size n triangle board, we obtain a size $n+1$ triangle board by placing an additional $(n+1) \times (n+1) \times 1$ board on the bottom so that the cells line up along $(1, 1, k)$ for $k = 1, .., n$. Using this method a size 5 board would appear as such.



The rook numbers for the triangle boards up to size 8 are as follows:

$n\backslash k$	0	1	2	3	4	5	6	7	8
0	1								
1	1	1							
2	1	5	1						
3	1	21	14	1					
4	1	85	147	30	1				
5	1	341	1408	627	55	1			
6	1	1365	13013	11440	2002	91	1		
7	1	5461	118482	196053	61490	5278	140	1	
8	1	21845	1071799	3255330	1733303	251498	12138	204	1

Sequence A008957 in [Sloane 2009], Triangle of central factorial numbers

Theorem. *The number of ways to place k rooks on a size n triangle board is $T(n+1, n+1-k)$ where $T(n, k) = T(n-1, k-1) + k^2 * T(n-1, k)$, $T(n, n) = 1$, and $T(n, 1) = 1$.*

Proof: We will prove this theorem by induction on n.

For the base case, $n = 1$, the rook polynomial is $1 + x$ and the corresponding central factorial numbers are $T(2, 2) = T(2, 1) = 1$. Hence the result is true for $n = 1$.

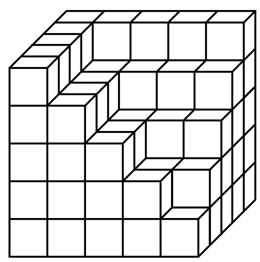
For our base step, we know that $T(n, n) = 1$ and $T(n, 1) = 1$, so $T(2, 1) = 1$ and $T(2, 2) = 1$. This corresponds with placing zero rooks on a 1×1 board and placing one rook on a 1×1 board.

For a three dimensional $(n-1) \times (n-1) \times (n-1)$ triangle board there is only one way to place $n-1$ non-attacking rooks on the board, that is by placing the rooks in the diagonal, meaning in the cells (i, i, i) for $i = 1..(n-1)$. This corresponds with $T(n, 1) = 1$.

We also know that there is only one way to place no rooks on an $(n-1) \times (n-1) \times (n-1)$ board and this corresponds with $T(n, n) = 1$.

Given that the number of ways to place k rooks on an $(n-2) \times (n-2) \times (n-2)$ triangle board is $T(n-1, n-1-k)$, we will show that the number of ways to place k rooks on an $(n-1) \times (n-1) \times (n-1)$ is $T(n, n-k)$. Let us consider placing k rooks on an $(n-1) \times (n-1) \times (n-1)$ triangle board. We can place k rooks on the top $n-2$ layers, which compose an $(n-2) \times (n-2) \times (n-2)$ triangle board within the $(n-1) \times (n-1) \times (n-1)$ triangle board, in $T(n-1, n-1-k)$ ways. The other case is placing $k-1$ rooks in the $(n-2) \times (n-2) \times (n-2)$ triangle board and one rook in the $(n-1) \times (n-1)$ layer. We can place $k-1$ rooks in the $(n-2) \times (n-2) \times (n-2)$ triangle board in $T(n-1, n-k)$ ways. Since we have placed $k-1$ rooks in the $(n-2) \times (n-2) \times (n-2)$ board, we have $n-1-(k-1) = n-k$ rows and $n-k$ columns to place our last rook in the $(n-1) \times (n-1)$ layer. This gives us $(n-k)^2 T(n-1, n-k)$ ways to place k rooks in an $(n-1) \times (n-1) \times (n-1)$ board with one rook lying in the bottom layer. Adding up the case where one rook lies in the last layer, and the case where a rook does not lie in the last layer, we have $T(n-1, n-1-k) + (n-k)^2 T(n-1, n-k)$ total ways to place k rooks on an $(n-1)\times(n-1)\times(n-1)$ board. We can see that this follows the same recursion as $T(n,k)$, and since we have shown that all special cases $T(n,1)$ and $T(n,n)$ corresponds with placing $n-1$ rooks on an $(n-1)\times(n-1)\times(n-1)$ board and placing no rooks on an $(n-1)\times(n-1)\times(n-1)$ board, we know that the number of ways to place k rooks on an $n\times n\times n$ triangle board is $T(n+1, n+1-k)$, by induction. □

3.3. **Genocchi Board.** During our search for the best generalization of a two dimensional triangle board into three dimensions, we developed a board related to the Genocchi numbers, we call this board the Genocchi board. A size m Genocchi board is constructed by letting the height of each tower equal the maximum of the corresponding wall and slab that the tower lies in. For example, the height of tower in the second wall and third slab would be three since the highest value of the two is three. An example of a size 5 Genocchi board is given below.



