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HYPERGEOMETRIC FUNCTIONS IN ARITHMETIC GEOMETRY

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HYPERGEOMETRIC FUNCTIONS IN ARITHMETIC GEOMETRY

by

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DISSERTATION

Presented to the Faculty of the Graduate School of

The University of Texas at Austin

in Partial Fulfillment

of the Requirements

for the Degree of

DOCTOR OF PHILOSOPHY

THE UNIVERSITY OF TEXAS AT AUSTIN

May 2009

A mis padres, Diana y Raúl.

Acknowledgments

This dissertation and my general graduate school experience would not have been as exciting and productive had it not been for the support from many different people. In no particular order, but perhaps in reverse chronological order, I will list those people who have made my academic and non-academic life so rich.

First, I would like to thank Fernando Rodríguez-Villegas for being such an outstanding math role model. He always taught me to look at the big picture, to be a self-sufficient researcher, and to not be afraid of checking formulas with the computer. I am also immensely grateful for how supportive he was of my non-mathematical endeavors, and for pointing me in the right direction when it became time to decide what to do after graduate school.

My mathematical mind was clearly shaped by the other members of my committee, John Tate, Jeffrey Vaaler, Felipe Voloch, and Douglas Ulmer. I can't remember a semester, other than the current one, in which I wasn't taking at least one class from the former trio. I want to thank Douglas Ulmer for valuable email discussions about my research in the last year.

I also want to thank David Savitt and everyone related to the organization of Arizona Winter School (some of whom have already been mentioned).

This program reignited my excitement for mathematics every year, and made me realize the value of working intensely with other people on new problems. I also met great people through this program that I'm lucky to count as my friends.

My life as a graduate student was greatly improved by the support of the mathematics department in general. Nancy Lamm in particular was instrumental in my feeling welcomed, appreciated and supported during my stay here at UT. I especially credit Mike Starbird and Ted Odell with my growth as a teacher.

Finally, I want to give special thanks to my friends and family who have celebrated with me through the good times and supported me through the hard times. I made great friends in graduate school, but I want to thank especially John, Brian, Pippa, Elizabeth, and Brandy, for enriching my social and emotional life, and my office mates throughout the years, particularly Katja, Todd, Neil, and Cody. Last but not least, I would like to thank my parents, Diana and Raúl, who taught me to be creative and curious, my sisters, Claudia and Patricia, who have been my role models and partners in everything, and my nieces, Camila and Isabel, whose cuteness decorated my desktop and brought a smile to my face every day.

HYPERGEOMETRIC FUNCTIONS IN ARITHMETIC GEOMETRY

Publication No. ______________

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The University of Texas at Austin, 2009

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Hypergeometric functions seem to be ubiquitous in mathematics. In this document, we present a couple of ways in which hypergeometric functions appear in arithmetic geometry.

First, we show that the number of points over a finite field $\mathbb{F}_q$ on a certain family of hypersurfaces, $N_{\mathbb{F}_q}(\lambda)$, is a linear combination of hypergeometric functions. We use results by Koblitz and Gross to find explicit relationships, which could be useful for computing Zeta functions in the future.

We then study more geometric aspects of the same families. A construction of Dwork's gives a vector bundle of deRham cohomologies equipped with a connection. This connection gives rise to a differential equation which is known to be hypergeometric. We developed an algorithm which computes the parameters of the hypergeometric equations given the family of hypersurfaces.

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Chapter 1

Introduction

Hypergeometric functions seem to be ubiquitous in mathematics, particularly when studying arithmetic and geometric properties of certain varieties. As we describe below, it is a classical result that both the number of points on a certain family of curves over a finite field and the solutions to a differential equation related to this family are related to the same hypergeometric function. In this document, we present some results we have obtained while trying to generalize these ideas to more varieties.

For each $\lambda \in \mathbb{P}^1 - \{0, 1, \infty\}$ we can define an elliptic curve

$$E_\lambda : y^2 = x(x-1)(x-\lambda).$$

These form the so-called Legendre family. For $\lambda \in \mathbb{Z}$ the number of $\mathbb{F}_p$-points on these curves modulo p, $N_{\mathbb{F}_p}(\lambda)$, is a hypergeometric function of the parameter λ (modulo p). In fact, a simple computation shows that

$$N_{\mathbb{F}_p}(\lambda) \equiv (-1)^{(p+1)/2} \sum_{r=0}^{\frac{p-1}{2}} \binom{-1/2}{r}^2 \lambda^r \bmod p$$

$$\equiv (-1)^{(p-1)/2} \sum_{r=0}^{\frac{p-1}{2}} \frac{(1/2)_r (1/2)_r}{r! r!} \lambda^r \bmod p,$$

where the last sum is the hypergeometric function

$${}_2F_1 \left(\frac{1}{2}, \frac{1}{2}; 1 \middle| \lambda \right)$$

truncated at $\frac{p-1}{2}$.

There is a differential equation satisfied by the periods of these elliptic curves known as the Picard-Fuchs differential equation, which happens to be a hypergeometric differential equation. The surprising result is that the holomorphic solution around zero is also

$${}_2F_1 \left(\frac{1}{2}, \frac{1}{2}; 1 \middle| \lambda \right).$$

Igusa was first to notice this, in [15]. Later, Manin proved a similar result for higher genus curves by using the algebro-geometric version of the Lefschetz fixed-point theorem on the Frobenius mapping. For more details on all these ideas, see [31] or [5].

We have focused on generalizing these results to varieties of the form:

$$X_\lambda : x_1^d + \cdots + x_n^d - d\lambda x_1^{h_1} \cdots x_n^{h_n} = 0 \tag{1.1}$$

where each $h_i \geq 0 \in \mathbb{Z}$, $\sum h_i = d$ and $g.c.d.(d, h_1, \ldots, h_n) = 1$. To generalize the situation described above, one must examine two questions:

1. Can we find a good formula for $N_{\mathbb{F}_q}(\lambda)$, for q a power of a prime p? If so, is this function related to some version of a hypergeometric function modulo p?

2. Can we find a differential equation related to the geometry of these varieties? If so, is this a hypergeometric differential equation?

The final challenge would be to see how the two objects ($N_{\mathbb{F}_q}(\lambda)$ and the differential equation) are related to each other, getting a result similar to Igusa's. Answering these questions would also be useful for computing Zeta- and L-functions of these varieties X_λ, which have been interesting to physicists working on string theory and mirror symmetry (cf. [4] and [7]).

We have obtained promising partial results and numerous examples suggesting that the classical result can be extended to these varieties, which are described in more detail in the upcoming chapters.

Chapter 2 contains preliminaries about the theory of hypergeometric functions. We establish basic definitions and list all the important results which will be used in the rest of the document.

The main goal of Chapters 3 and 4 is to establish the relationship between the number of $\mathbb{F}_q$-points of a family of hypersurfaces and hypergeometric functions, both the finite field and general versions. These results answer Question 1 for certain special cases.

Chapter 5 describes an algorithm we developed, and later implemented in Pari-GP, which computes hypergeometric differential equations associated to the geometry of certain families of varieties. The main tool used is a construction of Dwork's which gives a more manageable version of cohomology. We give some tables which explicitly show our computations.

The first Appendix contains some background on Ordinary Differential Equations which is mainly used in Chapter 5, and Appendix B contains all of the GP scripts used in our algorithm.

Table 1.1: Notation

$\mathbb{F}_q$	finite field of $q = p^f$ elements
K^*	multiplicative group of a field K
$\mathbb{Q}_p$	field of rational p-adic numbers
$\mathbb{C}_p$	p-adic completion of the algebraic closure of $\mathbb{Q}_p$
$\mathbb{P}^n$	n-dimensional complex projective space
$v_p(x)$	p-adic valuation of x
ω	Teichmüller character from $\mathbb{F}_q^* \to \mathbb{C}_p^*$
$\{x\}$	fractional part of $x = x - [x]$
A^T	the transpose of the matrix or vector A
$N_{\mathbb{F}_q}(V)$	the number of $\mathbb{F}_q$-points on a variety V
μ_n	the subgroup of $\mathbb{C}^*$ of n-th roots of unity
$char(G)$	the character group of an abelian group G

Chapter 2

Hypergeometric Functions

The series

$$\sum_{k\geq 0} \frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!}, \tag{2.1}$$

where we use the Pochhammer notation

$$(x)_k = x(x+1)\cdots(x+k-1) = \frac{\Gamma(x+k)}{\Gamma(x)},$$

is called the *Gauss hypergeometric function*.

The first person to use the term "hypergeometric" was John Wallis in his work *Arithmetica Infinitorum*(1655). He used it to denote a series which was beyond the ordinary geometric series

$$1 + x + x^2 + x^3 + \cdots$$

It was in Gauss' famous thesis, *Disquisitiones generales circa seriem infinitam*(1812), that the brilliant mathematician defined the series (2.1) and used the notation $F(a, b; c|z)$ for it. He proved important summation theorems and gave many relations between two or more of these series.

But it was Kummer, in 1836, who was the first to use the term "hypergeometric" for series of the type (2.1) only. For a more detailed history and a great summary of all the basics, see [32].

Many variants of the definition of a hypergeometric function arose afterwards, a few of which will be used throughout this work. In this chapter, we will introduce three versions of this function and present some of their most important features and properties.

2.1 The generalized hypergeometric function

The most classical definition is the extension of Gauss's hypergeometric function, with notation due to Barnes, as presented in [32].

Definition 2.1.1. Let, $A, B \in \mathbb{Z}$ and $\alpha_1, \ldots, \alpha_A, \beta_1, \ldots, \beta_B \in \mathbb{Q}$, with all of the $\beta_i \geq 0$. The **generalized hypergeometric function** is defined as the series (taking $z \in \mathbb{C}$)

$$ {}_AF_B(\alpha_1, \ldots, \alpha_A; \beta_1, \ldots, \beta_B|z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_A)_k z^k}{(\beta_1)_k \cdots (\beta_B)_k k!}. \tag{2.2} $$

The α_i will be referred to as "numerator parameters" and the β_i as "denominator parameters".

Notice that in this notation Gauss's hypergeometric function becomes ${}_2F_1(\alpha_1, \alpha_2; \beta_1|z)$.

Sometimes we will use the shortened notation

$$_AF_B(\alpha;\beta|z) = {_AF_B}(\alpha_1,\ldots,\alpha_A;\beta_1,\ldots,\beta_B|z).$$

2.1.1 Convergence

To understand the convergence of the series $_AF_B(\alpha;\beta|z)$, let $u_n z^n$ denote the n-th term of the series, so we have

$$\begin{aligned}\left|\frac{u_{n+1}}{u_n}\right||z| &= \frac{|\alpha_1+n|\cdots|\alpha_A+n||z|}{|\beta_1+n|\cdots|\beta_B+n|(1+n)}\\ &\leq \frac{|z|n^{A-B-1}(1+|\alpha_1|/n)\cdots(1+|\alpha_A|/n)}{(1+1/n)(1+|\beta_1|/n)\cdots(1+|\beta_B|/n)}.\end{aligned}$$

If $A \leq B$ the above clearly tends to zero as $n \to \infty$, so the series converges for all z.

If $A = B+1$, the series is convergent if $|z| < 1$. It also converges when $z = 1$ if

$$Re\left(\sum_{\nu=1}^{B}\beta_\nu - \sum_{\nu=1}^{A}\alpha_\nu\right) > 0,$$

and when $z = -1$ if

$$Re\left(\sum_{\nu=1}^{B}\beta_\nu - \sum_{\nu=1}^{A}\alpha_\nu\right) > -1.$$

Finally, if $A > B+1$, the series never converges except for when $z = 0$. Notice that in this case we could only define the function when the series terminates, that is, one or more of the numerator parameters must be zero or negative.

2.1.2 Differential Equations

Let θ denote the operator $z\dfrac{d}{dz}$. The series ${}_AF_B(\alpha;\beta|z)$ satisfies the differential equation

$$\{\theta(\theta+\beta_1-1)\cdots(\theta+\beta_B-1)-z(\theta+\alpha_1)\cdots(\theta+\alpha_A)\}\,y=0.$$

We will denote, as in [1],

$$D(\alpha_1,\ldots,\alpha_A;\beta_1,\ldots,\beta_B)=\theta(\theta+\beta_1-1)\cdots(\theta+\beta_B-1)-z(\theta+\alpha_1)\cdots(\theta+\alpha_A).$$

If $A=B+1$ this is a Fuchsian differential equation with regular singularities at $z=0,1,\infty$. If $A\leq B$, the equation has a regular singularity at $z=0$ and an irregular one at $z=\infty$.

To find the other series solutions (centered at 0) to this differential equation, we use the method of Frobenius. Suppose $y=z^r\sum_{n=0}^{\infty}a_nz^n$ is a solution to the differential equation. Substituting, and using the fact that $P(\theta)(z^n)=P(n)z^n$, where P is a polynomial expression, we get that the *indicial equation* is

$$r(r+\beta_1-1)(r+\beta_2-1)\cdots(r+\beta_B-1)=0,$$

which has solutions

$$r=0,1-\beta_1,\ldots,1-\beta_B.$$

The coefficients will satisfy a recurrence relation. For each r,

$$(r+n)(r+n+\beta_1-1)\cdots(r+n+\beta_B-1)a_n = (r+n+\alpha_1-1)\cdots(r+n+\alpha_A-1)a_{n-1}.$$

Hence

$$a_n = \frac{(r+\alpha)_n}{(r+\beta)_n(r+1)_n},$$

and so the complete set of solutions is given by

$$_AF_B(\alpha;\beta|z), \qquad \text{(the holomorphic solution around 0)}$$

and the functions

$$z^{1-\beta_i}\,{}_AF_B(1+\alpha_1-\beta_i,\ldots,1+\alpha_A-\beta_i;1+\beta_1-\beta_i,\check{\ldots},1+\beta_B-\beta_i,2-\beta_i|z),$$

where $\check{}$ denotes the omission of $1+\beta_i-\beta_i$ (in fact, this term becomes the factorial term in the series).

As before, the radius of convergence of the resulting series solutions is 0,1, or ∞, depending on whether $A > B+1, A = B+1, A \le B$ respectively.

Notice that the parameters α_i, β_i completely characterize the hypergeometric function and its corresponding differential equation.

2.1.3 The Monodromy Group of ${}_nF_{n-1}$

Let H be the fundamental group $\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}, z_0)$ where z_0 is some fixed base point, for example $z_0 = \frac{1}{2}$. Then clearly H is generated by g_0, g_1, g_∞ with the relation $g_\infty g_1 g_0 = 1$, as pictured below.

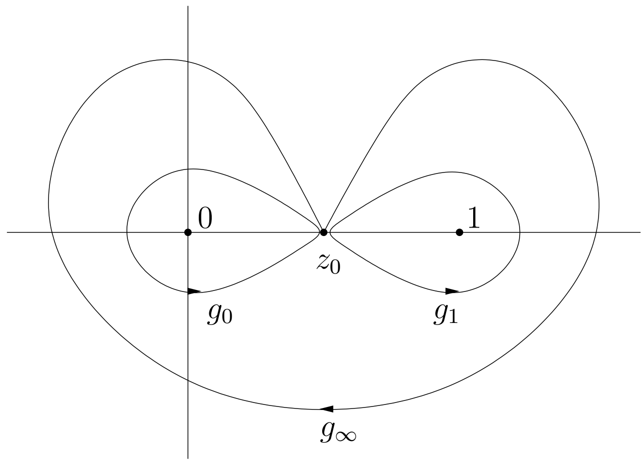

Figure 2.1: The generators of $\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}, z_0)$

Recall that the differential equation for a hypergeometric function of the form ${}_nF_{n-1}(\alpha; \beta|z)$ is Fuchsian with regular singular points $0, 1, \infty$. Around a regular point, for example $z_0 = \frac{1}{2}$, there are n linearly independent analytic solutions with a non-zero radius of convergence.

Take $f_1, \ldots, f_n$ to be n linearly independent solutions to the differential equation, and let V be the space of solutions. For any f_i, we can take its analytic continuation along a path γ in $\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}, z_0)$. Analytic continuation along a homotopically trivial loop yields the same function, but along a non-trivial loop we may get a different function (we have moved to a different branch). So $f_i \mapsto F_i$ by analytic continuation along γ not trivial. For example, take $\gamma = g_0$.

The coefficients of the differential equation are unaltered by letting the variable z describe this circuit around 0, so the functions $F_1, \ldots, F_n$ are also solutions. Hence we can express the F_i in terms of the basis of the space of solutions as

$$\begin{aligned} F_1 &= a_{11}f_1 + \cdots + a_{1n}f_n \\ F_2 &= a_{21}f_1 + \cdots + a_{2n}f_n \\ &\vdots \\ F_n &= a_{n1}f_1 + \cdots + a_{nn}f_n. \end{aligned}$$

And so in fact, we can associate to g_0 the matrix $B = (a_{ij})$. The same is true for the other generators. Let $A, B, C \in GL(V)$ be determined by analytic continuation, so that

$$\begin{array}{ccc} A & \leftrightarrow & g_\infty \\ B & \leftrightarrow & g_0 \\ C & \leftrightarrow & g_1 \end{array}$$

The group $\Gamma \subset GL(V)$ generated by A, B, C with the relation $ACB = I$ is called the *monodromy group*, and the map

$$H \to GL(V)$$

$$g_\infty, g_0, g_1 \mapsto A, B, C,$$

is a representation of H.

