Author(s): Pablo Beker and Subir Chattopadhyay

Article Title: Consumption Dynamics in General Equilibrium : A Characterisation when Markets are Incomplete

Year of publication: 2009

Link to published article:
http://www2.warwick.ac.uk/fac/soc/economics/research/papers_2009/twerp_921.pdf

Publisher statement: None

Consumption Dynamics in General Equilibrium :
A Characterisation when Markets are Incomplete

Pablo Beker and Subir Chattopadhyay

No 921

WARWICK ECONOMIC RESEARCH PAPERS

DEPARTMENT OF ECONOMICS

THE UNIVERSITY OF WARWICK

Consumption Dynamics in General Equilibrium: A Characterisation when Markets are Incomplete*

Pablo BEKER
University of Warwick
and
Subir CHATTOPADHYAY
University of York

September 2009 (Previous version—April 2006, November 2008)

ABSTRACT

We introduce a methodology for analysing infinite horizon economies with two agents, one good, and incomplete markets. We provide an example in which an agent's equilibrium consumption is zero eventually with probability one even if she has correct beliefs and is marginally more patient. We then prove the following general result: if markets are effectively incomplete forever then on any equilibrium path on which some agent's consumption is bounded away from zero eventually, the other agent's consumption is zero eventually—so either some agent vanishes, in that she consumes zero eventually, or the consumption of both agents is arbitrarily close to zero infinitely often. Later we show that (a) for most economies in which individual endowments are finite state time homogeneous Markov processes, the consumption of an agent who has a uniformly positive endowment cannot converge to zero and (b) the possibility that an agent vanishes is a robust outcome since for a wide class of economies with incomplete markets, there are equilibria in which an agent's consumption is zero eventually with probability one even though she has correct beliefs as in the example. In sharp contrast to the results in the case studied by Sandroni (2000) and Blume and Easley (2006) where markets are complete, our results show that when markets are incomplete not only can the more patient agent (or the one with more accurate beliefs) be eliminated but there are situations in which neither agent is eliminated.

Journal of Economic Literature Classification Numbers: D52, D61

Correspondence To:
Pablo F. Beker
Department of Economics
University of Warwick
Coventry CV4 7AL
UNITED KINGDOM
Telephone: +44 2476 150588 Fax: +44 2476 523032 e-mail: Pablo.Beker@warwick.ac.uk

* An earlier version of the paper was circulated with the title "Economic Survival when Markets are Incomplete". We thank M.Quinzii and M. Woodford for very useful exchanges, D. Easley for comments, and J. Thomas for an observation. We are also grateful to an Associate Editor who suggested that we consider the case of finite state time homogeneous Markov endowments which led to Theorem 2, and to a referee for comments that improved the paper. Beker acknowledges financial support from MEC project SEJ2004-02172.

1. INTRODUCTION

This paper introduces a methodology for analysing the asymptotic behaviour of individual consumption in general equilibrium in economies where the asset market is incomplete. The case where markets are dynamically complete and endowments are bounded has been analysed extensively and the picture that emerges is that the degree of impatience and the accuracy of beliefs are the key elements that determine whether an agent's consumption is eventually bounded away from zero, i.e. "survives", thereby ensuring that in the long run she matters for asset pricing; attitudes toward risk are irrelevant. This is significant because it appears to validate the market selection hypothesis (henceforth, MSH) which, in the weak form due to Alchian (1950) and Friedman (1953), requires that only agents whose behaviour is consistent with rational and informed maximization of returns can survive and affect prices in the long run.[1] The fact that survival depends only on discount factors and the accuracy of beliefs could reflect an intrinsic property of competitive markets; it could also be driven by the assumption that markets are dynamically complete. Very little is known about this and that is the question we address.

We consider an infinite horizon economy with only one good, two agents, a single short lived inside asset, and dynamically incomplete markets. Our assumptions on the structure of uncertainty are quite general since we only require that one of a fixed and finite number of states is realised each period and that the one period ahead conditional probability of the occurrence of a state is uniformly positive. Our assumptions on beliefs are also quite general (see Section 2.6). We use a standard notion of equilibrium in which agents maximise subject to a sequence of budget constraints and the requirement that the value of debt be uniformly bounded across dates and events.[2] Our formulation includes recursive equilibria that can be represented by a Markov chain (Duffie et al (1994) and Ljungqvist and Sargent (2004)), a particularly important case in macroeconomics. Our interest is in the asymptotic behaviour of equilibrium consumption and it is well known that studying that is equivalent to studying the evolution through time of the ratio of the values of the derivatives of the Bernoulli functions of the two agents, y_t.

For pedagogical reasons, we briefly return to the special case that arises when markets are dynamically complete and endowments are bounded. In such a framework, equilibrium allocations are Pareto optimal and so, at an interior allocation, the utility gradients of the different agents point in the same direction. When preferences are additively separable across time, the key implication is that the ratio of (the one-period ahead intertemporal) marginal rates of substitution of the two agents weighted by the discount factors is one independent of the date and event; equivalently, y_t can be written as the product of the ratio of the discount factors, the ratio of the beliefs, and an initial condition. So if both the agents have correct beliefs (or even identical incorrect beliefs) and the same discount factor then consumption of both is uniformly positive eventually, while if agents differ

[1]Cootner (1967) and Fama (1965) offered a stronger version of the MSH which claims that markets select for investors with correct beliefs, which can be inferred from long run equilibrium prices.

[2]For more on the boundedness property see Magill and Quinzii (1994) and Levine and Zame (1996).

in their degree of impatience, then only the most patient agent has uniformly positive consumption eventually—a result conjectured by Ramsey (1928 pp. 558-559) and proved by Becker (1980), Rader (1981) and Bewley (1982). With heterogeneous beliefs, Sandroni (2000) showed that among agents with the same discount factor, traders who eventually accurately predict infinite horizon events, and only those traders, have positive wealth eventually; in the absence of such accurate predictors, the entropy of beliefs determines survival and investors whose forecasts are persistently wrong vanish in the presence of a learner. Sandroni considered a Lucas-tree economy, a restriction that is inessential since Blume and Easley (2006) showed that Pareto optimality of the allocation is the key point; as we emphasized earlier, none of the results depends on agents' preferences towards risk.

We initiate our methodological innovation by writing y_t as the ratio of two stochastic processes where each is the product of conditional mean one random variables. Market incompleteness typically implies that the ratio of marginal rates of substitution of the two agents is not degenerate so that, with uniformly positive asset returns, y_t grows with positive conditional probability (since otherwise one of the two Euler equations would not hold with equality). That is the key ingredient in Theorems 1 and 4.

Our general approach suggests a conjecture about the implications of market incompleteness in infinite horizon economies where the Euler equations always hold with equality: the consumption of some agent comes arbitrarily close to zero infinitely often. That still allows for an intriguing possibility that an example illustrates. In it agent 1 has arbitrary CRRA preferences (but not logarithmic) and a positive stochastic endowment forever, and agent 2 has logarithmic preferences and a positive endowment only at date zero. We show that even if agents are equally patient and have correct beliefs, one can find a time invariant asset structure such that the consumption of the agent with logarithmic preferences converges to zero, i.e. "vanishes", with probability one in every equilibrium. A continuity argument shows that the same is true even if agent 2 is marginally more patient or if she holds correct beliefs and agent 1 does not. The example shows that the factors determining survival with complete markets have little relevance when markets are dynamically incomplete. It also suggests the conjecture: the consumption of some agent is zero eventually. Our theorems refine and strengthen the two conjectures.

Our first result is very intuitive since it is based on the observation that on almost every path one can have arbitrarily long strings of states where y_t keeps rising because y_t grows with positive conditional probability, and because we assume that the likelihood ratio is eventually uniformly bounded across paths. This fact can be shown to imply that if a prespecified agent has consumption that is bounded away from zero eventually, then every prespecified lower bound on the other agent's consumption is violated eventually; the technical tool used is Levy's conditional form of the Second Borel-Cantelli Lemma, see e.g. Freedman (1973). Theorem 1 shows that either (i) marginal rates of substitution are equalized in the limit or (ii) the ratio of marginal rates of substitution displays one period ahead conditional variability forever and then either (a) the equilibrium is complicated in that the consumption of both agents will be arbitrarily close to zero infinitely often, or (b) one of the two agents will cease to consume eventually, as in the example. The result applies equally regardless of whether beliefs are homogeneous or heterogeneous.[3] For Theorem 1 we assume that the asset pays a uniformly positive amount and that the one period ahead conditional probability of the occurrence of a state is uniformly positive, assumptions that are standard although they can be weakened.

That one of the two agents' consumption vanishes is surprising and one would like to identify situations where such a result cannot be true. In Theorem 2 we consider the particular case, often considered in the applied general equilibrium literature, where individual endowments follow a finite state time homogeneous Markov process. We show that, for most endowment distributions in such economies, if an agent's endowment is uniformly positive then the set of paths where her consumption converges to zero has measure zero. We remark that the result holds for all discount factors and all beliefs that are compatible with the Markov chain structure of endowments. The intuition for the result is that the agent who vanishes can face arbitrarily long sequences in which the same state is realized and in such an event her debt is uniformly bounded only if it is maintained at a specific constant value that may depend on the state. But only if endowments are suitably special will debt remain confined to such a finite set of sustainable debt levels.

One may read Theorem 2 as suggesting that an agent vanishes in only rather special situations. However, it might be more appropriate to bear in mind the restrictive assumptions under which Theorem 2 is proved: that debt is uniformly bounded, that endowments follow a finite Markov chain, and that endowments are uniformly positive. It is clear that the latter two are assumed for analytical convenience only; also, there are other notions of equilibrium in the literature in which debt is not uniformly bounded.

Theorem 2 also provides a different route to show that long run equilibrium behaviour depends on whether markets are complete. The result provides conditions under which an agent cannot vanish; yet, if the agent with a uniformly positive endowment is less patient or has incorrect beliefs then she would vanish if markets were to be complete.

Our final result provides sufficient conditions for an agent to vanish in equilibrium. We say that an agent's one period ahead marginal valuation of the asset is "predetermined" if the asset payoff times the value of the derivative of the agent's Bernoulli function is constant across immediate successor states. We first show that if beliefs are correct then, among feasible consumption processes which satisfy the Euler equations, those for which some agent's valuation is always predetermined lead to that agent consuming zero eventually on almost every path. We then propose a method that generates processes with the stated properties that are uniquely specified for each value of consumption at the initial date and we provide a condition under which the supporting prices are summable.

To show that there are equilibria with the "predeterminedness" property we proceed

[3] It also applies to economies with a retradable long lived asset with strictly positive returns—we do not consider such economies for notational simplicity. Duffie et al (1994) provide an existence theorem for Lucas-tree economies with incomplete markets in which consumption is uniformly bounded away from zero. For their result it is crucial that there are no short sales and no one period inside assets either.

in two stages. First, we provide two sets of sufficient conditions for verifying that given consumption processes can be supported as equilbria. The conditions include the requirements that the allocation is aggregate feasible, that the utility values are well defined, that the Euler equations are satisfied, that the supporting prices are summable (or a related weaker condition), and a condition that allows one to verify that the value of debt is uniformly bounded. That result, Theorem 3, could be of independent interest.[4]

Finally, Theorem 4 provides sufficient conditions under which there are allocations that satisfy both the "predeterminedness" property and the conditions of Theorem 3. This lets us identify a family of "no trade" equilibria that are supported with trivial, hence uniformly bounded, asset portfolios. We then show that for each such no trade equilibrium there is a family of endowment perturbations that reallocate the total endowment across the two agents so that each agent's endowment is uniformly positive but path dependent (so that it cannot have finite support) for which the initial allocation continues to be an equilibrium but now with asset trade. An implication of Theorem 4 is that an economy in which a real bond is the only asset has many endowment distributions which lead to equilibria in which a predetermined agent vanishes and this can happen even though her endowment is uniformly positive; this result is stated as Corollary 2.

Our analysis exposes the ways in which examples of economies like ours that have appeared in the literature are special (see Section 4.2). It also has implications for the MSH. Based on an example of an economy like ours in which an agent with correct beliefs is driven out while the agent who survives has wrong beliefs and a higher saving rate, Blume and Easley (2006) conclude that savings behaviour driven by beliefs is the key that explains survival, a point also raised by Sandroni (2005). Our Theorem 4, in which all agents hold correct beliefs, suggests that, at the margin, market incompleteness determines the fate of the trader. Theorem 4 and our example make very clear that even the version of the MSH due to Alchian (1950) and Friedman (1953) does not hold in general and that, in dynamically incomplete markets economies, no entropy measure that depends only on the truth, beliefs, and the market structure can be critical to understanding survival because any properly defined entropy measure must attain its maximum when beliefs are correct and yet, as per the example, survival is not guaranteed.[5]

To summarize, for infinite horizon economies with two agents and one short-lived asset we provide a complete characterization of limiting consumption behaviour when markets are incomplete, show that to get simple limiting behaviour one agent must be driven out of the market, and show that such a possibility is a robust outcome. By implication, the MSH is valid in a robust sense only if the equilibrium allocation is Pareto optimal.[6]

[4]The existence result in Magill and Quinzii (1994) imposes a uniform lower bound on individual endowments precluding its use in situations in which some agent's consumption approaches zero.

[5]This answers a question posed by Sandroni (2005).

[6]There is a literature on the asymptotic behaviour of consumption in a partial equilibrium framework. Chamberlain and Wilson (2000) provide sufficient conditions on discount factors and interest rate paths for consumption to have an unbounded subsequence with probability one. We show that such combinations cannot arise in equilibrium in the class of economies considered in this paper.

In Section 2 we introduce the model, define the relevant notions of survival, and discuss the scope of the paper. In Section 3 we develop the general approach to study the long run dynamics of equilibria and then present the leading example. Afterwards, in Section 4 we present Theorem 1 and our discussion of earlier examples in the literature. Section 5 shows that by suitably restricting aggregate and individual endowments one can ensure that no agent vanishes since otherwise her debt would fail to be uniformly bounded. Finally, in Section 6 we construct equilibria in which only one agent survives. Concluding remarks are presented in Section 7. All proofs are gathered in the Appendix.

2. MODEL

2.1 PROBABILITY NOTATION

We consider an infinite horizon with dates $t = 0, 1, 2, \cdots$. The temporal state space is $\mathcal{S} \equiv \{1, 2, \cdots, S\}$, $S < \infty$. $\mathcal{S}^t$ is the t-fold Cartesian product of $\mathcal{S}$ and $\Omega \equiv \mathcal{S}^\infty$ with typical element $\omega = (s_1, s_2, \cdots)$ where s_t is the realization at date $t \geq 1$. In fact, we shall write $\omega = (s_1(\omega), s_2(\omega), \cdots)$. Also $s^t \equiv (s_1, \cdots, s_t)$ and if we wish to make the dependence on ω explicit, we shall use $s^t(\omega) \equiv (s_1(\omega), \cdots, s_t(\omega))$. $\Omega(s^t) \equiv \{\omega \in \Omega : \omega = (s^t, s_{t+1}, \cdots),\ s^t \in \mathcal{S}^t\}$ is a *t-cylinder* and $\mathcal{F}_t$ is the σ-algebra obtained by considering finite unions of the sets $\Omega(s^t)$ for fixed t. This induces a sequence of σ-algebras on Ω denoted $\{\mathcal{F}_t\}_{t=1}^\infty$ where $\mathcal{F}_{t-1} \subset \mathcal{F}_t$ for all $t \geq 1$; we set $\mathcal{F}_0 \equiv \{\emptyset, \Omega\}$, and we set $\sigma\Big(\cup_{t\geq 0} \mathcal{F}_t\Big) \subset \mathcal{F}$. That is our *filtration* with $\mathcal{F}$ a σ-algebra on Ω. All statements will be made using $(\Omega, \mathcal{F})$.

Any function $X : \Omega \to R$ that is $\mathcal{F}$-measurable is a *random variable.* $\sigma(X)$ is the σ-algebra generated by X.

For $Q : \mathcal{F} \to [0, 1]$ a probability measure, let dQ_t be the $\mathcal{F}_t$ measurable function defined by $dQ_t(\omega) \equiv Q(\Omega(s^t(\omega)))$ for $t \geq 1$ and $dQ_0(\omega) \equiv 1$, i.e. $dQ_t(\omega)$ is the probability of the cylinder $\Omega(s^t(\omega))$. Define the probability that state $s_t(\omega)$ occurs, conditional on the occurance of $s^{t-1}(\omega)$, by $Q_t(\omega) \equiv \frac{dQ_t(\omega)}{dQ_{t-1}(\omega)}$; when $s = s_t(\omega)$, the one period ahead conditional probability that the state s occurs is $Q_t(\omega)$. $E_Q[X|\mathcal{G}]$ denotes the conditional expectation of the random variable X taken with respect to the measure Q where the σ-algebra $\mathcal{G}$ satisfies $\mathcal{G} \subset \mathcal{F}$. $E_Q[X|\mathcal{G}]$ is a $\mathcal{G}$-measurable random variable.

2.2 THE ECONOMY

There is only one perishable good at each date. An agent is denoted $i \in \mathcal{I}$. There are two agents, so $\mathcal{I} \equiv \{1, 2\}$, each of whom lives forever.

$\omega \in \Omega$ is chosen according to the objective probability measure P while agent i's subjective belief is denoted P_i. $(\Omega, \mathcal{F}, P)$ is the objective probability triple. $(\Omega, \mathcal{F}, P_i)$, $i = 1, 2$, are the triples used by the agents for their decisions. We shall assume that the one period ahead conditional probability that state s occurs is uniformly positive and agents correctly believe it to be so.[7] So, define $\underline{p} \equiv \inf_{t\geq 0} \inf_{\omega\in\Omega} P_t(\omega)$.

ASSUMPTION A.1: $0 < \underline{p} \leq \inf_{t\geq 0} \inf_{\omega\in\Omega} P_{i,t}(\omega)$.

[7]This assumption is standard in the literature (see Sandroni (2000) and Blume and Easley (2006)).

We have assumed that S is finite and that P satisfies A.1. These are minimal assumptions and allow for a very wide range of stochastic behaviour including nonstationarity.

Define $\Psi^t \equiv \{f : \Omega \to R : f \,\text{is}\, \mathcal{F}_t - \text{measurable}\}$. An element of $\times_{t=0}^{\infty}\Psi^t$ is a *process*. Also define[8] $\Psi \equiv \{(f_0, f_1, \cdots) \in \times_{t=0}^{\infty}\Psi^t : \sup_{t\geq 0} \parallel f_t \parallel_{\infty,P} < \infty\}$; since $S < \infty$ and A.1 taken together imply that $\sup_{t\geq 0} \parallel f_t \parallel_{\infty,P} = \sup_{t\geq 0} \parallel f_t \parallel_{\infty,P_i}$, we have used Ψ to denote the normed space of processes.[9] For the same reason, we define $\Psi_+^t \equiv \Psi^t \cap \{f(\omega) \geq 0 \,\text{for all}\, \omega \in \Omega\}$, and $\Psi_+ \equiv \Psi \cap \left(\times_{t=0}^{\infty} \Psi_+^t \right)$.

$E_Q[f] \equiv E_Q[f|\mathcal{F}_0](\omega)$ is the unconditional expectation where $Q \in \{P, P_1, P_2\}$.

The *aggregate endowment* process is denoted $Z \equiv \{Z_t\}_{t=0}^{\infty}$ and its range is $[\underline{z}, \bar{z}]$ so that for all $t \geq 0$, $Z_t(\omega) \in [\underline{z}, \bar{z}]$. i's *endowment* process is denoted $z_i \equiv \{z_{i,t}\}_{t=0}^{\infty}$, a nonnegative process, and $z_1 + z_2 = Z$.

ASSUMPTION A.2: $[\underline{z}, \bar{z}] \subset R_{++}$. $z_i \in \Psi_+$.

u_i is i's state independent Bernoulli utility function. β_i is agent i's discount factor.

ASSUMPTION A.3: For $i \in \mathcal{I}$ (i) $u_i : R_{++} \to R$ is strictly increasing, strictly concave, and C^2 with $\lim_{c\to 0^+} u_i'(c) = \infty$, with $u_i(0) \equiv \lim_{c\to 0^+} u_i(c) \in R \cup \{-\infty\}$, and (ii) $\beta_i \in (0, 1)$.

There is a single one period asset available in zero net supply. Its return is r, where r is a process with range $[\underline{r}, \bar{r}]$ so that for all $t \geq 0$, $r_t(\omega) \in [\underline{r}, \bar{r}]$. r is assumed to be uniformly positive so Arrow securities are ruled out; the role of this restriction will be discussed in Sections 4.1 and 4.2.

ASSUMPTION A.4: $[\underline{r}, \bar{r}] \subset R_{++}$.

For the result in section 5 we assume that individual endowments and asset returns follow finite state time homogeneous Markov processes. Formally

ASSUMPTION A.5: The image of Z is $\mathcal{S}$. If $s_{t-1}(\omega) = s_{t'-1}(\omega') \equiv s$ and $s_t(\omega) = s_{t'}(\omega') \equiv s'$ then $P_t(\omega) = P_{t'}(\omega') \equiv \pi_{s,s'}$, and $P_{i,t}(\omega) = P_{i,t'}(\omega') \equiv \pi_{i,s,s'}$, for $i = 1, 2$. Given $z_i : \mathcal{S} \to R_+$ and $r : \mathcal{S} \to R_+$, the individual endowment and the asset payoff are also time homogeneous Markov Processes defined as $z_{i,t}(\omega) \equiv z_i(Z_t(\omega))$ and $r_t(\omega) \equiv r(Z_t(\omega))$.

$\pi_{s,s'}$ induces Π, a Markov transition matrix. Π_i, $i = 1, 2$, are similarly obtained.

The next assumption will be used to prove that the consumption processes that we construct and use in Theorem 4 are supportable as equilibria. Notice that, under A.2-4, $1 < M < \infty$ where M is specified in A.6.

ASSUMPTION A.6: $\beta_2 < 1/M$ where $M \equiv \max \left\{ \frac{\bar{r} \cdot u_2'(\underline{z}/2)}{\underline{r} \cdot u_2'(\bar{z})}; \frac{\beta_1}{\beta_2} \frac{\bar{r} \cdot u_1'(\underline{z}/2)}{\underline{r} \cdot u_1'(\bar{z})} \right\}$.

[8] For $h : \Omega \to R$ an $\mathcal{F}$-measurable function, $\parallel h \parallel_{\infty,Q} \equiv \inf_{A\in\mathcal{F},Q(A)=0} \sup_{\omega\in\Omega/A} |f(\omega)|$ is the *essential supremum of* h, with respect to the measure Q.

[9] $S < \infty$ and A.1 also imply that Ψ coincides with the set of processes with the sup norm defined by considering the supremum over the range of the process which is at most a countable set.

We shall impose one further assumption that will be stated and discussed in Section 6.1 and is used to construct equilibria in which some agent necessarily vanishes.

REMARK 1: Assumption A.5 will be used only in the analysis of Section 5 to rule out the possibility that some agent vanishes. Assumption A.6 will be used only in Section 6. A weaker versions of A.6 that takes into account specific details of the endowment process and asset return process suffices for Theorem 4 to go through. It is not stated formally since the gain in generality is not justified by the notational complication.

An *economy* is a list $(P, Z, P_1, P_2, \beta_1, \beta_2, u_1, u_2, r)$. A *private ownership economy* is a list $(P, z_1, z_2, P_1, P_2, \beta_1, \beta_2, u_1, u_2, r)$ and is related to an economy by the relation $Z = z_1 + z_2$.

i's *consumption* process is denoted c_i. We require $c_i \in \Psi_+$ and for such a c_i, the *utility payoff* is given by $\lim_{T\to+\infty} \sum_{t=0}^{T} \beta_i^t E_{P_i}[u_i(c_{i,t})]$. i's holding of the asset is the portfolio *process* denoted θ_i. $\theta_{i,-1}(\omega) = 0$ is a convenient convention.

(c_1, c_2) is *feasible* if $c_i \in \Psi_+$ for $i \in \mathcal{I}$ and, for all (ω, t), $c_{1,t}(\omega) + c_{2,t}(\omega) = Z_t(\omega)$.

At each (ω, t) there is a spot market for the good with the price normalized to one and a market for the asset with prices given by the *price* process q. A *market clearing allocation* consists of $(c_1, c_2, \theta_1, \theta_2)$ such that (c_1, c_2) is feasible and, for all (ω, t), $\theta_{1,t}(\omega)+\theta_{2,t}(\omega) = 0$.

2.3 IDC EQUILIBRIUM

A notion of equilibrium in our model economy requires the specification of a budget set subject to which each agent maximizes. Evidently, the budget set will incorporate a sequence of budget constraints; an additional condition, in the form of a uniform bound on the value of debt, is imposed to guarantee that a maximizer exists.

i's *IDC (implicit debt constraint) budget set* is defined as

$$\begin{aligned} BC(q; z_i) \equiv \Big\{ & c_i \in \Psi_+ : \text{ there exists } \theta_i \in \times_{t=0}^{\infty} \Psi^t \text{ such that} \\ & \forall\, t \geq 0,\ c_{i,t}(\omega) + q_t(\omega) \cdot \theta_{i,t}(\omega) \leq z_{i,t}(\omega) + r_t(\omega) \cdot \theta_{i,t-1}(\omega) \quad \forall\, \omega \in \Omega \\ & q \cdot \theta_i \in \Psi \Big\}. \end{aligned}$$

The first set of conditions require that the consumption process be in i's consumption set, i.e. $\{c_{i,t}\}_{t=0}^{+\infty}$ is such that, for all t, $c_{i,t}$ is nonnegative, $\mathcal{F}_t$-measurable and uniformly bounded, the second that there exists a supporting portfolio process which together with the consumption process satisfies the sequence of spot market budget constraints, and the last condition is an *implicit debt constraint* that requires that the value of debt be uniformly bounded.

For i, c_i is an *IDC maximizer* given q if (i) $c_i \in BC(q; z_i)$ and (ii) there is no $\tilde{c}_i \in BC(q; z_i)$ for which

$$\lim_{T\to+\infty} \sum_{t=0}^{T} \beta_i^t E_{P_i}[u_i(\tilde{c}_{i,t})] > \lim_{T\to+\infty} \sum_{t=0}^{T} \beta_i^t E_{P_i}[u_i(c_{i,t})].$$

DEFINITION 1: An *IDC equilibrium* is a tuple $(c_1^*, c_2^*, \theta_1^*, \theta_2^*, q^*)$ that is a market clearing allocation and, for $i \in \mathcal{I}$, c_i^*, with supporting portfolio θ_i^*, is an IDC maximizer given q^*.

Implicit debt constraints have been treated extensively in earlier literature on incomplete market economies with an infinite time horizon, e.g. Magill and Quinzii (1994) who provide conditions such that in any equilibrium where a transversality condition holds at every date-event, the value of debt is uniformly bounded. The IDC budget set does not permit Ponzi schemes (see Magill and Quinzii (1994)).

2.4 EQUILIBRIUM—NECESSARY CONDITIONS

In our framework, at any interior solution to the maximization problem, a set of first order conditions necessarily hold with equality (they also form an important part of the sufficient conditions for identifying a maximizer). Say that c_i is an *Euler* process at the price process q if

$$\forall\, t \geq 0,\ q_t(\omega) = \beta_i \cdot \frac{E_{P_i}[r_{t+1} \cdot u_i'(c_{i,t+1})|\mathcal{F}_t](\omega)}{u_i'(c_{i,t}(\omega))} \quad \forall\, \omega \in \Omega.$$

2.5 SURVIVAL

We shall use various notions to describe the asymptotic behaviour of consumption. We follow the definitions that have been established in the literature.

DEFINITION 2: Fix a path ω.
Agent i dominates on ω if $\liminf_t c_{i,t}(\omega) > 0$.
Agent i survives on ω if $\liminf_t c_{i,t}(\omega) = 0$ and $\limsup_t c_{i,t}(\omega) > 0$.
Agent i vanishes on ω if $\limsup_t c_{i,t}(\omega) = 0$.

The definitions given are made operational by considering the behaviour of marginal utility. Given consumption processes for $i \in \mathcal{I}$, define

$$y_t(\omega) \equiv \frac{u_2'(c_{2,t}(\omega))}{u_1'(c_{1,t}(\omega))}.$$

The proof of the following lemma is straightforward hence omitted.

LEMMA 1: Assume A.2 and A.3. Then
agent 2 dominates on ω $\iff$ $0 \leq \liminf_t y_t(\omega) \leq \limsup_t y_t(\omega) < \infty$;
agent 2 survives on ω $\iff$ $0 \leq \liminf_t y_t(\omega) < \limsup_t y_t(\omega) = \infty$;
agent 2 vanishes on ω $\iff$ $\lim_t y_t(\omega) = \infty$.

The corresponding results for agent 1 are obtained by studying the behaviour of $1/y_t(\omega)$. Both the agents dominate on ω if and only if $0 < \liminf_t y_t(\omega) \leq \limsup_t y_t(\omega) < \infty$.

2.6 THE SCOPE OF THE PAPER—SUBJECTIVE BELIEFS

Although it might not be evident at first glance, our treatment of subjective beliefs is very general in that it accomodates several cases considered in the learning literature (see Blume and Easley (2006) and the references therein). Indeed, as Jackson, Kalai and

Smorodinsky (1999) show, an agent's subjective belief can always be represented through a probability space $(\Xi, \mathcal{B}, \mu_i)$, where μ_i represents "prior beliefs over a parameter", and "models" of the stochastic process generating the data specified through a parametric family of probability measures $P^\xi : \mathcal{F} \to [0, 1]$, where the mapping $\xi \to P^\xi$ is $\mathcal{B}-$measurable. This is because one can define agent i's subjective belief $P_i : \mathcal{F} \to [0, 1]$ as

$$P_i(A) \equiv \int P^\xi(A)\, \mu_i(d\xi) \quad \text{for all } A \in \mathcal{F}.$$

A well-known example of the procedure is the following. Let Ξ be a subset of the unit simplex in R^S. Any $\xi \in \Xi$ generates a probability measure, p^ξ, on $2^{\mathcal{S}}$. An agent's "prior belief" is a probability measure on $\mathcal{B}$, the Borel subsets of Ξ, and a "model" is the probability measure, P^ξ, generated by the rule $P^\xi(\Omega(s^t)) \equiv \prod_{\tau=1}^{t} p^\xi(s_\tau)$, i.e. one induced by i.i.d. draws according to the measure p^ξ.

Our theorems will be proved under three rather different restrictions on beliefs. The result in Theorem 1 (i) does not require us to constrain beliefs. The result in Theorem 1(ii) applies whenever the family of subjective beliefs is such that, with positive probability according to P, for each labelling of agents the likelihood ratio $\frac{dP_{i,t}}{dP_{j,t}}$ does not have zero as an accumulation point. In particular, it applies to a case in which agents' subjective beliefs are as in the example above and, in addition, the objective measure is a model in the same class, i.e. a P^{ξ^*}, where $\xi^* \in \Xi$, and ξ^* is in the support of each agent i's prior belief μ_i. This set up is considered in Blume and Easley (2006) and it can be shown that in such a situation the likelihood ratio has a positive limit $P - a.s.$[10] Importantly, Theorem 1(ii) does not require the likelihood ratio to converge although it does impose a restriction on the set of limit points of the likelihood ratio.

Theorem 2, on the other hand, imposes A.5, the condition that both the objective probability measure as well as the agents' subjective beliefs are generated by the Markov matrices Π and Π_i respectively. This condition is equivalent to requiring that the agent's prior places point mass on one matrix in a set of transition matrices.

Theorem 4 requires that agents' beliefs agree with the objective probability P.

3. WHAT HAPPENS IN THE LONG-RUN?—A METHODOLOGY

In this section we show that the dynamics of equilibrium consumption with incomplete markets when Euler equations hold with equality can be analysed systematically by studying the solution to an appropriate equation that generalizes the earlier method used in Sandroni (2000) and Blume and Easley (2006) in the case where markets are complete. We then turn to an example that illustrates the drastic change in the asymptotic behaviour of the system when markets fail to be complete. The section ends with a summary of what one might expect to obtain as general results with incomplete markets.

[10]Phillips and Ploberger (1996) provide much more general conditions on the family of subjective priors and true probability measures such that the likelihood ratio converges. It can be checked that those conditions are satisfied not only in the case where the draws are i.i.d. but also when the draws are from a Markov process.

3.1 FIRST ORDER CONDITIONS AND THEIR IMPLICATIONS

Since i's marginal utility at (t,ω) is given by the expression $\beta_i^t \cdot dP_{i,t}(\omega)\cdot u_i'(c_{i,t}(\omega))$, the first order necessary conditions at an interior Pareto optimal allocation can be summarized in the form

$$\frac{\beta_2^t \cdot dP_{2,t}(\omega)\cdot u_2'(c_{2,t}(\omega))}{\beta_1^t \cdot dP_{1,t}(\omega)\cdot u_1'(c_{1,t}(\omega))} = \frac{u_2'(c_{2,0}(\omega))}{u_1'(c_{1,0}(\omega))} \quad \Leftrightarrow \quad (\beta_2/\beta_1)^t \cdot \frac{dP_{2,t}(\omega)}{dP_{1,t}(\omega)} \cdot y_t(\omega) = y_0(\omega).$$

That equation determines the behaviour of the variable y_t as a function of the ratio of the discount factors, the ratio of the beliefs of agents (the likelihood ratio), and an initial condition. Lemma 1 implies that in the case where markets are complete, and so the first welfare theorem holds, an agent's survival prospects are identified by the equation. In the case where beliefs are homogeneous one obtains the result that both agents dominate if and only if $\beta_1 = \beta_2$ while i dominates and $-i$ vanishes if and only if $\beta_i > \beta_{-i}$. This turnpike result for complete market economies is well known (Becker (1980), Rader (1981), and Bewley (1982)). When beliefs are heterogeneous and $\beta_1 = \beta_2$ both agents dominate on a path if and only if $0 < \liminf \frac{dP_{2,t}(\omega)}{dP_{1,t}(\omega)}$ and $\limsup \frac{dP_{2,t}(\omega)}{dP_{1,t}(\omega)} < \infty$, sufficient conditions for which can be found in Phillips and Ploberger (1996) and Sandroni (2000). Notice that the equation can also be written in the form

$$(\beta_2/\beta_1) \cdot \frac{P_{2,t}(\omega)}{P_{1,t}(\omega)} \cdot y_t(\omega) = y_{t-1}(\omega).$$

We now show that, more generally, the behaviour of y_t can be captured succinctly using the ratio of two processes where each is the product of random variables with conditional mean one (taken with respect to the subjectively held belief) in addition to the ratio of the discount factors and an initial condition. That is the content of Proposition 2 (ii); the generalization permits the analysis of the case where markets are incomplete.

Given consumption processes for $i \in \mathcal{I}$, define

$$\hat{r}_{i,t}(\omega) \equiv \frac{r_t(\omega)\cdot u_i'(c_{i,t}(\omega))}{E_{P_i}\left[r_t \cdot u_i'(c_{i,t})|\mathcal{F}_{t-1}\right](\omega)}.$$

Our first result notes that $\hat{r}_{i,t}$ has conditional mean one, is positive, and is uniformly bounded. It also shows that $\prod_{t=1}^T \hat{r}_{i,t}$ almost surely converges since it is a martingale. Define $\bar{\hat{r}}_i \equiv \sup_{t\geq 0}\sup_{\omega\in\Omega} \hat{r}_{i,t}(\omega)$.