Using our computer program we generated rook polynomials for various sizes of this board. Comparing our numbers with different sequences from *The On-Line Encyclopedia of Integer Sequences* we found that the number of ways to place m rooks on a board of size m corresponds with the unsigned $(m+1)^{th}$ Genocchi number.

m	0	1	2	3	4	5	6	7
r_m	1	1	3	17	155	2073	38227	929569
$a(m+1)$	-1	1	-3	17	-155	2073	-38227	929569

Using these numbers we were able to come up with the following theorem.

Theorem. *The number of ways to place m non-attacking rooks on a size m Genocchi board is the unsigned m^{th} Genocchi number.*

Proof: The complement of a size m Genocchi board in an $m\times m\times m$ cube is a size $m-1$ triangle board. Hence, using the theorem of complementary boards, we can calculate the number of ways to place m rooks on a size m Genocchi board in terms of central factorial numbers which are the rook numbers for the triangle board. Recall that r_k for a size m triangle board is $T(n+1, n+1-k)$ and recall that the number of ways to place k rooks on a board A given the polynomial $R_{\bar{A}}(x) = r_0 + r_1 x + r_2 x^2 + ... + r_k x^k ...$ to its complement is $\sum_{i=0}^{k} \binom{m-i}{k-i}\binom{n-i}{k-i}\binom{r-i}{k-i}(k-i)!^2 r_i (-1)^i$. Using these we find that in general the number of ways to place m rooks on an $m \times m \times m$ Genocchi board is $\sum_{i=0}^{n} (-1)^i (i!)^2 T(n,i)$. This summation has been shown to equal the m^{th} Genocchi number [Dumont 1973], thus the number of ways to place m rooks on an $m \times m \times m$ Genocchi board is the m^{th} Genocchi number.

□

3.4. **Ferrers Boards in Three and Higher Dimensions.** In two dimensions the Ferrers board encompassed both the triangle and the rectangular boards. It is expected that a theory of Ferrers boards in three and higher dimensions can be developed to include the triangle and full boards as we defined above, as well as sweeps of two-dimensional Ferrers boards. Such a generalization is expected to yield boards whose rook polynomials can be expressed in terms of the values which generalize the heights of the columns in two dimensions.

4. Appendix

```
Rook:=proc(A,m,n,p,B,k,rem)
local C,i,j,h,g,l,count,v;
count:=0;
if k=1 then
 for i from 1 to m do
  for j from 1 to n do
   for g from 1 to p do
    if 'not'('in'([i,j,g],B)) then
     if add(add(A[i,a1,a2],a1=1..n),a2=1..p)=0 then
      if add(add(A[b1,j,b2],b1=1..m),b2=1..p)=0 then
       if add(add(A[c1,c2,g],c1=1..m),c2=1..n)=0 then
        count:=count+1
       end if
      end if
     end if
    end if
   end do
  end do
 end do
else
C:=Array(1..m,1..n,1..p);
 for i from 1 to m do
  for j from 1 to n do
   for g from 1 to p do
    if 'not'('in'([i,j,g],B)) then
     if add(add(A[i,a1,a2],a1=1..n),a2=1..p)=0 then
      if add(add(A[b1,j,b2],b1=1..m),b2=1..p)=0 then
       if add(add(A[c1,c2,g],c1=1..m),c2=1..n)=0 then
```

```
        for h from 1 to m do
         for l from 1 to n do
          for v from 1 to p do
           C[h,l,v]:=A[h,l,v]
          end do
         end do
        end do
        C[i,j,g]:=1;
        count:=count+Rook(C,m,n,p,B,k-1);
        C[i,j,g]:=0
       end if
      end if
     end if
    end if
   end do
  end do
 end do
end if
count:=count/k
end proc:
```

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