In Chapter 5, we will use a certain property of monodromy groups in order to relate them to hypergeometric functions. First, we need some definitions from [1].

Definition 2.1.2. Let V be a finite dimensional complex vector space. A linear map $g \in GL(V)$ is called a *reflection* if $g - Id$ has rank one. The determinant of a reflection is called the *special eigenvalue* of g.

Definition 2.1.3. Suppose $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{C}^*$ with $a_j \neq b_k$ for all $j = 1, \ldots, n$. A *hypergeometric group* with numerator parameters $a_1, \ldots, a_n$ and denominator parameters $b_1, \ldots, b_n$ is a subgroup of $GL(n, \mathbb{C})$ generated by elements $h_0, h_1, h_\infty \in GL(n, \mathbb{C})$ such that $h_\infty h_1 h_0 = \mathrm{Id}$,

$$\det(z - h_\infty) = \prod_{i=1}^{n} (z - a_j)$$
$$\det(z - h_0^{-1}) = \prod_{j=1}^{n} (z - b_j),$$

and h_1 is a reflection in the sense of Definition 2.1.2.

Then we have the following useful result.

Proposition 2.1.1. *Suppose $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{C}^*$ with $a_j \neq b_k$ for all $j, k = 1, \ldots, n$ and assume $b_n = 1$. Let $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_{n-1} \in \mathbb{C}$ be such that $a_j = e^{2\pi i \alpha_j}$ for $j = 1, \ldots, n$ and $b_k = e^{2\pi i \beta_k}$ for $k = 1, \ldots, n-1$. Then the monodromy group of the hypergeometric equation*

$$D(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_{n-1}) y = 0$$

is a hypergeometric group with parameters $a_1, \ldots, a_n, b_1, \ldots, b_n$.

The most important consequence of this result is that if we have a hypergeometric group Γ in the sense of the Definition 2.1.3, we can find a hypergeometric differential equation whose monodromy group is Γ.

2.2 Hypergeometric weight systems

One can think of (2.2) in terms of ratios of factorials (rather than Pochhammer symbols). In [28], Rodríguez-Villegas defines a **hypergeometric weight system** as a formal linear combination

$$\gamma = \sum_{\nu \geq 1} \gamma_\nu [\nu], \qquad \nu \in \mathbb{Z}, \tag{2.3}$$

where the $\gamma_\nu \in \mathbb{Z}$ are zero for all but finitely many ν, satisfying the following conditions:

1. $\sum_{\nu \geq 1} \nu \gamma_\nu = 0$
2. $d = d(\gamma) := -\sum_{\nu \geq 1} \gamma_\nu > 0$

We refer to condition (1) as the *regularity* of γ and d is the dimension of γ.

To γ we can associate the formal power series

$$u(\lambda) := \sum_{n \geq 0} u_n \lambda^n$$

where

$$u_n = \prod_{\nu \geq 1} (\nu n)!^{\gamma_\nu}.$$

Lemma 2.2.1. *u is a hypergeometric function, that is, for some minimal r we have*

$$u(\lambda) = {}_rF_{r-1}\left(\alpha_1, \ldots, \alpha_r; \beta_1, \ldots, \beta_{r-1} \,\middle|\, \frac{\lambda}{\lambda_0}\right)$$

where $\lambda_0^{-1} = \prod_{\nu \geq 1} \nu^{\nu\gamma_\nu}$ and $\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_{r-1}$ are rational numbers.

Proof. Recall that we can write the Pochhammer symbol in terms of the Γ function as follows:

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}$$

Using the Gauss-Legendre multiplication formula, we obtain

$$\begin{aligned}
\prod_{k=1}^{\nu-1} \left(\frac{k}{\nu}\right)_n &= \frac{\prod_{k=1}^{d-1} \Gamma\left(\frac{k}{\nu} + n\right)}{\prod_{k=1}^{d-1} \Gamma\left(\frac{k}{\nu}\right)} \\
&= \frac{\prod_{k=1}^{d-1} \Gamma\left(\frac{k+\nu n}{\nu}\right)}{\prod_{k=1}^{d-1} \Gamma\left(\frac{k}{\nu}\right)} \\
&= \nu^{1-\nu n} \frac{\Gamma(\nu n)}{\Gamma(n)} \\
&= \nu^{-\nu n} \frac{\Gamma(\nu n + 1)}{\Gamma(n+1)} \\
&= \nu^{-\nu n} \frac{(\nu n)!}{n!}.
\end{aligned}$$

Therefore we can write

$$(\nu n)! = n! \cdot \nu^{\nu n} \cdot \prod_{k=1}^{\nu-1} \left(\frac{k}{\nu}\right)_n.$$

And so

$$
\begin{aligned}
\prod_{\nu\geq 1}(\nu n)!^{\gamma_\nu} &= \prod_{\nu\geq 1}\left[(n!)^{\gamma_\nu}(\nu^{\nu\gamma_\nu})^n\prod_{n=1}^{\nu-1}\left(\frac{k}{\nu}\right)_n^{\gamma_\nu}\right] \\
&= (n!)^{\sum\gamma_\nu}\prod_{\nu\geq 1}\left[(\nu^{\nu\gamma_\nu})^n\prod_{n=1}^{\nu-1}\left(\frac{k}{\nu}\right)_n^{\gamma_\nu}\right] \\
&= (n!)^{-d}\left(\prod_{\nu\geq 1}(\nu^{\nu\gamma_\nu})\right)^n\prod_{\nu\geq 1}\prod_{n=1}^{\nu-1}\left(\frac{k}{\nu}\right)_n^{\gamma_\nu}.
\end{aligned}
$$

So we can let our α_i be the $\dfrac{k}{\nu}$for the $\gamma_\nu > 0$, where we repeat an α_i if $\gamma_\nu > 1$, and β_i to be the $\dfrac{k}{\nu}$ for the $\gamma_\nu < 0$, and 1 for $d-1$ of the $n!$ terms. There will be the same number of elements in each set of parameters because of the regularity condition. In other words, there are $(\nu-1)\gamma_\nu$ of the α_i, and $(\nu-1)\gamma_\nu + d$ of the β_i, and we know:

$$\sum_{\nu\geq 1}(\nu-1)\gamma_\nu = \sum_{\nu\geq 1}\nu\gamma_\nu - \sum_{\nu\geq 1}\gamma_\nu = 0 + d = d,$$

and by splitting into positives and negatives we get what we want.

If different ν's share common factors, we could cancel some of these terms out, but that wouldn't change the number we have on the numerator versus the number we have on the denominator. So the minimal r is the number of numerator parameters that survive after cancelation.

Finally, if instead of λ we substitute $\prod_{\nu\geq 1}(\nu^{\nu\gamma_\nu})\cdot\lambda$, we get the desired result.

$\square$

This lemma gives us a method for going from a hypergeometric weight system to a hypergeometric function. To go from a hypergeometric function to a hypergeometric weight system, we first need there to be complete sets of fractions for each denominator, i.e., if $1/5$ is one of the α_i's, then $2/5, 3/5, 4/5$ should also be numerator parameters, otherwise the method described above fails. Thus, we can think of a hypergeometric function as being associated to a hypergeometric weight system and viceversa, provided that certain conditions are satisfied.

There is a useful function associated to a hypergeometric weight system which we will now define. Most of this is taken from [28].

Definition 2.2.1. The *Landau function* associated to γ is defined by

$$\mathcal{L}(x) = \mathcal{L}_\gamma(x) := -\sum_{\nu \geq 1} \gamma_\nu \{\nu x\}, \qquad x \in \mathbb{R}$$

where $\{x\}$ denotes the fractional part of x. This function is periodic of period 1.

The Landau function is useful for checking whether the coefficients of the series $u(z)$ are integers.

Proposition 2.2.2 (Landau)**.** *$u_n \in \mathbb{Z}$ for all $n \geq 0$ if and only if $\mathcal{L}(x) \geq 0$ for all $x \in \mathbb{R}$.*

There is a complete proof of this proposition in [28]. We want to point out a crucial step of the proof because it will be used later , and that is the following

Lemma 2.2.3. *Let p be a prime and let $v_p(x)$ denote the p-adic valuation of x.*

$$v_p(u_n) = \sum_{k \geq 1} \mathfrak{L}\left(\frac{n}{p^k}\right)$$

So the Landau function encodes information about the p-adic valuation of the coefficients of the series.

This function has many other properties as listed in [28]. Here we list a few which will be useful in some of our computations later on.

Proposition 2.2.4. *1. The regularity condition is equivalent to $\mathfrak{L}$ being locally constant.*

2. $\mathfrak{L}$ is right continuous with discontinuity points exactly at $x \equiv \alpha_i \bmod 1$ or $x \equiv \beta_i \bmod 1$ for some $i = 1, \ldots, r$. More precisely,

$$\mathfrak{L} = \#\{j | \alpha_i \leq x\} - \#\{j | 0 < \beta_j \leq x\}.$$

3. $\mathfrak{L}$ takes only integer values.

4.

$$\int_0^1 \mathfrak{L}(x)dx = \frac{1}{2}d, \qquad \lim_{x \to 1^-} \mathfrak{L}(x) = d, \qquad \lim_{x \to 0^+} \mathfrak{L}(x) = 0.$$

In particular, for a general non-zero $\gamma = \sum_{\nu \geq 1} \gamma_\nu [\nu]$, integrality implies positive dimension.

5. *Away from the discontinuity points of* $\mathcal{L}$ *we have*

$$\mathcal{L}(-x) = d - \mathcal{L}(x)$$

and, in particular, for all x

$$\mathcal{L}(x) \le d, \qquad \textit{if } u_n \in \mathbb{Z} \textit{ for all } n.$$

2.3 The finite field analog

2.3.1 Gauss and Jacobi sums

Let $\chi_{1/(q-1)} : \mathbb{F}_q^* \to K^*$ be a fixed generator of the character group of $\mathbb{F}_q^*$ with values in an algebraically closed field K of characteristic zero (such as $\mathbb{C}$ or $\mathbb{C}_p$).

Examples:

1. If $K = \mathbb{C}$ fix a primitive root of $\mathbb{F}_q^*$ and determine $\chi_{1/(q-1)}$ by taking that root to $e^{2\pi i/(q-1)}$.

2. If $K = \mathbb{C}_p$ it's natural to take $\chi_{1/(q-1)}$ to be the Teichmüller character. Recall that $\omega : \mathbb{F}_q^* \to \mathbb{C}_p^*$ is the Teichmüller character where $\omega(x)$ is defined as the unique element of $\mathbb{C}_p^*$ which is a $(q-1)$-st root of unity and such that $\omega(x) \equiv x \bmod p$.

For $s \in \frac{1}{q-1}\mathbb{Z}/\mathbb{Z}$ we let $\chi_s = \left(\chi_{1/(q-1)}\right)^{s(q-1)}$, and for any s set $\chi_s(0) = 0$. Let $\psi : \mathbb{F}_q \to K^*$ be a (fixed) additive character.

Definition 2.3.1. For $s \in \frac{1}{(q-1)}\mathbb{Z}/\mathbb{Z}$ we let $g(s)$ denote the Gauss sum

$$g(s) = \sum_{x \in \mathbb{F}_q} \chi_s(x)\psi(x)$$

Lemma 2.3.1. *Gauss sums satisfy the following properties:*

1. $g(s)g(-s) = q\chi_s(-1)$ *if* $s \neq 0$, *and* $g(0) = -1$.
2. *If* $d|q-1$,
$$\prod_{j=0}^{d-1} g\left(s + \frac{j}{d}\right) = \chi_{-ds}(d)g(ds)\prod_{j=1}^{d-1} g\left(\frac{j}{d}\right)$$

Definition 2.3.2. If $s_1, \ldots, s_r \in \dfrac{1}{q-1}\mathbb{Z}/\mathbb{Z}$ and $\sum s_i \not\equiv 0 \bmod \mathbb{Z}$, we define the Jacobi sum

$$J(s_1, \ldots, s_r) = \sum_{\substack{x_1,\ldots,x_r \in \mathbb{F}_q \\ x_1+\cdots+x_r=1}} \chi_{s_1}(x_1)\cdots\chi_{s_r}(x_r), r > 1; J(s_1) = 1.$$

Jacobi sums can be expressed in terms of Gauss sums as follows:

$$J(s_1, \ldots, s_r) = \frac{g(s_1)\cdots g(s_r)}{g(s_1 + \cdots + s_r)}.$$

2.3.2 Katz's hypergeometric functions

There is a finite field analog of the hypergeometric function which was defined by Katz [20].

Definition 2.3.3. Let $t \in \mathbb{F}_q^*$. Define the set

$$V_t = \left\{x \in (\mathbb{F}_q^*)^n, y \in (\mathbb{F}_q^*)^m | x_1 \cdots x_n = t y_1 \cdots y_m\right\}$$

Also, let $\psi : \mathbb{F}_q \to K^*$, be a (fixed) additive character where K is an algebraically closed field (like $\mathbb{C}$ or $\mathbb{C}_p$), let χ denote, as in the previous section, a generator of the character group of $\mathbb{F}_q^*$, and $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m \in \dfrac{1}{q-1}\mathbb{Z}/\mathbb{Z}$ so that $\chi_{\alpha_1}, \ldots, \chi_{\alpha_n}, \chi_{\beta_1}, \ldots, \chi_{\beta_m} : \mathbb{F}_q^* \to K^*$ are multiplicative characters. Then we define the *finite field version of a hypergeometric function* as

$$\begin{aligned} H(\alpha;\beta|t) : &= \sum_{x,y \in V_t} \psi(x_1 + \cdots + x_n - (y_1 + \cdots + y_m))\chi_{\alpha_1}(x_1) \cdots \chi_{\alpha_n}(x_n) \\ & \quad \cdot \overline{\chi}_{\beta_1}(y_1) \cdots \overline{\chi}_{\beta_m}(y_m) \end{aligned}$$

It is natural to wonder what field H is defined over. First of all, since ψ is an additive character on $\mathbb{F}_q$ and χ is a multiplicative character on $\mathbb{F}_q^*$, we know that

$$H(\alpha;\beta|t) \in \mathbb{Q}(\xi_{(q-1)q}) = L$$

where $\xi_{(q-1)q}$ is a $(q-1)q$-th root of unity. Let l be such that $gcd(l, (q-1)q) = 1$, and define the following endomorphism of L:

$$\begin{aligned} \sigma_l \quad : \quad & L \to L \\ & \xi \mapsto \xi^l \end{aligned}$$

$H(\alpha;\beta|t)$ must lie in some subextension of L. We want to find l such that $H(\alpha;\beta|t)^{\sigma_l} = H(\alpha;\beta|t)$.

$$\begin{aligned} H(\alpha;\beta|t)^{\sigma_l} \quad = \quad & \sum_{x,y\in V_t} \psi(l(x_1+\cdots+x_n) - l(y_1+\cdots+y_m))\chi_{\alpha_1}(x_1^l)\cdots\chi_{\alpha_n}(x_n^l) \\ & \cdot\overline{\chi}_{\beta_1}(y_1^l)\cdots\overline{\chi}_{\beta_m}(y_m^l). \end{aligned}$$

And by definition $x_1\cdots x_n = ty_1\cdots y_m$. Define $x_i' = lx_i, y_i' = ly_i$ so that $x_1'\cdots x_n' = l^{m-n}ty_1'\cdots y_m'$. Therefore

$$\begin{aligned} H(\alpha;\beta|t)^{\sigma_l} \quad = \quad & H(\alpha_1+l,\ldots,\alpha_n+l;\beta_1+l,\ldots,\beta_m+l|l^{m-n}t) \\ & \cdot\overline{\chi_{\alpha_1}}\cdots\overline{\chi_{\alpha_n}}\chi_{\beta_1}\cdots\chi_{\beta_m}(l^l) \end{aligned}$$

We want the above to be equal to $H(\alpha;\beta|t)$, and so this restricts the possibilities for our choices of characters as well. If $m = n$ and taking l-th powers of the multiplicative characters only permutes them, i.e. $\{\chi_{\alpha_1},\ldots,\chi_{\alpha_n}\}$ and $\{\chi_{\beta_1},\ldots,\chi_{\beta_n}\}$ are complete sets of characters, then we get that

$$H(\alpha;\beta|t)^{\sigma_l} = H(\alpha;\beta|t)\overline{\chi_{\alpha_1}}\cdots\overline{\chi_{\alpha_n}}\chi_{\beta_1}\cdots\chi_{\beta_m}(l^l)$$

Conjugating also just permutes the characters in each set, so

$$\overline{\chi_{\alpha_1}}\cdots\overline{\chi_{\alpha_n}}\chi_{\beta_1}\cdots\chi_{\beta_m}$$

is either trivial or a quadratic character. Therefore $H(\alpha;\beta|t) \in \mathbb{Q}$ or $\mathbb{Q}(\sqrt{p})$.

It will be convenient to think of this definition in a different form which is given by its Fourier series expansion.