PROPOSITION 1: Assume A.1, A.2, A.3 and A.4. Then $E_{P_i}\left[\hat{r}_{i,t}|\mathcal{F}_{t-1}\right](\omega) = 1$, $0 < \hat{r}_{i,t}(\omega)$, and $\bar{\hat{r}}_i < \infty$. Also, there is a random variable R_i^* that is nonnegative and a.s. finite with $E_{P_i}\left[R_i^*\right] \leq 1$ such that $R_i^*(\omega) = \lim_{T\to\infty}\prod_{t=1}^T \hat{r}_{i,t}(\omega)$ P_i-a.s.

The next result encapsulates our methodological innovation.

PROPOSITION 2: Assume A.2, A.3, and A.4, and consider consumption processes c_i that are Euler processes at the price process q. Then

$$\text{(i)} \quad \prod_{t=1}^{T+1} \hat{r}_{i,t}(\omega) = \beta_i^{T+1}\cdot \frac{u_i'(c_{i,1+T}(\omega))}{u_i'(c_{i,0}(\omega))}\cdot \prod_{t=0}^{T}\left(\frac{r_{1+t}(\omega)}{q_t(\omega)}\right),$$

(ii) $\dfrac{\hat{r}_{2,t}(\omega)}{\hat{r}_{1,t}(\omega)} = \dfrac{\beta_2}{\beta_1} \cdot \dfrac{y_t(\omega)}{y_{t-1}(\omega)}$ and $y_T(\omega) = \left(\dfrac{\beta_1}{\beta_2}\right)^T \cdot \dfrac{\prod_{t=1}^T \hat{r}_{2,t}(\omega)}{\prod_{t=1}^T \hat{r}_{1,t}(\omega)} \cdot y_0(\omega)$,

(iii) $y_{t-1}(\omega) = \dfrac{\beta_2}{\beta_1} \cdot E_{P_2}\left[\hat{r}_{1,t} \cdot y_t | \mathcal{F}_{t-1}\right](\omega)$, $\dfrac{1}{y_{t-1}(\omega)} = \dfrac{\beta_1}{\beta_2} \cdot E_{P_1}\left[\hat{r}_{2,t} \cdot \dfrac{1}{y_t} \middle| \mathcal{F}_{t-1}\right](\omega)$.

REMARK 2: Proposition 2 applies even when the allocation is Pareto optimal; in particular, Proposition 2 (ii) implies that in such a case $\frac{\hat{r}_{2,t}(\omega)}{\hat{r}_{1,t}(\omega)} = \frac{P_{1,t}(\omega)}{P_{2,t}(\omega)}$.

Proposition 2 can be used to characterize the behaviour of asset prices, analyse survival and construct equilibria. In the rest of this subsection we discuss each of these issues.

Proposition 2 (i) shows that $\prod_{t=1}^T \hat{r}_{i,t}$ is exactly the discounted marginal value of the "reinvesting" strategy where the entire payoff from the asset is reinvested for T periods. It follows that if $\prod_{t=1}^T \hat{r}_{i,t}(\omega) \to 0 \quad P-$ a.s., then the perpetual "reinvesting" strategy induces a discounted return that converges to zero, $\prod_{t=0}^T \frac{\beta_i \cdot r_{t+1}(\omega)}{q_t(\omega)} \to 0 \quad P-$ a.s.

When beliefs are correct and agents are equally patient, Proposition 2 can be used to show that $\prod_{t=0}^T \frac{\beta_i \cdot r_{t+1}(\omega)}{q_t(\omega)} \to 0 \quad P-$ a.s., thus characterizing the behaviour of asset prices when markets are incomplete for an important class of economies. To see this, suppose that y_t has either zero or infinity or both as limit points. In such a situation, since Proposition 1 shows that $\prod_{t=1}^T \hat{r}_{i,t}$ almost surely converges to a finite value, by Proposition 2 (ii) $\prod_{t=1}^T \hat{r}_{i,t}$ converges to zero for some i. But then, by Proposition 2 (i), $\prod_{t=0}^T \frac{\beta_i \cdot r_{t+1}(\omega)}{q_t(\omega)} \to 0 \quad P-$ a.s. To complete the argument note that our characterisation result, Theorem 1 in Section 4.1, shows that when agents are equally patient and markets are incomplete forever, y_t does have zero and/or infinity as limit points as assumed above. This result on asset prices is stated as Corollary 1 in Section 4.1.

Let us now consider the analysis of survival. With homogeneous beliefs, market incompleteness can be expected to imply that $\frac{\beta_2}{\beta_1} \cdot y_t(\omega) \neq y_{t-1}(\omega)$ so that, by the first equation in Proposition 2 (ii), $\frac{\hat{r}_{2,t}}{\hat{r}_{1,t}} \neq 1$. Since $\hat{r}_{i,t}$ has conditional mean one, $\frac{\hat{r}_{2,t}}{\hat{r}_{1,t}}$ fluctuates around the value one and $\frac{\prod_{t=1}^T \hat{r}_{2,t}}{\prod_{t=1}^T \hat{r}_{1,t}}$ can be expected to fluctuate so much that it might even have either zero or infinity or both as limit points. By Proposition 2 (ii) fluctuations in $\frac{\prod_{t=1}^T \hat{r}_{2,t}}{\prod_{t=1}^T \hat{r}_{1,t}}$ are equivalent to fluctuations in $\frac{y_t}{y_0}$, allowing the use of Lemma 1 to make inferences about survival. For the formal analysis characterising the dynamics when $\beta_1 = \beta_2$, it turns out to be easier to use a different approach which includes the case of heterogeneous beliefs—Theorem 1 shows that market incompleteness implies that $\frac{P_{2,t}}{P_{1,t}} \cdot \frac{y_t}{y_{t-1}}$ fluctuates forever by at least a little and that in turn implies that $\frac{dP_{2,t}}{dP_{1,t}} \cdot \frac{y_t}{y_0}$ fluctuates a lot. Since Theorem 1 applies to those paths on which the ratio $\frac{dP_{2,t}}{dP_{1,t}}$ stays away from zero and infinity, it shows that market incompleteness implies that $\frac{y_t}{y_0}$, equivalently $\frac{\prod_{t=1}^T \hat{r}_{2,t}}{\prod_{t=1}^T \hat{r}_{1,t}}$, fluctuates a lot.

An alternative approach to survival, an approach that is better suited to the construction of equilibria with prespecified asymptotic behaviour, is based on studying the limit behaviour of each of the product processes separately, providing conditions so that the individual results can be combined, and then using Proposition 2 (ii). By Proposition 1, each of these processes converges; by Jensen's inequality, if there is enough variability in the tail of the process then the limit must be zero. Indeed

$$\frac{1}{T}\sum_{t=1}^{T}\log\hat{r}_{i,t}(\omega)\to\frac{1}{T}\sum_{t=1}^{T}E_{P_i}[\log\hat{r}_{i,t}|\mathcal{F}_{t-1}](\omega)<\frac{1}{T}\sum_{t=1}^{T}\log E_{P_i}[\hat{r}_{i,t}|\mathcal{F}_{t-1}](\omega)=0\quad P_i-\text{a.s.}$$

where the first result, with a.s. convergence, follows from a suitable Strong Law of Large Numbers, the second uses Jensen's inequality (which guarantees a weak inequality in the limit), and the third uses the defining property $E_{P_i}[\hat{r}_{i,t}|\mathcal{F}_{t-1}](\omega) = 1$. Proposition 5 in Section 6.1 provides conditions that ensure that the inequality is strict so that $\prod_{t=1}^{T}\hat{r}_{i,t}(\omega)\ \to\ 0\quad P_i-$a.s. Yet, determining the $P-$a.s. behaviour of $y_t(\omega)$ is tricky (Proposition 5 is limited to a statement made using the subjective belief P_i); we now identify a class of paths on which, nonetheless, an answer can be given. If the rate of convergence to zero of the two processes is comparable, then one would expect $\left(\frac{\beta_2}{\beta_1}\right)^t\cdot\frac{y_t}{y_0}$ to fluctuate a lot unless the processes are very finely tuned, e.g. the case of a Pareto optimal allocation when agents have homogeneous beliefs since in that case the two processes are actually colinear and so, as pointed out in Remark 2, their ratio is always equal to one. When beliefs coincide with the objective probability and one process converges to zero at a smaller rate than the other, or it converges to a positive number, then $\left(\frac{\beta_2}{\beta_1}\right)^t\cdot\frac{y_t}{y_0}$ must converge either to zero or infinity implying that when discount rates are homogeneous then only one agent survives. One particular case where this might happen is when $\hat{r}_2$ is a degenerate process. Indeed, in such a case if $\beta_1=\beta_2$ then to show that 2 vanishes a.s. it suffices to show that $\prod_{t=1}^{T}\hat{r}_{1,t}$ converges to zero. An example of this possibility is provided in Section 3.2. In Section 6 we provide a general constructive approach to equilibria in which the behaviour of consumption is prespecified so that an agent vanishes. We believe that this methodology, in which one specifies processes for $\hat{r}_{i,t}$ and then identifies those processes that are compatible with all the equilibrium restrictions, can also be used to construct equilibria in which no agent vanishes.

3.2 A LEADING EXAMPLE

We turn to our example which has five salient features. (i) $u_1(x) = (1/(1-a))x^{1-a}$ with $a > 0$ and $a \neq 1$, and $u_2(x) = \log x$. (ii) $z_{2,0}(\omega) = Z_0(\omega)$ and $z_{2,t}(\omega) = 0$ otherwise. (iii) The uncertainty in the model comes from 1's endowment which follows an i.i.d. process with two points in its support: $Z_t \in \{\underline{z}, \bar{z}\}$ with probability $p \in (0,1)$ and $(1-p)$ respectively. (iv) The asset payoff is perfectly correlated with the aggregate endowment, $r_t(\omega) = Z_t(\omega)$. (v) The beliefs of each agent are $(p_i,(1-p_i))$ with $p_i \in (0,1)$ and both could hold incorrect beliefs.

It is known that 2's optimal decision rule is

$c_{2,t}(\omega) = (1-\beta_2)\cdot w_{2,t}(\omega)$ and $\theta_{2,t}(\omega) = \beta_2\cdot[w_{2,t}(\omega)/q_t(\omega)]$, where $w_{2,t}(\omega) = r_t(\omega) \cdot \theta_{2,t-1}(\omega) = Z_t(\omega) \cdot \theta_{2,t-1}(\omega)$, which is independent of p_2. We have

$$\hat{r}_{2,t}(\omega) = \frac{r_t(\omega) \cdot \Big(c_{2,t}(\omega)\Big)^{-1}}{E_{P_2}[r_t \cdot (c_{2,t})^{-1}|\mathcal{F}_{t-1}]} = \frac{\Big((1-\beta_2)\cdot\theta_{2,t-1}(\omega)\Big)^{-1}}{E_{P_2}\Big[\Big((1-\beta_2)\cdot\theta_{2,t-1}(\omega)\Big)^{-1}\Big|\mathcal{F}_{t-1}\Big]} = 1,$$

the key point being that $r_t(\omega) \cdot u_2'(c_{2,t}(\omega))$ is an $\mathcal{F}_{t-1}-$measurable quantity.

As for 1, when agent 2 optimizes and the allocation is feasible, we must have $c_{1,t}(\omega) = Z_t(\omega) - c_{2,t}(\omega) = Z_t(\omega) - (1-\beta_2)\cdot w_{2,t}(\omega) = Z_t(\omega)[1-(1-\beta_2)\cdot\theta_{2,t-1}(\omega)]$. It follows that

$$\hat{r}_{1,t}(\omega) = \frac{r_t(\omega)\cdot\Big(c_{1,t}(\omega)\Big)^{-a}}{E_{P_1}[r_t\cdot(c_{1,t})^{-a}|\mathcal{F}_{t-1}]} = \frac{Z_t(\omega)\cdot\Big(Z_t(\omega)[(1-\beta_2)\cdot\theta_{2,t-1}(\omega)]\Big)^{-a}}{E_{P_1}\Big[Z_t\cdot\Big(Z_t[(1-\beta_2)\cdot\theta_{2,t-1}]\Big)^{-a}\Big|\mathcal{F}_{t-1}\Big]} = \frac{[Z_t(\omega)]^{1-a}}{E_{P_1}[Z_t^{1-a}]}.$$

The first order conditions for 1 and 2 are

$$q_{t-1}(\omega) = \beta_1\,\frac{E_{P_1}\Big[r_t\cdot(c_{1,t})^{-a}\Big|\mathcal{F}_{t-1}\Big](\omega)}{\Big(c_{1,t-1}(\omega)\Big)^{-a}} \qquad q_{t-1}(\omega) = \beta_2\cdot\frac{E_{P_2}\Big[r_t\cdot(c_{2,t})^{-1}\Big|\mathcal{F}_{t-1}\Big](\omega)}{\Big(c_{2,t-1}(\omega)\Big)^{-1}}$$

which can be simplified using the fact that $r_t(\omega) = Z_t(\omega)$ and the fact that $r_t(\omega)\cdot u_2'(c_{2,t}(\omega))$ is an $\mathcal{F}_{t-1}-$measurable quantity, and then equated to obtain

$$\beta_1\,\frac{E_{P_1}\Big[Z_t\cdot(c_{1,t})^{-a}\Big|\mathcal{F}_{t-1}\Big](\omega)}{\Big(c_{1,t-1}(\omega)\Big)^{-a}} = \beta_2\cdot\frac{Z_t(\omega)\cdot(c_{2,t}(\omega))^{-1}}{(c_{2,t-1}(\omega))^{-1}},$$

which, using the definition of $\hat{r}_{1,t}$, may be rewritten as

$$\frac{(c_{2,t-1}(\omega))^{-1}}{(c_{2,t}(\omega))^{-1}} = \frac{\beta_2}{\beta_1}\cdot\hat{r}_{1,t}(\omega)\cdot\frac{(c_{1,t-1}(\omega))^{-a}}{(c_{1,t}(\omega))^{-a}} \quad\Longleftrightarrow\quad \frac{(c_{1,t}(\omega))^{a}}{c_{2,t}(\omega)} = \frac{\beta_1}{\beta_2}\cdot\frac{1}{\hat{r}_{1,t}(\omega)}\cdot\frac{(c_{1,t-1}(\omega))^{a}}{c_{2,t-1}(\omega)}.$$

We have obtained the equation in Proposition 2 (ii). $c_{1,t}$ and $c_{2,t}$ must satisfy the equation whenever an allocation is feasible, is maximizing for 2, and satisfies the first order conditions for 1; therefore, the equation must hold in every equilibrium.[11]

By iterating we see that

$$\frac{(c_{1,T}(\omega))^{a}}{c_{2,T}(\omega)} = \Big(\frac{\beta_1}{\beta_2}\Big)^T\cdot\frac{1}{\prod_{t=1}^T\hat{r}_{1,t}(\omega)}\cdot\frac{(c_{1,0}(\omega))^{a}}{c_{2,0}(\omega)}.$$

We now show that $\prod_{t=1}^T\hat{r}_{1,t}(\omega) \to 0 \quad P-$a.s. for p_1 sufficiently close to p, for which it suffices to show that $\text{Lim}_{T\to\infty}\frac{1}{T}\sum_{t=1}^T \log\hat{r}_{1,t}(\omega) < 0 \quad P-$a.s. Since $\hat{r}_{1,t}(\omega) = [Z_t(\omega)]^{1-a}/E_{P_1}[Z_t^{1-a}]$, it suffices to show that

$$\Big(\lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^T\log\,[Z_t(\omega)]^{1-a}\Big) - \log\Big(E_{P_1}[Z_t^{1-a}]\Big) < 0 \quad P-\text{a.s.}$$

[11] Any feasible values of $c_{1,t-1}(\omega)$ and $c_{2,t-1}(\omega)$ induce a positive value for $y_{t-1}(\omega)$. Since $\hat{r}_{1,t}(\omega) = \frac{[Z_t(\omega)]^{1-a}}{E_{P_1}[Z_t^{1-a}]}$, the equation above induces a positive value for $y_t(\omega)$ which in turn induces feasible values of $c_{1,t}(\omega)$ and $c_{2,t}(\omega)$. This shows that the equation always has a real valued solution.

For $a > 1$, existence of an IDC equilibrium follows from our Theorem 4.

Since Z_t is an i.i.d. process, by the Strong Law of Large Numbers

$$\frac{1}{T}\sum_{t=1}^{T}\log\,[Z_t(\omega)]^{1-a} \to E_P[\log Z_t^{1-a}] \quad P-\text{a.s.},$$

and so, by Jensen's inequality and the fact that var $\left[\frac{[Z_t]^{1-a}}{E_P[Z_t^{1-a}]}\right] > 0$, there exists $\epsilon > 0$ such that

$$\left(\lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}\log\,[Z_t(\omega)]^{1-a}\right) - \log\Big(E_P[Z_t^{1-a}]\Big) < -\epsilon \quad P-\text{a.s.},$$

where, by continuity, the result is also true when $\log\Big(E_P[Z_t^{1-a}]\Big)$ is replaced by $\log\Big(E_{P_1}[Z_t^{1-a}]\Big)$ for p_1 sufficiently close to p.

Under A.3 and feasibility, $(c_{1,T}(\omega))^a$ is bounded above. Also, as shown above, for p_1 sufficiently close to p, $\prod_{t=1}^{T}\hat{r}_{1,t}(\omega) \to 0 \quad P-\text{a.s.}$ So when $p_1 = p$, so that 1's beliefs are correct, and $\beta_1 = \beta_2 = \beta$, so that both the agents are equally impatient, $c_{2,T}(\omega) \to 0 \quad P-\text{a.s.}$, i.e. in *every* equilibrium of the example, agent 2 vanishes with probability one.

As noted above, since the application of Jensen's inequality is strict, agent 2 could have correct beliefs and agent 1 could have incorrect ones in an open set around p and yet 2 vanishes almost surely in every equilibrium. The same reason lets us conclude that even if 1 is marginally more impatient than 2, and allowing for the possibility that 1 also has marginally incorrect beliefs, 2 vanishes almost surely in every equilibrium.

The example shows very clearly that no entropy measure that depends only on the truth, beliefs, and the market structure, can be critical to understanding survival because any properly defined entropy measure must attain its maximum when beliefs are correct.

The phenomenon in the example illustrates the crucial elements highlighted in Section 3.1: market incompleteness forever ensures that $\prod_{t=1}^{T}\hat{r}_{1,t}(\omega) \to 0 \quad P-\text{a.s.}$ and so the "reinvesting" strategy induces a discounted return that converges to zero, and the fact that agent 2's marginal valuation of the asset at date t is $\mathcal{F}_{t-1}-$measurable, i.e. $\hat{r}_{2,t}$ is degenerate, implies that her marginal utility diverges.

REMARK 3: We note the following features of the example. Since $c_{2,t}(\omega) = (1-\beta_2)\cdot w_{2,t}(\omega)$ and $\theta_{2,t}(\omega) = \beta_2\cdot[w_{2,t}(\omega)/q_t(\omega)]$, $q_t(\omega)\cdot\theta_{2,t}(\omega) = \beta_2\cdot w_{2,t}(\omega) = \beta_2\cdot(c_{2,t}(\omega)/(1-\beta_2))$ so debt is uniformly bounded in any equilibrium since consumption is nonnegative and bounded by the uniform upper bound on the aggregate endowment.

Also, since $\hat{r}_{1,t}(\omega) = [Z_t(\omega)]^{1-a}/E_{P_1}[Z_t^{1-a}]$ and, by A.1, var $\left[\frac{[Z_t]^{1-a}}{E_P[Z_t^{1-a}]}\middle|\mathcal{F}_{t-1}\right](\omega) =$ var $\left[\frac{[Z_t]^{1-a}}{E_P[Z_t^{1-a}]}\right] > 0$, we have var $\left[\hat{r}_{1,t}\middle|\mathcal{F}_{t-1}\right](\omega) > 0$. So assumption A.7, introduced in Section 6.1, holds in the example.

The analysis in this section depends heavily on the endowment structure where 2 has no endowment except in period 0. Theorem 4 will show that, in fact, the property we identify is robust to changes in the endowment process, preferences, and asset structure.

3.3 THE GENERAL LESSON

The analysis thus far is indicative of two very interesting phenomena that appear to be driven by the fact that markets are incomplete. They lead to the following conjectures about the implications of market incompleteness: a first conjecture, based on Proposition 2 (ii) together with the possibility that $\frac{\hat{r}_{2,t}}{\hat{r}_{1,t}} \neq 1$ so that $\frac{\prod_{t=1}^{T} \hat{r}_{2,t}}{\prod_{t=1}^{T} \hat{r}_{1,t}}$ fluctuates a lot, is the statement (a) that the consumption of some agent is arbitrarily close to zero infinitely often, and a second conjecture, based on the example in Section 3.2, is the statement (b) that the consumption of some agent stays close to zero eventually. We would like to pin down the extent to which these results are a general property of economies with dynamically incomplete markets. We will say that markets are effectively incomplete forever on a path if the ratio of the one period ahead marginal rates of substitution, $\frac{P_{2,t}}{P_{1,t}} \cdot \frac{y_t}{y_{t-1}}$, displays variability, i.e. $\frac{\hat{r}_{2,t}}{\hat{r}_{1,t}}$ displays variability. Theorem 1 (ii) shows that in an equilibrium of such an economy if some agent's consumption is uniformly positive eventually, the other agent's consumption is zero eventually. This shows that either (a) holds for both agents or (b) holds. Theorem 2 in Section 5 shows that if endowments and asset returns follow finite state Markov processes then (b) can be ruled out. Theorem 4 in Section 6.5 shows that, in a robust class of economies, there are equilibria in which the consumption of an agent stays close to zero eventually on every path. We remark that Theorem 1 holds regardless of whether beliefs are homogeneous or heterogeneous. Also, one expects a version of Theorem 1 to hold in specifications of infinite horizon economies with incomplete markets that are not covered by our analysis so long as the Euler equation holds with equality always; in particular, the asset could be retradable and long lived.

4. RULING OUT DOMINANCE

In this section we prove our first main result: if agents are equally impatient and markets are incomplete, then, on paths on which an agent's consumption is uniformly positive eventually, the other agent's consumption is zero eventually. So, in contrast to the case where markets are complete, both agents cannot consume uniformly positive quantities eventually when market are incomplete. To be able to prove the result, (a) we use an implication of the fact that the Euler equations hold with equality, namely, that if the ratio of one period ahead marginal rates of substitution, $\frac{P_{2,t}}{P_{1,t}} \cdot \frac{y_t}{y_{t-1}}$, displays conditional variability, then it increases with positive conditional probability, and (b) we restrict attention to those paths on which the accumulation points of the ratio $\frac{dP_{2,t}}{dP_{1,t}}$ do not include zero and infinity. The transformation introduced in Section 3.1 leads to Theorem 1 which is stated and discussed in Section 4.1. Theorem 1 also characterizes asset prices when markets are incomplete—this result is stated as Corollary 1. Section 4.2 relates our result to examples of infinite horizon economies with incomplete markets that have appeared in the literature.

4.1 THE RESULT

In this section we study the asymptotic behavior of agents' consumption processes on paths where the ratio of marginal rates of substitution (a) does not display one period ahead conditional variability in the limit, and (b) does display such variability infinitely often, i.e. markets are effectively incomplete forever.

To be more precise, we define

$$V_0 \equiv \Big\{\omega : \lim_t \operatorname{var}\Big[\frac{P_{2,t}}{P_{1,t}} \cdot \frac{y_t}{y_{t-1}} \Big| \mathcal{F}_{t-1}\Big](\omega) = 0\Big\},$$

and, for $\epsilon > 0$, we define

$$V_\epsilon \equiv \Big\{\omega : \limsup_t \operatorname{var}\Big[\frac{P_{2,t}}{P_{1,t}} \cdot \frac{y_t}{y_{t-1}} \Big| \mathcal{F}_{t-1}\Big](\omega) \geq \epsilon\Big\}.$$

Also, for $\omega \in V_\epsilon$, we define

$$\Delta_t^\epsilon(\omega) \equiv \inf\Big\{k \geq 1 : \operatorname{var}\Big[\frac{P_{2,t+k}}{P_{1,t+k}} \frac{y_{t+k}}{y_{t+k-1}} \Big| \mathcal{F}_{t+k-1}\Big](\omega) \geq \epsilon\Big\}.$$

Finally, for $T \in \{1, 2, 3, \cdots\}$, we define

$$V_{T,\epsilon} \equiv \Big\{\omega : \limsup_t \operatorname{var}\Big[\frac{P_{2,t}}{P_{1,t}} \frac{y_t}{y_{t-1}} \Big| \mathcal{F}_{t-1}\Big](\omega) \geq \epsilon \text{ and} \sup_t \Delta_t^\epsilon(\omega) = T\Big\}.$$

$\cup_{\epsilon>0} V_\epsilon = \Omega/V_0$ is the set of paths where the ratio of marginal rates of substitution displays one period ahead variability infinitely often. $\cup_{\epsilon>0} V_\epsilon$ can be further partitioned into two sets, one containing those paths where the ratio of one period ahead marginal rates of substitution displays variability on some bounded interval of time of length $T < \infty$, $\cup_{T,\epsilon>0} V_{T,\epsilon}$, and its complement, the set V_∞, on which, although the ratio of marginal rates of substitution displays one period ahead variability infinitely often, yet, for some subsequence of dates, the maximal length of the time interval until it displays variability again diverges on each path, i.e. $\sup_t \Delta_t^\epsilon(\omega) = +\infty$ where $\Delta_t^\epsilon(\omega)$ is the minimum number of periods it takes for the ratio of marginal rates of substitution to display one period ahead variability after date t.[12] The interest of studying paths in the set V_∞ is not evident and Theorem 1 does not apply to them.[13]

Let us also define

$$L_{\underline{\lambda},\overline{\lambda}} \equiv \Big\{\omega : \underline{\lambda} < \liminf \frac{dP_{j,t}(\omega)}{dP_{i,t}(\omega)} \leq \limsup \frac{dP_{j,t}(\omega)}{dP_{i,t}(\omega)} < \overline{\lambda}\Big\},$$

where $\underline{\lambda} > 0$ and $\overline{\lambda} < \infty$ are positive constants. Paths in the set $\Omega/\cup_{\underline{\lambda}>0,\overline{\lambda}<\infty} L_{\underline{\lambda},\overline{\lambda}}$ have the property that the ratio of the probability assigned to a cylinder by agent 2 to that assigned by agent 1, $\frac{dP_{j,t}(\omega)}{dP_{i,t}(\omega)}$, contains a subsequence that either converges to zero or diverges to infinity. Theorem 1 (ii) does not apply to such paths. As we noted in Section

[12]Clearly, $\Delta_t^\epsilon(\omega)$ is finite for every ϵ, t and $\omega \in V_\epsilon$; however, it may have a divergent subsequence.

[13]Results on the lack of collinearity of marginal utility vectors in generic finite horizon incomplete market economies suggest that the set V_∞ might even be null for generic economies.

2.6, although this does rule out some subjective beliefs, it is a very weak restriction. An important case where the restriction is violated is when the true process is i.i.d. and agents only consider i.i.d. models but their "prior beliefs" (as defined in Section 2.6) have disjoint supports, as in our example in Section 3.2. Our constructive approach in Section 6, extended through a continuity argument (see Remark 9), shows a different route by which limiting consumption behaviour can be studied in such economies.

We can now state our main result.

THEOREM 1: Consider an IDC equilibrium. Assume that $\beta_1 = \beta_2$, that A.1, A.2, A.3, and A.4 hold. Then,

(i) $\lim_t \left(\frac{P_{2,t}(\omega)}{P_{1,t}(\omega)} \cdot \frac{y_t(\omega)}{y_{t-1}(\omega)} \right) = 1$ P-a.s. $\omega \in V_0$.

(ii) For every $T \in \{1, 2, 3, \cdots\}$, $\epsilon > 0$, n, $\underline{\lambda} > 0$ and $\overline{\lambda} < \infty$, $\limsup_t c_{i,t}(\omega) \leq 1/n$ P-a.s. $\omega \in V_{T,\epsilon} \cap L_{\underline{\lambda},\overline{\lambda}} \cap \{\omega : \liminf_t c_{j,t}(\omega) > 1/n\}$.

Theorem 1 (i) shows that when one restricts attention to paths in V_0, marginal rates of substitution are equalized in the limit, $\lim_t \left(\frac{P_{2,t}(\omega)}{P_{1,t}(\omega)} \cdot \frac{y_t(\omega)}{y_{t-1}(\omega)} \right) = 1$.[14] Theorem 1 (ii) shows that when one restricts attention to paths that are in $\cup_{T,\epsilon>0} V_{T,\epsilon}$ as well as in $\cup_{\underline{\lambda}>0,\overline{\lambda}<\infty} L_{\underline{\lambda},\overline{\lambda}}$, if the consumption of some agent is uniformly positive eventually then the consumption of the other agent is zero eventually. Equivalently, if no agent vanishes then every positive lower bound on consumption is violated infinitely often for both agents. Theorem 1 (ii) can be read as showing that when markets are effectively incomplete forever, the only equilibria with asymptotically simple behaviour are the ones in which only one agent consumes in the limit as in the example and in Theorem 4.

The proof of Theorem 1 (i) is a relatively straightforward consequence of the fact that the ratio of marginal rates of substitution is at least one with positive conditional probability and on V_0 its conditional variance converges to zero. We turn to the proof of Theorem 1 (ii). In what follows, $\underline{\lambda}$ and $\overline{\lambda}$ are specified in the statement of Theorem 1 (ii) and are used to isolate the set of paths $L_{\underline{\lambda},\overline{\lambda}}$, and $\underline{y}_n$, $\overline{y}_n$, and $T_n(\gamma)$ are defined in the proof. First, Lemma 4 uses the fact that the Euler equations hold with equality and that markets are incomplete to conclude that whenever $\frac{P_{2,t}}{P_{1,t}} y_t$ displays sufficient variability conditional on the realization of y_{t-1}, captured by $\epsilon > 0$, $\frac{P_{2,t}}{P_{1,t}} \frac{y_t}{y_{t-1}}$ increases by a factor γ with uniformly positive conditional probability. It follows that, because in at most T periods $\frac{P_{2,t}}{P_{1,t}} y_t$ must display sufficient variability, the ratio of marginal rates of substitution must increase by the factor γ with positive conditional probability in any span of T dates. We use this result to show that, with positive conditional probability, starting from a consumption

[14] Since in a Pareto optimal allocation where both the agents have positive wealth (so that marginal rates of substitution are equalized at every date-event) and beliefs are homogeneous and correct, each agent's consumption is bounded away from zero, one is tempted to conclude that Theorem 1 (i) has a similar implication for the behaviour of consumption in such economies. However, this is not obvious. Although we do not have an example, we believe that on V_0 an agent might have consumption that is arbitrarily close to zero infinitely often or even zero eventually as in Theorem 1 (ii).

distribution where agent 1's consumption is bounded away from zero, $c_{1,t_0} > 1/n$, so that $\frac{P_{2,t_0}}{P_{1,t_0}} y_{t_0}$ is also bounded away from zero ($\frac{P_{2,t_0}}{P_{1,t_0}} y_{t_0} \geq \underline{\lambda y}_n$ in formal terms), in a finite number of periods $\frac{P_{2,t}}{P_{1,t}} y_t$ becomes large enough ($\frac{P_{2,t}}{P_{1,t}} y_t \geq \overline{\lambda} \overline{y}_n$) to let us conclude that y_t exceeds $\overline{y}_n$ so that agent 2's consumption falls below a pre-fixed threshold level, $c_{2,t} \leq 1/n$. To clinch the result we need to verify that such a possibility occurs infinitely often. Lemma EBC, which is a version of the Second Borel-Cantelli Lemma that does not require independence and appears in Freedman (1973), lets us prove that in fact on paths on which c_{1,t_0} is above $1/n$ infinitely often, $c_{2,t}$ must fall below $1/n$ infinitely often. It follows that if $c_{2,t}$ exceeds $1/n$ on every subsequence then c_{1,t_0} is below $1/n$ on every subsequence, as stated in Theorem 1 (ii). The difficult part of the proof is in specifying an appropriate sequence of events; we consider the events $\Omega_{1,t}$ wherein, starting from a date t_0 at which 1's consumption is above the threshold, the variable $\frac{P_{2,t}}{P_{1,t}} y_t$ never decreases strictly, increases by the factor γ a fixed number of times τ (where τ is arbitrarily large and so larger than $T_n(\gamma)$ which identifies the number of periods required to make the transition from $\underline{\lambda y}_n$ to $\overline{\lambda} \overline{y}_n$ when $\frac{P_{2,t}}{P_{1,t}} y_t$ grows by the factor γ in every span of T periods) and increases by that amount at the date that indexes the event. The analysis of such events suffices for our purposes.

Theorem 1 (ii) allows us to characterize the behaviour of asset prices when markets are incomplete since $\prod_{t=0}^{\widehat{T}} \frac{\beta_i \cdot r_{t+1}(\omega)}{q_t(\omega)} \to 0 \quad P-$a.s. (the argument was given in Section 3.1). In contrast, when markets are complete, asset prices might behave differently, e.g. when there is no aggregate risk and one considers the price of a riskless real bond, $r_t(\omega) = r$, we must have $\frac{\beta_i \cdot r_{t+1}(\omega)}{q_t(\omega)} = 1$. We state the result as a corollary to the theorem.

COROLLARY 1: Consider an IDC equilibrium. Assume that $P_1 = P_2 = P$, $\beta_1 = \beta_2$, and that A.1, A.2, A.3, and A.4 hold. For every $T < \infty$ and $\epsilon > 0$, asset prices satisfy:

$$\prod_{t=0}^{\widehat{T}} \frac{\beta_i \cdot r_{t+1}(\omega)}{q_t(\omega)} \to 0 \quad P-\text{a.s. } \omega \in V_{T,\epsilon}.$$

REMARK 4: We would like to draw the reader's attention to four facts about Theorem 1 and Corollary 1. (i) The results hold regardless of whether preferences exhibit "prudence" since they are proved under assumptions on preferences summarised in A.3 which do not make any reference to the behaviour of u'''. (ii) Since the results hold whenever y_t has zero or infinity as accumulation points, they also hold in the construction we propose in Section 6 where one of the agents vanishes. (iii) Although Corollary 1 shows that when the asset is a real bond, $r_t(\omega) = 1$ for all $t \geq 0$ and $\omega \in \Omega$, $\prod_{t=0}^{T} \frac{\beta_i}{q_t(\omega)} \to 0 \quad P-$a.s., this clearly does not require that $\frac{\beta_i}{q_t(\omega)} < 1$ for all $t \geq 0$ and $P-$a.s. ω; the latter is generally not true although it does hold in special cases, e.g. with prudence and no aggregate risk as shown by Levine and Zame (2002). (iv) Theorem 1 does not fully exploit the structure of an IDC equilibrium and holds for any consumption processes that are aggregate feasible Euler processes.

Theorem 1 (ii) holds even if the asset return is positive in only two states as that ensures the required variability. With a single Arrow security that pays in state s the only restrictions that the Euler equations impose is that $y_t(\omega) = y_{t+1}(\omega)$ if $s_{t+1}(\omega) = s$; as Remark 5 shows, this implies that the support of the equilibrium consumption process is typically finite in economies where individual endowments depend only on the current state.