Lemma 2.3.2. *The Fourier series expansion of $H(\alpha;\beta|t)$ is*

$$H(\alpha;\beta|t) = \frac{1}{q-1}\sum_{s\in\frac{1}{q-1}\mathbb{Z}/\mathbb{Z}} g(s+\alpha_1)\cdots g(s+\alpha_n)g(-s-\beta_1)\cdots g(-s-\beta_m)\overline{\chi_s}(t)$$

Proof. The Fourier series expansion of $H(\alpha;\beta|t)$ is given by

$$H(\alpha;\beta|t) = \sum_{s\in\frac{1}{q-1}\mathbb{Z}/\mathbb{Z}} c(s)\overline{\chi_s}(t),$$

where $c(s) = \frac{1}{q-1}\sum_{t\in K^*} H(\alpha;\beta|t)\chi_s(t)$. By definition of V_t, we get that $t = x_1\cdots x_n y_1^{-1}\cdots y_m^{-1}$, and we can do the following computation:

$$\begin{aligned}\sum_{t\in K^*} H(\alpha;\beta|t)\chi_s(t) &= \sum_{t\in K^*}\sum_{x,y\in V_t}\psi(x_1+\cdots+x_n-(y_1+\cdots+y_m)) \\ &\quad \cdot\frac{\chi_{\alpha_1}(x_1)\cdots\chi_{\alpha_n}(x_n)}{\chi_{\beta_1}(y_1)\cdots\chi_{\beta_m}(y_m)}\chi_s(t) \\ &= \sum_{\substack{x\in(K^*)^n\\ y\in(K^*)^m}}\psi(x_1+\cdots+x_n-(y_1+\cdots+y_m)) \\ &\quad \cdot\frac{\chi_{\alpha_1}(x_1)\cdots\chi_{\alpha_n}(x_n)}{\chi_{\beta_1}(y_1)\cdots\chi_{\beta_m}(y_m)}\lambda(x_1\cdots x_n y_1^{-1}\cdots y_m^{-1})\end{aligned}$$

$$
\begin{aligned}
&= \sum_{x,y} \frac{\psi(x_1)\cdots\psi(x_n)}{\psi(y_1)\cdots\psi(y_m)} \frac{\chi_{\alpha_1}(x_1)\cdots\chi_{\alpha_n}(x_n)}{\chi_{\beta_1}(y_1)\cdots\chi_{\beta_m}(y_m)} \frac{\chi_s(x_1)\cdots\chi_s(x_n)}{\chi_s(y_1)\cdots\chi_s(y_m)} \\
&= \sum_{x,y} (\chi_{s+\alpha_1})(x_1)\psi(x_1)\cdots(\chi_{s+\alpha_n})(x_n)\psi(x_n) \\
&\quad \cdot(\overline{\chi_{s+\beta_1}})(y_1)\overline{\psi}(y_1)\cdots(\overline{\chi_{s+\beta_m}})(y_m)\overline{\psi}(y_m) \\
&= \left(\sum_{x_1\in K^*}(\chi_{s+\alpha_1})(x_1)\psi(x_1)\right)\cdots\left(\sum_{x_n\in K^*}(\chi_{s+\alpha_n})(x_n)\psi(x_n)\right) \\
&\quad \cdot\left(\sum_{y_1\in K^*}(\overline{\chi_{s+\beta_1}})(y_1)\overline{\psi}(y_1)\right)\cdots\left(\sum_{y_m\in K^*}(\overline{\chi_{s+\beta_m}})(y_m)\overline{\psi}(y_m)\right) \\
&= g(s+\alpha_1)\cdots g(s+\alpha_n)g(-(s+\beta_1))\cdots g(-(s+\beta_m))
\end{aligned}
$$

□

Chapter 3

The number of $\mathbb{F}_q$-points of a family of hypersurfaces

This chapter shows a first approach to establishing a relationship between the number of points and hypergeometric functions.

First, let us define the families we will be working with for the rest of this document. A family of *monomial deformations of diagonal hypersurfaces* will be a family of varieties of the form:

$$X_\lambda : x_1^d + \cdots + x_n^d - d\lambda x_1^{h_1} \cdots x_n^{h_n} = 0 \tag{3.1}$$

where $\sum h_i = d, \text{g.c.d.}(d, h_1, \ldots, h_n) = 1$. For $\lambda \in \mathbb{Z}$, let $N_{\mathbb{F}_q}(\lambda)$ denote the number of points on the hypersurface in $\mathbb{P}^{n-1}_{\mathbb{F}_q}$.

The first two sections in this chapter are mostly a summary of a paper by Koblitz ([22]). In this paper, Koblitz gives formulas for the number of points on monomial deformations of diagonal hypersurfaces, in terms of Gauss and Jacobi sums. Much of the work is a generalization of the proofs and ideas in Weil's famous paper [34]. The last section in the chapter is devoted to showing an explicit relationship between $N_{\mathbb{F}_q}(\lambda)$ and Katz's finite field hypergeometric function.

3.1 Weil's theorem

Suppose we have an algebraic variety V defined over a finite field $\mathbb{F}_q$ and we want to determine the number $N_{\mathbb{F}_q}(V)$ of $\mathbb{F}_q$-points on it. Since these points are the $\overline{\mathbb{F}}_q$-points of V fixed by the q-th power Frobenius map $F : (\dots, X_i, \dots) \mapsto (\dots, X_i^q, \dots)$, it follows that

$$N_{\mathbb{F}_q}(V) = \#\{X \in V | F(X) = X\}.$$

If we have a group G acting on V, then it is convenient to split up $N_{\mathbb{F}_q}(V)$ into pieces $N_{\mathbb{F}_q}(V, \chi)$, where $\chi : G \to K^*$ is a character with values in an algebraically closed field K of characteristic zero. $N_{\mathbb{F}_q}(V, \chi)$ is defined as follows:

$$N_{\mathbb{F}_q}(V, \chi) = \frac{1}{\#G} \sum_{\xi \in G} \chi^{-1}(\xi) \#\{X \in V | F \circ \xi(X) = X\}.$$

In all of our examples, G will be abelian, so the only irreducible representations will be one-dimensional characters χ. In that case, we have the following lemma, which follows immediately from the previous definitions

Lemma 3.1.1.

$$N_{\mathbb{F}_q}(V) = \sum_{\chi \in char(G)} N_{\mathbb{F}_q}(V, \chi).$$

The simplest example of a variety V with a large group action is the diagonal hypersurface of degree d in $\mathbb{P}^{n-1}_{\mathbb{F}_q}$, where we assume $d|q-1$:

$$D_{d,n} : x_1^d + \cdots + x_n^d = 0$$

The group μ_d^n of n-tuples of d-th roots of unity in $\mathbb{F}_q^*$ acts on $D_{d,n}$ by $\xi = (\xi_1, \ldots, \xi_n)$ taking the point $(x_1, \ldots, x_n)$ to $(\xi_1 x_1, \ldots, \xi_n x_n)$. Let Δ be the diagonal elements of μ_d^n, i.e. elements of the form $(\xi, \cdots, \xi)$. Notice that Δ acts trivially on $D_{d,n}$ and μ_d^n/Δ acts faithfully. The character group of μ_d^n/Δ is in one-to-one correspondence with the n-tuples

$$w = (w_1, \ldots, w_n), 0 \leq w_i < d, \text{for which} \sum w_i \equiv 0 \bmod d,$$

where

$$\chi_w(\xi) := \chi(\xi^w), \qquad \xi^w = \xi_1^{w_1} \cdots \xi_n^{w_n}$$

and χ is a fixed primitive character of μ_d, which we can get for example by restricting $\chi_{1/(q-1)}$ to μ_d. In [34], Weil proves:

Theorem 3.1.2 (Weil).

$$N_{\mathbb{F}_q}(D_{d,n}, \chi_w) = \begin{cases} 0 & \textit{if some but not all } w_i = 0 \\ \dfrac{q^{n-1}-1}{q-1} & \textit{if all } w_i = 0 \\ -\dfrac{1}{q} J\left(\dfrac{w_1}{d}, \ldots, \dfrac{w_n}{d}\right) & \textit{if all } w_i \neq 0 \end{cases}$$

3.2 Koblitz's formula

The goal of Koblitz's paper is to use Weil's result and similar methods to find the number of points on the monomial deformation (3.1). Notice that these hypersurfaces allow an action of the subgroup $G \subset \mu_d^n/\Delta$, consisting of elements which preserve the monomial $x^h = x_1^{h_1} \cdots x_n^{h_n}$, that is,

$$G = \{\xi \in \mu_d^n | \xi^h = 1\}/\Delta.$$

The characters χ_w of μ_d^n/Δ which act trivially on G are precisely powers of χ_h. Thus, $char(G)$, the character group of G, corresponds to equivalence classes of w in

$$W = \{(w_1, \ldots, w_n) | 0 \leq w_i < d, \sum w_i \equiv 0 \mod d\},$$

where $w' \sim w$ if $w - w'$ is a multiple (mod d) of h. Notice each equivalence class contains d n-tuples w' because $g.c.d(d, h_1, \ldots, h_n) = 1$.

We are now ready to state the main theorem of the paper.

Assume $d|q-1$ and let $N_{\mathbb{F}_q}(0)$ be the number of $\mathbb{F}_q$-points on the diagonal hypersurface $D_{d,n}$.

Theorem 3.2.1 (Koblitz)**.**

$$N_{\mathbb{F}_q}(\lambda) = N_{\mathbb{F}_q}(0) + \frac{1}{q-1} \sum_{\substack{s \in \frac{d}{q-1}\mathbb{Z}/\mathbb{Z} \\ w \in W}} \frac{g\left(\dfrac{w+sh}{d}\right)}{g(s)} \chi_s(d\lambda),$$

where we denote $g\left(\dfrac{w+sh}{d}\right) = \prod_i g\left(\dfrac{w_i + sh_i}{d}\right)$.

3.3 The relation to Katz's hypergeometric function

In this section, we will see that $N_{\mathbb{F}_q}(\lambda) - N_{\mathbb{F}_q}(0)$ is related to the finite field version of a hypergeometric function by a method similar to one described in Koblitz's paper.

First, consider for some fixed w the sum

$$\sum_{\substack{s\in\frac{d}{q-1}\mathbb{Z}/\mathbb{Z}\\ w'\sim w}} \frac{g\left(\dfrac{w+sh}{d}\right)}{g(s)}\chi_s(d\lambda). \tag{3.2}$$

It is not hard to check that if we replace d by ds and sum over $s \in \frac{1}{(q-1)}\mathbb{Z}/\mathbb{Z}$ we obtain

$$(3.2) = \sum_{s\in\frac{1}{q-1}\mathbb{Z}/\mathbb{Z}} \frac{g\left(hs+\dfrac{w}{d}\right)}{g(ds)}\chi_{ds}(d\lambda). \tag{3.3}$$

Using property 2 of Gauss sums stated earlier, we can rewrite the previous statement as

$$(3.3) = \prod_{j=1}^{d-1} g\left(\frac{j}{d}\right) \sum_s \frac{g\left(h_1 s+\dfrac{w_1}{d}\right)\cdots g\left(h_n s+\dfrac{w_n}{d}\right)}{g(s)g\left(s+\dfrac{1}{d}\right)\cdots g\left(s+\dfrac{d-1}{d}\right)}\chi_{ds}(\lambda). \tag{3.4}$$

For each i, notice that we can use property (2) again, but we need to assume $dh_i|q-1$ for all i. Basically, this means that all of our upcoming computations will make sense for a large enough q.

$$g\left(h_i s + \frac{w_i}{d}\right) = g\left(h_i\left(s + \frac{w_i}{dh_i}\right)\right)$$
$$= \frac{\prod_{j=0}^{h_i-1} g\left(s + \frac{w_i}{dh_i} + \frac{j}{h_i}\right)}{\chi_{-(h_i s + \frac{w_i}{d})}(h_i)\prod_{j=1}^{h_i-1} g\left(\frac{j}{h_i}\right)}.$$

Combining, we get that for a fixed w,

$$(3.4) = \frac{c}{q-1}\sum_{s\in\frac{1}{q-1}\mathbb{Z}/\mathbb{Z}} \frac{\prod_{i=1}^{n}\prod_{j=0}^{h_i-1} g\left(s + \frac{w_i+dj}{dh_i}\right)}{g(s)g\left(s+\frac{1}{d}\right)\cdots g\left(s+\frac{d-1}{d}\right)}\chi_s\left(\prod_i h_i^{h_i}\lambda^d\right), \tag{3.5}$$

where

$$c = \frac{\prod_{j=1}^{d-1} g\left(\frac{j}{d}\right)}{\prod_{i=1}^{n}\prod_{j=1}^{h_i-1} g\left(\frac{j}{h_i}\right)}.$$

Notice that over $\frac{1}{q-1}\mathbb{Z}/\mathbb{Z}$, $g(-s) = g(1-s)$, and so property (1) of Gauss sums can be rewritten as

$$g(s)g(1-s) = q\chi_s(-1).$$

Using this, we can rewrite the products above as

$$\prod_{j=1}^{d-1} g\left(\frac{j}{d}\right) = q^{\frac{d-1}{2}}\xi_1$$

and

$$\prod_{i=1}^{n}\prod_{j=1}^{h_i-1} g\left(\frac{j}{h_i}\right) = q^{\sum \frac{h_i-1}{2}}\xi_2 = q^{\frac{d-n}{2}}\xi_2,$$

where ξ_1, ξ_2 are $q-1$ roots of unity. And so c becomes much simpler,

$$c = \xi q^{\frac{n-1}{2}},$$

where ξ is some root of unity which depends on d, n, h_i.

We want to relate this last expression to Katz's hypergeometric function. Notice that it is almost in the same form, except that we need to add over $\overline{\chi_s} = \chi_{-s}$, but we can change variables in the sum, so that we get

$$(3.5) = \frac{\xi q^{\frac{n-1}{2}}}{q-1} \sum_{s\in\frac{1}{q-1}\mathbb{Z}/\mathbb{Z}} \frac{\prod_{i=1}^{n}\prod_{j=0}^{h_i-1} g\left(-s+\dfrac{w_i+dj}{dh_i}\right)}{g(-s)g\left(-s+\dfrac{1}{d}\right)\cdots g\left(-s+\dfrac{d-1}{d}\right)} \chi_{-s}\left(\prod_i h_i^{h_i}\lambda^d\right)$$

$$= \frac{\xi q^{\frac{n-1}{2}}}{q-1} \sum_{s\in\frac{1}{q-1}\mathbb{Z}/\mathbb{Z}} \frac{\prod_{i=1}^{n}\prod_{j=0}^{h_i-1} g\left(-\left(s-\dfrac{w_i+dj}{dh_i}\right)\right)}{g(-s)\cdots g\left(-\left(s-\dfrac{d-1}{d}\right)\right)} \overline{\chi}_s\left(\prod_i h_i^{h_i}\lambda^d\right). \quad (3.6)$$

Now we can use property (1) of Gauss sums to change from expressions involving $g(-s)$ to expressions involving $g(s)$ and viceversa by

$$g(-s) = \frac{q\chi_s(-1)}{g(s)},$$

to get

$$
\begin{aligned}
(3.6) \quad &= \frac{c'}{q-1} \sum_{s \in \frac{1}{q-1}\mathbb{Z}/\mathbb{Z}} g(s) g\left(s - \frac{1}{d}\right) \cdots g\left(s - \frac{d-1}{d}\right) \\
&\quad \cdot \prod_{i=1}^{n} \prod_{j=0}^{h_i - 1} g\left(-\left(s - \frac{w_i + dj}{dh_i}\right)\right) \chi_{-s}\left((-1)^d \prod_i h_i^{h_i} \lambda^d\right) \\
&= \frac{c'}{q-1} \sum_{s \in \frac{1}{q-1}\mathbb{Z}/\mathbb{Z}} g(s) g\left(s + 1 - \frac{1}{d}\right) \cdots g\left(s + 1 - \frac{d-1}{d}\right) \\
&\quad \cdot \prod_{i=1}^{n} \prod_{j=0}^{h_i - 1} g\left(-\left(s + 1 - \frac{w_i + dj}{dh_i}\right)\right) \chi_{-s}\left((-1)^d \prod_i h_i^{h_i} \lambda^d\right) \\
&= c' H\left(0, \frac{1}{d}, \ldots, \frac{d-1}{d}; \ldots, 1 - \frac{w_i + dj}{dh_i}, \ldots \middle| \prod_{i=1}^{n} h_i^{h_i} (-\lambda)^d\right),
\end{aligned}
$$

where the lower exponents run through the h_i values $\frac{w_i+dj}{dh_i}, j = 0, \ldots, h_i - 1$ for each i, and no exponent appears if $h_j = 0$, and modulo some cancelation if some of the numerator and denominator terms are the same. Notice that there will be the same number of upper and lower exponents, since we required $\sum h_i = d$. The constant term is now

$$
c' = \xi q^{\frac{n-1}{2}} \cdot \frac{\chi_{\frac{d-1}{2}}(-1)}{q^d} = \xi q^{\frac{n-2d-1}{2}},
$$

where ξ still denotes a $q-1$ root of unity.

To get the total number of points we would need to add over equivalence class representatives, and so

$$N_{\mathbb{F}_q}(\lambda) - N_{\mathbb{F}_q}(0) = \xi q^{\frac{n-2d-1}{2}} \cdot$$
$$\sum_{[w]\in W/\sim} H\left(0, \frac{1}{d}, \ldots, \frac{d-1}{d}; \ldots, 1 - \frac{w_i + dj}{dh_i}, \ldots \middle| \prod_{i=1}^{n} h_i^{h_i}(-\lambda)^d\right).$$

Remark 3.3.1. Notice that the above formula implies that the hypergeometric function is independent of the choice of representative w. This is because the characters that define H were defined modulo integer powers, and $w' \sim w$ means that $w'_i \equiv w_i + kh_i \bmod d$, so substituting by an equivalent w gives the same characters for H.

3.3.1 Examples

THE 0-DIMENSIONAL CASE

The easiest example is the 0-dimensional family defined by

$$Z_\lambda : x_1^d + x_2^d - d\lambda x_1 x_2^{d-1} = 0.$$

Notice that to put this in the situation of Koblitz's theorem in the previous section, we have to assume $d(d-1)|q-1$, and we have $h = (1, d-1)$. Also, we can see that $W = \{(0,0), (1, d-1), \ldots, (d-1, 1)\}$, so in particular there is only one equivalence class, that of $(0,0)$. So using the last equation, we get that

$$N_{\mathbb{F}_q}(\lambda) - N_{\mathbb{F}_q}(0) = \xi q^{\frac{3-2d}{2}} \cdot$$
$$H\left(\frac{1}{d}, \ldots, \frac{d-1}{d}; 0, \frac{1}{d-1}, \ldots, \frac{d-2}{d-1} \middle| -(d-1)^{(d-1)}(-\lambda)^d\right).$$

In the case $d = 3$, the number of points is

$$N_{\mathbb{F}_q}(\lambda) - N_{\mathbb{F}_q}(0) = \xi q^{-\frac{3}{2}} H\left(\frac{1}{3}, \frac{2}{3}; 0, \frac{1}{2} \middle| 2^2\lambda^3\right).$$

THE DWORK FAMILY OF HYPERSURFACES

The Dwork family is a family of the type (3.1) with $n = d$ and $h_i = 1$ for all i. That is, the family

$$Y_\lambda : x_1^d + \cdots + x_d^d - d\lambda x_1 \cdots x_d = 0.$$

The cases $d = 3, 4$ were studied extensively by Dwork while he was studying the rationality of the zeta function, for example in [10].

In this case, for each equivalence class we get that

$$\sum_{s \in \frac{1}{q-1}\mathbb{Z}/\mathbb{Z}} \frac{g\left(hs + \frac{w}{d}\right)}{g(ds)} \chi_{ds}(d\lambda) =$$
$$\prod_{j=1}^{d-1} g\left(\frac{j}{d}\right) \sum_{s} \frac{g\left(h_1 s + \frac{w_1}{d}\right) \cdots g\left(h_n s + \frac{w_n}{d}\right)}{g(s)g\left(s + \frac{1}{d}\right) \cdots g\left(s + \frac{d-1}{d}\right)} \chi_{ds}(\lambda)$$

$$= \prod_{j=1}^{d-1} g\left(\frac{j}{d}\right) \sum_{s \in \frac{1}{q-1}\mathbb{Z}/\mathbb{Z}} \frac{g\left(s+\frac{w_1}{d}\right)\cdots g\left(s+\frac{w_n}{d}\right)}{g(s)g\left(s+\frac{1}{d}\right)\cdots g\left(s+\frac{d-1}{d}\right)} \chi_s(\lambda^d).$$

There will be cancelation when the w_i coincide with $0, 1, \ldots, d-1$. Again, we replace s by $-s$ and get that

$$N_{\mathbb{F}_q}(\lambda) - N_{\mathbb{F}_q}(0) = \xi q^{\frac{-d-1}{2}} \cdot \\ \sum_{[w] \in W/\sim} H\left(0, \frac{1}{d}, \ldots, \frac{d-1}{d}; 1-\frac{w_1}{d}, \ldots, 1-\frac{w_n}{d} \middle| (-\lambda)^d\right).$$

Let $d = 3$ (the family is actually a family of elliptic curves). In other words, the family with $d = 3 = n, h = (1, 1, 1)$.

We can see that

$$W = \{(0,0,0), (1,1,1), (2,2,2), (1,2,0), (2,0,1),$$

$$(0,1,2), (2,1,0), (0,2,1), (1,0,2)\}.$$

And, in fact, there are three equivalence class representatives, $(0,0,0)$, $(1,2,0)$, $(2,1,0)$, but the latter two are of the same "type", i.e., one is the permutation of the other. Therefore, we obtain

$$N_{\mathbb{F}_q}(\lambda) - N_{\mathbb{F}_q}(0) \quad = \quad \xi q^{-1} H\left(\frac{1}{3}, \frac{2}{3}; 1, 1 \middle| \lambda^3\right) + \frac{2\xi q}{(q-1)} \sum_{s \in \frac{1}{q-1}\mathbb{Z}/\mathbb{Z}} \overline{\chi}_s(\lambda^3).$$

For the terms corresponding to the "type" $(1,2,0)$, the w_i's completely cancel out with the list $0, 1, 2$, which means we have an empty parameter set.

This also means that H is the sum over all multiplicative characters of $\chi_s(\lambda^3)$, for $\lambda \in \mathbb{F}_q^*$, which is zero unless $\lambda^3 = 1$ in $\mathbb{F}_q^*$, in which case we get $(q-1)$.

Remark 3.3.2. We will see in Chapter 5 that the λ's that make Y_λ non-singular are exactly the non d-th roots of unity. And so for all λ such that Y_λ is non-singular, the second term in the above sum is zero, and we get that the number of points is written in terms of a hypergeometric function.

Chapter 4

A p-adic approach

The main goal of this chapter is to develop a p-adic version of Koblitz's formula for $N_{\mathbb{F}_p}(\lambda)$, where p is prime, so that we can find the relation between the number of solutions over $\mathbb{F}_p$ and generalized hypergeometric functions. We will first summarize the main ideas of the Gross-Koblitz formula, and then restrict our attention to two special examples.

4.1 The Gross-Koblitz Formula

The Gross-Koblitz formula was developed as a way of relating Gauss sums to the p-adic version of the Γ function. Most of this background follows a chapter in [26].

First, we will need to recall the following definition by Morita:

Definition 4.1.1. The *p-adic gamma function* is the continuous function

$$\Gamma_p : \mathbb{Z}_p \to \mathbb{Z}_p$$

that extends

$$f(n) := (-1)^n \prod_{1 \leq j < n, p \nmid j} j \qquad (n \geq 2).$$

This is well-defined because the function $a \mapsto f(a) : \mathbb{N} - \{0,1\} \to \mathbb{Z}$ is uniformly continuous for the p-adic topology and hence has a unique continuous extension $\mathbb{Z}_p \to \mathbb{Z}_p$.

This function has properties that are reminiscent of those of the classical gamma function.

Proposition 4.1.1. *Let p be an odd prime.*

1. $\Gamma_p(0) = 1, \quad \Gamma_p(1) = -1, \quad \Gamma_p(2) = 1,$

   $\Gamma_p(n+1) = (-1)^{n+1} n! \quad (1 \leq n < p).$
2. $\Gamma_p(x+1) = \begin{cases} -x\Gamma_p(x) & \text{if } x \in \mathbb{Z}_p^*, \\ -\Gamma_p(x) & \text{if } x \in p\mathbb{Z}_p \end{cases}$
3. $\Gamma_p(x)\Gamma_p(1-x) = (-1)^{R(x)}$, *where* $R(x) \in \{1, 2, \ldots, p\}$, $R(x) \equiv x \bmod p$.
4. *(Gauss multiplication formula) Let $m \geq 1$ be an integer prime to p. Then*

$$\prod_{0 \leq j < m} \Gamma_p\left(x + \frac{j}{m}\right) = \epsilon_m \cdot m^{1-R(mx)} \cdot (m^{p-1})^{s(mx)} \cdot \Gamma_p(mx),$$

where

$$\epsilon_m = \prod_{0 \leq j < m} \Gamma_p\left(\frac{j}{m}\right),$$

$$R(y) \in \{1, \ldots, p\}, R(y) \equiv y \bmod p,$$

$$s(y) = \frac{R(y) - y}{p} \in \mathbb{Z}_p.$$

Let $s = \dfrac{a}{q-1} \in \dfrac{1}{q-1}\mathbb{Z}/\mathbb{Z}$, ω be the Teichmüller character and ψ be an additive character of $\mathbb{F}_q$, as before. Consider now the Gauss sum

$$g(s) = \sum_{0 \neq x \in \mathbb{F}_q} \omega(x)^{-s(q-1)} \psi(x),$$

Suppose $q = p^f$. Let $\pi \in \mathbb{C}_p$ be a root of $\pi^{p-1} = -p$. Define for $0 \leq \dfrac{a}{q-1} = s < 1$, the sum $S_p(a) = \sum_{0 \leq j < f} a_j$ to be the sum of the digits in the p-adic expansion of a, and the integers $a^{(i)}$ as having p-adic expansions obtained from the cyclic permutations from the expansion of a (denoted $a^{(0)}$).

Theorem 4.1.2 (Gross-Koblitz)**.** *Let* $0 \leq s = \dfrac{a}{q-1} < 1$. *The value of the Gauss sum* $g(s)$ *is explicitely given by*

$$g(s) = -\pi^{S_p(a)} \prod_{0 \leq j < f} \Gamma_p\left(\frac{a^{(j)}}{q-1}\right).$$

Over $\mathbb{F}_p$, i.e. if we assume $f = 1$, the formula becomes much simpler, yielding

$$g(s) = -\pi^a \Gamma_p\left(\frac{a}{p-1}\right) = -\pi^{s(p-1)}\Gamma_p(s) = -(-p)^s \Gamma_p(s).$$

There is a very nice and straightforward proof of this theorem in [27].

Our intention is to use this theorem to produce new formulas which are computationally more manageable and will allow us to elucidate the relation between the number of points and generalized hypergeometric functions.

4.2 Special cases

For the rest of this chapter, we will be restricting ourselves to $\mathbb{F}_p$ rather than the more general finite fields.

4.2.1 The 0-dimensional example

As seen at the end of the previous chapter, the easiest example to deal with is the 0-dimensional variety

$$Z_\lambda : x_1^d + x_2^d - d\lambda x_1 x_2^{d-1} = 0.$$

Recall that Koblitz's theorem gives, in this case, that

$$N_{\mathbb{F}_p}(\lambda) - N_{\mathbb{F}_p}(0) = \frac{1}{p-1} \sum_{s \in \frac{1}{p-1}\mathbb{Z}/\mathbb{Z}} \frac{g(s)g((d-1)s)}{g(ds)} \chi_{ds}(\lambda).$$

Assume that the generator of the multiplicative character group is ω^{-1} and p is a prime such that $d(d-1)|p-1$. Using the Gross-Koblitz formula yields

Lemma 4.2.1.

$$N_{\mathbb{F}_p}(\lambda) = N_{\mathbb{F}_p}(0) + \frac{-1}{p-1} \sum_{a=0}^{p-2} \frac{(-p)^{\left(\frac{a}{p-1} + \{\frac{(d-1)a}{p-1}\} - \{\frac{da}{p-1}\}\right)} \Gamma_p\left(\frac{a}{p-1}\right) \Gamma_p\left(\left\{\frac{(d-1)a}{p-1}\right\}\right)}{\Gamma_p\left(\left\{\frac{da}{p-1}\right\}\right)} \omega(d\lambda)^{-da}$$

Proof. First, notice that we can rewrite $N_{\mathbb{F}_p}(\lambda) - N_{\mathbb{F}_p}(0)$ by changing its summation indices as follows:

$$\sum_{s\in\frac{1}{p-1}\mathbb{Z}/\mathbb{Z}} \frac{g(s)g((d-1)s)}{g(ds)}\omega(d\lambda)^{-ds(p-1)} = \sum_{a=0}^{p-2} \frac{g\left(\left\{\frac{(d-1)a}{p-1}\right\}\right) g\left(\frac{a}{p-1}\right)}{g\left(\left\{\frac{da}{p-1}\right\}\right)}\omega(d\lambda)^{-da}.$$

Now we are free to use the simpler version of the formula. Combining like terms, we get

$$\begin{aligned} N_{\mathbb{F}_p}(\lambda) - N_{\mathbb{F}_p}(0) &= \frac{-1}{p-1}\sum_{a=0}^{p-2} \frac{(-p)^{\left(\left\{\frac{(d-1)a}{p-1}\right\}\right)}\Gamma_p\left(\left\{\frac{(d-1)a}{p-1}\right\}\right)(-p)^{\left(\frac{a}{p-1}\right)}\Gamma_p\left(\frac{a}{p-1}\right)}{(-p)^{\left(\left\{\frac{da}{p-1}\right\}\right)}\Gamma_p\left(\left\{\frac{da}{p-1}\right\}\right)}\omega(d\lambda)^{-da} \\ &= \frac{-1}{p-1}\sum_{a=0}^{p-2} \frac{(-p)^{\left(\frac{a}{p-1}+\left\{\frac{(d-1)a}{p-1}\right\}-\left\{\frac{da}{p-1}\right\}\right)}\Gamma_p\left(\frac{a}{p-1}\right)\Gamma_p\left(\left\{\frac{(d-1)a}{p-1}\right\}\right)}{\Gamma_p\left(\left\{\frac{da}{p-1}\right\}\right)}\omega(d\lambda)^{-da} \end{aligned}$$

□

Suppose the have a hypergeometric weight system given by $\gamma = [d] - [1] - [d-1]$. This is clearly related to the power series with binomial coefficients $\binom{dn}{n}$. The Landau function associated to this system is

$$\mathcal{L}(x) = \{x\} + \{(d-1)x\} - \{dx\}.$$

Notice that the power of p that appears in the lemma is exactly determined by $\mathcal{L}\left(\frac{a}{p-1}\right)$. But this means that the valuation of the terms of the previous sum is very similar to the valuation of the terms in the hypergeometric series with coefficients $\binom{dn}{n}$.

Notice that

$$\sum_{n\geq 0}\binom{dn}{n}z^n = {}_{d-1}F_{d-2}\left(\frac{1}{d},\dots,\frac{d-1}{d};\frac{1}{d-1},\dots,\frac{d-1}{d-2}\,\middle|\,\frac{(d-1)^{(d-1)}}{d^d}z\right).$$

The discontinuities of $\mathcal{L}$ are therefore the α_i and β_i parameters. In fact, it is clear that the parameters interlace, that is, $0 < \frac{1}{d} < \frac{1}{d-1} < \cdots < \frac{d-2}{d-1} < \frac{d-1}{d} < 1$. By property (2) of the Landau function we get that $\mathcal{L}\left(\frac{a}{p-1}\right) = 1$ for $(p-1)\alpha_i \leq a < (p-1)\beta_i$ and 0 on the other intervals. Therefore the terms with $(p-1)\beta_i = \frac{(p-1)i}{d-1} \leq a < (p-1)\alpha_{i+1} = \frac{(p-1)(i+1)}{d}$ are the only ones that survive mod p. There are $d-1$ of these intervals. For a fixed i,

$$\sum_{a=\frac{(p-1)i}{d-1}}^{\frac{(p-1)(i+1)}{d}-1}\frac{(-p)^{\left(\frac{a}{p-1}+\left\{\frac{(d-1)a}{p-1}\right\}-\left\{\frac{da}{p-1}\right\}\right)}\Gamma_p\left(\frac{a}{p-1}\right)\Gamma_p\left(\left\{\frac{(d-1)a}{p-1}\right\}\right)}{\Gamma_p\left(\left\{\frac{da}{p-1}\right\}\right)}\omega(d\lambda)^{-da}$$

$$= \sum_{a=\frac{(p-1)i}{d-1}}^{\frac{(p-1)(i+1)}{d}-1}\frac{\Gamma_p\left(\frac{a}{p-1}\right)\Gamma_p\left(\left\{\frac{(d-1)a}{p-1}\right\}\right)}{\Gamma_p\left(\left\{\frac{da}{p-1}\right\}\right)}\omega(d\lambda)^{-da}$$

$$= \sum_{a=\frac{(p-1)i}{d-1}}^{\frac{(p-1)(i+1)}{d}-1}\frac{\Gamma_p\left(\frac{a}{p-1}\right)\Gamma_p\left(\frac{(d-1)a}{p-1}-i\right)}{\Gamma_p\left(\frac{da}{p-1}-i\right)}\omega(d\lambda)^{-da}$$

$$\equiv \sum_{a=\frac{(p-1)i}{d-1}}^{\frac{(p-1)(i+1)}{d}-1}\frac{\Gamma_p(-a)\Gamma_p(-(d-1)a-i)}{\Gamma_p(-da-i)}(d\lambda)^{-da} \bmod p$$

$$\equiv \sum_{a=\frac{(p-1)i}{d-1}}^{\frac{(p-1)(i+1)}{d}-1}\frac{\Gamma_p(da+i+1)}{\Gamma_p(a+1)\Gamma_p((d-1)a+i+1)}(d\lambda)^{-da} \bmod p \qquad (\star)$$

$$\equiv \sum_{a=\frac{(p-1)i}{d-1}}^{\frac{(p-1)(i+1)}{d}-1} \frac{(da+i)!}{a!((d-1)a+i)!}(d\lambda)^{-da} \bmod p$$

$$\equiv \sum_{a=\frac{(p-1)i}{d-1}}^{\frac{(p-1)(i+1)}{d}-1} \binom{da+i}{a}(d\lambda)^{-da} \bmod p.$$

At step $(\star)$ we used property (3) of the p-adic gamma function. And we have just shown that

$$N_{\mathbb{F}_p}(\lambda) - N_{\mathbb{F}_p}(0) \equiv \sum_{i=0}^{d-2} \sum_{a=(p-1)\beta_i}^{(p-1)\alpha_{i+1}-1} \binom{da+i}{a}(d\lambda)^{-da} \bmod p.$$