REMARK 5: Consider an economy with a single Arrow security that pays in state s. So $\hat{r}_{1,t}(\omega) = \hat{r}_{2,t}(\omega) \neq 0$ if $s_t(\omega) = s$, and $\hat{r}_{i,t}(\omega)$ is not defined otherwise. By Proposition 2 (ii), the only restrictions that the Euler equations impose is that $y_t(\omega) = y_{t+1}(\omega)$ if $s_{t+1}(\omega) = s$ (where we assume that $\beta_1 = \beta_2$). Consider a pair (ω, t) such that $s_t(\omega) = \tilde{s} \in \mathcal{S}/\{s\}$ and $s_{t+\tau}(\omega) = s$ for all $\tau \geq 1$, and let $y_t(\omega) = \bar{y}$. Then, $z_{t+\tau}(\omega) = z^s$ and so $y_{t+\tau}(\omega) = \bar{y}$ implies that $c_{i,t+\tau}(\omega) = c_i(\bar{y})$ and, therefore, $q_{t+\tau}(\omega) = \bar{q}$. One can verify that there is a supporting portfolio $\theta_{i,t+\tau}(\omega)$ that is constant, $\theta_i(\bar{y})$. Since $\tilde{s} \neq s$, $y_t(\omega) = \bar{y}$ and $z^{\tilde{s}}$ determine $c_{i,t}(\omega) = c_i^{\tilde{s}}(\bar{y})$ and hence $q_t(\omega) = q^{\tilde{s}}(\bar{y})$. The portfolio $\theta_{i,t}(\omega)$ must satisfy the budget equations (where, for $\tilde{s} \in \mathcal{S}/\{s\}$ the agents' wealth is their endowment) $c_i^{\tilde{s}}(\bar{y}) + q^{\tilde{s}}(\bar{y}) \cdot \theta_{i,t}(\omega) = z_i^{\tilde{s}}$ and $c_i(\bar{y}) + \bar{q} \cdot \theta_i(\bar{y}) = z_i^s + \theta_{i,t}(\omega)$. For each value of $\tilde{s} \in \mathcal{S}/\{s\}$, these equations have a unique solution $\bar{y}^{\tilde{s}}$ if $q^{\tilde{s}}(\bar{y})$ is either strictly monotone in $\bar{y}$ or is constant, which it must be if all risk is idiosyncratic; more generally, one expects the set of solutions to be finite typically. Evidently, for any (ω, t') either $s_t(\omega) = s$ for all $t \in \{0, 1, \cdots, t'\}$ and $\bar{y}^s$ can be obtained by an analogous argument since $\theta_{i,-1}(\omega) = 0$, or there exists $t \in \{1, 2, \cdots, t'-1\}$ such that $s_t \neq s$ and so $y_{t'}(\omega) = \bar{y}^{s_t}$.

4.2 RELATING TO EARLIER EXAMPLES

Blume and Easley (2006) provide an example that, to the best of our knowledge, is the only one in the literature that addresses the question of survival in economies like ours. Their example has three agents where agents one and two are identical except for the fact that agent one correctly believes that the economy is deterministic while agent two mistakenly believes it to be stochastic. Agent three also has correct beliefs and is the most patient of the three. Their approach is to construct the endowment process of the third agent so as to support prespecified prices for the asset and rules for consumption for all three agents. They show that their example economy has an equilibrium in which agent one, who has correct beliefs, is driven out and the other two agents, one with correct beliefs and the other with incorrect beliefs, survive. Clearly, if also agent two had correct beliefs then all agents would correctly believe that the economy is deterministic, markets would be complete, and the behaviour of consumption would be determined only by the rates of impatience. So their example does not yield any additional insight about the behaviour of consumption under incomplete markets when beliefs are homogeneous and this is in sharp contrast to our leading example; moreover, it is not obvious that their technique permits unequivocal conclusions in more general settings where, for example, saving rates across agents are not unambiguously ordered along a path as in their example.[15]

Krebs (2004a) considers a two agent economy with idiosyncratic risk and homogeneous beliefs, and shows that the range of the equilibrium consumption process cannot be a compact set with a strictly positive lower bound (the possibility that the lower bound is zero is ruled out by his assumption that the Bernoulli utility function is unbounded below); from his analysis one cannot conclude whether zero is or is not approached on a given path. Like us, he considers equilibria in which the Euler equations hold with equality; the only asset that he allows for is a Lucas-tree with uniformly positive dividends. By our result, very generally, not only is zero approached for some agent but infinitely often so.

Coury and Sciubba (2005) provide an example where both agents dominate. They start with a Pareto optimal allocation supportable with incomplete markets and then change beliefs in a manner that leaves demand unchanged. Market incompleteness makes this possible; however, the construction is clearly degenerate. The method they use to construct their example implies that for some labelling of agents, $\lim \frac{dP_{j,t}(\omega)}{dP_{i,t}(\omega)} = 0$ on each path and so $L_{\underline{\lambda},\overline{\lambda}}$ has zero measure for all prespecified $\underline{\lambda} > 0$ and $\infty > \overline{\lambda}$.

Levine and Zame (2002) provide an example in which both agents survive. They use a random selection from a static economy with multiple equilibria to construct a sunspot equilibrium in the infinite horizon economy. The sunspot realization is fixed once and for all at the first date so markets are effectively complete from then onwards.

Kubler and Schmedders (2002) provide examples of economies in which both agents survive. This is possible because they restrict attention to Arrow securities, which violates Assumption A.4, and individual endowments that depend only on the current state.

Constantinides and Duffie (1996) and Krebs (2004b) consider economies like ours but with a dividend paying asset. Since they allow endowments to grow without any upper bound, it is not clear that an analogue of Theorem 1 can be proved in their framework.

Becker and Foias (1987) consider a deterministic Ramsey economy with productive capital which must be held in nonnegative quantities, i.e. "short" sales are not permitted. Agents have heterogeneous discount rates. They show that the borrowing constraint bites in the following sense: if an impatient agent holds capital at every subsequence of dates then her Euler equation holds as an equality and she vanishes and yet she is not optimizing since her labour income remains unspent (labour income is uniformly positive because, eventually, capital is uniformly positive as otherwise its return is too high even for the least patient agent to not hold any of it.) It follows that in any Ramsey equilibrium, all but the most patient agent must hold zero capital at some subsequence of dates. They also show that if capital income is increasing in the amount of capital then in every Ramsey equilibrium, eventually, only the most patient agent holds capital. In their framework, no

[15] Furthermore, completing the market in their example leads to nonexistence, a fact that they note while in our leading example doing so leads to an equilibrium where the allocation is Pareto optimal and, by the result in Blume and Easley (2006), both the agents survive.

These authors present a second example that shows that there are situations in which relative entropy is simply the wrong measure of belief accuracy because it does not match well with the asset structure.

agent's consumption can approach zero since labour income is, eventually, positive and there are no other assets.

Becker and Zilcha (1997) provide an example of a Ramsey economy with production uncertainty and two agents with distinct discount factors in which both agents must hold capital in any stationary equilibrium. This negates the possibility of extending Becker's (1980) result on Ramsey's conjecture in the deterministic framework to a stochastic one.

5. SUFFICIENT CONDITIONS SO THAT NO ONE VANISHES

In this section we present our second main result. We show that in an important class of economies, a subset of those that are often considered in the applied general equilibrium literature, no agent vanishes even though markets are incomplete and so, by Theorem 1 (ii), the dynamics must be complicated in the sense that both agents' consumption approaches zero infinitely often. Theorem 2 reinforces the stark contrast in the behaviour of equilibrium consumption in the two market structures since the class of economies encompasses ones where some agent would vanish were markets complete.

We consider economies where the aggregate endowment follows a finite state time homogeneous Markov process, and both individual endowments and asset returns are functions of the current realization of the aggregate endowment. Agents may have incorrect and heterogeneous beliefs and may have different discount rates. In Theorem 2 we show that, for most endowment distributions, if an agent's endowment is uniformly positive then the set of paths where her consumption converges to zero has measure zero. The intuition is that, when some agent vanishes, the dynamics of debt become rather special in that for each state there is only one level of debt for which the resulting dynamics prevent debt from violating the uniform bound after a sufficiently long string in which that state is always realized. However, when successor states are not identical, debt will remain within this set of stable levels only for very special configurations of endowments.

It is important to be aware that the conclusion in Theorem 2 depends heavily on all three hypotheses: that the equilibrium concept considered imposes a uniform bound on debt although other alternatives have been considered in the literature, that endowment processes are finite state time homogeneous Markov processes, and that individual endowments are uniformly positive. The following three observations provide the reasons for caution: (a) for a robust class of economies that include those where individual endowments follow finite state Markov processes, the construction that we propose in Section 6 leads to equilibria in which debt although bounded is not uniformly bounded across all paths,[16] (b) as we show in Theorem 4 (ii), the constructive approach to no trade equilibria in which an agent vanishes can be extended to endowments obtained through perturbations that are not restricted to have finite support and by doing so one continues to obtain the same equilibrium consumption processes and trade with uniformly bounded debt and endowments uniformly bounded away from zero, and (c) as our exam-

[16]The result appeared in an earlier version of the paper and is available from the authors.

ple showed, individual endowments that are not uniformly positive are compatible with an agent vanishing.

To sketch the formal argument, let us define the set A as $A \equiv \{\omega : c_{i,t}(\omega) \to 0\}$, the set of paths where agent i's consumption converges to zero. Throughout the section we will assume A.5 so that Z_t is a time homogeneous finite state Markov process and the probability measures P, P_1 and P_2 are generated by the transition matrices Π, Π_1 and Π_2, respectively. Let us define

$$q_j(z) \equiv \beta_j \cdot E_{P_j}\left(\frac{u_j'(Z_{t+1}) \cdot r(Z_{t+1})}{u_j'(z)}\middle| Z_t = z\right),$$

where, for notational convenience, we condition directly on the realization of Z_t instead of on the $\sigma-$algebra generated by realizations of Z_t. q_j is the asset price process that would prevail in an economy in which only j consumes; q_j and q_i will be different unless agents have identical beliefs, discount rates and Bernoulli utilities.

Proposition 3 shows that for almost every path in A, the asset price converges to the asset price of an otherwise identical economy where agent j consumes all the endowment.

PROPOSITION 3: Consider an IDC equilibrium. Assume A.1, A.2, A.3, A.4 and A.5 hold. Then, $q_t(\omega) \to q_j(Z_t(\omega))$ $P-$a.s. $\omega \in A$.

Next, Proposition 4 uses the assumption that i's endowment is uniformly positive, where the lower bound is denoted $\underline{z}_i$, to show that on almost every path in A, the value of i's debt must be negative infinitely often.

PROPOSITION 4: Consider an IDC equilibrium. Assume A.1, A.2, A.3, A.4 and A.5 hold and $\underline{z}_i > 0$. Let $B_{i,t}(\omega) \equiv q_t(\omega) \cdot \theta_{i,t}(\omega)$. Then $\liminf B_{i,t}(\omega) < 0 \quad P-$ a.s. $\omega \in A$.

The result follows because, in an IDC equilibrium, the value of debt at date t equals the sum of (i) the present discounted value of the sums of the excess demand up to date t on each path, which must be negative for t large enough since i's excess demand must be negative eventually on paths where his consumption is zero eventually since his endowment is uniformly bounded away from zero, and (ii) the present discounted value of future debt, which is arbitrarily close to zero infinitely often since debt is uniformly bounded and the compounded rate of return is unbounded above. The fact that the compounded rate of return is unbounded follows from a no arbitrage argument in an economy where only j consumes, as reflected in the fact that the state price process for j converges uniformly to zero, and is proved separately as Lemma 5 and Lemma 6 in the Appendix.

Theorem 2 provides a set of equations that must hold if, on the set of paths in A, debt is to remain bounded. These equations capture the requirement that the value of i's debt move among points in the collection defined by the values $B(z, z)$ as z varies. One expects those equations to hold only for a very special configuration of individual endowments.

THEOREM 2: Consider an IDC equilibrium. Assume A.1, A.2, A.3, A.4 and A.5 hold, and that $\underline{z}_i > 0$. Suppose that for every $z' \in \mathcal{S}$ there exists $z \in \mathcal{S}$ such that

$$\frac{r(z)}{q_j(z')} \cdot B[z', z'] + z_i(z) \neq B(z, z) \qquad \text{where} \qquad B(z, z') \equiv -\frac{z_i(z')}{\frac{r(z')}{q_j(z)} - 1}.$$

Then, the set A has $P-$measure zero.

The argument to prove Theorem 2 is based on the result in Proposition 3, whereby it suffices to look directly at the limit where only j consumes. It also depends on the assumption that i's individual endowment is uniformly positive. The relation between the true dynamics and the dynamics in the limit economy can be seen in the fact that the values $B(z, z)$, defined below, which correspond to the limit economy, are actually the only accumulation points of debt in the true dynamics. That is the content of Lemma 8.

Lemma 7, stated and proved in the Appendix, shows that there is a $\overline{B} < 0$ such that, on almost every path in A, if the value of i's debt exceeds $\overline{B}$ then it becomes nonnegative eventually and remains nonnegative. Since that contradicts Proposition 4, Lemma 7 concludes that on almost every path in A, i's debt cannot exceed $\overline{B}$. With a similar argument Lemma 7 also shows that, necessarily, on almost every path in A the asset's rate of return after two dates in which the same state is realized, $r(z)/q(z)$, must be sufficiently larger than one—otherwise, after a sufficiently long string in which the same state is always realized, i will have fully repaid her debt which would then become positive and remain so, once again contradicting Proposition 4.

Lemma 8, also stated and proved in the Appendix, draws out the following strong implication of Lemma 7: given a state, except for an initial condition that is a single point, $B(z, z) < 0$ where $B(z, z') \equiv -z_i(z') / \left(\frac{r(z')}{q_j(z)} - 1\right)$, if the given state is realized repeatedly then either debt becomes positive or it violates the uniform lower bound. This is because, given that $r(z)/q(z) > 1$, either the outstanding debt exceeds $B(z, z)$, in which case it would be repaid in the event that a sufficiently long finite string of repetitions of the state z were to occur, or the outstanding debt is lower than $B(z, z)$, in which case, since the agent's endowment is bounded, i would not be able to repay the interest and therefore the debt would explode towards $-\infty$.

REMARK 6: Theorem 2 is proved under a condition on beliefs that is equivalent to requiring that the agent places point mass on one matrix in a set of transition matrices. It can be shown that the result holds more generally when the one period ahead conditional probabilities induced by P_j, where i vanishes, converge to an $S \times S$ transition matrix $\tilde{\Pi}_j$.

Theorem 2 reinforces the contrast between complete and incomplete markets. Consider the case where agent i is less patient or has inaccurate beliefs while agent j's beliefs are accurate. In such a situation, Pareto optimality dictates that i should vanish in every complete market equilibrium. This happens even though i has an endowment that is uniformly positive. Yet, by Theorem 2, when markets are incomplete, i cannot vanish.

6. EQUILIBRIA WHERE SOMEONE VANISHES

In this section we turn to our third main result. We will show that the property that the example displays, namely, that some agent vanishes with probability one, is a robust implication of market incompleteness. We do so by combining the following two results: (i) for equilibria where $\hat{r}_2$ is a degenerate process, agent 2 vanishes almost surely, and (ii) there exist endowment distributions for which one can construct equilibrium consumption processes with the property that $\hat{r}_2$ is degenerate as in the example.

Section 6.1 develops the first result which uses the Strong Law of Large Numbers for uncorrelated random variables with uniformly bounded second moments. Section 6.2 shows that it is possible to construct aggregate feasible consumption processes that satisfy the Euler equations, that have summable supporting prices, and that induce a degenerate process $\hat{r}_2$. Theorem 3 in Section 6.3 provides conditions that let us identify IDC equilibria. Finally, in Section 6.4 we provide our result. In Theorem 4 we show that for a continuum of endowment distributions, the processes that we construct are the equilibrium consumption processes in a no trade IDC equilibrium; we also provide a perturbation that leads to uniformly positive endowments but that is path dependent and yet supports the no-trade consumption process obtained earlier but now as an IDC equilibrium with asset trade.[17] Corollary 2 summarises our findings by showing that in an economy with a real bond in which the aggregate endowment has uniformly positive conditional variance, there are IDC equilibria with uniformly positive endowments in which agent 2 vanishes almost surely.

For the main results in this section we shall assume that beliefs are correct so $P_1 = P_2 = P$; when some result holds more generally, we make the more general statement.

6.1 THE STRONG LAW ARGUMENT

As discussed in Section 3.1, for the analysis of survival it helps to know the behaviour of the process $\prod_{t=1}^{T} \hat{r}_{i,t}$. By Proposition 1 we know that $\lim_{T\to\infty} \prod_{t=1}^{T} \hat{r}_{i,t}(\omega)$ exists P_i-a.s. We would like to provide a condition that guarantees that $\lim_{T\to\infty} \prod_{t=1}^{T} \hat{r}_{i,t}(\omega) = 0$ P_i-a.s. Assumption A.7 is the appropriate condition—it imposes the requirement that there is variability in the tail of the process $\{E_{P_i}[\log \hat{r}_{i,t}|\mathcal{F}_{t-1}](\omega)\}$.

It is important to note that A.7 is used only in Proposition 5; Corollary 2 provides conditions on fundamentals that ensure that A.7 holds.

ASSUMPTION A.7: $P\{\omega : \text{limsup}\, \frac{1}{T} \sum_{t=1}^{T} E_{P_i}[\log \hat{r}_{i,t}|\mathcal{F}_{t-1}](\omega) < 0\} = 1.$

[17] It is possible to show that around each no-trade equilibrium there is an open set of endowment distributions that leads to an equilibrium that is weaker in that there may be no uniform bound across paths on debt. This equilibrium concept requires maximization subject to a sequence of budget constraints and a single transversality condition at date zero, and market clearing. It can be shown that such an equilibrium concept does not permit Ponzi schemes. Santos and Woodford (1997) propose a notion of equilibrium without uniform bounds for a much more general set-up. Blume and Easley (2006) provide an example in which the equilibrium value of an agent's debt diverges according to the agent's subjectively held belief.

When $\hat{r}_j$, $j \neq i$, is a degenerate process, A.7 amounts to the requirement that on almost all paths, markets never become effectively complete so that complete risk sharing remains impossible. Jensen's inequality and $E_{P_i}[\hat{r}_{i,t}|\mathcal{F}_{t-1}](\omega) = 1$ lead to the weaker property where the set that appears in A.7 is defined with a weak inequality.

A.7 holds if the time average is uniformly below zero, a strong sufficient condition. As we noted in Remark 2, A.7 holds in our leading example for $i = 1$. With A.7 we are able to obtain the desired result by applying the Strong Law of Large Numbers for uncorrelated random variables with uniformly bounded second moments. Define the set $\mathcal{A}_i \equiv \{\omega \in \Omega : \text{liminf}\, \hat{r}_{i,t}(\omega) = 0\}$. We have

PROPOSITION 5: Assume A.1, A.2, A.3, A.4 and A.7. Then $\prod_{t=1}^{T} \hat{r}_{i,t}(\omega) \to 0$ $P_i -$ a.s. $\omega \in \Omega/\mathcal{A}_i$. Furthermore, given β_{-i} and $\epsilon > 0$, there exists $\delta \in (0,1)$ such that

$$\beta_i \in (\delta \cdot \beta_{-i}, \beta_{-i}) \quad \Rightarrow \quad P_i\left(\left\{\omega : \log(\beta_{-i}/\beta_i) + \lim_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} \log \hat{r}_{i,t}(\omega) < 0\right\}\right) = P_i(\Omega/\mathcal{A}_i) - \epsilon.$$

REMARK 7: In the case where A.7 is strengthened to require

$$P\left\{\omega : \text{limsup}\, \frac{1}{T}\sum_{t=1}^{T} E_{P_i}[\log \hat{r}_{i,t}|\mathcal{F}_{t-1}](\omega) \leq \epsilon < 0\right\} = 1,$$

the statement in the second part of Proposition 5 can be strengthened to:
given β_{-i}, there exists $\delta \in (0,1)$ such that

$$\beta_i \in (\delta \cdot \beta_{-i}, \beta_{-i}) \quad \Rightarrow \quad P_i\left(\left\{\omega : \log(\beta_{-i}/\beta_i) + \lim_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} \log \hat{r}_{i,t}(\omega) < 0\right\}\right) = P_i(\Omega/\mathcal{A}_i).$$

The second part of Proposition 5 can be used to show that, at the margin, the turnpike property fails when markets are incomplete since the less patient agent can survive as in our leading example. However, we shall not proceed further with a formal analysis of that case since we shall assume that $\beta_2 \leq \beta_1$ as that lets us show that, when $P_1 = P_2 = P$, $P(\mathcal{A}_1) = 0$.

6.2 A CONSTRUCTIVE APPROACH TO EQUILIBRIUM

In this section we propose a methodology for constructing feasible consumption processes that satisfy $\hat{r}_{2,t}(\omega) = 1$ for every $t \geq 0$ and all $\omega \in \Omega$ in addition to satisfying the Euler equations and having summable supporting prices.

Consider the function

$$f_{t-1,\omega,y}(\lambda) \equiv (\beta_1/\beta_2) \cdot E_P\left[r_t \cdot u_1'\left(Z_t - (u_2')^{-1}\left(\frac{y \cdot \lambda}{r_t}\right)\right)\middle|\mathcal{F}_{t-1}\right](\omega)$$

in the variable λ. Lemma 11 shows that, for each value of the parameters $(t-1, \omega, y)$, the function has a unique interior fixed point. For $\lambda = r_t \cdot u_2'(c_{2,t})/y_{t-1}$ the fixed point condition is

$$\frac{r_t(\omega) \cdot u_2'(c_{2,t}(\omega))}{y_{t-1}(\omega)} = (\beta_1/\beta_2) \cdot E_P[r_t \cdot u_1'(c_{1,t})|\mathcal{F}_{t-1}](\omega).$$

It is obvious that $r_t(\omega) \cdot u_2'(c_{2,t}(\omega))$ is $\mathcal{F}_{t-1}-$measurable, so that $\hat{r}_{2,t}(\omega) = 1$ for all $\omega \in \Omega$, and that the Euler equation holds for both agents. This is done in Lemma 12. A recursive argument lets us construct consumption processes. Proposition 6 summarises the basic properties of our construction.

PROPOSITION 6: Assume A.2, A.3, and A.4, and that $P_1 = P_2 = P$. For Z an aggregate endowment process, consider $c \in (0, Z_0(\omega))$. Then there exists a unique pair of feasible consumption processes, denoted $\{C_{i,t}(\omega)\}_{t\geq 0}$,[18] defined for all $\omega \in \Omega$ and with $C_{1,0}(\omega) = c$ such that the following statements are true for $t \geq 1$ and all $\omega \in \Omega$:
(i) $\hat{r}_{2,t}(\omega) = 1$;
(ii) $y_{t-1}(\omega) = (\beta_2/\beta_1) \cdot \hat{r}_{1,t}(\omega) \cdot y_t(\omega)$.

REMARK 8: It is possible to show that the construction satisfies the following additional properties: it is monotone increasing and continuous in the initial condition, and it has nice boundary behaviour with respect to the initial condition. These properties follow since further analysis of the fixed point map allows us to show that if $y_{t-1}(\omega) > y'_{t-1}(\omega)$ then the induced values satisfy $y_t(\omega) > y'_t(\omega)$, and that $y_t(\omega)$ is a continuous function of $y_{t-1}(\omega)$. These additional properties are useful in constructing consumption processes that are equilibria when the notion of equilibrium is one other than IDC.

To apply Proposition 5 to conclude that in our solution agent 2 vanishes a.s. we need to show that $P(\mathcal{A}_1) = 0$ where $\mathcal{A}_i \equiv \{\omega \in \Omega : \text{liminf}\, \hat{r}_{i,t}(\omega) = 0\}$. This is done by showing that since the induced process y does not have zero as a limit point, neither does C_1 have zero as a limit point which implies that zero cannot be a limit point of $\hat{r}_1$.

PROPOSITION 7: Assume A.2, A.3, and A.4, and that $\beta_2 \leq \beta_1$ and $P_1 = P_2 = P$. Then, in the proposed solution $P(\mathcal{A}_1) = 0$.

By combining Propositions 5 and 7 we can conclude that, when $\beta_2 \leq \beta_1$ and $P = P_1 = P_2$, $\sum_{t=0}^{T} \log \hat{r}_{1,t}(\omega) \to -\infty$.

We need to verify that the personalized prices (marginal utility valuations) that support the proposed allocation are summable. To do so we show that the one period undiscounted intertemporal rate of substitution for agent 2 is uniformly bounded by M, the number specified in A.6. Imposing A.6, $\beta_2 M < 1$, completes the proof. We also show that, under similar conditions, the consumption paths that we construct produce finite lifetime discounted utility when $u(c) = \frac{c^{1-a}}{1-a}$, $a > 1$, or $u(c) = \log c$ as in our example.

PROPOSITION 8: Assume A.2, A.3, A.4, and A.6, and $P_1 = P_2 = P$. Then
(i) $0 \leq \lim_{T\to\infty} \sum_{t=0}^{T} \beta_i^t \cdot E_P \left[\frac{u_i'(C_{i,t})}{u_i'(C_{i,0})}\right] \leq 1/(1-\beta_2 \cdot M)$ and $\lim_{T\to\infty} \beta_i^T \cdot E_P \left[\frac{u_i'(C_{i,T})}{u_i'(C_{i,0})}\right] = 0$,
(ii) $\lim_{T\to\infty} \sum_{t=0}^{T} \beta_i^t E_P \left[u_i\left(C_{i,t}\right)\right] > -\infty$ if $u_i(c) \equiv \log c$ or $u_i(c) \equiv \frac{c^{1-a}}{1-a}$, where $a > 1$.

[18] Although the construction is parameterised by the initial value c, we do not make that dependence explicit in the notation.

6.3 IDENTIFYING EQUILIBRIA

We turn to a result that lets us identify allocations as IDC equilibria. We provide two sets of sufficient conditions which we shall refer to as Theorem 3A and Theorem 3B; in order to avoid repetition a composite result is presented, Theorem 3, within which a distinction is drawn between the assumptions that change across the two sets of conditions.

With a single good and a single asset, given q and z_i, each c_i determines one and only one θ_i that satisfies every spot market budget constraint. Formally, given c_i, θ_i *is a supporting portfolio* process at the prices q and endowment process z_i if

(i) $\theta_{i,t} \in \Psi^t \quad \forall\, t \geq 0$ and
(ii) $\forall\, t \geq 0,\ c_{i,t}(\omega) + q_t(\omega) \cdot \theta_{i,t}(\omega) = z_{i,t}(\omega) + r_t(\omega) \cdot \theta_{i,t-1}(\omega)$ for all $\omega \in \Omega$.

THEOREM 3:[19] Assume A.2, A.3 and that beliefs are correct, $P_1 = P_2 = P$. Consider consumption processes $\widehat{c}_i$, $i \in \mathcal{I}$, and a price process $\widehat{q}$ such that (i) $(\widehat{c}_1, \widehat{c}_2)$ are feasible, and, for each $i \in \mathcal{I}$, (ii) $\lim_{T\to+\infty} \sum_{t=0}^{T} \beta_i^t\, E_P[u_i(\widehat{c}_{i,t})] > -\infty$, (iii) $\widehat{c}_i$ is supported by the portfolio $\widehat{\theta}_i$ at $(\widehat{q}, \widehat{z}_i)$, and (iv) $\widehat{c}_i$ is an Euler process at prices $\widehat{q}$. Then $(\widehat{c}_1, \widehat{c}_2, \widehat{\theta}_1, \widehat{\theta}_2, \widehat{q})$ constitute an IDC equilibrium if either of the following conditions also hold:

(v A) $\lim_{T\to+\infty} \beta_i^T\, E_P\Big[u_i'(\widehat{c}_{i,T})\Big] = 0$ and (vi A) $\widehat{q} \cdot \widehat{\theta}_i \in \Psi$,

or

(v B) $\lim_{T\to\infty} \sum_{t=0}^{T} \beta_i^t \cdot E_P\Big[u_i'(\widehat{c}_{i,t})\Big] < \infty$ and
(vi B) $\lim_{T\to+\infty} \beta_i^T \cdot E_P\Big[u_i'(\widehat{c}_{i,T}) \cdot \widehat{q}_T \cdot \widehat{\theta}_{i,T} | \mathcal{F}_t\Big](\omega) = 0$ for all $t \geq 0$ and for all $\omega \in \Omega$.

Theorem 3A is proved by first showing that (ii) and the first order conditions, (iv), together with an appropriate transversality condition are sufficient to identify a maximiser, Lemma 19. It is then shown that conditions (v A) and (vi A) together imply that the required transversality condition holds. By (i) and (iii) the consumption processes are budget feasible. (v A) is implied by the fact that marginal valuations are summable.

Theorem 3B is proved by showing that (ii) and the first order conditions, (iv), together with summability, (v B), and a transversality condition at date 0, (vi B) at date 0, are sufficient to identify a maximiser on an appropriate budget set, Lemma 20, 21, and 22. That and (i) and (iii) allow us to conclude that we have an equilibrium with a transversality condition at each node, as defined in Magill and Quinzii (1994). The proof is concluded by invoking Theorem 5.2 in Magill and Quinzii (1994) to show that any such equilibrium is also an IDC equilibrium.

Condition (ii) in Theorem 3 is implied by strengthening A.3 to include the condition that u_i is bounded below. There are cases in which condition (ii) can verified and our results hold even when u_i is unbounded below—Proposition 8 provided some examples.

6.4 THE RESULT

We turn to our third main result which restricts attention to the case where both agents have correct beliefs and shows that the phenomenon exhibited in the leading example and

[19]Theorem 3 can be generalized to the case with nonhomogeneous beliefs.

identified in Theorem 1 (ii), wherein an agent vanishes almost surely, occurs robustly.

Theorem 4 invokes Theorem 3 to conclude that quite generally an economy has an IDC equilibrium with consumption specified as in our construction. The conditions in Theorem 4 (i) include the special case where agent 2 has a logarithmic Bernoulli function and an endowment at only date 0. The element that is new in Theorem 4 (i) is a proof of the fact that under the conditions specified, a transversality condition can be shown to hold at every date and event, Lemma 23 and Lemma 24. Theorem 4 (ii) shows that there are a continuum of endowment distributions that are no trade IDC equilibria with consumption specified as in our construction (footnote 18 noted that the construction is parameterised by an initial value). Theorem 4 (iii) provides a mild condition (which must hold if 2 vanishes) under which each endowment distribution identified in Theorem 4 (ii) can be perturbed to produce uniformly positive endowments while maintaining consumption at (C_1, C_2) to induce an IDC equilibrium with trade.

THEOREM 4: Assume A.1-4, A.6, $\beta_2 \leq \beta_1$, and $P_1 = P_2 = P$. Also assume that $\lim_{T\to\infty}\sum_{t=0}^{T}\beta_i^t E_P\left[u_i\left(C_{i,t}\right)\right] > -\infty$. In each of the following cases an IDC equilibrium exists with (C_1, C_2) as equilibrium consumption processes:

(i) the economy is such that, $\forall\, t \geq 1$, $u_2'(C_{2,t}(\omega))\cdot\left(z_{2,t}(\omega)-C_{2,t}(\omega)\right) = \bar{c}_{2,t}$ for all $\omega \in \Omega$ and $u_2'(C_{2,0}(\omega))\cdot\left(z_{2,0}(\omega)-C_{2,0}(\omega)\right) = -\mathrm{Lim}_{T\to+\infty}\sum_{\tau=1}^{T}\beta_2^{\tau}\cdot\bar{c}_{2,\tau}$,

(ii) $(z_1, z_2) = (C_1, C_2)$ so that there is no trade in equilibrium,

(iii) there exists $\tilde{\tau}(\omega)$ such that $0 < C_{2,t}(\omega) \leq \frac{Z_t(\omega)}{2}$ for all $t \geq \tilde{\tau}(\omega)$, in which case the endowment could be any one of a continuum of perturbations of (C_1, C_2).

We remind the reader that Proposition 8 (ii) provides sufficient conditions on parametric Bernoulli functions that guarantee the condition in Theorem 4 and Corollary 2 which requires that the utility from C_i, $\lim_{T\to\infty}\sum_{t=0}^{T}\beta_i^t E_P\left[u_i\left(C_{i,t}\right)\right]$, is bounded below.

The condition in case (i) holds if $u_2(x) = \log x$ and $z_{2,t}(\omega) = 0$ for $t \geq 1$. So the leading example generalizes to arbitrary nonnegative asset payoffs and arbitrary characteristics for agent 1. Case (ii) guarantees that our construction is not vacuous. Case (iii) clearly shows that our construction can go through even though endowments are uniformly positive; however, the date beyond which the endowment is pertubed is path dependent. The perturbation is easy to construct: for $\bar{\epsilon} > 0$, decrease agent 2's endowment at date $\tilde{T}(\omega)$ by $\bar{\epsilon} > 0$ and increase it at each subsequent date by $\bar{\epsilon}\left[\left(r_t(\omega)/q_{t-1}^*(\omega)\right)-1\right]$ (agent 1 faces the symmetric change). This ensures that there is a portfolio, $\theta_{2,t}(\omega)$, that costs $\bar{\epsilon}$ at each $t \geq \tilde{T}(\omega)$, i.e. 2 borrows the initial decrease in her endowment and then repays only the interest. The "predeterminedness" property, Proposition 6 (i), and the mild hypothesis on the tail behaviour of $C_{2,t}(\omega)$, suffice to identify an open subset of possible values for $\bar{\epsilon}$.

Corollary 2 summarises our findings. It shows that in an economy with a real bond and uniformly positive endowments, there are IDC equilibria in which agent 2 vanishes almost surely if the aggregate endowment has uniformly positive conditional variance.

COROLLARY 2: Assume A.1-3, A.6, $r_t(\omega) = 1$ for all $t \geq 0$ and $\omega \in \Omega$, var $[Z_t|\mathcal{F}_{t-1}](\omega) > \epsilon > 0$ for all $t \geq 0$ and $P-$a.s. $\omega \in \Omega$, $\beta_2 \leq \beta_1$, and $P_1 = P_2 = P$. Also assume that $\lim_{T\to\infty} \sum_{t=0}^{T} \beta_i^t E_P [u_i (C_{i,t})] > -\infty$. Then there exists an IDC equilibrium where agent 2 vanishes $P-$a.s. $\omega \in \Omega$.

The result follows from the fact that when the asset is a real bond, $\hat{r}_{2,t}(\omega) = 1$ implies that $c_{2,t}$ is $\mathcal{F}_{t-1}-$measurable so that conditional variability in the endowment guarantees that $r_t \cdot u_1'(Z_t - c_{2,t})$ is nondegenerate, i.e. A.7 holds. By Theorem 4 we have an IDC equilibrium in which $\hat{r}_{2,t}(\omega) = 1$. Since A.7 holds, by Propositions 5 and 7, 2 must vanish on almost every path.

REMARK 9: One expects that a continuity argument can be used to provide an analogue of Theorem 4 in the case where $\beta_1 < \beta_2$ but sufficiently close; this generalizes a property that the example in Section 3 displayed.[20]

7. CONCLUDING REMARKS

We showed that, asymptotically, equilibrium consumption in an infinite horizon economy with incomplete markets with two agents and one good must exhibit very special behaviour: either one of the two agents will eventually cease to consume, or the equilibrium is complicated in the sense that the consumption of both the agents is arbitrarily close to zero infinitely often. We also provided two robustness checks: for most economies where individual endowments follow a finite state time homogeneous Markov process and an agent's endowment is uniformly positive, her consumption can converge to zero only on a set of paths that has measure zero, and for a distinct robust class of economies there are equilibria in which an agent's consumption is zero eventually with probability one even though she has correct beliefs and uniformly positive endowments.

Our results help to disentangle the role played by the heterogeneity of beliefs from that played by the market structure in determining the fate of an agent since we show that even when beliefs are homogeneous, the fact that markets are incomplete could imply that an agent vanishes. Evidently, the MSH and the Ramsey conjecture can hold in a robust sense only if the equilibrium allocation is Pareto optimal.