Notice that, for a fixed i,

$$\begin{aligned}
\binom{da+i}{a} &= \frac{(da+i)(da+i-1)\cdots(da+1)}{((d-1)a+i)((d-1)a+i-1)\cdots((d-1)a+1)}\binom{da}{a} \\
&= \frac{(da+i)\cdots(da+1)}{((d-1)a+i)\cdots((d-1)a+1)} \cdot \frac{d^{da}(\frac{1}{d})_a\cdots(\frac{d-1}{d})_a}{(d-1)^{(d-1)a}a!(\frac{1}{d-1})_a\cdots(\frac{d-2}{d-1})_a} \\
&= \frac{d^i(a+\frac{i}{d})\cdots(a+\frac{1}{d})}{(d-1)^i(a+\frac{i}{d-1})\cdots(a+\frac{1}{d-1})} \cdot \frac{d^{da}(\frac{1}{d})_a\cdots(\frac{d-1}{d})_a}{(d-1)^{(d-1)a}a!(\frac{1}{d-1})_a\cdots(\frac{d-2}{d-1})_a},
\end{aligned}$$

and we can combine the products so that the last expression equals

$$= \frac{d^i\frac{1}{d}\cdots\frac{i}{d}}{(d-1)^i\frac{1}{d-1}\cdots\frac{i}{d-1}} \cdot \frac{d^{da}(\frac{1}{d}+1)_a\cdots(\frac{i}{d}+1)_a(\frac{i+1}{d})_a\cdots(\frac{d-1}{d})_a}{(d-1)^{(d-1)a}(\frac{1}{d-1}+1)_a\cdots(\frac{i}{d-1}+1)_a(\frac{i+1}{d-1})_a\cdots(\frac{d-2}{d-1})_a}.$$

We have just proved:

Theorem 4.2.2. *Let* $\alpha^{(0)} = \left(\frac{1}{d}, \ldots, \frac{d-1}{d}\right)$ *and* $\beta^{(0)} = \left(\frac{1}{d-1}, \ldots, \frac{d-2}{d-1}\right)$.

$$N_{\mathbb{F}_p}(\lambda) - N_{\mathbb{F}_p}(0) \equiv \sum_{i=0}^{d-2} \left[{}_dF_{d-1}(\alpha^{(i)}; \beta^{(i)} | (d-1)^{-(d-1)}\lambda^{-d})\right]_{\frac{i(p-1)}{d-1}}^{\frac{(i+1)(p-1)}{d}-1} \bmod p,$$

where $\alpha^{(i)} = (\alpha_1 + 1, \ldots, \alpha_i + 1, \alpha_{i+1}, \ldots, \alpha_{d-1})$, *and* $\beta^{(i)} = (\beta_1 + 1, \ldots, \beta_i + 1, \beta_{i+1}, \ldots, \beta_{d-2})$, *that is we add 1 to the numerator and denominator parameters up to the* i*-th place.*

Notation. $[(u(z)]_i^j$ denotes the polynomial which is the truncation of a series $u(z)$ from $n = i$ to j.

So for example in the case $d = 3$ we get that

$$\begin{aligned} N_{\mathbb{F}_p}(\lambda) - N_{\mathbb{F}_p}(0) \quad &\equiv \quad \left[{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; \frac{1}{2}\middle| \frac{1}{2^2\lambda^3}\right)\right]_0^{\frac{p-1}{3}-1} \\ &\quad + \left[{}_2F_1\left(\frac{4}{3}, \frac{2}{3}; \frac{3}{2}\middle| \frac{1}{2^2\lambda^3}\right)\right]_{\frac{p-1}{2}}^{\frac{2(p-1)}{3}-1} \bmod p. \end{aligned}$$

From [32], we know that adding integers to the numerator and denominator parameters of a Gauss hypergeometric function is akin to taking a derivative. More specifically, we have the following

Lemma 4.2.3.

$$\frac{d}{dz}\left\{(1-z)^{a+n-1}{}_2F_1(a, b; c|z)\right\} = \frac{(a)_n(c-b)_n}{(-1)^n(c)_n}(1-z)^{a-1}{}_2F_1(a+n, b; c+n|z).$$

In particular, we can rewrite the second term in the expression for $N_{\mathbb{F}_p}(\lambda) - N_{\mathbb{F}_p}(0)$ above. Specifically,

$$
{}_2F_1\left(\frac{4}{3},\frac{2}{3};\frac{3}{2}\middle|\,\frac{1}{2^2\lambda^3}\right)=9(1-\lambda)^{\frac{2}{3}}\frac{d}{d\lambda}\left\{(1-\lambda)^{\frac{1}{3}}{}_2F_1\left(\frac{1}{3},\frac{2}{3};\frac{1}{2}\middle|\,\frac{1}{2^2\lambda^3}\right)\right\}.
$$

And finally, we can write

$$
\begin{aligned}
N_{\mathbb{F}_p}(\lambda)-N_{\mathbb{F}_p}(0)\equiv{}&\left[{}_2F_1\left(\frac{1}{3},\frac{2}{3};\frac{1}{2}\middle|\,\frac{1}{2^2\lambda^3}\right)\right]_0^{\frac{p-1}{3}-1}\\
+&\left[9(1-x)^{\frac{2}{3}}\frac{d}{d\lambda}\left\{(1-\lambda)^{\frac{1}{3}}{}_2F_1\left(\frac{1}{3},\frac{2}{3};\frac{1}{2}\middle|\,\frac{1}{2^2\lambda^3}\right)\right\}\right]_{\frac{p-1}{2}}^{\frac{2(p-1)}{3}-1}\mod p.
\end{aligned}
$$

The advantage of writing it in this form is that it makes it clear, in this particular example, that the number of points is actually related to only one hypergeometric function and its derivatives.

This is not so easy to do for a general d because there are no known formulas for generalized hypergeometric functions like the one in the lemma, although we expect that a similar relation does exist in general.

Remark 4.2.1. Notice the difference between Theorem 4.2.2 and Igusa's result: in our situation, more than one hypergeometric function appears. As far as we know, all of the known examples that have been computed have coincided with Igusa in the sense that only one hypergeometric function appears modulo p. In the next example (the Dwork family), we will show a known computation using our methods, in which only one hypergeometric function appears.

4.2.2 The Dwork family

We want to use the Gross-Koblitz formula in the same way as before, to find a formula that will more easily produce the relationship between the number of points and hypergeometric functions. We will focus on two examples, the cases $d = 3, 4$.

$d = 3$ **The elliptic curve case**

Recall that we have three equivalence classes, $(0,0,0), (1,2,0), (2,1,0)$, and so we can split the sum into three sums (although since the last two are permutations of each other the sums will be the same), so we get:

$$\begin{aligned} N_{\mathbb{F}_p}(\lambda) - N_{\mathbb{F}_p}(0) &= \frac{1}{p-1} \sum_{s \in \frac{1}{p-1}\mathbb{Z}/\mathbb{Z}} \frac{g(s)^3}{g(3s)} \chi_{3s}(3\lambda) \\ &+ \frac{2}{p-1} \sum_{s \in \frac{1}{p-1}\mathbb{Z}/\mathbb{Z}} \frac{g(s) g\left(s + \frac{1}{3}\right) g\left(s + \frac{2}{3}\right)}{g(3s)} \chi_{3s}(3\lambda) \end{aligned}$$

Again, before using the formula it is convenient to change the summation into something more manageable:

$$N_{\mathbb{F}_p}(\lambda) - N_{\mathbb{F}_p}(0) = \frac{1}{p-1} \sum_{s=0}^{p-2} \frac{g\left(\frac{s}{p-1}\right)^3}{g\left(\left\{\frac{3s}{p-1}\right\}\right)} \omega(3\lambda)^{-3s} +$$

$$+ \frac{2}{p-1} \sum_{s=0}^{p-2} \frac{g\left(\frac{s}{p-1}\right) g\left(\left\{\frac{s}{p-1} + \frac{1}{3}\right\}\right) g\left(\left\{\frac{s}{p-1} + \frac{2}{3}\right\}\right)}{g\left(\left\{\frac{3s}{p-1}\right\}\right)} \omega(3\lambda)^{-3s}$$

Substituting and simplifying, we get that

$$N_{\mathbb{F}_p}(\lambda) - N_{\mathbb{F}_p}(0) = \frac{1}{p-1} \sum_{s=0}^{p-2} \frac{(-p)^{\left(\frac{3s}{p-1} - \left\{\frac{3s}{p-1}\right\}\right)} \Gamma_p\left(\frac{s}{p-1}\right)^3}{\Gamma_p\left(\left\{\frac{3s}{p-1}\right\}\right)} \omega(3\lambda)^{-3s} +$$

$$+ \frac{2}{p-1} \sum_{s=0}^{p-2} \frac{(-p)^{\gamma(x)} \Gamma_p\left(\frac{s}{p-1}\right) \Gamma_p\left(\left\{\frac{s}{p-1} + \frac{1}{3}\right\}\right) \Gamma_p\left(\left\{\frac{s}{p-1} + \frac{2}{3}\right\}\right)}{\Gamma_p\left(\left\{\frac{3s}{p-1}\right\}\right)} \omega(3\lambda)^{-3s},$$

where $\gamma(x) = \left(\frac{s}{p-1} + \left\{\frac{s}{p-1} + \frac{1}{3}\right\} + \left\{\frac{s}{p-1} + \frac{2}{3}\right\} - \left\{\frac{3s}{p-1}\right\}\right)$.

Once more, the power of p in the first part of the sum is determined by $\mathcal{L}\left(\frac{a}{p-1}\right)$ where $\mathcal{L}(x)$ is the Landau function associated to the hypergeometric weight system $[3] - 3[1]$. The discontinuities are $0, 1/3, 2/3$ and the function is zero only when $0 \leq x < 1/3$.

The function $\gamma(x)$ is always equal to one, so mod$p$ we get

$$\begin{aligned}
N_{\mathbb{F}_p}(\lambda) - N_{\mathbb{F}_p}(0) &\equiv -\sum_{s=0}^{\frac{p-1}{3}-1} \frac{\Gamma_p(-s)^3}{\Gamma_p(-3s)}(3\lambda)^{-3s} \bmod p \\
&\equiv -\sum_{s=0}^{\frac{p-1}{3}-1} \frac{\Gamma_p(1+3s)}{\Gamma_p(1+s)^3}(3\lambda)^{-3s} \bmod p \\
&\equiv -\sum_{s=0}^{\frac{p-1}{3}-1} \frac{(3s)!}{s!^3}(3\lambda)^{-3s} \bmod p \\
&\equiv -\left[{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1 \,\middle|\, \lambda^{-3}\right)\right]_0^{\frac{p-1}{3}-1} \bmod p.
\end{aligned}$$

$d = 4$ **The $K3$-surface case**

This is the case

$$X_\lambda : x_1^4 + x_2^4 + x_3^4 + x_4^4 - 4\lambda x_1 x_2 x_3 x_4 = 0$$

In other words, the family with $d = 4 = n, h = (1, 1, 1, 1)$.

The set W is made up of 64 vectors, but we can split them up into 16 orbits, and of those there are only three "types". These are

$$(0, 0, 0, 0), (1, 1, 1, 1), (2, 2, 2, 2), (3, 3, 3, 3)$$

$$(0,1,1,2),(1,2,2,3),(2,3,3,0),(3,0,0,1)$$

$$(0,0,2,2),(1,1,3,3),(2,2,0,0),(3,3,1,1)$$

The rest are permutations of these. So there is one orbit of the first type, 12 orbits of the second type, and 3 orbits of the third type.

This makes the formula look as follows:

$$N_{\mathbb{F}_p}(\lambda) - N_{\mathbb{F}_p}(0) = \frac{1}{p-1} \sum_{s \in \frac{1}{p-1}\mathbb{Z}/\mathbb{Z}} \frac{g(s)^4}{g(4s)} \chi_{4s}(4\lambda) \qquad (S_1)$$

$$+\frac{12}{p-1} \sum_{s \in \frac{1}{p-1}\mathbb{Z}/\mathbb{Z}} \frac{g(s)g(s+\frac{1}{4})^2 g(s+\frac{1}{2})}{g(4s)} \chi_{4s}(4\lambda) \qquad (S_2)$$

$$+\frac{3}{p-1} \sum_{s \in \frac{1}{p-1}\mathbb{Z}/\mathbb{Z}} \frac{g(s)^2 g(s+\frac{1}{2})^2}{g(4s)} \chi_{4s}(4\lambda). \qquad (S_3)$$

Let's focus on the first term of the sum, denoted by S_1. Using Gross-Koblitz we get

$$S_1 = \frac{1}{p-1} \sum_{s=0}^{p-2} \frac{(-p)^{(\frac{4s}{p-1} - \left\{\frac{4s}{p-1}\right\})} \Gamma_p\left(\frac{s}{p-1}\right)^4}{\Gamma_p\left(\left\{\frac{4s}{p-1}\right\}\right)} \omega(4\lambda)^{-4s}$$

By inspecting the power of $-p$ we can see that again it is determined by $\mathfrak{L}_\gamma$ where $\gamma = [4] - 4[1]$. Thus, the only terms that survive mod p are those for which $0 \leq s < \frac{p-1}{4}$. So

$$S_1 \equiv -\sum_{s=0}^{\frac{p-1}{4}-1} \frac{\Gamma_p(-s)^4}{\Gamma_p(-4s)} (4\lambda)^{-4s} \bmod p$$

$$\equiv \sum_{s=0}^{\frac{p-1}{4}-1} \frac{\Gamma_p(1+4s)}{\Gamma_p(1+s)^4}(4\lambda)^{-4s} \bmod p$$

$$\equiv \sum_{s=0}^{\frac{p-1}{4}-1} \frac{(4s)!}{(s!)^4}(4\lambda)^{-4s} \bmod p$$

$$\equiv \left[{}_3F_2\left(\frac{1}{4}, \frac{1}{2}, \frac{3}{4}; 1, 1 \middle| \lambda^{-4} \right) \right]_0^{\frac{p-1}{4}-1} \bmod p.$$

Inspection shows that S_2 and S_3 are both zero modulo p.

Notice that in both examples, the only terms to survive mod p are the ones related to the class of $(0, \ldots, 0)$. Clearly some information is lost that might not be lost if we studied these cases modulo other powers of p. One of our future plans is to try using the Gross-Koblitz formula for the more general finite fields to compute these examples. In the case of the elliptic curve, we believe p^3 will be the right power, and we expect that for any d, we should study the number of solutions modulo p^d. This was actually checked by Rodríguez-Villegas, Candelas and de la Ossa for $d = 5$ in [8].

Chapter 5

Gauss-Manin connections and Differential Equations

In his work studying the Zeta functions of families of hypersurfaces, Dwork constructed certain modules which were essentially the middle deRham cohomology, equipped with an integrable connection equivalent to the Gauss-Manin connection. The upshot is that this connection is equivalent to differentiating with respect to the parameter, which gives us a system of differential equations.

We developed an algorithm which is quite similar to finding the rational canonical form of a matrix, using basic ideas from the theory of ordinary differential equations and some results by Brieskorn [3] and Beukers and Heckman [1], to convert the differential systems obtained from the connection into differential systems which were companion matrices to a higher order differential equation.

The main goal of this chapter is to describe the algorithm we implemented in Pari-GP, which takes as input a particular element in the cohomology and outputs the parameters of the hypergeometric differential equation which has that element as a "solution". The actual code can be found in Appendix B, and a summary of the ideas that we use from the theory of ODE's can be found in Appendix A.

5.1 Background

This section is mostly a summary of some ideas in lecture notes by Kiran Kedlaya [11].

5.1.1 Algebraic deRham cohomology

This construction was originally introduced by Grothendieck based on ideas of Atiyah and Hodge.

Let X be a smooth affine variety over a field K of characteristic 0. Since X is affine, we can think of it as $X = \mathrm{Spec}A$ where $A \subseteq K[x_1, \ldots, x_n]$.

The *module of Kähler differentials*, or $\Omega^1_{A/K}$, is the A-module generated by symbols da, $a \in A$, modulo the relations

1. da, $a \in K$ (i.e. the derivative of a constant is zero.)
2. $d(ab) - adb - bda$, $a, b, \in A$ (i.e. the product rule.)

Now define $\Omega^i_{A/K} = \wedge^i \Omega^1_{A/K}$, that is, $\Omega^i_{A/K}$ is the i-th alternating power of $\Omega^1_{A/K}$ over A. Also, we can think of $A = \Omega^0_{A/K}$. There are maps

$$\begin{aligned} d^{i+1} : \Omega^i_{A/K} &\rightarrow \Omega^{i+1}_{A/K} \\ f dx_1 \wedge \cdots \wedge dx_i &\mapsto df \wedge dx_1 \wedge \cdots \wedge dx_i \end{aligned}$$

In the same way as this was done in differential topology, we know that $d \circ d = 0$. The $\Omega^{\cdot}_{A/K}$ form a complex, and so we define

$$H^i_{dR}(X) = \text{Ker} d^{i+1} / \text{Im} d^i = \text{“closed forms”}/\text{“exact forms”}$$

to be the i-th algebraic deRham cohomology of X.

Suppose X is a hypersurface defined over $\mathbb{P}^n$, and so it is $n-1$ complex dimensional. The *middle* deRham cohomology will be the $n-1$-st cohomology. (Notice that there are $2(n-1)$ cohomology spaces, so it makes sense to call it the middle one.)

It is a classical result that the i-th deRham cohomology of a hypersurface, for $i \neq n-1$, is identical to the i-th deRham cohomology of $\mathbb{P}^{n-1}$. So in fact, the middle cohomology is the only “interesting” one.

5.1.2 Connections in general

Let V be a vector bundle over T. A *connection* on V is a bundle map $\nabla : V \to V \otimes \Omega^1_T$ that satisfies the Leibniz rule, that is, for any $U \subseteq T$ open, and any $f \in \mathcal{O}_T(U)$ (a regular function on U) and s a bundle section (a “rule” that assigns to every point of U a vector from the attached vector space), we have

$$\nabla(fs) = f\nabla(s) + s \otimes df.$$

A section s is said to be *horizontal* if $\nabla(s) = 0$.

Let $\nabla_1 : V \otimes \Omega^1_T \to V \otimes \Omega^2_T$ be the map which takes $s \otimes \omega \mapsto \nabla(s) \wedge \omega + s \otimes d\omega$, where the wedge denotes the map given by wedging the second and third factors. The *curvature* of V is the map $\nabla_1 \circ \nabla : V \to V \otimes \Omega^2_T$. If the curvature vanishes, we say the connection is *integrable*. (This is automatic if $\dim(T) = 1$.)

Remark 5.1.1. There is another way to think about the integrability of a connection. Let $z_1, \ldots, z_n$ be local coordinates for T at a point t. Then $dz_1, \ldots, dz_n$ form a basis of the cotangent bundle Ω^1_T on some neighborhood of t, and so it admits a dual basis $\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n}$ of tangent vector fields. We can contract ∇ with the vector field $\frac{\partial}{\partial z_i}$ to obtain a map from V to itself satisfying the Leibniz rule with respect to $\frac{\partial}{\partial z_i}$, which one can think of as an action of $\frac{\partial}{\partial z_i}$ on sections of V. In other words, we have a map

$$\begin{aligned} \frac{\partial}{\partial z_i} : \text{Sections of } V &\to \text{Sections of } V \\ fs &\mapsto \frac{\partial}{\partial z_i}(f)s + f\frac{\partial}{\partial z_i}(s) \end{aligned}$$

Then ∇ is integrable if and only if the $\frac{\partial}{\partial z_i}$ commute with each other. Again, this makes it obvious that if T is one-dimensional then ∇ is integrable.

The most important idea that we will use later is the fact that, if T is one-dimensional, having a connection ∇ on a vector bundle V is equivalent to having an action of the derivative with respect to z, the local coordinate around a point $t \in T$.

5.1.3 Gauss-Manin connections

Let $\pi : X \to S$ be a smooth proper morphism between smooth algebraic varieties over a field of characteristic zero. We define the *relative deRham cohomology* $H^i_{dR}(X/S)$ as the higher direct images $\mathbb{R}^i\Omega^{\cdot}_{X/S}$ of the complex of relative differentials. $\Omega^1_{X/S}$ is the quotient of $\Omega^1_{X/A}$ by the pullback of $\Omega^1_{S/A}$. Basically, these turn out to be vector bundles on S whose fibers can be identified with the cohomology of the fibers X_b.

Notice that this construction of the relative deRham cohomology throws away some information. This means that given a relative i-form $\omega \in \Omega^1_{X/S}$, if one lifts ω to an absolute i-form $\tilde{\omega} \in \Omega^i_{X/A}$ and differentiates the result, we may get something nonzero even if ω was closed. If we project this lift into $\Omega^i_{X/S} \otimes \Omega^1_{S/A}$ we have essentially constructed the *Gauss-Manin connection.*

This was defined more formally by Katz and Oda in [25].

5.2 Dwork's construction

Most of this section is a summary of a section in [19]. Recall the Dwork family of hypersurfaces defined by

$$Y_\lambda : x_1^n + \cdots + x_n^n - n\lambda x_1 \cdots x_n = 0.$$

Let $F_\lambda(x_1, \ldots, x_n) := x_1^n + \cdots + x_n^n - n\lambda x_1 \cdots x_n$.

The cases $n = 3, 4$ have been studied extensively by Dwork, and the $n = 5$ family of Calabi-Yau threefolds was studied by Rodríguez-Villegas, Candelas and de la Ossa [7].

Lemma 5.2.1. *X_λ is not smooth if and only if λ is an n-th root of unity.*

Proof. Recall that a variety is not smooth if its Jacobian does not have full rank. In this case, since we have only one polynomial, namely F_λ, defining our variety, we have to find the λ's for which all of the partial derivatives $\frac{\partial}{\partial x_i}$ vanish, i.e. we want to solve the following equations simultaneously:

$$nx_i^{n-1} - n\lambda x_1 \overset{\wedge}{\cdots} x_n = 0, \qquad \text{for each } i$$

where $^\wedge$ denotes the omission of x_i.

Clearly, this means

$$nx_i^{n-1} = n\lambda x_1 \overset{\wedge}{\cdots} x_n,$$

and so

$$x_i^{n-1} = \lambda x_1 \overset{\wedge}{\cdots} x_n.$$

Multiplying all the left- and right-hand sides together, we get the equation

$$x_1^{n-1} \cdots x_n^{n-1} = \lambda^n x_1^{n-1} \cdots x_n^{n-1},$$

and this implies that $\lambda^n = 1$.

This argument can easily be followed in reverse, and so we are done. $\square$

Let $T = \mathbb{C} - \mu_n$. It follows from the lemma that X_λ is non singular for $\lambda \in T$. Dwork constructed modules over $\mathbb{C}$ isomorphic to the relative deRham cohomology $H^{n-2}_{dR}(X_\lambda/T)$, which are quite combinatoric in nature. Recall, from the previous section, that this is the only "interesting" cohomology space.

Let $\mathfrak{L}$ be the free module (over $\mathbb{C}$) generated by the monomials

$$x_1^{w_1} \cdots x_n^{w_n} = x^w,$$

with all the $w_i \geq 0$ and $\sum_{i=1}^n w_i \equiv 0 \bmod n$.

Let $\mathfrak{L}^S$ be the submodule generated by monomials x^w with all $w_i \geq 1$. Let D_i be the T-linear mapping defined by

$$D_i : \mathfrak{L} \to \mathfrak{L}, \qquad D_i(x^w) = w_i x^w + x_i \frac{\partial F_\lambda}{\partial x_i} x^w.$$

Define

$$\mathcal{W} = \mathfrak{L}^S / \mathfrak{L}^S \bigcap \left(\sum_{i=1}^n D_i \mathfrak{L} \right).$$

$\mathcal{W}$ is a vector bundle over T equipped with an integrable connection ∇ defined by

$$\nabla(f(\lambda)x^w) = \frac{\partial}{\partial \lambda} f(\lambda) x^w + f(\lambda) \frac{\partial}{\partial \lambda} F_\lambda x^w.$$

Let $\mathcal{W}_\lambda$ be the vector space associated to $\lambda \in T$.

Proposition 5.2.2. *$\mathcal{W}_\lambda$ is generated over $\mathbb{C}$ by the set of monomials*

$$\mathcal{B} = \left\{ x_1^{w_1} \cdots x_n^{w_n} = x^w | 1 \leq w_i \leq n-1, \sum w_i \equiv 0 \bmod n \right\}.$$

The basic idea for the proof is that every time one encounters a monomial x^w where some of the $w_i \geq n$, we can use the relations given by the D_i to write it in terms of monomials with all of their powers less than or equal to $n-1$. In fact, this idea will be crucial in a moment.

It is not hard to see from the proposition that $\mathcal{W}_\lambda$ has dimension

$$(n-1)^{(n-1)} - (n-1)^{(n-2)} + (n-1)^{(n-3)} - \cdots \pm (n-1).$$

Notation. We will frequently represent a monomial x^w by its exponent w.

For example, if $n = 6$, for $\lambda \in T$ we have

$$\mathcal{W}_\lambda = \left\langle x^w | 1 \leq w_i \leq 5, \sum w_i \equiv 0 \bmod 6 \right\rangle,$$

and $\mathcal{W}_\lambda$ has dimension $5^5 - 5^4 + 5^3 - 5^2 + 5 = 2605$.

Theorem 5.2.3 (The Comparison Theorem)**.** *Let $w_0 = \frac{1}{n}\sum_{i=1}^n w_i$. There is a T-linear map $\mathcal{R} : \mathfrak{L}^S \to H_{dR}^{n-1}(\mathbb{P}^n - X_\lambda/T)$ given by*

$$\mathcal{R} : x^w \mapsto (-1)^{w_0}(w_0 - 1)! \frac{x^w}{F_\lambda^{w_0}} \frac{d(x_1/x_n)}{x_1/x_n} \wedge \cdots \wedge \frac{d(x_{n-1}/x_n)}{x_{n-1}/x_n}.$$

By the residue map (cf. [14]) we can also define a map

$$\Theta : \mathfrak{L}^S \xrightarrow{\mathcal{R}} H^{n-1}_{dR}(\mathbb{P}^n - X_\lambda/T) \xrightarrow{\sim} H^{n-2}_{dR}(X/T).$$

And we have the following:

Theorem 5.2.4. *The map Θ induces, by passage to quotients, an isomorphism*

$$\Theta : \mathcal{W} \xrightarrow{\sim} H^{n-2}_{dR}(X_\lambda/T),$$

which is compatible with the connection.

Θ transforms ∇ into the Gauss-Manin connection. Hence the space $\mathcal{W}$ obtained through Dwork's construction is isomorphic to the middle (relative) deRham cohomology.

5.3 Computing the connection matrix

We first need the following basic

Proposition 5.3.1. *The action of*

$$G = \{\xi \in \mu_n^n | \xi_1 \cdots \xi_n = 1\}/\Delta$$

on a fiber $\mathcal{W}_\lambda$, gives

$$\mathcal{W}_\lambda = \bigoplus_{\chi \in char(G)} \mathcal{W}_\lambda(\chi),$$

where $\mathcal{W}_\lambda(\chi)$ is an eigenspace with basis

$$\{w, w+\overline{1}(\text{mod} n), \ldots, w+\overline{n-1}(\text{mod} n)\},$$

but we exclude adding any vector $\overline{m}$ *such that* $m + w_i \equiv 0 \bmod n$ *for some* i.

Proof. Recall that the group

$$G = \{\xi \in \mu_n^n | \xi^h = 1\}/\Delta$$

acts on the points on the hypersurface. Let ξ be a primitive nth root of unity. Notice that we can also represent elements $\xi^m = (\xi^{m_1}, \ldots, \xi^{m_n}) \in G$ by $m = (m_1, \ldots, m_n)$ such that $\sum m_i \equiv 0 \bmod n$.

G acts on $\mathcal{W}_\lambda$ in the same fashion, that is, ξ^m acts on x^w by

$$\begin{aligned}
\xi^m \cdot x^w &= (\xi^{m_1}, \ldots, \xi^{m_n}) \cdot x_1^{w_1} \cdots x_n^{w_n} \\
&= (\xi^{m_1} x_1)^{w_1} \cdots (\xi^{m_n} x_n)^{w_n} \\
&= \xi^{m_1 w_1 + \cdots m_n w_n} x^w \\
&= \chi_w(\xi^m) x^w,
\end{aligned}$$

where χ is a generator of the character group $char(G)$.

But we know χ_w is equivalent to $\chi_{w'}$ if and only if $w - w'$ is a multiple of h modulo n. That is if $w - w' \equiv \alpha(1, \ldots, 1) \bmod n$, where $1 \leq \alpha \leq n-1$.

Therefore $w'_j = \alpha + w_j \bmod n$, for $j = 1, \ldots, n$. The number of such w' is then

$$= \#\{\{1, 2, \ldots, n\} \setminus \{w_1, \ldots, w_n\}\},$$

since every time we add $n - w_i + w_i = n \equiv 0 \bmod n$ and if any of the $w_i \equiv 0 \bmod n$ then x^w can be reduced in $\mathcal{W}_\lambda$.

$\square$

For example, for $n = 6$, take $w = (1, 1, 1, 2, 2, 5)$. The number of entries that coincide with one of the numbers in the list $\{1, 2, 3, 4, 5, 6\}$ is 3, so the dimension of the eigenspace related to w is 3. This is the eigenspace of $\mathcal{W}_\lambda$ for $\lambda \in T$ generated by

$$\mathcal{B}_{(1,1,1,2,2,5)} = \{(1, 1, 1, 2, 2, 5), (3, 3, 3, 4, 4, 1), (4, 4, 4, 5, 5, 2)\}.$$

Notice that the basis doesn't change while changing λ, so we can think of $\mathcal{W}$ as a $\mathbb{C}(\lambda)$-module with basis $\mathcal{B}$ as described earlier. To understand ∇'s effect on $\mathcal{W}$, it suffices to know what it does to elements in $\mathcal{B}$. From the definition of ∇ we see that

$$\nabla(x^w) = \frac{\partial}{\partial \lambda} F_\lambda x^w = -n x^{w+\overline{1}}$$

where $w + \overline{1} = (w_1 + 1, \ldots, w_n + 1)$.

Applying ∇ to a monomial adds one to all of its powers. If this process gives us a monomial which is outside the basis we can actually write that monomial in terms of the other basis vectors because of the relations on $\mathcal{W}$. Thus, the proposition implies that ∇ preserves eigenspaces.

Recall that the goal of this section is to compute the connection ∇ as a matrix. Because of the way in which ∇ preserves eigenspaces, the connection matrix will have blocks on its diagonal for each set of basis vectors of an eigenspace.

We have written an algorithm in Pari-GP (included in the Appendix) that takes any vector of integers as an input and outputs the block of the connection matrix that corresponds to that vector's eigenspace generators.

The main idea of the algorithm is to use the relations on $\mathcal{W}$ to methodically reduce the powers of the monomial until it is written in terms of basis vectors. This will be most easily described using an example.

5.3.1 Example with $n = 6$ and $w = (1, 1, 1, 2, 2, 5)$

Here is an example of the algorithm for computing the block in the matrix representation of ∇ for $n = 6$ and the eigenspace corresponding to the monomial $(1, 1, 1, 2, 2, 5)$, with basis denoted earlier by $\mathcal{B}_{(1,1,1,2,2,5)}$. I will denote this block by $\nabla_{\mathcal{B}_{(1,1,1,2,2,5)}}$.

1. Apply $\nabla(1, 1, 1, 2, 2, 5) = -6(2, 2, 2, 3, 3, 6)$. Using the relations we can write this last monomial in terms of the monomials in $\mathcal{B}_{(1,1,1,2,2,5)}$. We can represent the process of reducing the exponents using the relations graphically, as shown below:

$$\begin{tikzcd}
 & (2,2,2,3,3,6) \arrow[dl, "0"'] \arrow[dr, "\lambda"] & \\
(2,2,2,3,3,0) & & (3,3,3,4,4,1)
\end{tikzcd}$$

This means that $(2,2,2,3,3,6) = \lambda(3,3,3,4,4,1) + 0 \cdot (2,2,2,3,3,0)$, which is a monomial in $\mathcal{B}_{(1,1,1,2,2,5)}$. Thus, in the matrix representation of $\nabla_{\mathcal{B}_{(1,1,1,2,2,5)}}$, there will be a -6λ as the $(2,1)$ entry.

2. We repeat this process for the next monomial in the basis, $(3,3,3,4,4,1)$. Applying the connection we get $\nabla(3,3,3,4,4,1) = -6(4,4,4,5,5,2)$. Since this monomial is already in $\mathcal{B}$ we write -6 in the $(3,2)$ position in the block matrix.