When utility is unbounded below, Theorem 1 (ii) implies that the continuation utility is arbitrarily low infinitely often. This can be interpreted as showing that the implicit punishment required to ensure that an agent continues to participate in the market is the confiscation of her entire endowment, i.e. the maximal possible punishment.[21]

We believe that Theorem 1 holds in a wide class of models where markets are incomplete and the Euler equation always holds with equality. Since the result is based on pairwise comparisons of the agents' marginal rates of substitution, we conjecture that with any finite number of agents and goods, and numeraire assets, such that some asset has strictly positive returns in at least two states, at most one agent's consumption can

[20] We face a technical diffculty when $\beta_1 < \beta_2$ since in that case Proposition 7 does not go through.

[21] We are indebted to Emilio Espino for this observation.

be uniformly bounded away from zero eventually. Generalizations of Theorems 2 and 4 will require a different approach since the current proofs use the fact that there are only two agents.

Our approach does not cover models where the Euler condition holds as an inequality. Given the prevalence of such models in the literature on computational general equilibrium and macroeconomics, it would be useful to characterize the asymptotic properties of consumption in such models; perhaps our techniques can be adapted to such situations.

APPENDIX

PROOF OF PROPOSITION 1

That $E_{P_i}[\hat{r}_{i,t}|\mathcal{F}_{t-1}](\omega)=1$ follows from the definition of the process $\hat{r}_i$. The rest of the proof follows from Lemma 2 and 3. Lemma 2 shows that if the asset's return is positive and the one period ahead conditional probability that state s occurs is uniformly positive, A.1, then $\hat{r}_{i,t}(\omega)$ is positive and uniformly bounded above. Lemma 3 uses the martingale convergence theorem to show that $\lim_{T\to\infty}\prod_{t=1}^{T}\hat{r}_{i,t}(\omega)$ is P_i-a.s. finite.

LEMMA 2: Assume A.2, A.3 and $\underline{r}>0$. Then $0<\hat{r}_{i,t}(\omega)\leq 1/P_{i,t}(\omega)$. Hence, under A.1, A.2, A.3, and A.4, $\bar{\hat{r}}_i<\infty$.
PROOF: Under A.2 and A.3 $u_i'(c_{i,t}(\omega))$ is uniformly positive. So $\underline{r}>0$ implies that $\hat{r}_{i,t}(\omega)>0$. It follows that
$P_{i,t}(\omega)\leq P_{i,t}(\omega)+\frac{\sum_{\tilde{\omega}\in\Omega((s^{t-1}(\omega))/\Omega(s^t(\omega))}P_{i,t}(\tilde{\omega})\cdot r_t(\tilde{\omega})\cdot u_i'(c_{i,t})(\tilde{\omega})}{r_t(\omega)\cdot u_i'(c_{i,t}(\omega))}=\frac{E_{P_i}[r_t\cdot u_i'(c_{i,t})|\mathcal{F}_{t-1}](\omega)}{r_t(\omega)\cdot u_i'(c_{i,t}(\omega))}=\frac{1}{\hat{r}_{i,t}(\omega)}.$
The proof is completed by invoking A.1. ∎

LEMMA 3: Assume A.3 and $\underline{r}>0$. Then there is a random variable R_i^* that is nonnegative and a.s. finite with $E_{P_i}[R_i^*]\leq 1$ such that $R_i^*(\omega)=\lim_{T\to\infty}\prod_{t=1}^{T}\hat{r}_{i,t}(\omega)$ P_i-a.s.
PROOF: Under the stated condition, $\{\prod_{t=1}^{T}\hat{r}_{i,t}\}$ is a positive martingale since $E_{P_i}[\hat{r}_{i,t}|\mathcal{F}_{t-1}]=1$. Since $\sup_{T\geq 1}E_{P_i}[\prod_{t=1}^{T}\hat{r}_{i,t}]=1<+\infty$, the Martingale Convergence Theorem applies. ∎

PROOF OF PROPOSITION 2

(i) Since, by hypothesis, c_i satisfies the Euler equations for i at q, we have

$$q_{t-1}(\omega)=\beta_i\cdot\frac{E_{P_i}[r_t\cdot u_i'(c_{i,t})|\mathcal{F}_{t-1}](\omega)}{u_i'(c_{i,t-1}(\omega))}\quad\Leftrightarrow\quad\hat{r}_{i,t}(\omega)=\frac{\beta_i\cdot r_t(\omega)\cdot u_i'(c_{i,t}(\omega))}{q_{t-1}(\omega)\cdot u_i'(c_{i,t-1}(\omega))}$$

$$\Rightarrow\qquad\prod_{t=1}^{T+1}\hat{r}_{i,t}(\omega)=\beta_i^{T+1}\cdot\frac{u_i'(c_{i,1+T}(\omega))}{u_i'(c_{i,0}(\omega))}\cdot\prod_{t=0}^{T}\left(\frac{r_{1+t}(\omega)}{q_t(\omega)}\right).$$

(ii) By Proposition 1, under A.2, A.3, and A.4, we have $\hat{r}_{i,t}(\omega)>0$. Since

$$\frac{\hat{r}_{1,t}(\omega)}{\hat{r}_{2,t}(\omega)}=\frac{\frac{\beta_1\cdot r_t(\omega)\cdot u_1'(c_{1,t}(\omega))}{q_{t-1}(\omega)\cdot u_1'(c_{1,t-1}(\omega))}}{\frac{\beta_2\cdot r_t(\omega)\cdot u_2'(c_{2,t}(\omega))}{q_{t-1}(\omega)\cdot u_2'(c_{2,t-1}(\omega))}}=\frac{\beta_1}{\beta_2}\cdot\frac{\frac{u_1'(c_{1,t}(\omega))}{u_2'(c_{2,t}(\omega))}}{\frac{u_1'(c_{1,t-1}(\omega))}{u_2'(c_{2,t-1}(\omega))}}=\frac{\beta_1}{\beta_2}\cdot\frac{y_{t-1}(\omega)}{y_t(\omega)}.$$

It follows that

$$\Rightarrow\qquad y_T(\omega)=\frac{\left(\frac{\beta_1}{\beta_2}\right)^T}{\prod_{t=1}^{T}\left(\frac{\hat{r}_{1,t}(\omega)}{\hat{r}_{2,t}(\omega)}\right)}\cdot y_0(\omega)=\left(\frac{\beta_1}{\beta_2}\right)^T\cdot\frac{\prod_{t=1}^{T}\hat{r}_{2,t}(\omega)}{\prod_{t=1}^{T}\hat{r}_{1,t}(\omega)}\cdot y_0(\omega).$$

(iii) Finally, by rewriting the first property in (ii) we have

$$\hat{r}_{2,t}(\omega)\cdot y_{t-1}(\omega)=\frac{\beta_2}{\beta_1}\cdot\hat{r}_{1,t}(\omega)\cdot y_t(\omega)\quad\Leftrightarrow\quad E_{P_2}[\hat{r}_{2,t}\cdot y_{t-1}|\mathcal{F}_{t-1}](\omega)=\frac{\beta_2}{\beta_1}\cdot E_{P_2}[\hat{r}_{1,t}\cdot y_t|\mathcal{F}_{t-1}](\omega)$$

and the first result in (iii) follows by using the fact, noted in Proposition 1, that $E_{P_i}\left[\hat{r}_{i,t}|\mathcal{F}_{t-1}\right](\omega) = 1$. The second result in (iii) is proved in a similar manner. ■

LEMMA EBC

For $E \in \mathcal{F}$ an event, let 1_E denote the indicator function. Recall that if $\{\Omega_t\}_{t=0}^{\infty}$ is a sequence of events, then $\left\{\Omega_t \text{ i.o.}\right\} = \left\{\omega : \sum_{t=1}^{\infty} 1_{\Omega_t}(\omega) = +\infty\right\}$.

LEMMA EBC: Let $\{\Omega_t\}_{t=0}^{\infty}$ be adapted to the filtration $\{\mathcal{F}_t\}_{t=0}^{\infty}$. Then,

$$\forall N \geq 1 \quad \sum_{t=1}^{\infty} 1_{\Omega_t}(\widetilde{\omega}) = +\infty \quad P - \text{a.s. } \widetilde{\omega} \in \left\{\omega : \sum_{t=N}^{\infty} P\left(\Omega_t | \mathcal{F}_{t-N}\right)(\omega) = +\infty\right\}.$$

PROOF: For $\{\Omega_t\}_{t=0}^{\infty}$ a sequence of events adapted to the filtration $\{\mathcal{F}_t\}_{t=0}^{\infty}$, and $N \geq 1$, define Ω^N as

$$\Omega^N \equiv \left\{\omega : \sum_{t=N}^{\infty} P\left(\Omega_t | \mathcal{F}_{t-N}\right)(\omega) = +\infty\right\}.$$

We proceed inductively. For $N = 1$ the result is Levy's conditional form of the Borel-Cantelli Lemma and this follows from a more general result due to Freedman (1973 Proposition 39). Suppose that the result holds for n, that is

$$\sum_{t=1}^{\infty} 1_{\Omega_t}(\widetilde{\omega}) = +\infty \quad P - \text{a.s. } \widetilde{\omega} \in \Omega^n.$$

To show that it also holds for $n+1$, that is

$$\sum_{t=1}^{\infty} 1_{\Omega_t}(\widetilde{\omega}) = +\infty \quad P - \text{a.s. } \widetilde{\omega} \in \Omega^{n+1},$$

it suffices to show that

$$\widetilde{\omega} \in \Omega^n \quad P - \text{a.s. } \widetilde{\omega} \in \Omega^{n+1}$$

and, as we show at the end of the proof, for that it is sufficient that

$$P\left(\Omega_t | \mathcal{F}_{t-n}\right)(\widetilde{\omega}) \geq \underline{p}^n \quad \text{i.o.} \quad P - \text{a.s. } \widetilde{\omega} \in \Omega^{n+1},$$

a claim that we now prove.

If $\widetilde{\omega} \in \Omega^{n+1}$ then there exists a subsequence $\{t_k\}_{k=1}^{\infty}$ such that

$$P\left(\Omega_{t_k} | \mathcal{F}_{t_k-(n+1)}\right)(\widetilde{\omega}) > 0$$

and it follows by assumption $A.1$ that

$$P\left(\Omega_{t_k} | \mathcal{F}_{t_k-(n+1)}\right)(\widetilde{\omega}) \geq \underline{p}^{n+1}.$$

Therefore,

$$\begin{aligned} E\left[P\left(\Omega_{t_k} | \mathcal{F}_{t_k-n}\right) | \mathcal{F}_{t_k-(n+1)}\right](\widetilde{\omega}) &= E\left[E\left(1_{\Omega_{t_k}} \middle| \mathcal{F}_{t_k-n}\right) \middle| \mathcal{F}_{t_k-(n+1)}\right](\widetilde{\omega}) \\ &= E\left(1_{\Omega_{t_k}} \middle| \mathcal{F}_{t_k-(n+1)}\right)(\widetilde{\omega}) \\ &= P\left(\Omega_{t_k} | \mathcal{F}_{t_k-(n+1)}\right)(\widetilde{\omega}) \geq \underline{p}^{n+1}, \end{aligned}$$

where the first equality uses $P\left(\Omega_{t_k} \middle| \mathcal{F}_{t_k-n}\right)(\widetilde{\omega}) = E\left(1_{\Omega_{t_k}} \middle| \mathcal{F}_{t_k-n}\right)(\widetilde{\omega})$ and the second one uses the law of iterated expectations. It follows that

$$P\left[P\left(\Omega_{t_k} \middle| \mathcal{F}_{t_k-n}\right) \geq \underline{p}^{n+1} \middle| \mathcal{F}_{t_k-(n+1)}\right](\widetilde{\omega}) > 0$$

and so, by assumption $A.1$,

$$P\left[P\left(\Omega_{t_k} \middle| \mathcal{F}_{t_k-n}\right) > \underline{p}^{n+1} \middle| \mathcal{F}_{t_k-(n+1)}\right](\widetilde{\omega}) \geq \underline{p},$$

and, since $P\left(\Omega_{t_k} \middle| \mathcal{F}_{t_k-n}\right)(\widetilde{\omega}) > \underline{p}^{n+1} > 0$, it follows, once again by assumption $A.1$, that

$$\left\{\omega : P\left(\Omega_{t_k} \middle| \mathcal{F}_{t_k-n}\right)(\omega) > \underline{p}^{n+1}\right\} = \left\{\omega : P\left(\Omega_{t_k} \middle| \mathcal{F}_{t_k-n}\right)(\omega) > \underline{p}^{n}\right\}.$$

Thus,

$$P\left[P\left(\Omega_{t_k} \middle| \mathcal{F}_{t_k-n}\right) > \underline{p}^{n} \middle| \mathcal{F}_{t_k-(n+1)}\right](\widetilde{\omega}) \geq \underline{p}. \tag{1}$$

Now consider the event $\Omega_t^P \equiv \left\{\omega : P\left(\Omega_{t+n} \middle| \mathcal{F}_t\right)(\omega) > \underline{p}^n\right\}$. Let $t'_k = t_k - n$. It follows from (1) that

$$P\left(\Omega_{t'_k}^P \middle| \mathcal{F}_{t'_k-1}\right)(\widetilde{\omega}) \geq \underline{p}$$

and

$$\sum_{t=1}^{\infty} P\left(\Omega_t^P \middle| \mathcal{F}_{t-1}\right)(\widetilde{\omega}) \geq \sum_{k=1}^{\infty} P\left(\Omega_{t'_k}^P \middle| \mathcal{F}_{t'_k-1}\right)(\widetilde{\omega}) = +\infty \quad \forall \widetilde{\omega} \in \Omega^{n+1}.$$

We can invoke Lemma EBC with $n = 1$ to conclude that

$$\sum_{t=1}^{\infty} 1_{\Omega_t^P}(\widetilde{\omega}) = +\infty \qquad P-\text{a.s. } \widetilde{\omega} \in \Omega^{n+1}.$$

By the definition of Ω_t^P it follows that

$$P\left(\Omega_t \middle| \mathcal{F}_{t-n}\right)(\widetilde{\omega}) > \underline{p}^n \quad \text{i.o.} \qquad P-\text{a.s. } \widetilde{\omega} \in \Omega^{n+1}.$$

It remains to show that the last statement implies that $\widetilde{\omega} \in \Omega^n$. For $P-$a.s. $\widetilde{\omega} \in \Omega^{n+1}$, there exists a subsequence $\{t''_k\}_{k=1}^{\infty}$ such that $P\left(\Omega_{t''_k} \middle| \mathcal{F}_{t''_k-n}\right)(\widetilde{\omega}) > \underline{p}^n$. Then,

$$\sum_{t=1}^{\infty} P\left(\Omega_t \middle| \mathcal{F}_{t-n}\right)(\widetilde{\omega}) \geq \sum_{k=1}^{\infty} P\left(\Omega_{t''_k} \middle| \mathcal{F}_{t''_k-n}\right)(\widetilde{\omega}) = +\infty \quad P-\text{a.s. } \widetilde{\omega} \in \Omega^{n+1},$$

which is the desired conclusion. ■

PROOF OF THEOREM 1

(i) By definition, on the set V_0

$$\lim_t \left[\frac{P_{2,t}(\omega)}{P_{1,t}(\omega)} \cdot \frac{y_t(\omega)}{y_{t-1}(\omega)} - E_P\left[\frac{P_{2,t}}{P_{1,t}} \cdot \frac{y_t}{y_{t-1}} \middle| \mathcal{F}_{t-1}\right](\omega)\right] = 0.$$

Equivalently, using Proposition 2 (ii),

$$\lim_t \left[\frac{P_{2,t}(\omega)}{P_{1,t}(\omega)} \cdot \frac{\hat{r}_{2,t}(\omega)}{\hat{r}_{1,t}(\omega)} - E_P\left[\frac{P_{2,t}}{P_{1,t}} \cdot \frac{\hat{r}_{2,t}}{\hat{r}_{1,t}} \,\middle|\, \mathcal{F}_{t-1}\right](\omega)\right] = 0.$$

So there exists a process $\{\lambda_t\}_{t\geq 0}$ such that λ_t is $\mathcal{F}_t-$measurable and for every $\epsilon > 0$ there exists $t(\epsilon, \omega)$ such that $t > t(\epsilon, \omega)$ implies $\left|\frac{P_{2,t}(\omega)}{P_{1,t}(\omega)} \cdot \frac{\hat{r}_{2,t}(\omega)}{\hat{r}_{1,t}(\omega)} - \lambda_{t-1}(\omega)\right| < \epsilon$. It follows that $t > t(\epsilon,\omega) \Rightarrow (\lambda_{t-1}(\omega)-\epsilon)\cdot P_{1,t}(\omega)\cdot\hat{r}_{1,t}(\omega) < P_{2,t}(\omega)\cdot\hat{r}_{2,t}(\omega) < (\lambda_{t-1}(\omega)+\epsilon)\cdot P_{1,t}(\omega)\cdot\hat{r}_{1,t}(\omega)$. Since λ_{t-1} is $\mathcal{F}_{t-1}-$measurable, we have $t > t(\epsilon,\omega)$ implies
$(\lambda_{t-1}(\omega) - \epsilon) \cdot E_{P_1}[\hat{r}_{1,t}|\mathcal{F}_{t-1}](\omega) < E_{P_2}[\hat{r}_{2,t}|\mathcal{F}_{t-1}](\omega) < (\lambda_{t-1}(\omega) + \epsilon) \cdot E_{P_1}[\hat{r}_{1,t}|\mathcal{F}_{t-1}](\omega)$. Since $E_{P_i}[\hat{r}_{i,t}|\mathcal{F}_{t-1}](\omega) = 1$ and $\epsilon > 0$ is arbitrary, we have $\lim_t\lambda_{t-1} = 1$ $P-$a.s. $\omega \in V_0$. It follows from an application of Proposition 2 (ii) that $\lim_t \left(\frac{P_{2,t}(\omega)}{P_{1,t}(\omega)} \cdot \frac{y_t(\omega)}{y_{t-1}(\omega)}\right) = 1$.

(ii) We first put bounds on the conditional probability with which there is variability in $\frac{P_{2,t}}{P_{1,t}} \cdot \frac{y_t}{y_{t-1}}$.

LEMMA 4: Assume A.1. Then
(i) $\forall\, t \geq 1$, $P\left[\frac{P_{2,t}}{P_{1,t}} \cdot \frac{y_t}{y_{t-1}} \geq 1 \,\middle|\, \mathcal{F}_{t-1}\right](\omega) \geq \underline{p} > 0$ and $P\left[\frac{P_{2,t}}{P_{1,t}} \cdot \frac{y_t}{y_{t-1}} \leq 1 \,\middle|\, \mathcal{F}_{t-1}\right](\omega) \geq \underline{p} > 0$, $\forall\, \omega \in \Omega$.
(ii) $\text{var}\left[\frac{P_{2,t}}{P_{1,t}} \cdot \frac{y_t}{y_{t-1}} \,\middle|\, \mathcal{F}_{t-1}\right](\omega) \geq \epsilon > 0$ implies that

$$P\left[1-\frac{\sqrt{\epsilon}}{2} \geq \frac{P_{2,t}}{P_{1,t}}\cdot\frac{y_t}{y_{t-1}} \,\middle|\, \mathcal{F}_{t-1}\right](\omega) \geq \underline{p} > 0 \quad \text{OR} \quad P\left[\frac{P_{2,t}}{P_{1,t}}\cdot\frac{y_t}{y_{t-1}} \geq 1+\frac{\sqrt{\epsilon}}{2} \,\middle|\, \mathcal{F}_{t-1}\right](\omega) \geq \underline{p} > 0.$$

(iii) Assume A.2. If $\text{var}\left[\frac{P_{2,t}}{P_{1,t}} \cdot \frac{y_t}{y_{t-1}} \,\middle|\, \mathcal{F}_{t-1}\right](\omega) \geq \epsilon > 0$ and $y_{t-1}(\omega) > \underline{y}$, there exists $\gamma > 0$ such that

$$P\left[1-\gamma \geq \frac{P_{2,t}}{P_{1,t}}\cdot\frac{y_t}{y_{t-1}} \,\middle|\, \mathcal{F}_{t-1}\right](\omega) \geq \underline{p} > 0 \quad \text{AND} \quad P\left[\frac{P_{2,t}}{P_{1,t}}\cdot\frac{y_t}{y_{t-1}} \geq 1+\gamma \,\middle|\, \mathcal{F}_{t-1}\right](\omega) \geq \underline{p} > 0.$$

PROOF: (i) Suppose $P\left[\frac{P_{2,t}}{P_{1,t}} \cdot \frac{y_t}{y_{t-1}} \geq 1 \,\middle|\, \mathcal{F}_{t-1}\right](\omega) < \underline{p}$ for some ω on a subset of Ω with positive measure. Then, it follows by A.1 that $\frac{P_{2,t}(\widetilde{\omega})}{P_{1,t}(\widetilde{\omega})} \cdot \frac{y_t(\widetilde{\omega})}{y_{t-1}(\widetilde{\omega})} < 1$ for all $\widetilde{\omega} \in \Omega\left(s^{t-1}(\omega)\right)$. By Proposition 2 (ii), $\left(y_t(\widetilde{\omega})/y_{t-1}(\widetilde{\omega})\right) = \left(\hat{r}_{2,t}(\widetilde{\omega})/\hat{r}_{1,t}(\widetilde{\omega})\right)$. Hence, $E_{P_2}[\hat{r}_{2,t}|\mathcal{F}_{t-1}](\omega) < E_{P_1}[\hat{r}_{1,t}|\mathcal{F}_{t-1}](\omega) = 1$. Since for all $t \geq 1$ and for all $\omega \in \Omega$, $E_{P_i}[\hat{r}_{i,t}|\mathcal{F}_{t-1}](\omega) = 1$ for all $i = 1, 2$, a contradiction is reached.

An analogous argument shows that $P\left[\frac{P_{2,t}}{P_{1,t}} \cdot \frac{y_t}{y_{t-1}} \leq 1 \,\middle|\, \mathcal{F}_{t-1}\right](\omega) \geq \underline{p} > 0$ for all $\omega \in \Omega$.
(ii) Suppose

$$P\left[1 - \frac{\sqrt{\epsilon}}{2} < \frac{P_{2,t}}{P_{1,t}} \cdot \frac{y_t}{y_{t-1}} < 1 + \frac{\sqrt{\epsilon}}{2} \,\middle|\, \mathcal{F}_{t-1}\right](\omega) = 1.$$

Then,

$$1 - \frac{\sqrt{\epsilon}}{2} < E_P\left[\frac{P_{2,t}}{P_{1,t}} \cdot \frac{y_t}{y_{t-1}} \,\middle|\, \mathcal{F}_{t-1}\right](\omega) < 1 + \frac{\sqrt{\epsilon}}{2},$$

and so

$$\left[\frac{P_{2,t}}{P_{1,t}} \cdot \frac{y_t}{y_{t-1}} - E_P\left[\frac{P_{2,t}}{P_{1,t}} \cdot \frac{y_t}{y_{t-1}} \middle| \mathcal{F}_{t-1}\right](\omega)\right]^2 < \left(1 + \frac{\sqrt{\epsilon}}{2} - \left(1 - \frac{\sqrt{\epsilon}}{2}\right)\right)^2 = \epsilon,$$

implying that

$$\text{var}\left[\frac{P_{2,t}}{P_{1,t}} \cdot \frac{y_t}{y_{t-1}} \middle| \mathcal{F}_{t-1}\right](\omega) < \epsilon.$$

a contradiction.
(iii) By (ii)

$$P\left[1 - \frac{\sqrt{\epsilon}}{2} \geq \frac{P_{2,t}}{P_{1,t}} \cdot \frac{y_t}{y_{t-1}} \middle| \mathcal{F}_{t-1}\right](\omega) \geq \underline{p} > 0 \quad \text{OR} \quad P\left[\frac{P_{2,t}}{P_{1,t}} \cdot \frac{y_t}{y_{t-1}} \geq 1 + \frac{\sqrt{\epsilon}}{2} \middle| \mathcal{F}_{t-1}\right](\omega) \geq \underline{p} > 0.$$

Without loss of generality suppose

$$P\left[\frac{P_{2,t}}{P_{1,t}} \cdot \frac{y_t}{y_{t-1}} \geq 1 + \frac{\sqrt{\epsilon}}{2} \middle| \mathcal{F}_{t-1}\right](\omega) \geq \underline{p} > 0 \tag{2}$$

and let $\underline{c}(\epsilon, y)$ be the unique solution (by A.2) to

$$\frac{u_2'(\underline{z} - c)}{u_1'(c)} = \left(1 - \frac{\sqrt{\epsilon}}{2}\right) \cdot \frac{\underline{p}}{1 - \underline{p}} \cdot y$$

Suppose there exists $\omega' \in \Omega(s^{t-1}(\omega))$ such that $c_{1,t}(\omega') \leq \underline{c}\left(\epsilon, \underline{y}\right)$. Then

$$\frac{P_{2,t}(\omega')}{P_{1,t}(\omega')} \cdot \frac{y_t(\omega')}{y_{t-1}(\omega')} \leq \frac{1 - \underline{p}}{\underline{p}} \cdot \frac{y_t(\omega')}{\underline{y}} = \frac{1 - \underline{p}}{\underline{p}} \cdot \frac{1}{\underline{y}} \cdot \frac{u_2'(Z_t(\omega') - c_{1,t}(\omega'))}{u_1'(c_{1,t}(\omega'))} \leq 1 - \frac{\sqrt{\epsilon}}{2}$$

where the first inequality follows by A.1 and the last inequality follows by A.2 and the assumption that $c_{1,t}(\omega') \leq \underline{c}\left(\epsilon, \underline{y}\right)$.

Suppose $c_{1,t}(\widetilde{\omega}) > \underline{c}\left(\epsilon, \underline{y}\right)$ for all $\widetilde{\omega} \in \Omega(s^{t-1}(\omega))$. By Proposition 2 (ii), $\left(y_t(\widetilde{\omega})/y_{t-1}(\widetilde{\omega})\right) = \left(\hat{r}_{2,t}(\widetilde{\omega})/\hat{r}_{1,t}(\widetilde{\omega})\right)$ for all $\widetilde{\omega} \in \Omega$. By (2) there exists $\omega' \in \Omega(s^{t-1}(\omega))$ such that

$$\frac{P_{2,t}(\omega')}{P_{1,t}(\omega')} \cdot \frac{\hat{r}_{2,t}(\omega')}{\hat{r}_{1,t}(\omega')} \geq 1 + \frac{\sqrt{\epsilon}}{2}.$$

Let $S_1 = \left\{\widetilde{\omega} \in \Omega(s^{t-1}(\omega)) : \frac{P_{2,t}(\widetilde{\omega})}{P_{1,t}(\widetilde{\omega})} \frac{\hat{r}_{2,t}(\widetilde{\omega})}{\hat{r}_{1,t}(\widetilde{\omega})} \leq 1\right\}$. Note that $S_1 \neq \emptyset$ by (i). Then,

$$\begin{aligned}
\sum_{\widetilde{\omega} \in S_1} P_{2,t}(\widetilde{\omega}) \cdot \hat{r}_{2,t}(\widetilde{\omega}) &= 1 - \sum_{\widetilde{\omega} \in \Omega \setminus S_1} P_{2,t}(\widetilde{\omega}) \cdot \hat{r}_{2,t}(\widetilde{\omega}) \\
&< 1 - \sum_{\widetilde{\omega} \in \Omega \setminus S_1, \widetilde{\omega} \neq \omega'} P_{1,t}(\widetilde{\omega}) \cdot \hat{r}_{1,t}(\widetilde{\omega}) - \left(1 + \frac{\sqrt{\epsilon}}{2}\right) \cdot P_{1,t}(\omega') \cdot \hat{r}_{1,t}(\omega') \\
&= \sum_{\widetilde{\omega} \in S_1} P_{1,t}(\widetilde{\omega}) \cdot \hat{r}_{1,t}(\widetilde{\omega}) - \frac{\sqrt{\epsilon}}{2} \cdot P_{1,t}(\omega') \cdot \hat{r}_{1,t}(\omega')
\end{aligned}$$

where the first and third lines use the fact that $E_{P_i}\left[\hat{r}_{i,t}\left|\mathcal{F}_{t-1}\right.\right](\omega)=1$. It follows that

$$\sum_{\widetilde{\omega}\in S_1} P_{2,t}\left(\widetilde{\omega}\right)\cdot\hat{r}_{2,t}\left(\widetilde{\omega}\right)-\sum_{\widetilde{\omega}\in S_1} P_{1,t}\left(\widetilde{\omega}\right)\cdot\hat{r}_{1,t}\left(\widetilde{\omega}\right)<-\frac{\sqrt{\epsilon}}{2}\cdot P_{1,t}\left(\omega'\right)\cdot\hat{r}_{1,t}\left(\omega'\right)$$

and so there exists $\omega''\in\Omega\left(s^t\left(\omega\right)\right)$ such that

$$P_{2,t}\left(\omega''\right)\cdot\hat{r}_{2,t}\left(\omega''\right)-P_{1,t}\left(\omega''\right)\cdot\hat{r}_{1,t}\left(\omega''\right)<-\frac{\sqrt{\epsilon}}{2\cdot\#S_1}\cdot P_{1,t}\left(\omega'\right)\cdot\hat{r}_{1,t}\left(\omega'\right)\leq-\frac{\sqrt{\epsilon}}{2\cdot S}\cdot P_{1,t}\left(\omega'\right)\cdot\hat{r}_{1,t}\left(\omega'\right)$$

and then

$$\begin{aligned}\frac{P_{2,t}\left(\omega''\right)}{P_{1,t}\left(\omega''\right)}\cdot\frac{\hat{r}_{2,t}\left(\omega''\right)}{\hat{r}_{1,t}\left(\omega''\right)} &< 1-\frac{\sqrt{\epsilon}}{2\cdot S}\cdot\frac{P_{1,t}\left(\omega'\right)}{P_{1,t}\left(\omega''\right)}\cdot\frac{\hat{r}_{1,t}\left(\omega'\right)}{\hat{r}_{1,t}\left(\omega''\right)}\\ &\leq 1-\frac{\sqrt{\epsilon}}{2\cdot S}\cdot\frac{\underline{p}}{1-\underline{p}}\cdot\frac{u_1'\left(c_{1,t}\left(\omega'\right)\right)}{u_1'\left(c_{1,t}\left(\omega''\right)\right)}\\ &< 1-\frac{\sqrt{\epsilon}}{2\cdot S}\cdot\frac{\underline{p}}{1-\underline{p}}\cdot\frac{u_1'\left(\overline{z}\right)}{u_1'\left(\underline{c}\left(\epsilon,\underline{y}\right)\right)}\end{aligned}$$

where the second inequality follows by the definiton of $\hat{r}_{1,t}$ while the third follows by A.2 and the assumption that $c_{1,t}\left(\widetilde{\omega}\right)>\underline{c}\left(\epsilon,\underline{y}\right)$ for all $\widetilde{\omega}\in\Omega\left(s^{t-1}\left(\omega\right)\right)$.

Now let $\gamma=\min\left\{\frac{\sqrt{\epsilon}}{2},\frac{\sqrt{\epsilon}}{2\cdot S}\cdot\frac{\underline{p}}{1-\underline{p}}\cdot\frac{u_1'(\overline{z})}{u_1'\left(\underline{c}\left(\epsilon,\underline{y}\right)\right)}\right\}$. The desired result follows by A.1. ■

The gist of the argument underlying the proof of Theorem 1 (ii) was given in Section 4.2. We present the formal details.

Fix the values of ϵ, T, n, $\underline{\lambda}$, and $\overline{\lambda}$. Without loss of generality we identify agent 2 as j. We need to prove that

$$\limsup_t c_{1,t}(\omega)\leq 1/n \quad P-\text{a.s. }\omega\in V_{T,\epsilon}\cap L_{\underline{\lambda},\overline{\lambda}}\cap\{\omega:\liminf_t c_{2,t}(\omega)>1/n\}$$

$$\iff \quad P\Big(\{\omega:\limsup_t c_{1,t}(\omega)>1/n\}\cap V_{T,\epsilon}\cap L_{\underline{\lambda},\overline{\lambda}}\cap\{\omega:\liminf_t c_{2,t}(\omega)>1/n\}\Big)=0$$

$$\iff \quad \liminf_t c_{2,t}(\omega)\leq 1/n \quad P-\text{a.s. }\omega\in V_{T,\epsilon}\cap L_{\underline{\lambda},\overline{\lambda}}\cap\{\omega:\limsup_t c_{1,t}(\omega)>1/n\}.$$

We will prove the last statement.

Set $\underline{y}_n\equiv\Big(u_2'(\overline{z}-1/n)/u_1'(1/n)\Big)$ and $\overline{y}_n\equiv\Big(u_2'(1/n)/u_1'(\overline{z}-1/n)\Big)$. Also fix the value of $\gamma>0$, identified in Lemma 4 (iii) and induced by ϵ and $\underline{y}\equiv\left(\underline{\lambda}/\overline{\lambda}\right)\cdot\underline{y}_n>0$. Let $T_n(\gamma)$ satisfy $\underline{\lambda}\cdot\underline{y}_n\cdot(1+\gamma)^{T_n(\gamma)}>\overline{\lambda}\cdot\overline{y}_n$. Note that for $\omega\in L_{\underline{\lambda},\overline{\lambda}}$, there exists $T^{\underline{\lambda},\overline{\lambda}}(\omega)$ such that $\underline{\lambda}<\frac{dP_{2,t}(\omega)}{dP_{1,t}(\omega)}<\overline{\lambda}$ for all $t\geq T^{\underline{\lambda},\overline{\lambda}}(\omega)$.

For $\tau\geq 1$ and $t>\tau\cdot T$, where $T\geq 1$, define the event

$$\Omega_{1,t}^{\tau}\equiv\left\{\omega:\begin{array}{l} c_{1,t_0}(\omega)\geq 1/n,\frac{P_{2,t'}(\omega)}{P_{1,t'}(\omega)}\cdot\frac{y_{t'}(\omega)}{y_{t'-1}(\omega)}\geq 1\ \forall\, t'=t_0+1,\cdots,t,\\ \frac{P_{2,t}(\omega)}{P_{1,t}(\omega)}\cdot\frac{y_t(\omega)}{y_{t-1}(\omega)}\geq 1+\gamma,\\ \#\left\{\frac{P_{2,t'}(\omega)}{P_{1,t'}(\omega)}\cdot\frac{y_{t'}(\omega)}{y_{t'-1}(\omega)}\geq 1+\gamma,\ t'=t_0+1,\cdots,t-1\right\}=\tau-1,\\ \text{where } t-1\geq t_0\geq t-\tau\cdot T\end{array}\right\}.$$

For $\omega \in \Omega^{\tau}_{1,t}$, $y_{t_0}(\omega) \geq \underline{y}_n$ and

$$\frac{dP_{2,t}(\omega)}{dP_{1,t}(\omega)} \cdot y_t(\omega) \cdot \frac{1}{\frac{dP_{2,t_0}(\omega)}{dP_{1,t_0}(\omega)} \cdot y_{t_0}(\omega)} = \prod_{k=t_0+1}^{t} \frac{P_{2,k}(\omega)}{P_{1,k}(\omega)} \cdot \frac{y_k(\omega)}{y_{k-1}(\omega)} \geq (1+\gamma)^{\tau}.$$

Also, $\omega \in L_{\underline{\lambda},\overline{\lambda}}$ and $t_0 \geq T^{\underline{\lambda},\overline{\lambda}}(\omega)$ implies that $\frac{dP_{2,t_0}(\omega)}{dP_{1,t_0}(\omega)} > \underline{\lambda}$. Combining the above, we have

$$\omega \in \Omega^{T_n(\gamma)}_{1,t} \cap L_{\underline{\lambda},\overline{\lambda}} \quad \text{and} \quad t_0 \geq T^{\underline{\lambda},\overline{\lambda}}(\omega) \quad \Rightarrow \quad \frac{dP_{2,t}(\omega)}{dP_{1,t}(\omega)} \cdot y_t(\omega) > \underline{\lambda} \cdot \underline{y}_n \cdot (1+\gamma)^{T_n(\gamma)},$$

where, using the definition of $T_n(\gamma)$, the last term exceeds $\overline{\lambda} \cdot \overline{y}_n$. But then, since $t \geq t_0$, we have

$$\omega \in \Omega^{T_n(\gamma)}_{1,t} \cap L_{\underline{\lambda},\overline{\lambda}} \quad \text{and} \quad t_0 \geq T^{\underline{\lambda},\overline{\lambda}}(\omega) \quad \Rightarrow \quad y_t(\omega) > \overline{y}_n,$$

i.e. for $\omega \in \Omega^{T_n(\gamma)}_{1,t} \cap L_{\underline{\lambda},\overline{\lambda}}$ and $t_0 \geq T^{\underline{\lambda},\overline{\lambda}}(\omega)$ we must have $c_{2,t}(\omega) \leq 1/n$.

It follows that

$$\left\{\Omega^{T_n(\gamma)}_{1,t} \text{ i.o.}\right\} \cap L_{\underline{\lambda},\overline{\lambda}} \subset \{\omega : \liminf_t c_{2,t}(\omega) \leq 1/n\}.$$

We will show that

$$\left\{\Omega^{T_n(\gamma)}_{1,t} \text{ i.o.}\right\} \cap L_{\underline{\lambda},\overline{\lambda}} \quad \text{occurs } P-\text{a.s.}\, \widetilde{\omega} \in V_{T,\epsilon} \cap L_{\underline{\lambda},\overline{\lambda}} \cap \{\omega : \limsup_t c_{1,t}(\omega) > 1/n\}$$

so that

$$\liminf_t c_{2,t}(\omega) \leq 1/n \quad P-\text{a.s.}\ \widetilde{\omega} \in V_{T,\epsilon} \cap L_{\underline{\lambda},\overline{\lambda}} \cap \{\omega : \limsup_t c_{1,t}(\omega) > 1/n\}$$

as required.

The proof will be by induction on τ. To ease the burden of notation, define $C^{1,n}_t \equiv \{\omega : c_{1,t}(\omega) > 1/n\}$ so that $\left\{C^{1,n}_t \text{ i.o.}\right\} \supseteq \{\omega : \limsup_t c_{1,t}(\omega) > 1/n\}$. Also define $\Omega^N_{2,t} \equiv \left\{\omega : \frac{P_{2,t'}(\omega)}{P_{1,t'}(\omega)} \cdot \frac{y_{t'}(\omega)}{y_{t'-1}(\omega)} \geq 1,\ \forall\, t' = t+1-N, \cdots, t\right\}$.

The following two facts are used. Fact 1 says that if we consider a path $\widetilde{\omega} \in \Omega^{T}_{2,t'_k} \cap V_{T,\epsilon}$, so that $\frac{P_{2,t}}{P_{1,t}} \cdot \frac{y_t}{y_{t-1}}$ is at least one on exactly T dates starting with $t = t'_k + 1 - T$ and ending with $t = t'_k$, then there is some date t_k between $t'_k + 1 - T$ and t'_k such that $\frac{P_{2,t}}{P_{1,t}} \cdot \frac{y_t}{y_{t-1}}$ is at least one for $t_k - (t'_k + 1 - T) = t_k - 1 - (t'_k - T)$ periods followed by a date, t_k, at which the conditional variance of $\frac{P_{2,t_k}}{P_{1,t_k}} \cdot \frac{y_{t_k}}{y_{t_k-1}}$ at $t_k - 1$ exceeds ϵ. The proof follows directly from the fact that, in any span of T periods, the conditional variance must exceed ϵ at least once and that $\frac{P_{2,t}}{P_{1,t}} \cdot \frac{y_t}{y_{t-1}}$ is at least one at every date in that span of T periods.

FACT 1: If $\widetilde{\omega} \in \Omega^{T}_{2,t'_k} \cap V_{T,\epsilon}$ then, necessarily, $\widetilde{\omega} \in \Omega^{t_k-1-(t'_k-T)}_{2,t_k-1} \cap \left\{\omega : \text{var}\left[\frac{P_{2,t_k}}{P_{1,t_k}} \cdot \frac{y_{t_k}}{y_{t_k-1}} \middle| \mathcal{F}_{t_k-1}\right](\omega) \geq \epsilon\right\}$ for some $t_k \in \{t'_k - (T-1), \cdots, t'_k\}$. This can be proved by noting that (i) by the definition of T, for every t'_k there necessarily exists $t_k \in \{t'_k - (T-1), \cdots, t'_k\}$

such that $\tilde{\omega} \in \left\{\omega : \text{var}\left[\frac{P_{2,t_k}}{P_{1,t_k}} \cdot \frac{y_{t_k}}{y_{t_k-1}}\middle|\mathcal{F}_{t_k-1}\right](\omega) \geq \epsilon\right\}$, and (ii) $\tilde{\omega} \in \Omega^T_{2,t'_k}$ implies that $\tilde{\omega} \in \Omega^{t_k-1-\left(t'_k-T\right)}_{2,t_k-1}$ also for $t'_k-(T-1) \leq t_k \leq t'_k$.

Fact 2 uses Fact 1 to note that if we consider a path $\tilde{\omega} \in \Omega^\tau_{1,t'_k-T} \cap \Omega^T_{2,t'_k} \cap V_{T,\epsilon} \cap L_{\underline{\lambda},\overline{\lambda}}$, so that τ upward moves of $\frac{dP_{2,t}}{dP_{1,t}} \cdot y_t$ with no downward moves are followed by T periods where $\frac{P_{2,t}}{P_{1,t}} \cdot \frac{y_t}{y_{t-1}}$ is at least one and in at least one of those periods there is one period ahead variability, then there is a date t_k, that satisfies the conditions in Fact 1, such that $\frac{P_{2,t}}{P_{1,t}} \cdot \frac{y_t}{y_{t-1}}$ is at least $1+\gamma$ with conditional probability at t_k-1 that is at least $\underline{p}$.

FACT 2: By convention, $\Omega^0_{1,t} = C^{1,n}_t$ and $\Omega^0_{2,t} = \Omega$. For $\tilde{\omega} \in \Omega^\tau_{1,t'_k-T} \cap \Omega^T_{2,t'_k} \cap V_{T,\epsilon} \cap L_{\underline{\lambda},\overline{\lambda}}$, there is $t_k > t'_k$ such that $\tilde{\omega} \in \Omega^\tau_{1,t'_k-T} \cap \Omega^{t_k-1-(t'_k-T)}_{2,t_k-1} \cap \left\{\omega : \text{var}\left[\frac{P_{2,t_k}}{P_{1,t_k}} \cdot \frac{y_{t_k}}{y_{t_k-1}}\middle|\mathcal{F}_{t_k-1}\right](\omega) \geq \epsilon\right\} \cap L_{\underline{\lambda},\overline{\lambda}}$, and if $t'_k \geq T^{\underline{\lambda},\overline{\lambda}}(\tilde{\omega})$ then, by Lemma 4 (i),

$$\overline{\lambda} \cdot y_{t_{k-1}}(\tilde{\omega}) > \frac{dP_{2,t_k-1}(\tilde{\omega})}{dP_{1,t_k-1}(\tilde{\omega})} \cdot y_{t_{k-1}}(\tilde{\omega}) \geq \frac{dP_{2,t_0-1}(\tilde{\omega})}{dP_{1,t_0-1}(\tilde{\omega})} \cdot y_{t_0-1}(\tilde{\omega}) \geq \underline{\lambda} \cdot \underline{y}_n \Leftrightarrow y_{t_{k-1}}(\tilde{\omega}) > \left(\underline{\lambda}/\overline{\lambda}\right) \cdot \underline{y}_n \equiv \underline{y}$$

and so, by Lemma 4 (iii),

$$P\left[\frac{P_{2,t_k}}{P_{1,t_k}} \cdot \frac{y_{t_k}}{y_{t_k-1}} \geq 1+\gamma \middle| \mathcal{F}_{t_k-1}\right](\tilde{\omega}) \geq \underline{p} > 0.$$

So in particular, on any $\omega \in \left\{\Omega^\tau_{1,t} \text{ i.o.}\right\} \cap L_{\underline{\lambda},\overline{\lambda}}$, there is a subsequence of dates beyond which the hypothesis of Lemma 4 (iii) is satisfied.

We turn to the first step in the proof by induction. Consider $\tilde{\omega} \in \left\{C^{1,n}_t \text{ i.o.}\right\}$. By Lemma 4 (i), there exists a sequence $\{t_k\}_{k=1}^\infty$ such that $P\left[C^{1,n}_{t_k-T} \cap \Omega^T_{2,t_k} \middle| \mathcal{F}_{t_k-T}\right](\tilde{\omega}) \geq \left(\underline{p}\right)^T$, and so, by Lemma EBC,

$$\sum_{t=1}^\infty 1_{C^{1,n}_{t-T} \cap \Omega^T_{2,t}}(\tilde{\omega}) = +\infty \quad P-\text{a.s. } \tilde{\omega} \in \left\{C^{1,n}_t \text{ i.o.}\right\}.$$

Therefore, for P-a.s. $\tilde{\omega} \in \left\{C^{1,n}_t \text{ i.o.}\right\}$, there exists a sequence $\{t'_k\}_{k=1}^\infty$ such that $\tilde{\omega} \in C^{1,n}_{t'_k-T} \cap \Omega^T_{2,t'_k}$. So, for P-a.s. $\tilde{\omega} \in V_{T,\epsilon} \cap L_{\underline{\lambda},\overline{\lambda}} \cap \left\{C^{1,n}_t \text{ i.o.}\right\}$ there exists a sequence $\{t'_k\}_{k=1}^\infty$ such that $\tilde{\omega} \in C^{1,n}_{t'_k-T} \cap \Omega^T_{2,t'_k} \cap V_{T,\epsilon} \cap L_{\underline{\lambda},\overline{\lambda}}$. It follows from Fact 2 that for P-a.s. $\tilde{\omega} \in V_{T,\epsilon} \cap L_{\underline{\lambda},\overline{\lambda}} \cap \left\{C^{1,n}_t \text{ i.o.}\right\}$, there exists a $\gamma > 0$ and a subsequence $\{t_k\}_{k=1}^\infty$ such that $\tilde{\omega} \in C^{1,n}_{t'_k-T} \cap \Omega^\tau_{1,t'_k-T} \cap \Omega^{t_k-1-(t'_k-T)}_{2,t_k-1} \cap L_{\underline{\lambda},\overline{\lambda}}$ and $P\left[\frac{P_{2,t_k}}{P_{1,t_k}} \cdot \frac{y_{t_k}}{y_{t_k-1}} \geq 1+\gamma \middle| \mathcal{F}_{t_k-1}\right](\tilde{\omega}) \geq \underline{p}$. Therefore, $\sum_{t=1}^\infty P\left[\Omega^1_{1,t} \middle| \mathcal{F}_{t-1}\right](\tilde{\omega}) \geq \sum_{k=1}^\infty P\left[\Omega^1_{1,t_k} \middle| \mathcal{F}_{t_k-1}\right](\tilde{\omega}) = +\infty$ and, therefore, by Lemma EBC, $\sum_{t=1}^\infty 1_{\Omega^1_{1,t}}(\tilde{\omega}) = +\infty$ P-a.s. $\tilde{\omega} \in V_{T,\epsilon} \cap L_{\underline{\lambda},\overline{\lambda}} \cap \left\{C^{1,n}_t \text{ i.o.}\right\}$. Finally, since $\tilde{\omega} \in L_{\underline{\lambda},\overline{\lambda}}$, we have

$$\left\{\Omega^1_{1,t} \text{ i.o.}\right\} \cap L_{\underline{\lambda},\overline{\lambda}} \quad \text{occurs } P-\text{a.s.}\, \tilde{\omega} \in V_{T,\epsilon} \cap L_{\underline{\lambda},\overline{\lambda}} \cap \left\{C^{1,n}_t \text{ i.o.}\right\}.$$

We turn to the second step. So suppose it is true that for some τ

$$\left\{\Omega_{1,t}^{\tau} \text{ i.o.}\right\} \cap L_{\underline{\lambda},\overline{\lambda}} \quad \text{occurs } P-\text{a.s.}\, \widetilde{\omega} \in V_{T,\epsilon} \cap L_{\underline{\lambda},\overline{\lambda}} \cap \left\{C_t^{1,n} \text{ i.o.}\right\}.$$

Then, using Lemma 4 (i), there exists a subsequence $\left\{\widetilde{t}_k\right\}_{k=1}^{\infty}$ such that $P\left[\Omega_{1,\widetilde{t}_k-T}^{\tau} \cap \Omega_{2,\widetilde{t}_k}^{T} \middle| \mathcal{F}_{\widetilde{t}_k-T}\right](\widetilde{\omega}) \geq \left(\underline{p}\right)^T$ for all $k \geq 1$. By Lemma EBC,

$$\sum_{t=1}^{\infty} 1_{\Omega_{1,t-T}^{\tau} \cap \Omega_{2,t}^{T}}(\widetilde{\omega}) = +\infty \quad P-\text{a.s.}\, \widetilde{\omega} \in V_{T,\epsilon} \cap L_{\underline{\lambda},\overline{\lambda}} \cap \left\{C_t^{1,n} \text{ i.o.}\right\}.$$

Therefore, for P-a.s. $\widetilde{\omega} \in V_{T,\epsilon} \cap L_{\underline{\lambda},\overline{\lambda}} \cap \left\{C_t^{1,n} \text{ i.o.}\right\}$ there exists a sequence $\{t'_k\}_{k=1}^{\infty}$ such that $\widetilde{\omega} \in \Omega_{1,t'_k-T}^{\tau} \cap \Omega_{2,t'_k}^{T}$ and so there exists a sequence $\{t'_k\}_{k=1}^{\infty}$ such that $\widetilde{\omega} \in \Omega_{1,t'_k-T}^{\tau} \cap \Omega_{2,t'_k}^{T} \cap V_{T,\epsilon} \cap L_{\underline{\lambda},\overline{\lambda}}$. It follows from Fact 2 that for P-a.s. $\widetilde{\omega} \in V_{T,\epsilon} \cap L_{\underline{\lambda},\overline{\lambda}} \cap \left\{C_t^{1,n} \text{ i.o.}\right\}$, there exists a $\gamma > 0$ and a subsequence $\{t_k\}_{k=1}^{\infty}$ such that $\widetilde{\omega} \in C_{t'_k-T}^{1,n} \cap \Omega_{1,t'_k-T}^{\tau} \cap \Omega_{2,t_k-1}^{t_k-1-(t'_k-T)} \cap L_{\underline{\lambda},\overline{\lambda}}$ and $P\left[\frac{P_{2,t_k}}{P_{1,t_k}} \cdot \frac{y_{t_k}}{y_{t_k-1}} \geq 1+\gamma \middle| \mathcal{F}_{t_k-1}\right](\widetilde{\omega}) \geq \underline{p}$. Therefore,

$$\sum_{t=1}^{\infty} P\left[\Omega_{1,t}^{\tau+1} \middle| \mathcal{F}_{t-1}\right](\widetilde{\omega}) \geq \sum_{k=1}^{\infty} P\left[\Omega_{1,t_k}^{\tau+1} \middle| \mathcal{F}_{t_k-1}\right](\widetilde{\omega}) = \sum_{k=1}^{\infty} P\left[\frac{P_{2,t_k}}{P_{1,t_k}} \cdot \frac{y_{t_k}}{y_{t_k-1}} \geq 1+\gamma \middle| \mathcal{F}_{t_k-1}\right](\widetilde{\omega}) = +\infty$$

and it follows from Lemma EBC that $\sum_{t=1}^{\infty} 1_{\Omega_{1,t}^{\tau+1}}(\widetilde{\omega}) = +\infty$ P-a.s. $\widetilde{\omega} \in V_{T,\epsilon} \cap L_{\underline{\lambda},\overline{\lambda}} \cap \left\{C_t^{1,n} \text{ i.o.}\right\}$. Finally, since $\widetilde{\omega} \in L_{\underline{\lambda},\overline{\lambda}}$, we have

$$\left\{\Omega_{1,t}^{\tau+1} \text{ i.o.}\right\} \cap L_{\underline{\lambda},\overline{\lambda}} \quad \text{occurs } P-\text{a.s.}\, \widetilde{\omega} \in V_{T,\epsilon} \cap L_{\underline{\lambda},\overline{\lambda}} \cap \left\{C_t^{1,n} \text{ i.o.}\right\}.$$

This completes the induction on τ.

Hence, for every $\tau \geq 0$,

$$\left\{\Omega_{1,t}^{\tau+1} \text{ i.o.}\right\} \cap L_{\underline{\lambda},\overline{\lambda}} \quad \text{occurs } P-\text{a.s.}\, \widetilde{\omega} \in V_{T,\epsilon} \cap L_{\underline{\lambda},\overline{\lambda}} \cap \left\{C_t^{1,n} \text{ i.o.}\right\};$$

in particular,

$$\left\{\Omega_{1,t}^{T_n(\gamma)} \text{ i.o.}\right\} \cap L_{\underline{\lambda},\overline{\lambda}} \quad \text{occurs } P-\text{a.s.}\, \widetilde{\omega} \in V_{T,\epsilon} \cap L_{\underline{\lambda},\overline{\lambda}} \cap \{\omega : \limsup_t c_{1,t}(\omega) > 1/n\}$$

as required since, as already noted, $\left\{C_t^{1,n} \text{ i.o.}\right\} \supseteq \{\omega : \limsup_t c_{1,t}(\omega) > 1/n\}$.

■

PROOF OF COROLLARY 1

Since $P_1 = P_2 = P$ and $\beta_1 = \beta_2$, Theorem 1 implies that for $P-$ a.s. $\omega \in V_{T,\epsilon}$, where $T < \infty$ and $\epsilon > 0$, $y_t(\omega)$ has zero and/or infinity as limit points. Also, by Proposition 1 $\prod_{t=1}^{\widehat{T}} \hat{r}_{i,t}(\omega) \to R_i^*(\omega) \quad P-$a.s., since $P_i = P$. Therefore, by Proposition 2 (ii) and $\beta_1 = \beta_2$, we must have $R_i^*(\omega) = 0 \quad P-$ a.s. $\omega \in V_{T,\epsilon}$ for some i. The argument is completed by noting that under A.2 and A.3, $u_i'(c)$ is bounded below and so, by Proposition 2 (i), $\prod_{t=0}^{\widehat{T}} \frac{\beta_i \cdot r_{t+1}(\omega)}{q_t(\omega)} \to 0 \quad P-\text{a.s. } \omega \in V_{T,\epsilon}$. ■

PROOF OF PROPOSITION 3

For $\varepsilon > 0$, define the set

$$A_\varepsilon \equiv \Big\{ \{\omega : |q_t(\omega) - q_j(Z_t(\omega))| > \varepsilon\} \quad \text{i.o.} \Big\} \cap A. \tag{3}$$

On $\omega \in A$, $c_{j,t}(\omega) \to Z_t(\omega)$ and so there exists $T(\omega)$ such that if $t \geq T(\omega)$ then

$$\frac{1 - \frac{1}{2} \cdot \frac{\varepsilon}{\beta_j \overline{r}} \cdot \frac{u_j'(\overline{z})}{u_j'(\underline{z})}}{1 - \frac{\varepsilon}{\beta_j \overline{r}} \cdot \frac{u_j'(\overline{z})}{u_j'(\underline{z})}} \geq \frac{u_j'(c_{j,t}(\omega))}{u_j'(Z_t(\omega))} \geq 1. \tag{4}$$

Notice that

$$q_t(\omega) - q_j(Z_t(\omega)) = \beta_j \cdot E_{P_j}\left[\left(\frac{u_j'(c_{j,t+1})}{u_j'(c_{j,t})} - \frac{u_j'(Z_{t+1})}{u_j'(Z_t)} \right) \cdot r_{t+1} \middle| \mathcal{F}_t \right](\omega)$$

and so (3) and assumption A.1 imply that there is a subsequence $\{t_k\}_{k=1}^\infty$ with $t_1 \geq T(\omega)$ such that

$$P_j\left[\left(\frac{u_j'(c_{j,t_k+1})}{u_j'(c_{j,t_k})} - \frac{u_j'(Z_{t_k+1})}{u_j'(Z_{t_k})} \right) \cdot r_{t_k+1} > \frac{\varepsilon}{\beta_j} \middle| \mathcal{F}_{t_k} \right](\omega) > \underline{p} \ \text{ for all } \ k \geq 1$$

or

$$P_j\left[\left(\frac{u_j'(c_{j,t_k+1})}{u_j'(c_{j,t_k})} - \frac{u_j'(Z_{t_k+1})}{u_j'(Z_{t_k})} \right) \cdot r_{t_k+1} < -\frac{\varepsilon}{\beta_j} \middle| \mathcal{F}_{t_k} \right](\omega) > \underline{p} \ \text{ for all } \ k \geq 1.$$

Suppose $P\left[\left(\frac{u_j'(c_{j,t_k+1})}{u_j'(c_{j,t_k})} - \frac{u_j'(Z_{t_k+1})}{u_j'(Z_{t_k})} \right) \cdot r_{t_k+1} > \frac{\varepsilon}{\beta_j} \middle| \mathcal{F}_{t_k} \right](\omega) > \underline{p}$ for all $k \geq 1$. Then there is $\omega' \in \Omega(s^{t_k}(\omega))$,

$$\frac{u_j'(c_{j,t_k+1}(\omega'))}{u_j'(c_{j,t_k}(\omega'))} - \frac{u_j'(Z_{t_k+1}(\omega'))}{u_j'(Z_{t_k}(\omega'))} > \frac{\varepsilon}{\beta_j r_{t_k+1}(\omega')}$$

$$\begin{aligned}
\Leftrightarrow \quad & \frac{u_j'(c_{j,t_k+1}(\omega'))}{u_j'(c_{j,t_k}(\omega'))} > \frac{\varepsilon}{\beta_j r_{t_k+1}(\omega')} + \frac{u_j'(Z_{t_k+1}(\omega'))}{u_j'(Z_{t_k}(\omega'))} \\
\Leftrightarrow \quad & \frac{u_j'(c_{j,t_k+1}(\omega'))}{u_j'(c_{j,t_k}(\omega'))} > \frac{u_j'(Z_{t_k+1}(\omega'))}{u_j'(Z_{t_k}(\omega'))} \cdot \left(\frac{\varepsilon}{\beta_j r_{t_k+1}(\omega')} \cdot \frac{u_j'(Z_{t_k}(\omega'))}{u_j'(Z_{t_k+1}(\omega'))} + 1 \right) \\
\Leftrightarrow \quad & \frac{u_j'(c_{j,t_k+1}(\omega'))}{u_j'(Z_{t_k+1}(\omega'))} > \frac{u_j'(c_{j,t_k}(\omega'))}{u_j'(Z_{t_k}(\omega'))} \cdot \left(\frac{\varepsilon}{\beta_j r_{t_k+1}(\omega')} \cdot \frac{u_j'(Z_{t_k}(\omega'))}{u_j'(Z_{t_k+1}(\omega'))} + 1 \right)
\end{aligned}$$

and since $t_1 \geq T(\omega)$, it follows from (4) that there is $\omega' \in \Omega(s^{t_k}(\omega))$ such that,

$$\frac{u_j'(c_{j,t_k+1}(\omega'))}{u_j'(Z_{t_k+1}(\omega'))} > 1 + \frac{\varepsilon}{\beta_j \overline{r}} \cdot \frac{u_j'(\overline{z})}{u_j'(\underline{z})}.$$

Defining $\varepsilon' \equiv \frac{1}{2} \cdot \frac{\varepsilon}{\beta_j \overline{r}} \cdot \frac{u_j'(\overline{z})}{u_j'(\underline{z})} > 0$ one concludes that

$$P\left[\frac{u_j'(c_{j,t_k+1})}{u_j'(Z_{t_k+1})} > 1 + \varepsilon' \middle| \mathcal{F}_{t_k} \right](\omega) > \underline{p} \;\text{ for every } k \geq 1 \text{ on } \omega \in A_\varepsilon. \tag{5}$$

Suppose $P\left[\left(\frac{u_j'(c_{j,t_k+1})}{u_j'(c_{j,t_k})} - \frac{u_j'(Z_{t_k+1})}{u_j'(Z_{t_k})} \right) \cdot r_{t_k+1} < \frac{-\varepsilon}{\beta_j} \middle| \mathcal{F}_{t_k} \right](\omega) > \underline{p}$ for all $k \geq 1$. Then there is $\omega' \in \Omega\left(s^{t_k}(\omega)\right)$,

$$\frac{u_j'(c_{j,t_k+1}(\omega'))}{u_j'(c_{j,t_k}(\omega'))} - \frac{u_j'(Z_{t_k+1}(\omega'))}{u_j'(Z_{t_k}(\omega'))} < \frac{-\varepsilon}{\beta_j r_{t_k+1}(\omega')}$$

$$\Leftrightarrow \qquad \frac{u_j'(c_{j,t_k+1}(\omega'))}{u_j'(c_{j,t_k}(\omega'))} < \frac{-\varepsilon}{\beta_j r_{t_k+1}(\omega')} + \frac{u_j'(Z_{t_k+1}(\omega'))}{u_j'(Z_{t_k}(\omega'))}$$

$$\Leftrightarrow \qquad \frac{u_j'(c_{j,t_k+1}(\omega'))}{u_j'(c_{j,t_k}(\omega'))} < \frac{u_j'(Z_{t_k+1}(\omega'))}{u_j'(Z_{t_k}(\omega'))} \cdot \left(\frac{-\varepsilon}{\beta_j r_{t_k+1}(\omega')} \cdot \frac{u_j'(Z_{t_k}(\omega'))}{u_j'(Z_{t_k+1}(\omega'))} + 1 \right)$$

$$\Leftrightarrow \qquad \frac{u_j'(c_{j,t_k+1}(\omega'))}{u_j'(Z_{t_k+1}(\omega'))} < \frac{u_j'(c_{j,t_k}(\omega'))}{u_j'(Z_{t_k}(\omega'))} \cdot \left(1 - \frac{\varepsilon}{\beta_j r_{t_k+1}(\omega')} \cdot \frac{u_j'(Z_{t_k}(\omega'))}{u_j'(Z_{t_k+1}(\omega'))} \right)$$

$$\Leftrightarrow \qquad \frac{u_j'(c_{j,t_k+1}(\omega'))}{u_j'(Z_{t_k+1}(\omega'))} < \frac{u_j'(c_{j,t_k}(\omega'))}{u_j'(Z_{t_k}(\omega'))} \cdot \left(1 - \frac{\varepsilon}{\beta_j \overline{r}} \cdot \frac{u_j'(\overline{z})}{u_j'(\underline{z})} \right)$$

and since $t_1 \geq T(\omega)$, it follows from (4) that there is $\omega' \in \Omega\left(s^{t_k}(\omega)\right)$ such that

$$\frac{u_j'(c_{j,t_k+1}(\omega'))}{u_j'(Z_{t_k+1}(\omega'))} < 1 - \frac{1}{2} \cdot \frac{\varepsilon}{\beta_j \overline{r}} \cdot \frac{u_j'(\overline{z})}{u_j'(\underline{z})}.$$

One concludes that

$$P\left[\frac{u_j'(c_{j,t_k+1})}{u_j'(Z_{t_k+1})} < 1 - \varepsilon' \middle| \mathcal{F}_{t_k} \right](\omega) > \underline{p} \quad \text{for every } \; k \geq 1 \text{ on } \; \omega \in A_\varepsilon. \tag{6}$$

Conditions (5) and (6) imply that for each $\omega \in A_\varepsilon$ either

$$\sum_{t=0}^{\infty} P\left[\frac{u_j'(c_{j,t+1})}{u_j'(Z_{t+1})} > 1 + \varepsilon' \middle| \mathcal{F}_t \right](\omega) \geq \sum_{k=1}^{\infty} P\left[\frac{u_j'(c_{j,t_k+1})}{u_j'(Z_{t_k+1})} > 1 + \varepsilon' \middle| \mathcal{F}_{t_k} \right](\omega) = +\infty$$

or

$$\sum_{t=0}^{\infty} P\left[\frac{u_j'(c_{j,t+1})}{u_j'(Z_{t+1})} < 1 - \varepsilon' \middle| \mathcal{F}_t \right](\omega) \geq \sum_{k=1}^{\infty} P\left[\frac{u_j'(c_{j,t_k+1})}{u_j'(Z_{t_k+1})} < 1 - \varepsilon' \middle| \mathcal{F}_{t_k} \right](\omega) = +\infty$$

By Lemma *EBC*, for $P-$a.s. $\omega \in A_\varepsilon$ we have

$$\omega \in \left\{ \left\{ \tilde{\omega} : \frac{u_j'(c_{j,t}(\tilde{\omega}))}{u_j'(Z_t(\tilde{\omega}))} \geq 1 + \varepsilon' \right\} \quad \text{i.o.} \right\} \cup \left\{ \left\{ \tilde{\omega} : \frac{u_j'(c_{j,t}(\tilde{\omega}))}{u_j'(Z_t(\tilde{\omega}))} \leq 1 - \varepsilon' \right\} \quad \text{i.o.} \right\}. \tag{7}$$

Since u_j' is continuous (by A.3), (7) implies $c_{j,t}(\omega)$ does not converge to $Z_t(\omega)$ $P-$a.s. $\omega \in A_\varepsilon$. Since $A_\varepsilon \subset A$ then $P(A_\varepsilon) = 0$, as desired. ∎

PROOF OF PROPOSITION 4

First we show that on paths on which an agent vanishes, the rate of return violates any prespecified bound. This is done in two steps.

Define the event Ω_t^n as

$$\Omega_t^n \equiv \left\{ \omega : \prod_{\tau=t-n}^{t-1} \frac{r(Z_{\tau+1}(\omega))}{q_j(Z_\tau(\omega))} \geq \frac{u_j'(\overline{z})}{u_j'(\underline{z})} \cdot \left(\frac{1}{\beta_j}\right)^n \right\}.$$

LEMMA 5: Assume A.1, A.2 and A.3. Then, $P-$a.s. ω,

$$\omega \in \{\Omega_t^n \text{ i.o.}\} \text{ for all } n \in \{1, 2, 3, \cdots\}.$$

PROOF: From the Euler equation of agent $j \neq i$, in the economy where only j consumes, and the law of iterated expectations, it follows that

$$E_{P_j}\left(\frac{u_j'(Z_t)}{u_j'(Z_{t-n})} \cdot \prod_{\tau=t-n}^{t-1} \frac{r(Z_{\tau+1})}{q_j(Z_\tau)} \right| Z_{t-n} = z \Bigg)(\omega) = \left(\frac{1}{\beta_j}\right)^n \text{ for all } z \in \mathcal{S} \text{ and } n \geq 1,$$

and then by A.3

$$E_{P_j}\left(\prod_{\tau=t-n}^{t-1} \frac{r(Z_{\tau+1})}{q_j(Z_\tau)} \right| Z_{t-n} = z \Bigg)(\omega) \geq \frac{u_j'(\overline{z})}{u_j'(\underline{z})} \cdot \left(\frac{1}{\beta_j}\right)^n \text{ for all } z \in \mathcal{S} \text{ and } n \geq 1.$$

It follows that

$$P_j\left(\prod_{\tau=t-n}^{t-1} \frac{r(Z_{\tau+1})}{q_j(Z_\tau)} \geq \frac{u_j'(\overline{z})}{u_j'(\underline{z})} \cdot \left(\frac{1}{\beta_j}\right)^n \right| Z_{t-n} = z \Bigg)(\omega) > 0,$$

and so, assumption A.1 implies that

$$P\left(\prod_{\tau=t-n}^{t-1} \frac{r(Z_{\tau+1})}{q_j(Z_\tau)} \geq \frac{u_j'(\overline{z})}{u_j'(\underline{z})} \cdot \left(\frac{1}{\beta_j}\right)^n \right| Z_{t-n} = z \Bigg)(\omega) > \left(\underline{p}\right)^n.$$

Therefore,

$$P(\Omega_t^n | Z_{t-n} = z)(\omega) \geq \left(\underline{p}\right)^n > 0 \text{ for all } z \in \mathcal{S} \text{ and } n \geq 1,$$

where the inequality follows by A.1, and it follows by Lemma *EBC* that $\omega \in \{\Omega_t^n \text{ i.o.}\}$ for $P-$a.s. ω. ∎

LEMMA 6: Consider an IDC equilibrium. Assume A.1, A.2, A.3 and A.5 holds. Then, $P-$a.s. $\omega \in A$,

$$\omega \in \left\{ \left\{ \omega' : \prod_{\tau=t-n}^{t-1} \frac{r_{\tau+1}(\omega')}{q_\tau(\omega')} > \frac{1}{2} \cdot \frac{u_j'(\overline{z})}{u_j'(\underline{z})} \cdot \left(\frac{1}{\beta_j}\right)^n \right\} \text{ i.o.} \right\} \text{ for all } n \in N_+.$$

PROOF: By Proposition 2 there exists $T_n(\omega)$ such that for all $t \geq T_n(\omega)$,

$$\prod_{\tau=t-n}^{t-1} \frac{r_{\tau+1}(\omega)}{q_\tau(\omega)} \geq \frac{1}{2} \cdot \prod_{\tau=t-n}^{t-1} \frac{r(Z_{\tau+1}(\omega))}{q_j(Z_\tau(\omega))}.$$

By Lemma 5, $P-$a.s. $\omega \in A$, there exists a subsequence $\{t_k\}_{k=1}^{\infty}$ such that

$$\prod_{\tau=t_k-n}^{t_k-1} \frac{r(Z_{\tau+1}(\omega))}{q_j(Z_\tau(\omega))} > \frac{u_j'(\overline{z})}{u_j'(\underline{z})} \cdot \left(\frac{1}{\beta_j}\right)^n.$$

Then, there exists k' such that $t_{k'} \geq T_n(\omega)$. It follows that for all $k \geq k'$

$$\prod_{\tau=t_k-n}^{t_k-1} \frac{r_{\tau+1}(\omega)}{q_\tau(\omega)} \geq \frac{1}{2} \cdot \frac{u_j'(\overline{z})}{u_j'(\underline{z})} \cdot \left(\frac{1}{\beta_j}\right)^n.$$

■

We proceed with the proof of Proposition 4.