3. Take $\nabla(4,4,4,5,5,2) = -6(5,5,5,6,6,3)$. We have to do the reduction process again, represented below.

$$\begin{tikzcd}[column sep=tiny]
 & (5,5,5,6,6,3) \arrow[dl, "0"'] \arrow[dr, "\lambda"] & & & & & & \\
(5,5,5,0,6,3) & & (6,6,6,1,7,4) \arrow[dl, "-1/6"'] \arrow[dr, "\lambda"] & & & & & \\
 & (6,6,6,1,1,4) \arrow[dl, "0"'] \arrow[dr, "\lambda"] & & (7,7,7,2,2,5) \arrow[dl, "-1/6"'] \arrow[dr, "\lambda"] & & & & \\
(0,6,6,1,1,4) & & (1,7,7,2,2,5) \arrow[dl, "-1/6"'] \arrow[dr, "\lambda"] & & (2,8,8,3,3,6) \arrow[dl, "-2/6"'] \arrow[dr, "\lambda"] & & & \\
 & (1,1,7,2,2,5) \arrow[dl, "-1/6"'] \arrow[dr, "\lambda"] & & (2,2,8,3,3,6) \arrow[dl, "-2/6"'] \arrow[dr, "\lambda"] & & (3,3,9,4,4,7) \arrow[dl, "-3/6"'] \arrow[dr, "\lambda"] & & \\
(1,1,1,2,2,5) & & (2,2,2,3,3,6) \arrow[dl, "0"'] \arrow[dr, "\lambda"] & & (3,3,3,4,4,7) \arrow[dl, "-1/6"'] \arrow[dr, "\lambda"] & & (4,4,4,5,5,8) \arrow[dl, "-2/6"'] \arrow[dr, "\lambda"] & \\
 & (2,2,2,3,3,0) & & (3,3,3,4,4,1) & & (4,4,4,5,5,2) & & (5,5,5,6,6,3)
\end{tikzcd}$$

This is a bit harder to unravel than the other cases, but it works in exactly the same way. The diagram shows us that

$$
\begin{aligned}
(5,5,5,6,6,3) &= -\frac{\lambda^2}{108}(1,1,1,2,2,5)+\frac{17\lambda^4}{36}(3,3,3,4,4,1) \\
&\quad -\frac{3\lambda^5}{2}(4,4,4,5,5,2)+\lambda^6(5,5,5,6,6,3)
\end{aligned}
$$

And solving for $(5,5,5,6,6,3)$ we get that

$$
\begin{aligned}
&\nabla(4,4,4,5,5,2) = -6(5,5,5,6,6,3) \\
&= -\frac{\lambda^2}{18(\lambda^6-1)}(1,1,1,2,2,5)+\frac{17\lambda^4}{6(\lambda^6-1)}(3,3,3,4,4,1) \\
&-\frac{9\lambda^5}{\lambda^6-1}(4,4,4,5,5,2)
\end{aligned}
$$

4. Combining all of these steps, we can write $\nabla_{\mathcal{B}_{(1,1,1,2,2,5)}}$ as

$$
\nabla_{\mathcal{B}_{(1,1,1,2,2,5)}} = \begin{pmatrix} 0 & 0 & -\dfrac{\lambda^2}{18(\lambda^6-1)} \\ -6\lambda & 0 & \dfrac{17\lambda^4}{6(\lambda^6-1)} \\ 0 & -6 & -\dfrac{9\lambda^5}{\lambda^6-1} \end{pmatrix}.
$$

5.4 The differential equation associated to the connection

In this section, we will show, for a few examples, that the differential equation associated to the connection ∇ is a hypergeometric differential equation. Basically, we wrote an algorithm in Pari-GP which outputs the parameters α, β given the degree n, for each block representative. This has been checked by

Dwork for $n = 3, 4$ [10] and by Candelas, de la Ossa, and Rodríguez-Villegas for $n = 5$ [7]. We have included, in Table 1, all of the possibilities for $n = 6$.

A more general result was recently proved by Nicholas Katz in [21], and Daqing Wan and Antonio Rojas-Leon in [30]. They use powerful algebraic geometry tools and l-adic methods. Our approach is more direct and computational in nature.

First, we should explain how a system of first order differential equations arises from the connection. Recall from Section 5.1 that, on a vector bundle, being equipped with a connection ∇ is equivalent to being equipped with an action of $\frac{d}{d\lambda}$. In short, we have a first-order system defined by

$$\frac{d}{d\lambda}y = Ay,$$

where A is actually the transpose of the matrix we found in the previous section. In fact, each block defines its own differential system. Notice that the system

$$\frac{d}{d\lambda}y = \begin{pmatrix} 0 & -6\lambda & 0 \\ 0 & 0 & -6 \\ -\frac{\lambda^2}{18(\lambda^6-1)} & \frac{17\lambda^4}{6(\lambda^6-1)} & -\frac{9\lambda^5}{\lambda^6-1} \end{pmatrix} y,$$

which we found earlier, is equivalent to solving the three simultaneous equations, for $y = (y_1, y_2, y_3)^T$,

$$
\begin{aligned}
\frac{d}{d\lambda}y_1 &= -6\lambda y_2,\\
\frac{d}{d\lambda}y_2 &= -6y_3,\\
\frac{d}{d\lambda}y_3 &= -\frac{\lambda^2}{18(\lambda^6-1)}y_1+\frac{17\lambda^4}{6(\lambda^6-1)}y_2-\frac{9\lambda^5}{\lambda^6-1}y_3.
\end{aligned}
$$

The vector with

$$y_1=(1,1,1,2,2,5), y_2=(3,3,3,4,4,1), y_3=(4,4,4,5,5,2)$$

is a solution.

From the Cyclic Vector Lemma (see Appendix A), we know that any first order system is equivalent to a system which comes from a differential equation. This means that if $y=(y_1,y_2,\dots,y_n)^T$ is a solution for the system, we can find an equivalent system with solutions of the form $z=\left(z_1,\frac{dz_1}{d\lambda},\dots,\frac{d^{n-1}z_1}{d\lambda^{n-1}}\right)^T$. In fact, using the system, we can represent the derivatives $z_1^{(k)}$ as a linear combination of $y_1,\dots,y_n$. This determines a matrix S such that $Sy=z$.

The vector $Sy=z$ satisfies a differential system of the form

$$\frac{d}{d\lambda}z=\left(SAS^{-1}+\frac{dS}{d\lambda}S^{-1}\right)z,$$

and this last system is the companion matrix to a higher order differential equation. In our situation, since the basis vectors are basically already derivatives of each other, any vector in the basis, for example y_1, is a cyclic vector, and so S is easy to determine.

Our algorithm starts with the transpose of the connection matrix ∇, A, corresponding to a basis monomial, and then computes the matrix S. In our running example, the matrix S such that

$$S \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_1' \\ y_1'' \end{pmatrix},$$

is

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -6\lambda & 0 \\ 0 & -6 & 36\lambda \end{pmatrix}.$$

The algorithm then computes $\left(SAS^{-1} + \frac{dS}{d\lambda}S^{-1}\right)$. For $(1,1,1,2,2,5)$,

$$\left(SAS^{-1} + \frac{dS}{d\lambda}S^{-1}\right) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{2\lambda^3}{1-\lambda^6} & \frac{10\lambda^6 - 2}{\lambda^2(1-\lambda^6)} & \frac{7\lambda^6+2}{\lambda(1-\lambda^6)} \end{pmatrix},$$

which, as we expected, is the companion matrix for an order 3 differential equation. Solving high order differential equations is not a simple task, but all we need are the defining parameters to know exactly which differential equation it is (and what the holomorphic solution around 0 will be.) Therefore, we will invoke some results that will save us the trouble of actually solving the differential equation written above.

First, there is an algorithm by Brieskorn which relates Gauss-Manin connections to monodromy group generators [3]. Let A be the matrix representation of the connection. The algorithm uses the fact that if A has a simple pole around a given point, i.e. can be written as

$$A = A_{-1}(z - z_0)^{-1} + A_0 + A_1(z - z_0) + \cdots$$

then $h_0 = e^{2\pi i A_{-1}}$ gives the monodromy around z_0. So if we can write the connection so that it has a pole at 0, we can get the monodromy around zero. Notice that the monodromy group described by Beukers and Heckman in [1] is generated by the monodromy matrices around zero, one, and infinity, and the parameters are determined by the eigenvalues of these matrices. In fact, we can get the parameters of the hypergeometric differential equation directly from A_{-1}, the residue around zero, and $\tilde{A}_{-1}$, where $\tilde{A}$ is the system at ∞. The eigenvalues of A_{-1} will be the β's and the eigenvalues of $\tilde{A}_{-1}$ will be the α's.

The challenge is to get the system to have just a simple pole at zero. This can be done by changing the basis of solutions again slightly, as explained in Appendix A. So, in our example, we get that the system is equivalent to

$$\frac{d}{d\lambda}y = \frac{1}{\lambda}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ \dfrac{2\lambda^6}{1-\lambda^6} & \dfrac{10\lambda^6 - 2}{1-\lambda^6} & 2 - \dfrac{7\lambda^6 + 2}{1-\lambda^6} \end{pmatrix} y.$$

Notice we can change variables by setting $z = \lambda^6$. The change of variables leaves us with a system

$$\frac{d}{d\lambda}y = \frac{1}{z}\begin{pmatrix} 0 & 1/6 & 0 \\ 0 & 1/6 & 1/6 \\ \dfrac{z}{3(1-z)} & \dfrac{5z-1}{3(1-z)} & \dfrac{5z+4}{6(1-z)} \end{pmatrix} y.$$

Notice that this last system has regular singular points at $0, 1, \infty$ and no other singularities, and so it is Fuchsian, as we expected.

The residue at zero is

$$A_{-1} = \begin{pmatrix} 0 & 1/6 & 0 \\ 0 & 1/6 & 1/6 \\ 0 & -1/3 & 2/3 \end{pmatrix},$$

which has eigenvalues $0, 1/2, 1/3$. Let $h_0 = e^{2\pi i A_{-1}}$.

To study the system at ∞, we change variables from z to $1/\zeta$. The associated system is, as explained in Appendix A,

$$\frac{d\tilde{y}}{d\zeta} = -\frac{\tilde{A}(\zeta)}{\zeta^2}\tilde{y}.$$

In the example,

$$\frac{d\tilde{y}}{d\zeta} = \frac{1}{\zeta}\begin{pmatrix} 0 & -1/6 & 0 \\ 0 & -1/6 & -1/6 \\ \dfrac{1}{3(1-\zeta)} & \dfrac{5-\zeta}{3(1-\zeta)} & \dfrac{5+4\zeta}{6(1-\zeta)} \end{pmatrix} \tilde{y},$$

which has residue (at $\zeta = 0$) of

$$\tilde{A}_{-1} = \begin{pmatrix} 0 & -1/6 & 0 \\ 0 & -1/6 & -1/6 \\ 1/3 & 5/3 & 5/6 \end{pmatrix},$$

and thus yields the eigenvalues $1/3, 1/6, 1/6$. Let $h_\infty = e^{2\pi i \tilde{A}_{-1}}$.

Finally, we can look at the residue at 1 of $\frac{1}{z}A$, which is

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1/3 & -4/3 & -3/2 \end{pmatrix}.$$

The matrix $h_1 = e^{2\pi i D}$ is clearly a reflection in the sense described by Beukers and Heckman. Therefore, the matrices h_∞, h_1, h_0 generate a hypergeometric group. This group is the monodromy group of the differential equation

$$D\left(\frac{1}{6}, \frac{1}{6}, \frac{1}{3}; \frac{1}{2}, \frac{2}{3}\right) y = 0.$$

To sum it up, the block of the matrix ∇ corresponding to the eigenspace of $(1,1,1,2,2,5)$ gives rise to the hypergeometric differential equation which has

$${}_2F_3\left(\frac{1}{6}, \frac{1}{6}, \frac{1}{3}; \frac{1}{2}, \frac{2}{3} \middle| z\right)$$

as its holomorphic solution around 0.

Table 1 below shows some numerical examples for $n = 6$. Notice that given a vector $(w_1, w_2, \ldots, w_n)$, if we cancel out the numbers which it has in common with the list $(0, 1, 2, \ldots, n-1)$, then $\alpha_i = \frac{w_j}{n}$ for each w_j that survives the cancelation, and $\beta_i = \frac{k}{n}$ for each k that survives in the second vector.

Table 5.1: Parameters for $n = 6$

Vector	α_i	β_i
$[1, 1, 1, 1, 1, 1]$	$\left[\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right]$	$\left[\frac{1}{2}, \frac{2}{3}, \frac{5}{6}, \frac{1}{3}\right]$
$[5, 3, 1, 1, 1, 1]$	$\left[\frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right]$	$\left[\frac{2}{3}, \frac{1}{3}\right]$
$[4, 4, 1, 1, 1, 1]$	$\left[\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{2}{3}\right]$	$\left[\frac{1}{3}, \frac{1}{2}, \frac{5}{6}\right]$
$[5, 2, 2, 1, 1, 1]$	$\left[\frac{1}{3}, \frac{1}{6}, \frac{1}{6}\right]$	$\left[\frac{1}{2}, \frac{2}{3}\right]$
$[4, 3, 2, 1, 1, 1]$	$\left[\frac{1}{6}, \frac{1}{6}\right]$	$\left[\frac{5}{6}\right]$
$[3, 3, 3, 1, 1, 1]$	$\left[\frac{1}{2}, \frac{1}{2}, \frac{1}{6}, \frac{1}{6}\right]$	$\left[\frac{2}{3}, \frac{5}{6}, \frac{1}{3}\right]$
$[4, 2, 2, 2, 1, 1]$	$\left[\frac{1}{3}, \frac{1}{3}, \frac{1}{6}\right]$	$\left[\frac{1}{2}, \frac{5}{6}\right]$
$[3, 3, 2, 2, 1, 1]$	$\left[\frac{1}{2}, \frac{1}{3}, \frac{1}{6}\right]$	$\left[\frac{2}{3}, \frac{5}{6}\right]$
$[3, 2, 2, 2, 2, 1]$	$\left[\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right]$	$\left[\frac{2}{3}, \frac{5}{6}\right]$
$[5, 5, 3, 3, 1, 1]$	$\left[\frac{1}{6}, \frac{1}{2}, \frac{5}{6}\right]$	$\left[\frac{1}{3}, \frac{2}{3}\right]$
$[5, 5, 4, 2, 1, 1]$	$\left[\frac{1}{6}, \frac{5}{6}\right]$	$\left[\frac{1}{2}\right]$
$[5, 4, 4, 3, 1, 1]$	$\left[\frac{1}{6}, \frac{2}{3}\right]$	$\left[\frac{1}{3}\right]$
$[5, 4, 3, 3, 2, 1]$	$\left[\frac{1}{2}\right]$	$[]$
$[4, 4, 4, 3, 2, 1]$	$\left[\frac{2}{3}, \frac{2}{3}\right]$	$\left[\frac{5}{6}\right]$

Appendix

Appendix A

Ordinary Differential Equations

Before we describe the algorithm, it would be useful to summarize some of the key ideas from the theory of ordinary differential equations that we will use. These ideas come mainly from [2], [6] and [16], and most of the proofs will be omitted.

Consider the nth order equation

$$\sum_{m=0}^{n} a_{n-m}(z) y^{(m)} = 0, \qquad (a_0(z) \equiv 1) \tag{A.1}$$

where the $a_k(z)$ are single-valued and analytic in a punctured neighborhood of a point z_0. Recall that if any of the a_k have a singularity at z_0, then z_0 is called a *singular point* for (A.1), otherwise it is called an analytic point. z_0 is a *regular singular point* if

$$a_k(z) = (z - z_0)^{-k} b_k(z), \qquad (k = 1, \ldots, n),$$

where b_k is analytic at z_0.

A *system of n first order equations* over $\mathbb{C}(z)$ has the form

$$y' = Ay \tag{A.2}$$

in the unknown column vector $y = (y_1, \ldots, y_n)^T$ and where A is an $n \times n$-matrix with entries in $\mathbb{C}(z)$. The entries are assumed to be single-valued and analytic at a neighborhood of a point z_0, and will at most have a pole at that point.

If A has a singularity at z_0, then z_0 is a *singular point* for the system (A.2). z_0 is a *regular singular point* if

$$A(z) = (z - z_0)^{-1}\tilde{A}(z)$$

where $\tilde{A}$ is analytic for a neighborhood of z_0 (including z_0), and $\tilde{A}(z_0) \neq 0$.

A differential system or a differential equation for which all singularities are regular is called *Fuchsian*.

Notice that if we replace y by Sy in (A.2), where $S \in GL(n, \mathbb{C}(z))$, we obtain a new system for the new y,

$$y' = (SAS^{-1} + S'S^{-1})y,$$

where S' denotes the matrix of derivatives of each entry in S. Two $n \times n$ systems with matrices A, B are called *equivalent over* $\mathbb{C}(z)$ is there exists an S such that $B = SAS^{-1} + S'S^{-1}$.

It is not difficult to see that a differential equation like (A.1) can be rewritten as a system by setting $y_1 = y, y_2 = y', \ldots, y_n = y^{(n-1)}$. Notice that this means $y_1' = y_2, y_2' = y_3, \ldots, y_{n-1}' = y_n$, and y_n' is given by the differential equation. So the differential system is determined by a companion matrix, as follows:

$$\frac{d}{dz}\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}. \qquad \text{(A.3)}$$

The converse is, surprisingly, also true.

Theorem A.0.1 (Cyclic vector Lemma)**.** *Any system of linear first order differential equations is equivalent to a system which comes from a differential equation.*

Basically, this theorem says that in the space of solutions of a system (A.2) there is a cyclic vector, that is, a vector such that $v, Av, A^2v, \ldots, A^{n-1}v$ is a basis. This means that the matrix S mentioned earlier would be a change of basis matrix.

We now have a way of changing from a differential equation to a system and viceversa, but a regular singular point of (A.1), z_0, *may not* be a regular singular point of the system associated with it. This happens only when the a_k have at most simple poles at z_0.

However, there is an equivalent first-order system with the property that if z_0 is a regular singular point of (A.1) then z_0 is a regular singular point of the system.

Suppose (A.1) has a regular singularity at z_0, and let ϕ be a solution of (A.1). Define $\hat{\phi}$ to be the vector with components $\phi_1, \ldots, \phi_n$ by

$$\phi_k = (z - z_0)^{k-1}\phi^{(k-1)}, \qquad (k = 1, \ldots, n).$$

Then clearly

$$\begin{aligned}
(z - z_0)\phi_k' &= (z - z_0)((z - z_0)^{k-1}\phi^{(k-1)})' \\
&= (z - z_0)((k-1)(z - z_0)^{k-2}\phi^{(k-1)} + (z - z_0)^{k-1}\phi^{(k)}) \\
&= (k-1)(z - z_0)^{k-1}\phi^{(k-1)} + (z - z_0)^{k}\phi^{(k)} \\
&= (k-1)\phi_k + \phi_{k+1} \qquad (k = 1 \ldots, n-1)
\end{aligned}$$

And, finally,

$$(z - z_0)\phi_n' = (n-1)\phi_n - \sum_{m=1}^{n} b_{n-m+1}(z)\phi_m.$$

Therefore $\hat{\phi}$ is a solution of the linear system

$$y' = \hat{A}(z)y \tag{A.4}$$

where $\hat{A}$ has the structure

$$\hat{A}(z) = (z-z_0)^{-1}\begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & 1 & \cdots & 0 \\ 0 & 0 & 0 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -b_n & -b_{n-1} & -b_{n-2} & -b_{n-3} & \cdots & (n-1)-b_1 \end{pmatrix}$$

This system clearly has a regular singularity at z_0.