From agent i's sequential budget constraint it follows that

$$\frac{1}{\frac{r_{t+1}(\omega)}{q_t(\omega)}} \cdot [c_{i,t+1}(\omega) - z_{i,t+1}(\omega)] + \frac{1}{\frac{r_{t+1}(\omega)}{q_t(\omega)}} \cdot B_{i,t+1}(\omega) = B_{i,t}(\omega).$$

Solving forward, we have that

$$B_{i,t}(\omega) = \frac{c_{i,t+1}(\omega) - z_{i,t+1}(\omega)}{\frac{r_{t+1}(\omega)}{q_t(\omega)}} + \sum_{s=2}^{T} \left(\frac{c_{i,t+s}(\omega) - z_{i,t+s}(\omega)}{\prod_{\tau=t}^{t+s-1} \frac{r_{\tau+1}(\omega)}{q_\tau(\omega)}} \right) + \frac{B_{i,t+T}(\omega)}{\prod_{\tau=t}^{t+T-1} \frac{r_{\tau+1}(\omega)}{q_\tau(\omega)}}. \quad (8)$$

Let $K = \frac{\frac{1}{2}\underline{z}_i}{\frac{\overline{r}}{\min_z q_j(z)}}$. By A.4 and since $\underline{z}_i > 0$, Proposition 3 implies that, $P-$a.s. $\omega \in A$, there is $T(\omega)$ such that

$$\frac{1}{\frac{r_{t+1}(\omega)}{q_t(\omega)}} \cdot [c_{i,t+1}(\omega) - z_{i,t+1}(\omega)] < -\frac{1}{2} \cdot \frac{\underline{z}_i}{\frac{\overline{r}}{\min_z q_j(z)}} = -K \quad \text{for every } t \geq T(\omega). \quad (9)$$

By the uniform bounds condition, there exists $U > 0$ such that $B_{i,t+T}(\omega) \leq U$ for all $T \geq 1$. Since $\beta_j < 1$, there is n such that $\frac{1}{2} \cdot \frac{u_j'(\overline{z})}{u_j'(\underline{z})} \cdot \left(\frac{1}{\beta_j}\right)^n > 2 \cdot \frac{U}{K}$ and by Lemma 6 there is $\{t_k'\}_{k=1}^{\infty}$ with $t_1' \geq T(\omega) + n$ such that

$$\prod_{\tau=t_k'-n}^{t_k'-1} \frac{r_{\tau+1}(\omega)}{q_\tau(\omega)} > 2 \cdot \frac{U}{K} \quad \text{for every } k \geq 1 \ P-\text{a.s. } \omega \in A. \quad (10)$$

For each $k \in \{1, 2, ...\}$, set $t_k \equiv t_k' - n$. Then, (8), (9) and (10) imply that

$$B_{i,t_k}(\omega) \leq -K + \frac{K}{2} = -\frac{K}{2} < 0 \qquad P-\text{a.s. } \omega \in A.$$

Therefore,

$$\liminf B_{i,t}(\omega) \leq -\frac{K}{2} < 0 \qquad P-\text{a.s. } \omega \in A.$$

■

PROOF OF THEOREM 2

Before proving Theorem 2, we show that the debt of the agent who vanishes cannot have a nonnegative accumulation point and that the limiting asset return when a state is repeated exceeds one for every state (Lemma 7). We then show that these two facts have the following strong implication: given a state, except for an initial condition that is a single point, if that state keeps repeating then either debt becomes positive or it violates the uniform lower bound (Lemma 8).

LEMMA 7: Consider an IDC equilibrium with implicit bound $U > 0$. Assume A.1, A.2, A.3, A.4 and A.5 hold and $\underline{z}_i > 0$. Then,
(i) $\limsup B_{i,t}(\omega) < 0$ for $P-$a.s. $\omega \in A$.
(ii) If the set A has positive measure, then $\frac{r(z)}{q_j(z)} \geq 1 + \frac{1}{2}\frac{z_i(z)}{U}$ for all $z \in \mathcal{S}$.
PROOF: Define $A_q \equiv A \cap \{\omega : q_t(\omega) \to q_j(Z_t(\omega))\}$. By Proposition 3 it suffices to show that the Lemma holds for $P-$a.s. $\omega \in A_q$. Suppose $\omega \in A_q$. Then there is $T^*(\omega)$ such that $z_{i,t+1}(\omega) - c_{i,t+1}(\omega) > 0$ for all $t \geq T^*(\omega)$. Therefore, for every $t \geq T^*(\omega)$,

$$B_{i,t+1}(\omega) = \frac{r_{t+1}(\omega)}{q_t(\omega)} \cdot B_{i,t}(\omega) + z_{i,t+1}(\omega) - c_{i,t+1}(\omega) > \frac{r_{t+1}(\omega)}{q_t(\omega)} \cdot B_{i,t}(\omega).$$

It follows that,

$$B_{i,t}(\omega) \geq 0 \quad \text{and} \quad t \geq T^*(\omega) \quad \Rightarrow \quad B_{i,t+k}(\omega) > 0 \quad \text{for all } k \geq 1. \tag{11}$$

Recall that $B(z,z') \equiv -\frac{z_i(z')}{\frac{r(z')}{q_j(z)}-1}$ and define $\overline{B} \equiv \max_{z,z'\in\mathcal{S}} \{B(z,z') : B(z,z') < 0\}$. Consider B such that $\overline{B} < B < 0$ and $z, z' \in \mathcal{S}$. Suppose first that $\frac{r(z')}{q_j(z)} - 1 \leq 0$. Since $B < 0$, then $\left(\frac{r(z')}{q_j(z)} - 1\right) \cdot B + z_i(z') > 0$. Suppose now that $\frac{r(z')}{q_j(z)} - 1 > 0$. Since $\overline{B} < B$, it follows from the definitions of $B(z,z')$ and $\overline{B}$ that $-\frac{z_i(z')}{\frac{r(z')}{q_j(z)}-1} < B$ and we conclude that

$$\overline{B} < B < 0 \quad \Rightarrow \quad \left(\frac{r(z')}{q_j(z)} - 1\right) \cdot B + z_i(z') > 0 \quad \text{for all } z, z' \in \mathcal{S}. \tag{12}$$

(12) implies that, for every $\omega \in A_q$, there exists $\varepsilon' > 0$ and $T(\omega) \geq T^*(\omega)$ such that for all $t \geq T(\omega)$

$$\overline{B} < B < 0 \quad \Rightarrow \quad \left(\frac{r_{t+1}(\omega)}{q_t(\omega)} - 1\right) \cdot B + z_{i,t+1}(\omega) - c_{i,t+1}(\omega) > \varepsilon' > 0. \tag{13}$$

Also, for $\omega \in A_q$, $T(\omega) \geq T^*(\omega)$ can be chosen so that for all $t \geq T(\omega)$ and $z \in \mathcal{S}$,

$$\frac{1}{4} \cdot z_i(z) \geq c_{i,t}(\omega), \tag{14}$$

$$\frac{r(z)}{q_j(z)} < 1 + \frac{1}{2}\frac{z_i(z)}{U} \quad \text{and} \quad Z_{t+1}(\omega) = Z_t(\omega) = z \quad \Rightarrow \quad \frac{r_{t+1}(\omega)}{q_t(\omega)} < 1 + \frac{1}{2}\frac{z_i(z)}{U}. \tag{15}$$

Consider $A' \subset A_q$. If one were able to show that, $P-$a.s. $\omega \in A'$, there exists a date $t \geq T(\omega)$ such that $B_{i,t}(\omega) \geq 0$ then it would follow from (11) that, $P-$a.s. $\omega \in A'$, $\liminf B_{i,t}(\omega) \geq 0$ and by Proposition 3 one would conclude that A' has zero measure. To show (i) we argue that for $P-$a.s. $\omega \in A_q \cap \{\limsup B_{i,t}(\omega) \geq 0\}$, there exists a date $t \geq T(\omega)$ such that $B_{i,t}(\omega) \geq 0$. To show (ii) we argue that if $\frac{r(z)}{q_j(z)} < 1 + \frac{1}{2}\frac{z_i(z)}{U}$ for some $z \in \mathcal{S}$, then for $P-$a.s. $\omega \in A_q$ there exists a date $t \geq T(\omega)$ such that $B_{i,t}(\omega) \geq 0$.

(i) Suppose $\omega \in A_q \cap \{\limsup B_{i,t}(\omega) \geq 0\}$. Since $\limsup B_{i,t}(\omega) \geq 0 > \overline{B}$, there exists $\tau \geq T(\omega)$ such that $\overline{B} < B_{i,\tau}(\omega)$. If $B_{i,\tau}(\omega) \geq 0$ then we are done; so assume that $B_{i,\tau}(\omega) < 0$. Since

$$B_{i,t+1}(\omega) - B_{i,t}(\omega) = \left(\frac{r_{t+1}(\omega)}{q_t(\omega)} - 1\right) \cdot B_{i,t}(\omega) + z_{i,t+1}(\omega) - c_{i,t+1}(\omega),$$

(13) implies that

$$B_{i,\tau+1}(\omega) - B_{i,\tau}(\omega) \geq \varepsilon' > 0 \quad \text{and} \quad B_{i,\tau+T}(\omega) \geq B_{i,\tau}(\omega) + T \cdot \varepsilon'.$$

Since $B_{i,\tau}(\omega) \geq \overline{B} > -\infty$, there exists $t \geq T(\omega)$ such that $B_{i,t}(\omega) > 0$ as we wished to prove.

(ii) Suppose $\frac{r(z)}{q_j(z)} < 1 + \frac{1}{2}\frac{z_i(z)}{U}$ for some $z \in \mathcal{S}$. Consider an $\omega \in A_q$ such that for some $t \geq T(\omega)$, $Z_{t+1}(\omega) = Z_t(\omega) = z$ and $B_{i,t}(\omega) \leq 0$. Then, on $\omega \in A_q$, conditions (14) and (15) imply that

$$\begin{aligned}
B_{i,t+1}(\omega) - B_{i,t}(\omega) &= \left(\frac{r_{t+1}(\omega)}{q_t(\omega)} - 1\right) \cdot B_{i,t}(\omega) + z_{i,t+1}(\omega) - c_{i,t+1}(\omega) \\
&\geq \left(\frac{r_{t+1}(\omega)}{q_t(\omega)} - 1\right) \cdot B_{i,t}(\omega) + \frac{3}{4} \cdot z_i(z) \\
&\geq \left(\frac{r_{t+1}(\omega)}{q_t(\omega)} - 1\right) \cdot (-U) + \frac{3}{4} \cdot z_i(z) \\
&> -\frac{1}{2}z_i(z) + \frac{3}{4} \cdot z_i(z) = \frac{z_i(z)}{4}.
\end{aligned} \tag{16}$$

For $\widetilde{\varepsilon} > 0$, let $N_{z,\widetilde{\varepsilon}}$ be the smallest positive integer such that $N \cdot \frac{z_i(z)}{4} > U + \widetilde{\varepsilon}$. Define the event

$$\Omega_t^{N_{z,\widetilde{\varepsilon}}} \equiv \left\{\omega : Z_\tau(\omega) = z \quad \text{for} \quad t - N_{z,\widetilde{\varepsilon}} \leq \tau \leq t\right\}.$$

It follows by A.1 that $P\left(\Omega_t^{N_{z,\widetilde{\varepsilon}}}\middle|\mathcal{F}_{t-N}\right)(\omega) \geq \left(\underline{p}\right)^{N_{z,\widetilde{\varepsilon}}}$ and so by Lemma EBC one concludes that, $P-$a.s. $\omega \in A$, $\omega \in \left\{\Omega_t^{N_{z,\widetilde{\varepsilon}}} \quad \text{i.o.}\right\}$. So, $P-$a.s. $\omega \in A_q$, there is a subsequence $\{t_k\}_{k=1}^{\infty}$ where $t_1 \geq T(\omega) + N_{z,\widetilde{\varepsilon}}$ and $Z_\tau(\omega) = z$ for all $t_k - N_{z,\widetilde{\varepsilon}} \leq \tau \leq t_k$ and for all $k \geq 1$. It follows by (14) and (15) that, $P-$a.s. $\omega \in A_q$, either there exists some τ such that $t_k - N_{z,\widetilde{\varepsilon}} \leq \tau \leq t_k - 1$ and $B_{i,\tau}(\omega) \geq 0$ or

$$\begin{aligned}
B_{i,t_k}(\omega) &= B_{i,t_k-N_{z,\widetilde{\varepsilon}}}(\omega) + \sum_{\tau=t_k-N_{z,\widetilde{\varepsilon}}}^{t_k-1} \left[B_{i,\tau+1}(\omega) - B_{i,\tau}(\omega)\right] \\
&> -U + \sum_{\tau=t_k-N_{z,\widetilde{\varepsilon}}}^{t_k-1} \left[B_{i,\tau+1}(\omega) - B_{i,\tau}(\omega)\right] \\
&\geq -U + N_{z,\widetilde{\varepsilon}} \cdot \frac{z_i(z)}{4} > \widetilde{\varepsilon} > 0,
\end{aligned}$$

where the inequality in the last line follows from (16). It follows that, $P-$a.s. $\omega \in A$, there exists a date $t \geq T(\omega)$ such that $B_{i,t}(\omega) \geq 0$. ■

For $z \in \mathcal{S}$, $\varepsilon > 0$, and $N \in \{0, 1, 2, \cdots\}$, define
$\Omega_t^+(z, \varepsilon, N) \equiv \{\omega \in A : B(z, z) + \varepsilon \leq B_{i,t-N}(\omega) \quad Z_{t-N}(\omega) = \cdots = Z_t(\omega) = z\}$,
$\Omega_t^-(z, \varepsilon, N) \equiv \{\omega \in A : B_{i,t-N}(\omega) \leq B(z, z) - \varepsilon \quad Z_{t-N}(\omega) = \cdots = Z_t(\omega) = z\}$.

LEMMA 8: Consider an IDC equilibrium. Assume A.1, A.2, A.3, A.4 and A.5, $\underline{z}_i > 0$ and let $z \in \mathcal{S}$ be such that $\frac{r(z)}{q_j(z)} > 1$. Then, for every $\varepsilon > 0$
(a) $\limsup B_{i,t}(\omega) \geq 0$ $P-$a.s. $\omega \in \left\{\Omega_t^+(z, \varepsilon, 0) \quad \text{i.o.}\right\}$,
(b) $\liminf B_{i,t}(\omega) < -U$ $P-$a.s. $\omega \in \left\{\Omega_t^-(z, \varepsilon, 0) \quad \text{i.o.}\right\}$.
PROOF: (a) Let $A_q \equiv A \cap \{\tilde{\omega} : q_t(\tilde{\omega}) \to q_j(Z_t(\tilde{\omega}))\}$. Throughout the proof, z is an element of $\mathcal{S}$ such that $\frac{r(z)}{q_j(z)} > 1$.

By the definition of $B(z, z)$, $\left(\frac{r(z)}{q_j(z)} - 1\right) \cdot B(z, z) + z_i(z) = 0$. So, for $\varepsilon > 0$, we have

$$\left(\frac{r(z)}{q_j(z)} - 1\right) \cdot (B(z, z) + \varepsilon) + z_i(z) > 0.$$

Therefore, for $\omega \in A_q$, there is $T(\omega)$ and $\varepsilon' > 0$ such that if $\omega \in \Omega_{t+1}^+(z, \varepsilon, 1)$ for some $t + 1 \geq T(\omega)$ then

$$B_{i,t+1}(\omega) - B_{i,t}(\omega) = \left[\frac{r_{t+1}(\omega)}{q_t(\omega)} - 1\right] \cdot B_{i,t}(\omega) + z_{i,t+1}(\omega) - c_{i,t+1}(\omega) \geq \varepsilon' > 0.$$

If the initial condition is appropriate and the state z is realized at consecutive dates, then the argument may be applied repeatedly. Formally, by iterating the argument, if $t + 1 \geq T(\omega)$ then, for $N = 1, 2, 3, \cdots$,

$$A_q \cap \Omega_{t+N}^+(z, \varepsilon, N) \subset A_q \cap \left(\cap_{k=1}^N \Omega_{t+k}^+(z, \varepsilon, 1)\right)$$

so that if $\omega \in A_q \cap \Omega^+_{t+N}(z,\varepsilon,N)$ and $t+1 \geq T(\omega)$, then

$$B_{i,t+N}(\omega) = B_{i,t}(\omega) + \sum_{\tau=t}^{t+N-1} [B_{i,\tau+1}(\omega) - B_{i,\tau}(\omega)] \geq B(z,z) + N \cdot \varepsilon'. \qquad (*)$$

Let N_z be the smallest positive integer such that $B(z,z) + N_z \cdot \varepsilon' > 0$. By (*), if $\omega \in A_q \cap \Omega^+_{t+N_z}(z,\varepsilon,N_z)$ and $t+1 \geq T(\omega)$ then $B_{i,t+N_z}(\omega) \geq B(z,z) + N_z \cdot \varepsilon' > 0$.

So, to show that $\limsup B_{i,t}(\omega) > 0$ for $P-$a.s. $\omega \in \left\{\Omega^+_t(z,\varepsilon,0) \quad \text{i.o.}\right\}$, it suffices to show that for $P-$a.s. $\omega \in \left\{\Omega^+_t(z,\varepsilon,0) \quad \text{i.o.}\right\}$, $\omega \in A_q \cap \left\{\Omega^+_t(z,\varepsilon,N_z) \quad \text{i.o.}\right\}$.

By hypothesis, for each $\omega \in \left\{\Omega^+_t(z,\varepsilon,0) \quad \text{i.o.}\right\}$, there exists $\{t'_k\}_{k=1}^{\infty}$ such that $B(z,z) + \varepsilon \leq B_{i,t'_k}(\omega)$ and $Z_{t'_k}(\omega) = z$ for all $k \geq 1$. Given a positive integer N, define $t_k = t'_k + N$. Therefore, for $\omega \in \left\{\Omega^+_t(z,\varepsilon,0) \quad \text{i.o.}\right\}$ and any positive integer N, there exists $\{t_k\}_{k=1}^{\infty}$ such that

$$\begin{aligned}\sum_{t=0}^{\infty} P\left[\Omega^+_t(z,\varepsilon,N)\middle| \mathcal{F}_{t-N}\right](\omega) &\geq \sum_{k=1}^{\infty} P\left[\Omega^+_{t_k}(z,\varepsilon,N)\middle| \mathcal{F}_{t_k-N}\right](\omega) \\ &= \sum_{k=1}^{\infty} P\left[Z_{t_k+1-N}(\omega) = ... = Z_{t_k}(\omega) = z \middle| \mathcal{F}_{t_k-N}\right](\omega) = +\infty\end{aligned}$$

where the last equality follows by A.1. Lemma EBC lets us conclude that for every positive integer N, $P-$a.s. $\omega \in \left\{\Omega^+_t(z,\varepsilon,0) \quad \text{i.o.}\right\}$, $\omega \in \left\{\Omega^+_t(z,\varepsilon,N) \quad \text{i.o.}\right\}$. In particular, since $P(A/A_q) = 0$ by Proposition 3, $P-$a.s. $\omega \in \left\{\Omega^+_t(z,\varepsilon,0) \quad \text{i.o.}\right\}$, $\omega \in A_q \cap \left\{\Omega^+_t(z,\varepsilon,N_z) \quad \text{i.o.}\right\}$. That is the desired result.

An analogous argument proves (b). ∎

We proceed with the proof of Theorem 2. By way of contradiction, suppose that $P(A) > 0$, where A is the set on which i vanishes. Then, by Lemma 7 (ii), on A, for every state z, $\frac{r(z)}{q_j(z)} > 1$. By Lemma 8, when such a state z keeps repeating, the only stable point is the value $B(z,z)$. We now show that transitions across states do not restore stability except under the exacting conditions ruled out in the statement of the theorem. Let $A^* \subset A$ be the set of paths with each of which is associated a $z^*(\omega)$ with the property that the smallest accumulation point of debt on the path coincides with $B(z^*(\omega), z^*(\omega))$ (which, by Proposition 4, must be negative). We first show that $P(A) = P(A^*)$ and so $P(A^*) > 0$. We then show that if the condition in Theorem 2 is violated, then, on any path ω such that the value of debt is at or near $B(z^*(\omega), z^*(\omega))$ infinitely many times, there will be infinitely many periods in which both $B(z,z)$ is not hit and z is realized, where $z \neq z^*(\omega)$. On any such path, by Lemma 8, either $B_{i,t}$ has a positive accumulation point or the lower bound on debt is violated infinitely often. Since we are at an IDC equilibrium, and invoking Lemma 7 (i), both are impossible. Since the conditioning event contains A^*, it follows that A^* must have zero measure thereby providing a contradiction. But then A must have zero measure as claimed.

We turn to the details of the proof. Suppose that A has positive measure. By Lemma 7 (ii) we know that $\frac{r(z)}{q_j(z)} > 1$ for all $z \in \mathcal{S}$, equivalently, $B(z,z) < 0$ for all $z \in \mathcal{S}$.

Step (i) Let

$$A^* \equiv \{\omega \in A : \liminf B_{i,t}(\omega) = B[z^*(\omega), z^*(\omega)], \text{some } z^*(\omega) \in \mathcal{S}\}.$$

We now show that $A \backslash A^*$ has $P-$measure zero. It suffices to show that for every $\varepsilon > 0$,

$$A_\varepsilon \equiv \{\omega \in A : |\liminf B_{i,t}(\omega) - B(\tilde{z}, \tilde{z})| \geq \varepsilon, \forall \tilde{z} \in \mathcal{S}\}$$

is a $P-$measure zero subset of A. Consider $z \in \mathcal{S}$. By Proposition 4, for $P-$a.s. $\omega \in A_\varepsilon$, there exists $\{t'_k\}_{k=1}^{\infty}$ such that $z_{t'_k} = z$ and either $B(z,z) + \varepsilon \leq B_{i,t'_k}(\omega) < 0$ or $B_{i,t'_k}(\omega) \leq B(z,z) - \varepsilon$ for all $k \geq 1$. Since we know that $\frac{r(z)}{q_j(z)} > 1$, by Lemma 8 $P-$a.s. $\omega \in A_\varepsilon$, either $\limsup B_{i,t}(\omega) \geq 0$ or $\liminf B_{i,t}(\omega) < -U$. By the uniform bounds condition and Lemma 7 (i) we conclude that A_ε is a $P-$measure zero subset of A.

So if A has positive $P-$measure then A^* must have positive $P-$measure.

Step (ii) For $z \in \mathcal{S}$ and $n \in N$, consider the set $A_{z,n}$ defined as

$$A_{z,n} \equiv \left\{\omega \in A^* : \left|\frac{r(z)}{q_j(z^*(\omega))} \cdot B[z^*(\omega), z^*(\omega)] + z_i(z) - B(z,z)\right| > \frac{2}{n}\right\}.$$

By the hypothesis of Theorem 2, $A^* = \cup_{n \in N} \cup_{z \in \mathcal{S}} A_{z,n}$. Notice that for each $\omega \in A_{z,n}$, there exists $\delta > 0$ such that, for $z \neq z^*(\omega)$,

$$|B - B(z^*(\omega), z^*(\omega))| \leq \delta \Rightarrow \left|\frac{r(z)}{q_j(z^*(\omega))} \cdot B + z_i(z) - B(z,z)\right| > \frac{2}{n}. \tag{17}$$

Consider the event

$$\Omega_t^{z,n} \equiv \left\{\omega : |B_{i,t}(\omega) - B(z,z)| > \frac{1}{n}, Z_t(\omega) = z\right\}.$$

If for $P-$a.s. $\omega \in A_{z,n}$ one could find a subsequence $\{t_k\}_{k=1}^{\infty}$ such that $P\left(\Omega_{t_k+1}^{z,n}\middle|\mathcal{F}_{t_k}\right)(\omega) \geq \underline{p}$ for every k, then it would be the case that

$$\sum_{t=1}^{\infty} P\left(\Omega_t^{z,n}\middle|\mathcal{F}_{t-1}\right)(\omega) \geq \sum_{k=1}^{\infty} P\left(\Omega_{t_k+1}^{z,n}\middle|\mathcal{F}_{t_k}\right)(\omega) = +\infty.$$

Therefore, one would be able to invoke Lemma EBC to argue that, $P-$a.s. $\omega \in A_{z,n}$, $\omega \in \{\Omega_t^{z,n} \text{ i.o.}\}$. By Lemma 8, one would conclude that, $P-$a.s. $\omega \in A_{z,n}$, either $\limsup B_{i,t}(\omega) \geq 0$ or $\liminf B_{i,t}(\omega) < -U$. Since the latter is a violation of the uniform bounds condition, we would conclude that, $P-$a.s. $\omega \in A_{z,n}$, $\limsup B_{i,t}(\omega) \geq 0$ which implies that, $P-$a.s. $\omega \in A^*$, $\limsup B_{i,t}(\omega) \geq 0$. But then Lemma 7 (i) would let us conclude that A^* has $P-$measure zero.

Step (iii) We now show that indeed, $P-$a.s. $\omega \in A_{z,n}$, there is a subsequence $\{t_k\}_{k=1}^{\infty}$ such that $P\left(\Omega_{t_k+1}^{z,n}\middle|\mathcal{F}_{t_k}\right)(\omega) \geq \underline{p}$ for every k.

By Proposition 3 and the uniform bounds on debt, $P-$a.s. $\omega \in A$,

$$\left(\frac{r_{t+1}(\omega)}{q_t(\omega)} - \frac{r(Z_{t+1}(\omega))}{q_j(Z_t(\omega))}\right) \cdot B_{i,t}(\omega) \to 0$$

so that

$$\left(\frac{r_{t+1}(\omega)}{q_t(\omega)} \cdot B_{i,t}(\omega) + z_i(Z_{t+1}(\omega))\right) - \left(\frac{r(Z_{t+1}(\omega))}{q_j(Z_t(\omega))} \cdot B_{i,t}(\omega) + z_i(Z_{t+1}(\omega))\right) \to 0.$$

Also, on $\omega \in A$,

$$B_{i,t+1}(\omega) - \left(\frac{r_{t+1}(\omega)}{q_t(\omega)} \cdot B_{i,t}(\omega) + z_i(Z_{t+1}(\omega))\right) = -c_{i,t+1}(\omega) \to 0.$$

It follows that $P-$a.s. $\omega \in A$, there exists $T(\omega)$ such that for every $t \geq T(\omega)$,

$$\left|B_{i,t+1}(\omega) - \left(\frac{r(Z_{t+1}(\omega))}{q_j(Z_t(\omega))} \cdot B_{i,t}(\omega) + z_i(Z_{t+1}(\omega))\right)\right| \leq \frac{1}{n}. \tag{18}$$

In particular, since $A_{z,n} \subset A^* \subset A$, (18) holds for $P-$a.s. $\omega \in A_{z,n}$.

A straightforward application of Lemma EBC together with Lemma 8 (b) implies that, $P-$a.s. $\omega \in A^*$, there is a subsequence $\{t_k\}_{k=1}^{\infty}$ such that $Z_{t_k}(\omega) = z^*(\omega)$, $|B_{i,t_k}(\omega) - B[z^*(\omega), z^*(\omega)]| \leq \delta$, and $t_k \geq T(\omega)$ for all $k \geq 1$. By the triangular inequality and (18), $P-$a.s. $\omega \in A_{z,n}$, for every $k \geq 1$ it must be the case that

$$\begin{aligned} |B_{i,t_k+1}(\omega) - B(z,z)| &> \left|\left(\frac{r(Z_{t_k+1}(\omega))}{q_j(Z_{t_k}(\omega))} \cdot B_{i,t_k}(\omega) + z_i(Z_{t_k+1}(\omega))\right) - B(z,z)\right| - \\ &\quad \left|B_{i,t_k+1}(\omega) - \left(\frac{r(Z_{t_k+1}(\omega))}{q_j(Z_{t_k}(\omega))} \cdot B_{i,t_k}(\omega) + z_i(Z_{t_k+1}(\omega))\right)\right| \\ &\geq \left|\left(\frac{r(Z_{t_k+1}(\omega))}{q_j(z^*(\omega))} \cdot B_{i,t_k}(\omega) + z_i(Z_{t_k+1}(\omega))\right) - B(z,z)\right| - \frac{1}{n}, \end{aligned}$$

and since $\omega \in A_{z,n}$ and $|B_{i,t_k}(\omega) - B[z^*(\omega), z^*(\omega)]| \leq \delta$, it follows by (17) that, $P-$a.s. $\omega \in A_{z,n}$, one has

$$P\left(\Omega^{z,n}_{t_k+1} \middle| \mathcal{F}_{t_k}\right)(\omega) \geq P\left(Z_{t_k+1}(\omega) = z \middle| \mathcal{F}_{t_k}\right)(\omega) \geq \underline{p}$$

for every $k \geq 1$, as desired.

Step (iv) To complete the proof of Theorem 2 note that Steps (ii) and (iii) prove that A^* has $P-$ measure zero, while Step (i) proved that if A has positive $P-$measure then so does A^*. So our initial hypothesis about A is contradicted and we can conclude that A must have $P-$ measure zero as asserted in Theorem 2. ■

PROOF OF PROPOSITION 5

Let $i = 1$. This is without loss of generality.

Let us define a sequence of truncated processes parameterized by $\epsilon > 0$ by setting $g^{\epsilon}_{1,t}(\omega) \equiv \log(\max\{\hat{r}_{1,t}(\omega), \epsilon\})$ and $\mathcal{B}_{1,\epsilon} \equiv \{\omega : \limsup \frac{1}{T}\sum_{t=1}^{T} E_{P_1}[g^{\epsilon}_{1,t}|\mathcal{F}_{t-1}](\omega) < 0\}$.

Ω can be partitioned into three sets: $\cup_{n\geq 1}\mathcal{B}_{1,1/n}$, $\mathcal{A}_1$, and $\Omega/(\mathcal{A}_1 \cup (\cup_{n\geq 1}\mathcal{B}_{1,1/n}))$, where $\mathcal{A}_1 \equiv \{\omega \in \Omega : \liminf \hat{r}_{1,t}(\omega) = 0\}$. We first show that under A.7 the third set is null.

LEMMA 9: Assume A.7. Then $\Omega/\mathcal{A}_1 \subset \cup_{n\geq 1}\mathcal{B}_{1,1/n}$, where $\mathcal{A}_1 \equiv \{\omega : \liminf \hat{r}_{1,t}(\omega) = 0\}$, so that for all $\omega \in \Omega/\mathcal{A}_1$ there exists $\epsilon(\omega)$ such that $\omega \in \mathcal{B}_{1,\epsilon(\omega)}$.

PROOF: Consider $\tilde{\omega} \in \Omega/\mathcal{A}_1$. So liminf $\hat{r}_{1,t}(\tilde{\omega}) = 2 \cdot \epsilon(\tilde{\omega}) > 0$ and there exists $t(\tilde{\omega})$ such that $t \geq t(\tilde{\omega}) \quad \Rightarrow \quad \hat{r}_{1,t}(\tilde{\omega}) \geq \epsilon(\tilde{\omega})$. Since, by A.7,

$$\text{limsup}\left(\frac{1}{T}\sum_{t=1}^{T} E_{P_1}[\log \hat{r}_{1,t}|\mathcal{F}_{t-1}](\tilde{\omega})\right) < 0,$$

there exists $\epsilon'(\tilde{\omega}) < 0$ such that

$$\text{limsup}\left(\frac{1}{T}\sum_{t=1}^{T} E_{P_1}[\log \hat{r}_{1,t}|\mathcal{F}_{t-1}](\tilde{\omega})\right) < \epsilon'(\tilde{\omega}).$$

So for every sequence $\{T_k\}_{k=1}^{\infty}$, there exists k' such that for all $k \geq k'$,

$$\frac{1}{T_k}\sum_{t=1}^{T_k} E_{P_1}[\log \hat{r}_{1,t}|\mathcal{F}_{t-1}](\tilde{\omega}) < \epsilon'(\tilde{\omega}).$$

Clearly, for each such sequence, there also exists $\bar{k}$ such that for all $k \geq \bar{k}$,

$$\frac{1}{T_k}\sum_{t=1}^{t(\tilde{\omega})} E_{P_1}[\log \hat{r}_{1,t}|\mathcal{F}_{t-1}](\tilde{\omega}) \geq \frac{\epsilon'(\tilde{\omega})}{4}.$$

It follows that, for every sequence $\{T_k\}_{k=1}^{\infty}$, for all $k \geq \max\{k', \bar{k}\}$,

$$\begin{aligned}\frac{1}{T_k}\sum_{t=t(\tilde{\omega})+1}^{T_k} E_{P_1}[\log \hat{r}_{1,t}|\mathcal{F}_{t-1}](\tilde{\omega}) &= \frac{1}{T_k}\sum_{t=1}^{T_k} E_{P_1}[\log \hat{r}_{1,t}|\mathcal{F}_{t-1}](\tilde{\omega}) - \frac{1}{T_k}\sum_{t=1}^{t(\tilde{\omega})} E_{P_1}[\log \hat{r}_{1,t}|\mathcal{F}_{t-1}](\tilde{\omega}) \\ &< \frac{3}{4}\epsilon'(\tilde{\omega}).\end{aligned}$$

We conclude that

$$\text{limsup}\left(\frac{1}{T}\sum_{t=t(\tilde{\omega})+1}^{T} E_{P_1}[\log \hat{r}_{1,t}|\mathcal{F}_{t-1}](\tilde{\omega})\right) < 0.$$

But then we must have

$$\text{limsup}\left(\frac{1}{T}\sum_{t=t(\tilde{\omega})+1}^{T} E_{P_1}[\log\left(\max\left\{\hat{r}_{1,t}, \epsilon(\tilde{\omega})\right\}\right)|\mathcal{F}_{t-1}](\tilde{\omega})\right) < 0.$$

Since limsup $\left(\frac{1}{T}\sum_{t=1}^{t(\tilde{\omega})} E_{P_1}[\log\left(\max\left\{\hat{r}_{1,t}, \epsilon(\tilde{\omega})\right\}\right)|\mathcal{F}_{t-1}](\tilde{\omega})\right) = 0$, and, for arbitrary sequences $\{a_n\}$ and $\{b_n\}$, limsup $(a_n + b_n) \leq (\text{limsup}\, a_n) + (\text{limsup}\, b_n)$,

$$\text{limsup}\,\frac{1}{T}\left(\sum_{t=1}^{T} E_{P_1}[g_{1,t}^{\epsilon(\tilde{\omega})}|\mathcal{F}_{t-1}](\tilde{\omega})\right) < 0$$

so that $\tilde{\omega} \in \mathcal{B}_{1,\epsilon(\tilde{\omega})}$ as required. ∎

We continue with the proof of Proposition 5.

Since $\epsilon < \epsilon' \quad \Rightarrow \quad g^{\epsilon}_{1,t}(\omega) \leq g^{\epsilon'}_{1,t}(\omega) \quad \forall\, t, \quad \forall\, \omega$, it follows that $\epsilon < \epsilon' \quad \Rightarrow$ $\mathcal{B}_{1,\epsilon'} \subset \mathcal{B}_{1,\epsilon}$. So $\mathcal{B}_{1,1/n} \subset \mathcal{B}_{1,1/(n+1)} \subset \cdots$, and we set $\mathcal{B}_{1,0} \equiv \cup_{n\geq 1}\mathcal{B}_{1,1/n}$. It follows that $P_1(\mathcal{B}_{1,1/n}/\mathcal{A}_1)$ increases monotonically to $P_1(\mathcal{B}_{1,0}/\mathcal{A}_1)$. So for all $p > 0$, there exists $\epsilon(p)$ such that $P_1(\mathcal{B}_{1,\epsilon(p)}/\mathcal{A}_1) \geq P_1(\mathcal{B}_{1,0}/\mathcal{A}_1) - p$.

For fixed p and corresponding $\epsilon(p)$, consider the truncated process $\{g^{\epsilon(p)}_{1,t}\}_{t=0}^{+\infty}$ defined earlier. It is uniformly bounded below and, under A.1, A.2, A.3, and A.4, by Lemma 2, it is also uniformly bounded above. Hence the process $\{\, E_{P_1}\, [g^{\epsilon(p)}_{1,t}|\mathcal{F}_{t-1}]\}_{t=0}^{+\infty}$ is uniformly bounded below and above.