We need to say something about the solutions. A system as in (A.2) with a regular singularity at z_0 may be written as

$$y' = \left((z-z_0)^{-1}R + \sum_{m=0}^{\infty}(z-z_0)^m A_m\right) y \tag{A.5}$$

where $R \neq 0$, A_m are constant matrices, and the power series converges for a neighborhood of z_0.

Theorem A.0.2 ([6]). *In the system (A.5), if R has characteristic roots which do not differ by positive integers, then (A.5) has a fundamental matrix Φ of the form*

$$\Phi = P(z-z_0)^R \qquad (0 < |z-z_0| < c, c > 0)$$

where P is a power series

$$P(z) = \sum_{m=0}^{\infty}(z-z_0)^m P_m \qquad P_0 = I.$$

In order to study the behavior of a system (A.2) or an nth order equation (A.1) around an isolated singularity at $z = \infty$, we make the subtitution $z = \frac{1}{\zeta}$, and obtain a new system or equation with solutions which are functions of ζ. The point $z = \infty$ is a regular singularity if $\zeta = 0$ is a regular singularity of the induced system.

For example, in the case of the system, if $z = \frac{1}{\zeta}$, $\tilde{y}(\zeta) = y(\frac{1}{\zeta})$, $\tilde{A}(\zeta) = A(\frac{1}{\zeta})$, then the induced system corresponding to (A.2) is

$$\frac{d\tilde{y}}{d\zeta} = -\frac{\tilde{A}(\zeta)}{\zeta^2}\tilde{y}.$$

Appendix B

GP Scripts

B.1 Computing the connection matrix

This is a function that counts the number of coordinates with entries bigger than or equal to d.

```
count(a)=
{
   local(t);
   t=0;
   for(k=1,length(a), if(a[k]>=length(a),t=t+1));
   t
}
```

This function (from [29]) can tell if a given element is in a vector, and gives the "position" of the element.

```
memb(g,v)=for(k=1,length(v),if(g==v[k],return(k)));0
```

The following function takes a vector a and an integer m (its coefficient) and subtracts one to all the entries and adds d to one of them until it gets to

0, a vector in the basis, or the original vector. It saves the leftovers in a vector v. It is one of the two possible reductions coming from the relations on $\mathcal{W}$.

```
red1(a,m)=
{
    local(j,l,b,h,t,d,s,v,u);
    h=m;
    b=0;
    t=vector(length(a));
    d=length(a);
    l=1;
    j=1;
    s=vector(length(a));
    v=vector(0);
    t=a;
    if(count(t)>0,until(count(t)==0 || t==a,
        for(k=1,length(t), if(t[k]<t[j], j=k));
        for(k=1,length(t),s[k]=t[k]-1);
        for(k=1,length(t), if(k==j,t[k]=s[k]+d, t[k]=s[k]));
        l=h*s[j]/(d*n);
        v=concat(v,[[l,s]]);
        h=h*1/n));
    [h,t]
}
```

```
red1leftovers(a,m)=
{   local(j,l,b,h,t,d,s,v,u);
    h=m;
    b=0;
    t=vector(length(a));
    d=length(a);
    l=1;
    j=1;
    s=vector(length(a));
    v=vector(0);
    t=a;
    if(count(t)>0,until(count(t)==0 || t==a,
        for(k=1,length(t), if(t[k]<t[j], j=k));
        for(k=1,length(t),s[k]=t[k]-1);
        for(k=1,length(t), if(k==j,t[k]=s[k]+d, t[k]=s[k]));
        l=h*s[j]/(d*n);
        v=concat(v,[[l,s]]);
        h=h*1/n));
    v
}
```

Here is the other possible reduction. This one subtracts 5 from one spot and adds one to everything afterwards. Saves leftovers in vector v.

```
red2(a,m)=
{
    local(j,l,b,h,t,d,s,v,u);
    h=m;
    b=0;
    d=length(a);
    l=1;
    t=vector(length(a));
    v=vector(0);
    j=1;
    s=vector(length(a));
    t=a;
    if(count(t)>0,until(count(t)==0 || t==a,
            for(k=1,length(t), if(t[j]<t[k], j=k));
            for(k=1,length(t),if(k==j,s[k]=t[k]-d,s[k]=t[k]));
            for(k=1,length(t), t[k]=s[k]+1);
            l=h*(-s[j])/d;
            v=concat(v,[[l,s]]);
            h=h*n));
    [h,t]
}

red2leftovers(a,m)=
{
```

```
 local(j,l,b,h,t,d,s,v,u);
 h=m;
 b=0;
d=length(a);
l=1;
t=vector(length(a));
v=vector(0);
j=1;
s=vector(length(a));
t=a;
if(count(t)>0,until(count(t)==0 || t==a,
          for(k=1,length(t), if(t[j]<t[k], j=k));
          for(k=1,length(t),if(k==j,s[k]=t[k]-d,s[k]=t[k]));
          for(k=1,length(t), t[k]=s[k]+1);
          l=h*(-s[j])/d;
          v=concat(v,[[l,s]]);
          h=h*n));
    v
}
```

Now we combine these two reductions and loop until we get the right kind of vector (a monomial in $\mathcal{W}$). The input of this function is a vector of any length and the output will be the "linear combination" of that vector in terms of the basis vectors (vectors with entries between 1 and the length).

```
reduction(a)=
{
local(d,b,c,u,v,w,uu, bb,j, t, s, g,r);
u=vector(0);
v=vector(0);
d=length(a);
j=1;
if(count(a)==d,  u=[red1(a,1)]; v=red1leftovers(a,1),
                    u=[red2(a,1)];v=red2leftovers(a,1));
for(k=1, 10^d,
    if(k<=length(v),
        if(count(v[k][2])==0,
            if(v[k][1]==0, ,b=0;
                for(i=1,length(u),
                    if(v[k][2]==u[i][2],
                        u[i][1]=u[i][1]+v[k][1],
                        b=b+1));
                if(b==length(u),u=concat(u,[v[k]]))),
            if(v[k][1]==0, ,
                uu=red2(v[k][2],v[k][1]);
                v=concat(v,red2leftovers(v[k][2],v[k][1]));
                b=0;
                for(i=1,length(u),
                    if(uu[2]==u[i][2],
```

```
                        u[i][1]=u[i][1]+uu[1],
                        b=b+1));
                    if(b==length(u),u=concat(u,[uu])))),
            break));
b=0;
for(k=1, length(u),
    if(u[k][2]==a,
        w=vector(length(u)-1);
        for(j=1,k-1,w[j]=[u[j][1]/(1-u[k][1]),u[j][2]]);
        for(j=k,length(u)-1,
            w[j]=[u[j+1][1]/(1-u[k][1]),u[j+1][2]]),
        b=b+1));
        if(b==length(u), r=u, r=w);
    r
}
```

The step is to write the connection matrix from this, that is, write a function that gives the derivatives of each vector in terms of the basis. In fact, there is an easy way to write the derivative of any vector using the reduction function.

```
derivative(a)=
{
    local(d,t,w);
    d=length(a);
```

```
    t=vector(d);
    for(k=1,d,t[k]=a[k]+1);
    if(count(t)==0,[[-d,t]],
        w=reduction(t);
        for(k=1,length(w),
        w[k][1]=w[k][1]*(-d)); w)
}
```

Given a basis vector, we can find all the other basis vectors that will be a basis for the same eigenspace.

```
orbit(a)=
{   local(l,m, c,ss);
    d=length(a);
    l=0;
    c=0;
    ss=1;
    for(k=1,d, for(t=1,d, if(k==d-a[t], l=l+1;break)));
    m=d-l;
    b=vector(m); for(k=1,m, b[k]=vector(d));
    b[1]=a;
    for(s=1,d-1,for(t=1,d, if(s==d-a[t], , c=c+1));
        if(c==d,ss=ss+1;for(t=1,d, b[ss][t]=(a[t]+s)%d));c=0);
    b;
}
```

The following the matrix representation of the block of the Gauss-Manin connection associated to a particular basis vector (i.e., it gives a block of the whole matrix, which is related to the eigenspace related to this basis vector).

```
connectionmatrix(a)=
{   local(v,w,M);
    v=orbit(a);
    M=matrix(length(v),length(v));
    for(j=1,length(v),
        w=derivative(v[j]);
        for(k=1,length(w),M[memb(w[k][2],v),j]=w[k][1]));
    M=mattranspose(M);
    M
}
```

B.2 The algorithm to find the differential equation

The following finds the derivative with respect to λ of a vector with a coefficient. Basically, it's the product rule.

```
derivn(a)=
{   local(b,z, ww, vv);
    b=deriv(a[1]);
    z=derivative(a[2]);
```

```
    ww=[[b,a[2]]];
    vv=vector(length(z));
    for(k=1,length(z), vv[k]=[a[1]*z[k][1],z[k][2]]);
    for(k=1,length(vv),
        if(vv[k][2]==a[2], ww[1][1]=ww[1][1]+vv[k][1],
            ww=concat(ww,[vv[k]])));
    ww
}
```

We would like to have the derivative of a vector which is a linear combination of these monomials. This should use ideas like the function above. The first function finds the derivative of a vector (with a coefficient) in a prescribed basis determined by the orbit of b. The second does the same, but only outputs the vector of coordinates, without writing the basis down.

```
derivv(a,b)=
{   local(v,w,z);
    w=orbit(b);
    v=vector(length(w));
    for(i=1,length(w), v[i]=[0,w[i]]);
    for(k=1,length(a),
        z=derivn(a[k]);
        for(j=1,length(z),
            v[memb(z[j][2],w)][1]=v[memb(z[j][2],w)][1]+z[j][1]
        );
```

```
    );
    v
}

derivv2(a,b)=
{   local(v,w,z);
    w=orbit(b);
    v=vector(length(w));
    for(k=1,length(a),
        z=derivn(a[k]);
        for(j=1,length(z),
            v[memb(z[j][2],w)]=v[memb(z[j][2],w)]+z[j][1]
        );
    );
    v
}
```

We want to change basis, and we need a matrix that changes from our basis obtained by using "connection" to a basis obtained from derivatives.

```
cob(a)=
{
    local(z, vv, uu, w);
    uu=orbit(a);
    r=vector(length(uu),k,if(k==1,1));
```

```
    vv=vector(length(uu), k, [r[k],uu[k]]);
    z=[r];
        for(k=1,length(uu)-1,
                w=derivv2(vv,a);
                vv=derivv(vv,a);
                z=concat(z,[w])
        );
    S=Mat(z~);
    S
}
```

Sometimes the vectors in the basis are not cyclic. By the cyclic vector theorem we know that there is one in this space, though, so we just create a random vector in this basis and hope for the best. The following function finds the change of basis for a random vector.

```
cobrandom(a,bd=10)=
{
    local(z, vv, w, r, S, uu, rand);
    uu=orbit(a);
    r=vector(length(uu),k,random(bd)-bd);
    vv=vector(length(uu), k, [r[k],uu[k]]);
    z=[r];
        for(k=1,length(uu)-1,
                w=derivv2(vv,a);
```

```
            vv=derivv(vv,a);
            z=concat(z,[w])
        );
    S=Mat(z~);
    S
}
```

This function takes two input vectors, one is a basis and the other a vector indicating a linear combination of elements in this basis.

```
cobv(uu,r)=
{
    local(z, vv, w);
    vv=vector(length(uu), k, [r[k],uu[k]]);
    z=[r];
        for(k=1,length(uu)-1,
            w=derivv2(vv,a);
            vv=derivv(vv,a);
            z=concat(z,[w])
        );
    S=Mat(z~);
    S
}
```

The following computes what a change of basis does to the system of differential equations, where we change from a basis found by using the connection function to a basis of all the derivatives of a specific vector.

```
cobsystem(A, S)=
{
local(dS, C);
dS=matrix(length(A), length(A), X, Y, deriv(S[X,Y]));
C=S*A*1/S+dS*1/S;
    C
}
```

Now, as seen in Appendix A, we can change this system into an equivalent one with regular singularities at 0. Note that we need a matrix that gives a simple system like the ones given by the cobsystem function. That is, we need a companion matrix to make any of this work.

```
 regform(A)=
{
local(m,Areg);
m=length(A);
Areg=A;
for(k=1,m, Areg[m,k]=Areg[m,k]*n^(m-k+1));
for(k=1,m,
   for(i=1,m, if (i==k, Areg[k,i]=Areg[k,i]+k-1)));
Areg
}
```

We need to change variables to have it in terms of x instead of n^d.

```
varchange(A,d)=
{
for(k=1,length(A),
        for(i=1,length(A),
            A[k,i]=substpol(A[k,i],n^d,x)/d));
    A
}
```

Finally we can compute the residue of the matrix at $x = 0$.

```
residuezero(A)=
{
for(k=1,length(A),
for(i=1,length(A),
A[k,i]=subst(A[k,i],x,0)));
A
}
```

We want a function that finds the rational roots of a polynomial with rational coefficients (because all of our polynomials are of that form and only have rational roots). We are using the rational roots theorem.

```
ratlroots(f)=
```

```
{
    local(p,q,z,vv,a,b,c,n,r,j,i);
    c=poldegree(f);
    vv=vector(c+1);
    r=vector(0);
    z=vector(0);
        for(k=1,c+1, vv[k]=polcoeff(f,k-1));
    n=denominator(vv);
    f=f*n;
    a=substpol(f,x,0);
        if(a==0, r=concat(r,0); f=f/x; a=substpol(f,x,0));
    b=pollead(f);
    p=concat(divisors(a),-divisors(a));
    q=concat(divisors(b),-divisors(b));
    for(k=1,length(p),
        for(i=1,length(q),
            if(memb(p[k]/q[i],z)==0, z=concat(z,p[k]/q[i]))));
    for(i=1,length(z),
        if(substpol(f,x,z[i])==0, r=concat(r,z[i]);
                        f=f/(x-z[i]);i=1));
    r
}
```

We now want to combine all these steps to find the hypergeometric parameters given a vector (or monomial) in $\mathcal{W}$.

```
hypergcoeff(a)=
{
    local(A, B, S, Azero, Ainf,f,g, vv, uu, d, m,r,t);
    d=length(a);
    A=connectionmatrix(a);
    S=cob(a);
    A=cobsystem(A,S);
    A=regform(A);
    A=varchange(A,d);
    Azero=residuezero(A);
        f=charpoly(Azero);
    B=matrix(length(A),length(A));
    Ainf=matrix(length(A),length(A));
    for(k=1,length(A),
        for(i=1,length(A),
            B[k,i]=-substpol(A[k,i],x,1/y)));
    for(k=1,length(A),
        for(i=1,length(A),
            Ainf[k,i]=subst(B[k,i],y,0)));
        g=charpoly(Ainf);
    r=ratlroots(f);
    m=vector(length(r));
    for(k=1,length(r), m[k]=1-r[k]);
\\ print("alphas  ",ratlroots(g)," betas ",m)
```

```
    t=[ratlroots(g),m];
    t
}
```

We want to check these coefficients somewhat systematically, so we have to find a good way to generate basis vectors (or representatives up to permutations of the variables).

This function (also from [29]) finds all the partitions of a number m.

```
part(m)=
{
  local(k,j,sm,sj,s, S = []);
  k = j = 1;
  sm = sj = vector(m+1);

  while(k,
      s = sm[k]+j;
      if (s > m,
      until(j <= m, j = sj[k]+1; k--);
      next);

      k++; sm[k]=s; sj[k]=j;

      if (s == m,
      S = concat(S, [vector(k-1,l, sj[k-l+1])])));
```

```
  S
}
```

This one uses the previous one to find the partitions of a number m of length c and into numbers that are less than c.

```
part2(m,c)=
{
    local(v,w,t);
    v=part(m);
    w=vector(0);
    for(k=1,length(v),
        if(length(v[k])==c,
            t=count(v[k]);
            if(t==0,
                w=concat(w,[v[k]]))));
    w
}
```

Now we can put this together to get representatives of the basis. We don't have strict representatives, but at least we eliminate the cases in which none of the entries are equal to one, because those obviously are in an eigenspace with a vector with entries equal to one.

```
basisreps(m)=
{
    local(v,w);
    v=vector(0);
    for(k=1,ceil((m-1)/2),
        w=part2(m*k,m);
        for(j=1, length(w),
            if(memb(1,w[j])==0, ,v=concat(v,[w[j]]))));
    v
}
```

We now want to be able to put out a table with the basis vectors and the hypergeometric parameters associated to them given a number d. It should turn out to be the list of numbers in the vector that remain after canceling out with the list of numbers between 0 and d.

```
hypergtable(d)=
{
    local(v,u);
    v=basisreps(d);
    for(k=1,length(v), u=hypergcoeff(v[k]);
    print(v[k],"       ", "alphas       ", u[1],
                           "     betas    ", u[2]))
}
```

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This dissertation was typeset with LaTeX[‡] by the author.

[‡]LaTeX is a document preparation system developed by Leslie Lamport as a special version of Donald Knuth's TeX Program.