Define

$$\bar{g}^{\epsilon(p)}_{1,t}(\omega) \equiv g^{\epsilon(p)}_{1,t}(\omega) - E_{P_1}\, [g^{\epsilon(p)}_{1,t}|\mathcal{F}_{t-1}](\omega).$$

It follows that the process $\{\bar{g}^{\epsilon(p)}_{1,t}\}_{t=0}^{+\infty}$ is uniformly bounded above and below. Furthermore, $E_{P_1}\, [\, \bar{g}^{\epsilon(p)}_{1,t}\, \bar{g}^{\epsilon(p)}_{1,t+k}|\mathcal{F}_{t-1}] = 0$ for all $k \geq 1$, for all $t \geq 0$. Therefore, by the Strong Law of Large Numbers for uncorrelated random variables with uniformly bounded second moments (Chung 1974, page 103),

$$\lim_{T\to+\infty} \frac{1}{T}\sum_{t=1}^{T} \bar{g}^{\epsilon(p)}_{1,t}(\omega) = 0 \quad P_1 - \text{a.s.}$$

$$\Rightarrow \qquad \text{limsup}\, \frac{1}{T}\sum_{t=1}^{T} g^{\epsilon(p)}_{1,t}(\omega) \leq \text{limsup}\, \frac{1}{T}\sum_{t=1}^{T} E_{P_1}\, [g^{\epsilon(p)}_{1,t}|\mathcal{F}_{t-1}](\omega).$$

Since $\omega \in \mathcal{B}_{1,\epsilon(p)}/\mathcal{A}_1$ implies limsup $\frac{1}{T}\sum_{t=1}^{T} E_{P_1}\, [g^{\epsilon(p)}_{1,t}|\mathcal{F}_{t-1}](\omega) < 0$, it follows that $\forall\, \omega \in \mathcal{B}_{1,\epsilon(p)}/\mathcal{A}_1$, $\sum_{t=1}^{T} g^{\epsilon(p)}_{1,t}(\omega) \to -\infty$ so that $\forall\, \omega \in \mathcal{B}_{1,\epsilon(p)}/\mathcal{A}_1$, $\sum_{t=1}^{T} \log \hat{r}_{1,t}(\omega) =\!\to -\infty$ since $\sum_{t=1}^{T} \log \hat{r}_{1,t}(\omega) = \sum_{t=1}^{T} g^{0}_{1,t}(\omega) \leq \sum_{t=1}^{T} g^{\epsilon(p)}_{1,t}(\omega) \to -\infty$. The proof of the first part is completed by noting that as p goes to zero, we approximate the set $\mathcal{B}_{i,0}/\mathcal{A}_i$ and, by Lemma 9, that set coincides with $\Omega/\mathcal{A}_1$.

For the second part we set $\mathcal{C}_{1,\delta} \equiv \{\omega \in \Omega : \text{limsup}\, \frac{1}{T}\, \sum_{t=1}^{T} \log \hat{r}_{1,t}(\omega) < \log \delta\} \cap (\Omega/\mathcal{A}_1)$. Clearly, $\delta' < \delta''$ implies that $\mathcal{C}_{1,\delta'} \subset \mathcal{C}_{1,\delta''}$. It follows that $\cup_{n\geq 1}\mathcal{C}_{1,1-1/n} = \Omega/\mathcal{A}_1$ and hence that $P_1(\mathcal{C}_{1,1-1/n})$ increases monotonically to $P_1(\Omega/\mathcal{A}_1)$ so that for all $\epsilon > 0$, there exists $\delta = 1 - 1/n$ such that $P_1(\mathcal{C}_{1,\delta}) \geq P_1(\Omega/\mathcal{A}_1) - \epsilon$. ■

PROOF OF PROPOSITION 6

We give an outline of the proof. In Lemma 10 we show that one can work with the process c_1 and the process y interchangeably. Lemma 11 is the crucial step in which we study the parameterized fixed point of a special one dimensional map. Lemma 12 takes the fixed point found in Lemma 11 and deduces properties induced by it on consumption, marginal utility, Euler equations, etc. A recursive application of Lemma 12 going forward leads us to the properties listed in Proposition 6.

For $Z > 0$, let the function $\mathcal{Y}_Z : (0, Z) \to (0, \infty)$ be defined by $\mathcal{Y}_Z(c_1) = \frac{u_2'(Z-c_1)}{u_1'(c_1)}$.

LEMMA 10: Assume A.3. $\mathcal{Y}_Z$ is increasing in c_1, it is onto, and continuous with a continuous inverse.

PROOF: The result is a consequence of A.3; in particular, we use the fact that u_i are strictly concave, continuously differentiable, and satisfy the Inada condition at $c = 0$. ■

Given Z and feasible consumption processes, by Lemma 10, for any (t, ω) we have $y_t(\omega) = \mathcal{Y}_{Z_t(\omega)}(c_{1,t}(\omega))$. The inverse of $\mathcal{Y}_Z$ is denoted $(\mathcal{Y}_Z)^{-1}(y)$; by Lemma 10, it is well defined and continuous.

Proposition 6 is proved by using a recursive construction in the variable $y_t(\omega)$ which, by Lemma 10, is equivalent to using the variable $c_{1,t}(\omega)$. However, to establish the basic properties of the construction, it is easier to work with the variable $\lambda \equiv r \cdot u_2'(c_2)/y$. Lemma 11 studies the existence and some properties of the fixed point in λ of a special function.

LEMMA 11: Assume A.2, A.3, and A.4. For $t \geq 1$ and $\omega \in \Omega$, and $y > 0$, define $\underline{\lambda}(t-1, \omega, y) \equiv \max_{\omega' \in \Omega(s^{t-1}(\omega))} \frac{r_t(\omega') \cdot u_2'(Z_t(\omega'))}{y}$ and consider the function $f_{t-1,\omega,y} : [\underline{\lambda}(t-1, \omega, y), +\infty) \to [(\beta_1/\beta_2) \cdot \underline{r} \cdot u_1'(\bar{z}), +\infty)$ in the variable λ defined by

$$f_{t-1,\omega,y}(\lambda) \equiv (\beta_1/\beta_2) \cdot E_P\left[r_t \cdot u_1'\left(Z_t - (u_2')^{-1}\left(\frac{y \cdot \lambda}{r_t}\right)\right)\middle|\mathcal{F}_{t-1}\right](\omega).$$

Then (i) $f_{t-1,\omega,y}$ has a unique fixed point denoted $\lambda^*(t-1, \omega, y)$,
(ii) $\lambda^*(t-1, \omega, y) > \max_{\omega' \in \Omega(s^{t-1}(\omega))} \frac{r_t(\omega') \cdot u_2'(Z_t(\omega'))}{y}$ and $\lambda^*(t-1, \omega, y) > (\beta_1/\beta_2) \cdot \underline{r} \cdot u_1'(\bar{z})$.
PROOF: Evidently, the domain of the function $f_{t-1,\omega,y}$ and the function are both $\mathcal{F}_{t-1}-$measurable.
(i) Under A.4, $\underline{r} > 0$ so $\underline{\lambda}(t-1, \omega, y) \geq 0$. It can be verified that $f_{t-1,\omega,y}(\underline{\lambda}(t-1, \omega, y)) \geq (\beta_1/\beta_2) \cdot \underline{p} \cdot \underline{r} \cdot u_1'(0) = \infty$, where we use the Inada condition; furthermore, $f_{t-1,\omega,y}$ is continuous and strictly decreasing. Under A.2 and A.3 $(\beta_1/\beta_2) \cdot \bar{r} \cdot u_1'(\bar{z}) < \infty$; therefore, $\mathrm{Lim}_{\lambda \to \infty} f_{t-1,\omega,y}(\lambda) < \infty$. It follows that $f_{t-1,\omega,y}$ has a unique fixed point.
(ii) As noted at the beginning of the proof, $f_{t-1,\omega,y}$ is $\mathcal{F}_{t-1}-$measurable and, therefore, also the fixed point $\lambda^*(t-1, \omega, y)$ is $\mathcal{F}_{t-1}-$measurable. Since $f_{t-1,\omega,y}(\underline{\lambda}(t-1, \omega, y)) = \infty$, we must have $\lambda^*(t-1, \omega, y) > \underline{\lambda}(t-1, \omega, y)$. The second part follows from the fact that $f_{t-1,\omega,y}$ is strictly decreasing. ■

The next result induces values for consumption at the fixed point identified in Lemma 11 and specifies the implications on intertemporal marginal utilities induced by those values.

LEMMA 12: Assume A.2, A.3, and A.4, and $P = P_1 = P_2$. Let $y_{t-1} : \Omega \to R_+$ be an $\mathcal{F}_{t-1}-$measurable function. Set

$$c_{2,t}(\omega) \equiv (u_2')^{-1}\left(\frac{y_{t-1} \cdot \lambda^*(t-1, \omega, y_{t-1}(\omega))}{r_t}\right), \ c_{1,t}(\omega) \equiv Z_t(\omega) - c_{2,t}(\omega), \ y_t(\omega) = \mathcal{Y}_{Z_t(\omega)}(c_{1,t}(\omega)).$$

Then (i) $c_{i,t}(\omega) \geq 0$ and is $\mathcal{F}_t-$measurable, (ii) $\frac{r_t(\omega) \cdot u_2'(c_{2,t}(\omega))}{y_{t-1}(\omega)} = (\beta_1/\beta_2) \cdot E_P[r_t \cdot u_1'(c_{1,t})|\mathcal{F}_{t-1}](\omega)$ so $r_t(\omega) \cdot u_2'(c_{2,t}(\omega))$ is $\mathcal{F}_{t-1}-$measurable and $\hat{r}_{2,t}(\omega) = 1$ for all $\omega \in \Omega$, and (iii) $y_t(\omega) = \frac{\beta_1}{\beta_2} \cdot \frac{1}{\hat{r}_{1,t}(\omega)} \cdot y_{t-1}(\omega)$.

PROOF: (i) As per the definition in the hypothesis $\lambda^*(t-1,\omega,y_{t-1}(\omega)) = \frac{r_t(\omega)\cdot u_2'(c_{2,t}(\omega))}{y_{t-1}(\omega)}$. So using Lemma 11 (ii) we have $\lambda^*(t-1,\omega,y_{t-1}(\omega)) \geq \underline{\lambda}(t-1,\omega,y_{t-1}(\omega))$

$$\Leftrightarrow \quad \frac{r_t(\omega)\cdot u_2'(c_{2,t}(\omega))}{y_{t-1}(\omega)} \geq \frac{r_t(\omega)\cdot u_2'(Z_t(\omega))}{y_{t-1}(\omega)} \quad \Leftrightarrow \quad u_2'(c_{2,t}(\omega)) \geq u_2'(Z_t(\omega))$$

so that using the fact that u_2 is concave we can conclude that $c_{2,t}(\omega) \leq Z_t(\omega)$ so that $c_{1,t}(\omega) \geq 0$. The Inada condition guarantees that $c_{2,t}(\omega) \geq 0$. Since the measurability property is evident, the proof of (i) is complete.
(ii) Follows from the fixed point property since

$$\frac{r_t(\omega)\cdot u_2'(c_{2,t}(\omega))}{y_{t-1}(\omega)} = \lambda^*(t-1,\omega,y_{t-1}(\omega)) = f_{t-1,\omega,y_{t-1}(\omega)}(\lambda^*(t-1,\omega,y_{t-1}(\omega)))$$

$$= (\beta_1/\beta_2)\cdot E_P[r_t\cdot u_1'(c_{1,t})|\mathcal{F}_{t-1}](\omega).$$

This shows that $r_t(\omega)\cdot u_2'(c_{2,t}(\omega))$ is $\mathcal{F}_{t-1}-$measurable and so $\hat{r}_{2,t}(\omega)=1$ for all $\omega\in\Omega$.
(iii) By manipulating the fixed point condition, we obtain

$$\frac{u_2'(c_{2,t}(\omega))}{u_1'(c_{1,t}(\omega))} = y_{t-1}(\omega)\cdot\frac{\beta_1}{\beta_2}\cdot\frac{E_P[r_t\cdot u_1'(c_{1,t})|\mathcal{F}_{t-1}](\omega)}{r_t(\omega)\cdot u_1'(c_{1,t}(\omega))} \quad \Leftrightarrow \quad y_t(\omega) = \frac{\beta_1}{\beta_2}\cdot\frac{1}{\hat{r}_{1,t}(\omega)}\cdot y_{t-1}(\omega),$$

where we invoke $P=P_1$, proving (iii). ∎

Proposition 6 is proved by recursively applying Lemma 12. For existence we assume that we are given a pair $(y,\omega)\in R_{++}\times\Omega$, we set $y_0(\omega)\equiv y$ and treat it as a parameter and apply Lemma 12 (i) to induce a unique process for $\{y_t(\omega)\}_{t\geq 0}$ and for all $\omega\in\Omega$. By Lemma 10 this is equivalent to starting with a pair $(c,\omega)\in R_{++}\times\Omega$ with the additional condition that $c\in(0,Z_0(\omega))$, setting $c_{1,0}(\omega)\equiv c$ and treating it as a parameter and generating a unique pair of processes c_i that are feasible and solve the fixed point problem at each date $t\geq 1$ and for all $\omega\in\Omega$.

The notation $\{C_{i,t}(\omega)\}_{t\geq 0}$, where the process is defined for all $\omega\in\Omega$, was introduced in the statement of Proposition 6.

PROOF OF PROPOSITION 7

The proof follows from Lemma 13-15.

LEMMA 13: Assume A.3, $\underline{r}\geq 0$, $\beta_2\leq\beta_1$, and $P=P_1=P_2$. In the proposed solution, $P\{\omega : \liminf y_t(\omega)=0\}=0$.
PROOF: Since $y_T(\omega) = \left(\frac{\beta_1}{\beta_2}\right)^T\cdot\frac{1}{\prod_{t=1}^T\left[\hat{r}_{1,t}(\omega)\right]}\cdot y_0(\omega)$ and since, by Lemma 3, we know that $\prod_{t=1}^T[\hat{r}_{1,t}(\tilde{\omega})]$ is a.s. bounded, we conclude that $\liminf y_T(\omega)>0$ a.s. ∎

LEMMA 14: Assume $\underline{z}>0$, $\underline{r}\geq 0$, A.3, $\beta_2\leq\beta_1$, and $P=P_1=P_2$. In the proposed solution, $P(\underline{\mathcal{C}}_1)=0$ where $\underline{\mathcal{C}}_i\equiv\{\omega\in\Omega : \liminf C_{i,t}(\omega)=0\}$.
PROOF: Given y_0, choose $K>0$. For any such K let $c_K>0$ solve the equation

$u_2'(\underline{z} - c_K) = u_1'(c_K) \cdot y_0(\tilde{\omega})/K$.

For any $\tilde{\omega} \in \underline{\mathcal{C}}_1$ and such a K there exists a sequence $\{t_\tau^K\}$ of periods such that $C_{1,t_\tau^K} \le c_K$ so $y_{t_\tau^K}(\tilde{\omega}) \le y_0(\tilde{\omega})/K$. Then Lemma 13 implies that $P(\underline{\mathcal{C}}_1) = 0$. ∎

LEMMA 15: Assume A.2, A.3, $\underline{r} \ge 0$, $\beta_2 \le \beta_1$ and $P = P_1 = P_2$. In the proposed solution $P(\mathcal{A}_1) = 0$.

PROOF: Since $\bar{z} < \infty$, if, for some $\tilde{\omega}$, $\liminf \hat{r}_{1,t}(\tilde{\omega}) = 0$ then, since the numerator of $\hat{r}_{i,t}$ is strictly positive, $\limsup E_P[r_t \cdot u_1'(c_{1,t})|\mathcal{F}_{t-1}](\tilde{\omega}) = \infty$. We shall argue that if such an event occurs in the proposed solution, necessarily $\liminf C_{1,t}(\tilde{\omega}) = 0$ which, by Lemma 14, is a zero probability event.

So suppose $\tilde{\omega}$ is such that $\limsup E[r_t \cdot u_1'(C_{1,t})|\mathcal{F}_{t-1}](\tilde{\omega}) = \infty$ and $\liminf C_{1,t}(\tilde{\omega}) = 2\epsilon$ for some $\epsilon > 0$. It follows that there exists $\tilde{t}$ such that for $t \ge \tilde{t}$, $C_{1,\tilde{t}}(\tilde{\omega}) \ge \epsilon$. Choose $\delta(\epsilon)$ to satisfy $u_1'(\underline{z} - \delta(\epsilon)) < \Big(u_1'(\epsilon)/u_2'(\bar{z})\Big) \cdot u_2'(\delta(\epsilon))$. Since $\limsup E[r_t \cdot u_1'(C_{1,t})|\mathcal{F}_{t-1}](\tilde{\omega}) = \infty$, necessarily, for some $t' \ge \tilde{t}$,

$$E[r_{t'} \cdot u_1'(C_{1,t'})|\mathcal{F}_{t'-1}](\tilde{\omega}) > \bar{r} \cdot \frac{u_1'(\epsilon)}{u_2'(\bar{z})} \cdot u_2'(\delta(\epsilon)), \qquad (*)$$

and in the solution proposed

$$r_t(\omega) \cdot u_2'(C_{2,t}(\omega)) = \frac{\beta_1}{\beta_2} \cdot \frac{u_2'(C_{2,t-1}(\omega))}{u_1'(C_{1,t-1}(\omega))} \cdot E[r_t \cdot u_1'(C_{1,t})|\mathcal{F}_{t-1}](\omega)$$

so that for $(\tilde{\omega}, t')$

$$r_{t'}(\tilde{\omega}) \cdot u_2'(C_{2,t'}(\tilde{\omega})) \ge \frac{\beta_1}{\beta_2} \cdot \frac{u_2'(Z_{t'-1}(\tilde{\omega}) - \epsilon)}{u_1'(\epsilon)} \cdot E[r_{t'} \cdot u_1'(C_{1,t'})|\mathcal{F}_{t'-1}](\tilde{\omega})$$

$$> \frac{\beta_1}{\beta_2} \cdot \frac{u_2'(\bar{z})}{u_1'(\epsilon)} \cdot \bar{r} \cdot \frac{u_1'(\epsilon)}{u_2'(\bar{z})} \cdot u_2'(\delta(\epsilon)) = \frac{\beta_1}{\beta_2} \cdot \bar{r} \cdot u_2'(\delta(\epsilon)) \ge \bar{r} \cdot u_2'(\delta(\epsilon)),$$

since $\beta_2 \le \beta_1$.

Since $r_t(\omega) \cdot u_2'(C_{2,t}(\omega))$ is $\mathcal{F}_{t-1}$-measurable,

$$r_{t'}(\omega') \cdot u_2'(C_{2,t'}(\omega')) > \bar{r} \cdot u_2'(\delta(\epsilon)) \qquad \omega' \in \Omega\Big((s^{t'-1}(\tilde{\omega})\Big).$$

So $C_{2,t'}(\omega') < \delta(\epsilon)$ for all $\omega' \in \Omega\Big((s^{t'-1}(\tilde{\omega})\Big)$ and therefore, by feasibility, $C_{1,t'}(\omega') > Z_{t'}(\omega') - \delta(\epsilon)$ for all $\omega' \in \Omega\Big((s^{t'-1}(\tilde{\omega})\Big)$. It follows that

$$E[r_{t'} \cdot u_1'(C_{1,t'})|\mathcal{F}_{t'-1}](\tilde{\omega}) \le \bar{r} \cdot u_1'(\underline{z} - \delta(\epsilon))$$

which, using the definition of $\delta(\epsilon)$, contradicts $(*)$. We have shown that $\liminf \hat{r}_{1,t}(\tilde{\omega}) = 0$ implies that $\tilde{\omega} \in \underline{\mathcal{C}}_i$, a set that has measure zero according to Lemma 14. ∎

PROOF OF PROPOSITION 8

The proof follows from Lemma 16-17 and Lemma 18.

LEMMA 16: Assume A.2, A.3, and A.4. Then, for the solution proposed

$$\sup_{t\geq 0}\sup_{\omega\in\Omega}\frac{u_2'(C_{2,t+1}(\omega))}{u_2'(C_{2,t}(\omega))}\leq M\equiv\max\left\{\frac{\bar{r}\cdot u_2'(\underline{z}/2)}{\underline{r}\cdot u_2'(\bar{z})};\frac{\beta_1}{\beta_2}\frac{\bar{r}\cdot u_1'(\underline{z}/2)}{\underline{r}\cdot u_1'(\bar{z})}\right\}.$$

PROOF: If not then there is a pair $(\tilde{t},\tilde{\omega})$, such that

$$\frac{u_2'(C_{2,\tilde{t}+1}(\tilde{\omega}))}{u_2'(C_{2,\tilde{t}}(\tilde{\omega}))}>M \qquad \Rightarrow \qquad \frac{u_2'(C_{2,\tilde{t}+1}(\tilde{\omega}))}{u_2'(C_{2,\tilde{t}}(\tilde{\omega}))}>\frac{\bar{r}\cdot u_2'(\underline{z}/2)}{\underline{r}\cdot u_2'(\bar{z})}$$

$$\Rightarrow \qquad \frac{r_{\tilde{t}+1}(\tilde{\omega})\cdot u_2'(C_{2,\tilde{t}+1}(\tilde{\omega}))}{u_2'(C_{2,\tilde{t}}(\tilde{\omega}))}>\frac{\bar{r}\cdot u_2'(\underline{z}/2)}{u_2'(\bar{z})}. \qquad (*)$$

As shown in the proof of Lemma 12 (ii),

$$r_{t+1}(\omega)\cdot\frac{u_2'(C_{2,t+1}(\omega))}{u_2'(C_{2,t}(\omega))}=\frac{\beta_1}{\beta_2}\cdot\frac{E_P[r_{t+1}\cdot u_1'(C_{1,t+1})|\mathcal{F}_t](\omega)}{u_1'(C_{1,t}(\omega))},$$

so we must also have

$$\frac{\beta_1}{\beta_2}\cdot\frac{E_P[r_{\tilde{t}+1}\cdot u_1'(C_{1,\tilde{t}+1})|\mathcal{F}_{\tilde{t}}](\tilde{\omega})}{r_{\tilde{t}+1}(\tilde{\omega})\cdot u_1'(C_{1,\tilde{t}}(\tilde{\omega}))}>M$$

$$\Rightarrow \qquad \frac{\beta_1}{\beta_2}\cdot\frac{E_P[r_{\tilde{t}+1}\cdot u_1'(C_{1,\tilde{t}+1})|\mathcal{F}_{\tilde{t}}](\tilde{\omega})}{r_{\tilde{t}+1}(\tilde{\omega})\cdot u_1'(C_{1,\tilde{t}}(\tilde{\omega}))}>\frac{\beta_1}{\beta_2}\frac{\bar{r}\cdot u_1'(\underline{z}/2)}{\underline{r}\cdot u_1'(\bar{z})}$$

so that, since $C_{1,\tilde{t}}(\tilde{\omega})\leq\bar{z}$ and $u_1''<0$,

$$\Rightarrow \qquad \frac{E_P[r_{\tilde{t}+1}\cdot u_1'(C_{1,\tilde{t}+1})|\mathcal{F}_{\tilde{t}}](\tilde{\omega})}{r_{\tilde{t}+1}(\tilde{\omega})\cdot u_1'(\bar{z})}>\frac{\bar{r}\cdot u_1'(\underline{z}/2)}{\underline{r}\cdot u_1'(\bar{z})}$$

$$\Rightarrow \qquad E_P[r_{\tilde{t}+1}\cdot u_1'(C_{1,\tilde{t}+1})|\mathcal{F}_{\tilde{t}}](\tilde{\omega})>\bar{r}\cdot u_1'(\underline{z}/2)$$

since $\underline{r}\leq r_{\tilde{t}+1}(\tilde{\omega})$. It follows that for some $\omega'\in\Omega(s^{\tilde{t}}(\tilde{\omega}))$,

$$u_1'(C_{1,\tilde{t}+1}(\omega'))>u_1'(\underline{z}/2) \qquad \Leftrightarrow \qquad C_{1,\tilde{t}+1}(\omega')<\underline{z}/2\leq Z_t/2$$

$$\Leftrightarrow \quad C_{2,\tilde{t}+1}(\omega')>Z_t/2\geq\underline{z}/2 \quad \Rightarrow \quad r_{\tilde{t}+1}(\omega')\cdot u_2'(C_{2,\tilde{t}+1}(\omega'))<\bar{r}\cdot u_2'(\underline{z}/2)$$

$$\Leftrightarrow \qquad \frac{r_{\tilde{t}+1}(\omega')\cdot u_2'(C_{2,\tilde{t}+1}(\omega'))}{u_2'(\bar{z})}<\frac{\bar{r}\cdot u_2'(\underline{z}/2)}{u_2'(\bar{z})}.$$

But, using the fact that $u_2'(C_{2,\tilde{t}}(\tilde{\omega}))>u_2'(\bar{z})$, the last inequality contradicts the fact that $r_t(\omega)\cdot u_2'(C_{2,t}(\omega))$ is always $\mathcal{F}_{t-1}$-measurable since, according to (*), we must have $\frac{r_{\tilde{t}+1}(\tilde{\omega})\cdot u_2'(C_{2,\tilde{t}+1}(\tilde{\omega}))}{u_2'(C_{2,\tilde{t}}(\tilde{\omega}))}>\frac{\bar{r}\cdot u_2'(\underline{z}/2)}{u_2'(\bar{z})}$. ∎

LEMMA 17: Assume A.2, A.3, A.4, and A.6, and $P=P_1=P_2$. Then, for the solution proposed

$$0\leq\lim_{T\to\infty}\sum_{t=0}^{T}\beta_i^t\cdot E_P\Big[\frac{u_i'(C_{i,t})}{u_i'(C_{i,0})}\Big]\leq 1/(1-\beta_2\cdot M).$$

PROOF: We prove the result for $i = 1$ since it is trivial for $i = 2$.

Since, by Proposition 6, in the proposed solution

$$y_t(\omega) = \left(\frac{\beta_1}{\beta_2}\right)^T \cdot \frac{1}{\prod_{\tau=1}^t [\hat{r}_{1,\tau}(\omega)]} \cdot y_0(\omega) \quad \Leftrightarrow \quad \frac{u_2'(C_{2,t}(\omega))}{u_1'(C_{1,t}(\omega))} = \left(\frac{\beta_1}{\beta_2}\right)^T \cdot \frac{1}{\prod_{\tau=1}^t [\hat{r}_{1,\tau}(\omega)]} \cdot \frac{u_2'(C_{2,0}(\omega))}{u_1'(C_{1,0}(\omega))}$$

$$\Leftrightarrow \qquad \beta_1^t \cdot \frac{u_1'(C_{1,t}(\omega))}{u_1'(C_{1,0}(\omega))} = \beta_2^t \cdot \prod_{\tau=1}^t [\hat{r}_{1,\tau}(\omega)] \cdot \frac{u_2'(C_{2,t}(\omega))}{u_2'(C_{2,0}(\omega))}$$

$$\Rightarrow \qquad 0 \le \sum_{t=0}^T \beta_1^t \cdot E_P\Big[\frac{u_1'(C_{1,t})}{u_1'(C_{1,0})}\Big] = \sum_{t=0}^T \beta_2^t \cdot E_P\Big[\prod_{\tau=1}^t [\hat{r}_{1,\tau}] \cdot \frac{u_2'(C_{2,t})}{u_2'(C_{2,0})}\Big]$$

$$\le \sum_{t=0}^T \beta_2^t \cdot (M)^t \cdot E_P\Big[\prod_{\tau=1}^t [\hat{r}_{1,\tau}]\Big] = \sum_{t=0}^T \beta_2^t \cdot (M)^t$$

where we use the fact that $E_P[\hat{r}_{2,t}|\mathcal{F}_{t-1}](\omega) = 1$, that $P = P_1 = P_2$ together with the law of iterated expectations. The result follows by taking the limit. ∎

Finally, we verify that, for the proposed allocation, the payoff is finite if u_i is in the CRRA class of functions with coefficient greater than or equal to one. As the proof makes clear, the key condition is $\beta_2 M^{\frac{a-1}{a}} < 1$ which is implied by A.6 when $a \ge 1$. Since it is an open condition, the result is also true for some values of $a \in (0,1)$.

LEMMA 18: Assume $u_i(c) \equiv \log c$ or $u_i(c) \equiv \frac{c^{1-a}}{1-a}$, where $a > 1$, and also assume A.2, A.3, A.4, and A.6. Then $\lim_{T\to\infty} \sum_{t=0}^T \beta_i^t E_P[u_i(C_{i,t})] > -\infty$.
PROOF: First, we consider agent 2. By Lemma 16,

$$\sup_{t\ge0} \sup_{\omega\in\Omega} \frac{u_2'(C_{2,t+1}(\omega))}{u_2'(C_{2,t}(\omega))} \le M \qquad \Leftrightarrow \qquad \sup_{\omega\in\Omega} \frac{u_2'(C_{2,t}(\omega))}{u_2'(C_{2,0}(\omega))} \le M^t$$

$$\Rightarrow \quad \inf_{\omega\in\Omega} \left(\frac{C_{2,t}(\omega)}{C_{2,0}(\omega)}\right)^a \ge \frac{1}{M^t} \;\Leftrightarrow\; \inf_{\omega\in\Omega} \left(\frac{C_{2,t}(\omega)}{C_{2,0}(\omega)}\right) \ge \left(\frac{1}{M^t}\right)^{\frac{1}{a}}, \quad \text{where } a > 0.$$

Then,

$$\sum_{t=0}^T \beta_2^t \cdot u_2(C_{2,t}(\omega)) \ge \begin{cases} \frac{(C_{2,0}(\omega))^{1-a}}{1-a} + \sum\limits_{t=1}^T \beta_2^t \cdot \left(\frac{1}{M^t}\right)^{\frac{1-a}{a}} \cdot \frac{(C_{2,0}(\omega))^{1-a}}{1-a} & \text{if } a \ne 1 \\ \log C_{2,0}(\omega) + \sum\limits_{t=1}^T \beta_2^t \cdot \log\left(C_{2,0}(\omega) \cdot \frac{1}{M^t}\right) & \text{if } a = 1 \end{cases}$$

and it follows that

$$\lim_{T\to\infty} \sum_{t=0}^T \beta_2^t E_P[u_2(C_{2,t})] \ge \begin{cases} \frac{(C_{2,0}(\omega))^{1-a}}{1-a} \cdot \frac{1}{1-\beta_2 M^{\frac{a-1}{a}}} & \text{if } a > 1 \\ \frac{\log C_{2,0}(\omega)}{1-\beta_2} + \frac{\beta_2}{(1-\beta_2)^2} \cdot \log\left(\frac{1}{M}\right) & \text{if } a = 1 \end{cases}$$

since for $M > 1$ and $a > 1$ we have $M^{\frac{a-1}{a}} \le M$ so that, by using A.6, $\beta_2 M^{\frac{a-1}{a}} < 1$, while for $a = 1$ we use $\sum\limits_{t=1}^{\infty} (\beta_2^t \cdot t) = \frac{\beta_2}{(1-\beta_2)^2}$.

Now consider agent 1. By construction

$$\beta_1^t \cdot \frac{u_1'(C_{1,t}(\omega))}{u_1'(C_{1,0}(\omega))} = \beta_2^t \cdot \prod_{\tau=1}^{t} \left[\hat{r}_{1,\tau}(\omega)\right] \cdot \frac{u_2'(C_{2,t}(\omega))}{u_2'(C_{2,0}(\omega))}$$

and so, for $a > 0$,

$$\beta_1^t \cdot \left(\frac{C_{1,t}(\omega)}{C_{1,0}(\omega)}\right)^{-a} = \beta_2^t \cdot \left(\prod_{\tau=1}^{t} \hat{r}_{1,\tau}(\omega)\right) \cdot \frac{u_2'(C_{2,t}(\omega))}{u_2'(C_{2,0}(\omega))}.$$

It follows from Lemma 16 that, for $a > 0$,

$$C_{1,t}(\omega) = C_{1,0}(\omega)\left(\frac{\beta_2^t}{\beta_1^t} \cdot \prod_{\tau=1}^{t} \hat{r}_{1,\tau}(\omega)\right)^{-\frac{1}{a}} \cdot \left(\frac{u_2'(C_{2,t}(\omega))}{u_2'(C_{2,0}(\omega))}\right)^{-\frac{1}{a}} \geq C_{1,0}(\omega)\left(\frac{\beta_2 M}{\beta_1}\right)^{-\frac{t}{a}} \cdot \left(\prod_{\tau=1}^{t} \hat{r}_{1,\tau}(\omega)\right)^{-\frac{1}{a}}$$

and so

$$E_P\left[u_1\left(C_{1,t}\right)\right] \geq \begin{cases} C_{1,0}(\omega)^{1-a} \cdot \left(\frac{\beta_2 M}{\beta_1}\right)^{\frac{(a-1)t}{a}} \cdot E_P\left[\frac{1}{1-a} \cdot \left(\prod_{\tau=1}^{t} \hat{r}_{1,\tau}\right)^{\frac{a-1}{a}}\right] & \text{if } a \neq 1 \\ \log C_{1,0}(\omega) - t \cdot \log \frac{\beta_2 M}{\beta_1} - E_P\left[\sum_{\tau=1}^{t} \log \hat{r}_{1,\tau}\right] & \text{if } a = 1 \end{cases}$$

$$\Rightarrow \qquad E_P\left[u_1\left(C_{1,t}\right)\right] \geq \begin{cases} \frac{C_{1,0}(\omega)^{1-a}}{1-a} \cdot \left(\frac{\beta_2 M}{\beta_1}\right)^{\frac{(a-1)t}{a}} & \text{if } a \neq 1 \\ \log C_{1,0}(\omega) - t \cdot \log \frac{\beta_2 M}{\beta_1} & \text{if } a = 1 \end{cases}$$

where the first line uses Jensen's inequality and the fact that the function $\frac{x^{\frac{a-1}{a}}}{1-a}$ is convex for $a > 0$, and both lines use the conditional mean one property of $\hat{r}_{1,t}$.

Hence,

$$\beta_1^t \cdot E_P\left[u_1\left(C_{1,t}\right)\right] \geq \begin{cases} \frac{C_{1,0}(\omega)^{1-a}}{1-a} \cdot \left(\beta_1^{\frac{1}{a}} \cdot \left(\beta_2 \cdot M\right)^{\frac{a-1}{a}}\right)^t & \text{if } a \neq 1 \\ \beta_1^t \cdot \log C_{1,0}(\omega) - \beta_1^t \cdot t \cdot \log \frac{\beta_2 M}{\beta_1} & \text{if } a = 1 \end{cases}$$

and it follows that

$$\lim_{T\to\infty} \sum_{t=0}^{T} \beta_1^t \cdot E_P\left[u_1\left(C_{1,t}\right)\right] \geq \begin{cases} \frac{(C_{1,0}(\omega))^{1-a}}{1-a} \cdot \frac{1}{1-\beta_1^{\frac{1}{a}} \cdot (\beta_2 \cdot M)^{\frac{a-1}{a}}} & \text{if } a > 1 \\ \frac{\log C_{1,0}}{1-\beta_1} - \log \frac{\beta_2 M}{\beta_1} \cdot \frac{\beta_1}{(1-\beta_1)^2} & \text{if } a = 1 \end{cases}$$

where we use the facts that $\beta_1^{\frac{1}{a}} \cdot (\beta_2 \cdot M)^{\frac{a-1}{a}} < 1$ whenever $\beta_2 \cdot M < 1$ and $a > 1$, and that $\sum_{t=1}^{\infty} (\beta_0^t \cdot t) = \frac{\beta_1}{(1-\beta_1)^2}$. ■

PROOF OF THEOREM 3A

First we state and prove Lemma 19.

LEMMA 19: Assume A.3. Given prices q, let c_i be (i) a budget feasible consumption process, $c_i \in BC(q; z_i)$, supported by the portfolio process θ_i at (q, z_i), (ii) an Euler process, so that the first order conditions hold, and (iii) let $\lim_{T\to+\infty} \sum_{t=0}^{T} \beta_i^t \, E_{P_i}[u_i(c_{i,t})] > -\infty$ hold. If for every $\tilde{\theta}_i$ that supports a $\tilde{c}_i \in BC(q; z_i)$ the transversality condition at date 0 holds,

$$\lim_{T\to+\infty} \beta_i^T \, E_{P_i}\Big[u_i'(c_{i,T}) \cdot q_T \cdot \Big(\tilde{\theta}_{i,T} - \theta_{i,T}\Big)\Big] \geq 0,$$

then c_i is the maximiser on $BC(q; z_i)$.

PROOF: Using the budget constraints, which may be assumed to hold with equality since, by A.3, u_i is strictly increasing,

$$\tilde{c}_{i,t}(\omega) - c_{i,t}(\omega) = \Big(z_{i,t}(\omega) + r_t(\omega)\cdot\tilde{\theta}_{i,t-1}(\omega) - q_t(\omega)\cdot\tilde{\theta}_{i,t}(\omega)\Big) - \Big(z_{i,t}(\omega) + r_t(\omega)\cdot\theta_{i,t-1}(\omega) - q_t(\omega)\cdot\theta_{i,t}(\omega)\Big)$$

we obtain

$$\sum_{t=0}^{T} \beta_i^t \, E_{P_i}\Big[u_i'(c_{i,t}) \cdot \Big(\tilde{c}_{i,t} - c_{i,t}\Big)\Big] = \sum_{t=0}^{T} \beta_i^t \, E_{P_i}\Big[u_i'(c_{i,t}) \cdot \Big(r_t \cdot (\tilde{\theta}_{i,t-1} - \theta_{i,t-1}) - q_t \cdot (\tilde{\theta}_{i,t} - \theta_{i,t})\Big)\Big]$$

$$= E_{P_i}\Big[u_i'(c_{i,0}) \cdot \Big(r_0 \cdot (\tilde{\theta}_{i,-1} - \theta_{i,-1})\Big)\Big]$$

$$+ \sum_{t=1}^{T} \beta_i^{t-1} \, E_{P_i}\Bigg[E_{P_i}\Big[\Big(\beta u_i'(c_{i,t}) \cdot r_t - u_i'(c_{i,t-1}) \cdot q_{t-1}\Big)\Big|\mathcal{F}_{t-1}\Big] \cdot \Big(\tilde{\theta}_{i,t-1} - \theta_{i,t-1}\Big)\Bigg]$$

$$- \beta_i^T \, E_{P_i}\Big[u_i'(c_{i,T}) \cdot q_T \cdot \Big(\tilde{\theta}_{i,T} - \theta_{i,T}\Big)\Big]$$

which, upon using the condition $\tilde{\theta}_{i,-1}(\omega) = \theta_{i,-1}(\omega)$, and the first order conditions, becomes

$$\sum_{t=0}^{T} \beta_i^t \, E_{P_i}\Big[u_i'(c_{i,t}) \cdot \Big(\tilde{c}_{i,t} - c_{i,t}\Big)\Big] = - \beta_i^T \, E_{P_i}\Big[u_i'(c_{i,T}) \cdot q_T \cdot \Big(\tilde{\theta}_{i,T} - \theta_{i,T}\Big)\Big].$$

Since $c \in BC(q; z_i)$ and $\tilde{c} \in BC(q; z_i)$, both are uniformly bounded. The same is true of $q \cdot \theta$ and $q \cdot \tilde{\theta}$. So if we let $T \to \infty$ both the terms have limits.

Since for f a C^1 and concave function,

$$Df(y) \cdot (x - y) \geq f(x) - f(y), \qquad \forall\, x, y,$$

we see that

$$- \beta_i^T \, E_{P_i}\Big[u_i'(c_{i,T}) \cdot q_T \cdot \Big(\tilde{\theta}_{i,T} - \theta_{i,T}\Big)\Big] \geq \sum_{t=0}^{T} \beta_i^t \, E_{P_i}\Big[u_i(\tilde{c}_{i,t})\Big] - \sum_{t=0}^{T} \beta_i^t \, E_{P_i}\Big[u_i(c_{i,t})\Big].$$

By A.3 and the fact that c_i is uniformly bounded, the second term on the right hand side either has a limit or diverges to $-\infty$; by $\lim_{T\to+\infty} \sum_{t=0}^{T} \beta_i^t \, E_{P_i}[u_i(c_{i,t})] > -\infty$ the latter is ruled out. By hypothesis, $\lim_{T\to+\infty} \beta_i^T \, E_{P_i}\Big[u_i'(c_{i,T}) \cdot q_T \cdot \Big(\tilde{\theta}_{i,T} - \theta_{i,T}\Big)\Big] \geq 0$. So, by rearranging,

$$\lim_{T\to+\infty} \sum_{t=0}^{T} \beta_i^t \, E_{P_i}\Big[u_i(c_{i,t})\Big] \;\geq\; \text{limsup}_{T\to+\infty} \sum_{t=0}^{T} \beta_i^t \, E_{P_i}\Big[u_i(\tilde{c}_{i,t})\Big],$$

so that c_i is a maximiser on $BC(q; z_i)$. By strict concavity of u_i, it is the unique maximiser.
■

PROOF OF THEOREM 3A: Recall that $\widehat{\theta}_i$ is the portfolio that supports $\widehat{c}_i$ at the price process $\widehat{q}$ and endowment process $\widehat{z}_i$. Since the consumption processes are aggregate feasible we have $\widehat{c}_i \in \Psi_+$ so that, by conditions (iii) and (vi A), $\widehat{c}_i \in BC(\widehat{q}; \widehat{z}_i)$. Again, since the consumption processes are aggregate feasible, (iii) implies that at every $t \geq 0$, $\widehat{\theta}_{1,t}(\omega) + \widehat{\theta}_{2,t}(\omega) = 0$ for all $\omega \in \Omega$ where we use the fact that, since u_i is strictly increasing, the spot market budget constraints are satisfied with equality. It remains to verify that $\widehat{c}_i$ is also the maximiser on the budget set $BC(\widehat{q}; \widehat{z}_i)$. Condition (vi A) implies that $\widehat{q} \cdot (\tilde{\theta}_i - \widehat{\theta}_i) \in \Psi$, so that it is uniformly bounded, where $\tilde{\theta}$ is a portfolio that supports the budget feasible consumption $\tilde{c}_i \in BC_i(\widehat{q}; \widehat{z}_i)$. That, together with condition (v A), implies that the transversality condition specified in Lemma 19 is satisfied for both the agents,

$$\lim_{T\to+\infty} \beta_i^T \, E_P\Big[u_i'(\widehat{c}_{i,T}) \cdot \widehat{q}_T \cdot \Big(\tilde{\theta}_{i,T} - \widehat{\theta}_{i,T}\Big)\Big] = 0,$$

and therefore $\widehat{c}_i$ are indeed maximizers. ■

PROOF OF THEOREM 3B

Given c_i, define the *personalized supporting price* process for agent i, denoted p_i^c, by $p_{i,t}^c(\omega) \equiv \beta_i^t \cdot u_i'(c_{i,t}(\omega))/u_i'(c_{i,0}(\omega))$.

For $p \in \times_{t=0}^{\infty} \Psi_+^t$ with $\lim_{T\to+\infty} \sum_{t=0}^{T} E_P\Big[p_t\Big] < \infty$, and $z_i \in \Psi_+$, define

$$BC^{\text{AD}}(p; z_i) \equiv \Big\{\tilde{c}_i \in \Psi_+ : \lim_{T\to+\infty} \textstyle\sum_{t=0}^{T} E_P\Big[p_t \cdot \tilde{c}_{i,t}\Big] \leq \lim_{T\to+\infty} \sum_{t=0}^{T} E_P\Big[p_t \cdot z_{i,t}\Big]\Big\}.$$

LEMMA 20: Assume A.3 and that $P_1 = P_2 = P$. Consider a consumption process c_i, so $c_i \in \Psi_+$, such that $\lim_{T\to+\infty} \sum_{t=0}^{T} \beta_i^t \, E_{P_i}[u_i(c_{i,t})] > -\infty$, and p_i^c satisfies $\lim_{T\to+\infty} \sum_{t=0}^{T} E_P\Big[p_{i,t}^c\Big] < \infty$. If $z_i \in \Psi_+$ and $\lim_{T\to+\infty} \sum_{t=0}^{T} E_P\Big[p_{i,t}^c \cdot (c_{i,t} - z_{i,t})\Big] = 0$, then c_i is a maximizer for i on the set $BC^{\text{AD}}(p_i^c; z_i)$.

PROOF: Since $\lim_{T\to+\infty} \sum_{t=0}^{T} E_P\Big[p_{i,t}^c\Big] < \infty$ and $z_i \in \Psi_+$, $\lim_{T\to+\infty} \sum_{t=0}^{T} E_P\Big[p_{i,t}^c \cdot z_{i,t}\Big] < \infty$. Since $\lim_{T\to+\infty} \sum_{t=0}^{T} E_P\Big[p_{i,t}^c \cdot (c_{i,t} - z_{i,t})\Big] = 0$, $c_i \in BC^{\text{AD}}(p_i^c; z_i)$.

Define $\mu_i \equiv u_i'(c_{i,0}(\omega))$. $\mu_i > 0$. Clearly, c_i is the unique solution to the system of first order conditions $\beta_i^t \cdot u_i'(c_{i,t}(\omega)) = \mu_i \cdot p_{i,t}^c(\omega)$. Also, the Lagrangean function

$$\lim_{T\to+\infty} \left\{ \sum_{t=0}^{T} E_P\Big[\beta_i^t \cdot u_i(\tilde{c}_{i,t})\Big] + \mu_i \cdot \sum_{t=0}^{T} E_P\Big[p_{i,t}^c \cdot (\tilde{c}_{i,t} - z_{i,t})\Big] \right\}$$

is strictly concave in $\tilde{c}_i$ and is well defined at the point c_i. It follows (e.g. Luenberger (1969) Theorem 1 in Section 8.4 and Lemma 1 in Section 8.7) that the first order conditions are sufficient to identify a global maximizer and c_i maximizes the Lagrangean function. Therefore c_i solves the constrained optimization problem. ∎

Given a price process q, the set of *Arrow price* processes is defined as

$$\mathcal{P}(q) \equiv \Big\{p \in \times_{t=0}^{\infty}\Psi_{+}^{t} \ : \ \forall\, t \geq 0, \quad p_t(\omega)\cdot q_t(\omega) = E_P[p_{t+1}\cdot r_{t+1}|\mathcal{F}_t](\omega) \quad \forall\, \omega \in \Omega\Big\}.$$

Clearly, if c_i is an Euler process at the price process q then $p_i^c \in \mathcal{P}(q)$. Define

$$\mathcal{P}^1(q) \equiv \Big\{p \,\in\, \mathcal{P}(q) \ : \ \lim_{T\to+\infty} \textstyle\sum_{t=0}^{T} E_P\Big[p_t\Big] < \infty\Big\}$$

the set of Arrow price processes that are summable.

LEMMA 21: Assume A.3. Given q let $p \in \mathcal{P}^1(q)$. Let $z_i \in \Psi_+$ and let the portfolio θ_i supports c_i at (q, z_i). Then, given ω, c_i satisfies $\lim_{T\to+\infty} \sum_{t=0}^{T} E_P\Big[p_t\cdot(c_{i,t} - z_{i,t})\Big] = 0$ if and only if $\lim_{T\to+\infty} E_P\Big[p_T\cdot q_T\cdot\theta_{i,T}\Big] = 0$.
PROOF: Since θ_i supports c_i at (q, z_i), we can write

$$\lim_{T\to+\infty} \sum_{t=0}^{T} E_P\Big[p_t\cdot(c_{i,t} - z_{i,t})\Big] = \lim_{T\to+\infty} \sum_{t=0}^{T} E_P\Big[p_t\cdot(r_t\cdot\theta_{i,t-1} - q_t\cdot\theta_{i,t})\Big].$$

Since p is an Arrow price process we may use the argument given in Lemma 19 to reduce the right hand side to a single term. By doing so we obtain
$0 = \lim_{T\to+\infty} \sum_{t=0}^{T} E_P\Big[p_t\cdot(c_{i,t} - z_{i,t})\Big] = \lim_{T\to+\infty} E_P\Big[-p_T\cdot q_T\cdot\theta_{i,T}\Big]$. ∎

Given q and $p \in \mathcal{P}(q)$, define

$$\begin{aligned} BC^{\mathrm{TC}}(q,p;z_i) \equiv \Big\{ & c_i \in \Psi_+ : \text{ there exists } \theta_i, \text{ with } \theta_{i,t} \in \Psi^t\ \forall\, t\geq 0, \text{ such that} \\ & \forall\, t \geq 0,\ c_{i,t}(\omega) + q_t(\omega)\cdot\theta_{i,t}(\omega) \leq z_{i,t}(\omega) + r_t(\omega)\cdot\theta_{i,t-1}(\omega) \quad \forall\,\omega\in\Omega, \\ & \liminf_{T\to+\infty} E_P\Big[p_T\cdot q_T\cdot\theta_{i,T}\Big] \geq 0\Big\}. \end{aligned}$$

LEMMA 22: Assume A.3. Given q and any $p \in \mathcal{P}^1(q)$, if $z_i \in \Psi_+$ then $BC^{\mathrm{TC}}(q,p;z_i) \subset BC^{\mathrm{AD}}(p;z_i)$.
PROOF: Consider $c_i \in BC^{\mathrm{TC}}(q,p;z_i)$ and let θ_i denote the supporting portfolio process. We would like to show that

$$\lim_{T\to+\infty} \sum_{t=0}^{T} E_P\Big[p_t\cdot c_{i,t}\Big] \leq \lim_{T\to+\infty} \sum_{t=0}^{T} E_P\Big[p_t\cdot z_{i,t}\Big].$$

Using the sequence of budget constraints in the definition of the set $BC^{\mathrm{TC}}(q,p;z_i)$, we have

$$\sum_{t=0}^{T} E_P\Big[p_t\cdot(c_{i,t} - z_{i,t})\Big] \leq \sum_{t=0}^{T} E_P\Big[p_t\cdot(r_t\cdot\theta_{i,t-1} - q_t\cdot\theta_{i,t})\Big].$$

By the argument already used in Lemma 19 and 21 we conclude that for all $T \geq 0$ we have

$$\sum_{t=0}^{T} E_P\Big[p_t\cdot(c_{i,t} - z_{i,t})\Big] \leq\ E_P\Big[-p_T\cdot q_T\cdot\theta_{i,T}\Big].$$

The right hand side of the inequality is nonpositive since $c_i \in BC^{\rm TC}(q,p;z_i)$ implies that $\liminf_{T\to+\infty} E_P\Big[p_T \cdot q_T \cdot \theta_{i,T}\Big] \geq 0$. Also the left hand side has a limit since $p \in \mathcal{P}^1(q)$ so p is summable while $c_i \in \Psi_+$ and $z_i \in \Psi_+$ so that $(c_i - z_i)$ is uniformly bounded. We can conclude that $c_i \in BC_i^{\rm AD}(p;z_i)$. ■

PROOF OF THEOREM 3B: We shall use a result from Magill and Quinzii (1994). We establish that our construction satisfies their definition of an equilibrium with a transversality condition at each node. Then we invoke their Theorem 5.2 to conclude that such an equilibrium is also an IDC equilibrium.

Magill and Quinzii's definition of an equilibrium with transversality conditions requires that, in addition to feasibility, the following five conditions hold (a) $\widehat{\theta}_i$ supports $\widehat{c}_i$ at $(\widehat{q}, \widehat{z}_i)$, (b) $\widehat{p}_i \in \mathcal{P}^1(\widehat{q})$, (c) $\widehat{c}_i$ is a maximizer on $BC^{AD}(\widehat{p}_i;\widehat{z}_i)$, (d) a transversality condition holds at each node, and (e) $\widehat{c}_i$ is also a maximiser on the budget set defined by the first two conditions. As we now show, all five of thses requirements are met under the hypotheses of Theorem 3B.

Recall that $\widehat{\theta}_i$ is the portfolio that supports $\widehat{c}_i$ at the price process $\widehat{q}$ and endowment process $\widehat{z}_i$ so (a) holds. To simplify the notation we set $\widehat{p}_i \equiv p_i^{\widehat{c_i}}$. By hypothesis (iv) of Theorem 3, $\widehat{c}_i$ is an Euler process at $\widehat{q}$ and so $\widehat{p}_i \in \mathcal{P}(\widehat{q})$; by hypothesis (v B), summability, $\widehat{p}_i \in \mathcal{P}^1(\widehat{q})$ so (b) holds. Hypotheses (iii), supportability, (iv) and (v B) let us invoke Lemma 21 and Lemma 22. By hypotheses (i), feasibility, (ii) and (v B), Lemma 20 holds and so $\widehat{c}_i$ is a maximiser on $BC^{AD}(\widehat{p}_i;\widehat{z}_i)$ if $z_i \in \Psi_+$ and $\lim_{T\to+\infty} \sum_{t=0}^{T} E_P\Big[\widehat{p}_{i,t} \cdot (\widehat{c}_{i,t} - \widehat{z}_{i,t})\Big] = 0$. The latter requirement follows from Lemma 21 and hypothesis (vi B), transversality conditions, at date $t = 0$ so (c) holds. Lemma 21 also lets us claim that $\widehat{c}_i \in BC^{\rm TC}(\widehat{q}, \widehat{p}_i; \widehat{z}_i)$ while, by Lemma 22, $BC^{\rm TC}(\widehat{q}, \widehat{p}_i; \widehat{c}_i) \subset BC^{\rm AD}(\widehat{p}_i; \widehat{z}_i)$ so that $\widehat{c}_i$ is a maximizer on $BC_i^{\rm TC}(\widehat{q}, \widehat{p}_i; \widehat{z}_i)$. If, in addition, hypothesis (vi B) holds then $\widehat{c}_i$ is a maximiser on a budget set where a transversality condition holds at each node so (d) and (e) hold.

Since the consumption processes are aggregate feasible, (iii) implies that at every $t \geq 0$, $\widehat{\theta}_{1,t}(\omega) + \widehat{\theta}_{2,t}(\omega) = 0$ for all $\omega \in \Omega$ where we use the fact that, since u_i is strictly increasing, the spot market budget constraints are satisfied with equality. To complete the proof, notice that preferences with discounted additively separable expected utility representations satisfy the assumption of uniform impatience and so, by Theorem 5.2 in Magill and Quinzii (1994), we can conclude that there is a uniform bound on the value of debt. It follows that $\widehat{c}_i$ is a maximizer on $BC(\widehat{q}; \widehat{z}_i)$ and we have an IDC equilibrium. ■

PROOF OF THEOREM 4

By Lemma 12 (ii), in the proposed solution, for all $t \geq 1$

$$\beta_1 \cdot \frac{E_P[r_t \cdot u_1'(C_{1,t})|\mathcal{F}_{t-1}](\omega)}{u_1'(C_{1,t-1}(\omega))} = \beta_2 \cdot \frac{E_P[r_t \cdot u_2'(C_{2,t})|\mathcal{F}_{t-1}](\omega)}{u_2'(C_{2,t-1}(\omega))} \quad \text{for all } \omega \in \Omega.$$

Define an asset price process q

$$q_{t-1}(\omega) \equiv \beta_i \cdot \frac{E[r_t \cdot u_i'(C_{i,t})|\mathcal{F}_{t-1}](\omega)}{u_i'(C_{i,t-1}(\omega))}.$$

It follows that the consumption processes satisfy the Euler equations with the price process q. Also, Proposition 8 holds. Using the spot market budget constraints with asset prices q and consumption process C_i, we can construct the supporting portfolio θ_i.

To complete the proof of Theorem 4 we shall apply Theorem 3B for case (i) and Theorem 3A for cases (ii) and (iii).

For Case (i) we will apply Theorem 3B and so we need to verify (vi B), that a transversality condition is satisfied at each node. Lemma 23 and Lemma 24 provide the required verification.

LEMMA 23: If c_i is an Euler process at q and θ_i is a supporting portfolio then

$$\beta_i^T \cdot u_i'(c_{i,T}(\omega)) \cdot q_T(\omega) \cdot \theta_{i,T}(\omega) = \beta_i^T \cdot u_i'(c_{i,T}(\omega)) \cdot \Big(z_{i,T}(\omega) - c_{i,T}(\omega)\Big)$$
$$+ \sum_{\tau=0}^{T-1} \beta_i^\tau \cdot u_i'(c_{i,\tau}(\omega)) \cdot \left(\prod_{s=\tau}^{T-1} \hat{r}_{i,s+1}(\omega)\right) \cdot \Big(z_{i,\tau}(\omega) - c_{i,\tau}(\omega)\Big)$$

where $\hat{r}_i$ is the process induced by c_i.

PROOF: By Proposition 1 (i),

$$\prod_{s=\tau}^{T-1} \frac{r_{s+1}(\omega)}{q_s(\omega)} = \frac{1}{\beta_i^{T-\tau}} \cdot \Big(\prod_{s=\tau}^{T-1} \hat{r}_{i,s+1}(\omega)\Big) \cdot \frac{u_i'(c_{i,\tau}(\omega))}{u_i'(c_{i,T}(\omega))}.$$

The spot market budget constraints, which, by u_i strictly increasing, hold as equalities, are

$$c_{i,t}(\omega) + q_t(\omega) \cdot \theta_{i,t}(\omega) = z_{i,t}(\omega) + r_t(\omega) \cdot \theta_{i,t-1}(\omega).$$

Iterating on the equation we obtain

$$q_T(\omega) \cdot \theta_{i,T}(\omega) = z_{i,T}(\omega) - c_{i,T}(\omega) + \sum_{\tau=0}^{T-1} \left(\prod_{s=\tau}^{T-1} \frac{r_{s+1}(\omega)}{q_s(\omega)}\right) \cdot \Big(z_{i,\tau}(\omega) - c_{i,\tau}(\omega)\Big).$$

After carrying out the substitution we can evaluate

$$\beta_i^T \cdot u_i'(c_{i,T}(\omega)) \cdot q_T(\omega) \cdot \theta_{i,T}(\omega) = \beta_i^T \cdot u_i'(c_{i,T}(\omega)) \cdot \Big(z_{i,T}(\omega) - c_{i,T}(\omega)\Big)$$
$$+ \sum_{\tau=0}^{T-1} \beta_i^\tau \cdot u_i'(c_{i,\tau}(\omega)) \cdot \left(\prod_{s=\tau}^{T-1} \hat{r}_{i,s+1}(\omega)\right) \cdot \Big(z_{i,\tau}(\omega) - c_{i,\tau}(\omega)\Big). \qquad \blacksquare$$

LEMMA 24: Assume that the economy is such that in the proposed solution, $\forall\ t \geq 1$, $u_2'(C_{2,t}(\omega)) \cdot \Big(z_{2,t}(\omega) - C_{2,t}(\omega)\Big) = \bar{c}_{2,t}$ for all $\omega \in \Omega$ and $C_{2,0}(\omega)$ solves

$$u_2'(C_{2,0}(\omega)) \cdot \Big(z_{2,0}(\omega) - C_{2,0}(\omega)\Big) = -\mathrm{Lim}_{T\to+\infty} \sum_{\tau=1}^{T} \beta_2^\tau \cdot \bar{c}_{2,\tau}.$$

Then, for every $t \geq 0$, $\quad \mathrm{Lim}_{T\to+\infty} E[\beta_i^T \cdot u_i'(C_{i,T}) \cdot q_T \cdot \theta_{i,T}|\mathcal{F}_t](\omega) = 0 \quad$ for all $\omega \in \Omega$ and the transversality conditions for both the agents are satisfied.

PROOF: Consider $i = 2$. Since $\hat{r}_{2,t}(\omega) = 1 \quad \forall\, t \geq 0$ $P-\text{a.s.}\,\omega$, the expression obtained in Lemma 23 takes the form

$$\beta_2^T \cdot u_2'(C_{2,T}(\omega)) \cdot q_T(\omega) \cdot \theta_{2,T}(\omega) = \sum_{\tau=0}^{T} \beta_2^\tau \cdot u_2'(C_{2,\tau}(\omega)) \cdot \Big(z_{2,\tau}(\omega) - C_{2,\tau}(\omega)\Big)$$
$$= \sum_{\tau=1}^{T} \beta_2^\tau \cdot \bar{c}_{2,\tau} + u_2'(C_{2,0}(\omega)) \cdot \Big(z_{2,0}(\omega) - C_{2,0}(\omega)\Big).$$

So $\beta_2^T \cdot u_2'(C_{2,T}) \cdot q_T \cdot \theta_{2,T}$ is a deterministic quantity, and by the hypothesis

$$\mathrm{Lim}_{T\to+\infty} E_P[\beta_2^T \cdot u_2'(C_{2,T}) \cdot q_T \cdot \theta_{2,T}|\mathcal{F}_t](\omega) = 0.$$

Note that $\beta_2^T \cdot u_2'(C_{2,T}) \cdot q_T \cdot \theta_{2,T} = -\sum_{\tau=T+1}^{\infty} \beta_2^\tau \cdot \bar{c}_{2,\tau}$, a deterministic quantity.

We turn to agent 1. In the proposed solution $\theta_{1,t}(\omega) = -\theta_{2,t}(\omega)$ for all $t \geq 0$ and for all $\omega \in \Omega$. Since $\hat{r}_{2,t}(\omega) = 1$, by Proposition 1 (ii),

$$y_T(\omega) = \left(\frac{\beta_1}{\beta_2}\right)^T \cdot \frac{1}{\prod_{s=1}^{T} \hat{r}_{1,s}(\omega)} \cdot y_0(\omega) \;\Leftrightarrow\; \frac{u_1'(C_{1,T}(\omega))}{u_2'(C_{2,T}(\omega))} = \left(\frac{\beta_2}{\beta_1}\right)^T \cdot \prod_{s=1}^{T} [\hat{r}_{1,s}(\omega)] \cdot \frac{u_1'(C_{1,0}(\omega))}{u_2'(C_{2,0}(\omega))}.$$

It follows that

$$\mathrm{Lim}_{T\to+\infty} E_P[\beta_1^T \cdot u_1'(C_{1,T}) \cdot q_T \cdot \theta_{1,T}|\mathcal{F}_t](\omega)$$
$$= -\mathrm{Lim}_{T\to+\infty} E_P\left[\beta_2^T \cdot \prod_{s=1}^{T} [\hat{r}_{1,s}] \cdot \frac{u_1'(C_{1,0})}{u_2'(C_{2,0})} \cdot u_2'(C_{2,T}) \cdot q_T \cdot \theta_{2,T} \middle| \mathcal{F}_t\right](\omega)$$
$$= -\mathrm{Lim}_{T\to+\infty} \frac{u_1'(C_{1,0})}{u_2'(C_{2,0})} \cdot \beta_2^T \cdot u_2'(C_{2,T}) \cdot q_T \cdot \theta_{2,T} E_P\left[\prod_{s=1}^{T} [\hat{r}_{1,s}] \middle| \mathcal{F}_t\right](\omega).$$

Since

$$\beta_2^T \cdot u_2'(C_{2,T}) \cdot q_T \cdot \theta_{2,T} = -\sum_{\tau=T+1}^{\infty} \beta_2^\tau \cdot \bar{c}_{2,\tau}$$
$$\Rightarrow \qquad \mathrm{Lim}_{T\to+\infty} E[\beta_1^T \cdot u_1'(C_{1,T}) \cdot q_T \cdot \theta_{1,T}|\mathcal{F}_t](\omega)$$
$$= -\mathrm{Lim}_{T\to+\infty} \frac{u_1'(C_{1,0})}{u_2'(C_{2,0})} \Big(-\sum_{\tau=T+1}^{\infty} \beta_2^\tau \cdot \bar{c}_{2,\tau}\Big) \cdot E_P\left[\prod_{s=1}^{T} [\hat{r}_{1,s}] \middle| \mathcal{F}_t\right](\omega) = 0$$

where we use the fact that $E\left[\hat{r}_{i,t}|\mathcal{F}_{t-1}\right](\omega) = 1$ together with the law of iterated expectations and the fact that $\mathrm{Lim}_{T\to\infty} \sum_{\tau=T+1}^{\infty} \beta_2^\tau \cdot \bar{c}_{2,\tau} = 0$. ■

That completes the proof of Theorem 4 in case (i).

In case (ii) the asset is not traded and so it suffices to note that, by Proposition 8, condition (v A) in Theorem 3A is satisfied and that completes the proof of Theorem 4 (ii).

We turn to case (iii), the case with an endowment perturbation. We show that there is an endowment perturbation for which (C_1, C_2) and the associated supporting portfolios continue to satisfy the conditions in Theorem 3A thus proving that we have an IDC equilibrium.

For the consumption processes C_i, let the personalized supporting price process for agent i be

$p^C_{i,t}(\omega) \equiv \beta_i^t \cdot u_i'(C_{i,t}(\omega))/u_i'(C_{i,0}(\omega))$.

For some $c \in (0, Z_0(\omega))$, consider the economy with endowment distribution (C_1, C_2). By (ii) there exists a no-trade IDC equilibrium $(c_1^*, c_2^*, \theta_1^*, \theta_2^*, q^*)$ such that $c_i^* = C_i$ and $\theta^*_{i,t}(\omega) = 0$ for all $t \geq 0$ and $i = 1, 2$. Notice the following three properties of our construction:

(a) $p^C_{2,t+1}(\omega) \cdot \frac{r_{t+1}(\omega)}{q_t^*(\omega)} = p^C_{2,t}(\omega)$ for all $t \geq 0$ and $\omega \in \Omega$.

(b) $\frac{r_{t+1}(\omega)}{q_t^*(\omega)} = \frac{u_2'(C_{2,t}(\omega))}{\beta_2 u_2'(C_{2,t+1}(\omega))} > \frac{1}{\beta_2 M} > 1$ for all $t \geq 0$ and $\omega \in \Omega$.

(c) There exists $\overline{M} < +\infty$ such that $\frac{r_{t+1}(\omega)}{q_t^*(\omega)} \leq \overline{M}$ for all $t \geq 0$ and $\omega \in \Omega$.

Property (a) restates the Euler equation for 2 using the definitions of p_2^C and $\widehat{r}_2$ and the fact that in our construction $\widehat{r}_{2,t}(\omega) = 1$ always. Property (b) follows by property (a) and Lemma 16. Finally, property (c) follows from the Euler equations of the agents

$$\beta_2 \frac{u_2'(C_{2,t+1}(\omega)) \cdot r_{t+1}(\omega)}{u_1'(C_{2,t}(\omega))} = q_t^*(\omega) = \beta_1 \frac{E_P[r_{t+1} \cdot u_1'(C_{1,t+1}) | \mathcal{F}_t](\omega)}{u_1'(C_{1,t}(\omega))} \tag{19}$$

and the fact that assumption A.3 implies that the numerator is bounded below by $\underline{r} \cdot u_i'(\overline{z})$, which is bounded away from zero by assumptions A.2 and A.4. Indeed, if property (c) were not true then one could find a path ω where $q_t^*(\omega)$ is arbitrarily close to zero. But then, (19) would imply that both $C_{1,t}(\omega)$ as well as $C_{2,t}(\omega)$ are arbitrarily close to zero which would contradict feasibility since $Z_t(\omega) \geq \underline{z} > 0$ for all $t \geq 0$ by assumption A.2.

Now we construct an alternative endowment distribution by perturbing the no trade endowment distribution (C_1, C_2). By hypothesis, there exists $\widetilde{\tau}(\omega)$ such that $0 < C_{2,t}(\omega) \leq \frac{Z_t(\omega)}{2}$ for all $t \geq \widetilde{\tau}(\omega)$. Set $\widetilde{T}(\omega) \equiv \max\{1, \widetilde{\tau}(\omega)\}$. By Lemma 16 one has that

$$u'\left(C_{2,\widetilde{T}(\omega)}(\omega)\right) \leq M \cdot u'\left(C_{2,\widetilde{T}(\omega)-1}(\omega)\right) \leq M \cdot u'\left(\frac{\underline{z}}{2}\right)$$

and it follows that

$$C_{2,\widetilde{T}(\omega)}(\omega) \geq (u')^{-1}\left(M \cdot u'\left(\frac{\underline{z}}{2}\right)\right)$$

Pick $0 < \overline{\varepsilon} < \min\left\{C_{2,0}, (u')^{-1}(M \cdot u'(\underline{z}/2))\right\}$ so that $\overline{\varepsilon} \cdot \left(\overline{M} - 1\right) \leq \underline{z}/2$. It follows that $0 < \overline{\varepsilon} < C_{2,\widetilde{T}(\omega)}(\omega)$.

Define perturbed individual endowments $(\widetilde{z}_1, \widetilde{z}_2)$ as follows:

$$\widetilde{z}_{1,t}(\omega) \quad = \quad \begin{cases} C_{1,t}(\omega) & \text{if } t < \widetilde{T}(\omega) \\ C_{1,\widetilde{T}(\omega)}(\omega) + \overline{\varepsilon} & \text{if } t = \widetilde{T}(\omega) \\ C_{1,t}(\omega) - \overline{\varepsilon} \cdot \left[\frac{r_t(\omega)}{q^*_{t-1}(\omega)} - 1\right] & \text{otherwise} \end{cases}$$

$$\widetilde{z}_{2,t}(\omega) = \begin{cases} C_{2,t}(\omega) & \text{if } t < \widetilde{T}(\omega) \\ C_{2,\widetilde{T}(\omega)}(\omega) - \overline{\varepsilon} & \text{if } t = \widetilde{T}(\omega) \\ C_{2,t}(\omega) + \overline{\varepsilon} \cdot \left[\frac{r_t(\omega)}{q^*_{t-1}(\omega)} - 1\right] & \text{otherwise} \end{cases}$$

It follows by properties (b) and (c) above, and our choice of $\overline{\varepsilon}$, that $(\widetilde{z}_1, \widetilde{z}_2)$ is feasible and that $\widetilde{z}_{i,t}(\omega) \in (0, Z_t(\omega))$ for all $i = 1, 2$. Now, consider portfolios (θ_1, θ_2) where $\theta_1 = -\theta_2$ and

$$\theta_{2,t}(\omega) = \begin{cases} 0 & \text{if } t \leq \widetilde{T}(\omega) - 1 \\ -\frac{\overline{\varepsilon}}{q^*_t(\omega)} & \text{if } t = \widetilde{T}(\omega) \\ \frac{\overline{\varepsilon}}{q^*_t(\omega)} \cdot \left[\frac{r_t(\omega)}{q^*_{t-1}(\omega)} - 1\right] + \frac{r_t(\omega)}{q^*_t(\omega)} \cdot \theta_{2,t-1}(\omega) & \text{otherwise} \end{cases}$$

Then, it is easy to verify that for every agent $i = 1, 2$, portfolio θ_i supports consumption process C_i at prices q^* since by construction

$$\widetilde{z}_{i,t}(\omega) + r_t(\omega) \cdot \theta_{i,t-1}(\omega) - q^*_t(\omega) \cdot \theta_{i,t}(\omega) = C_{i,t}(\omega) \quad \text{for all } t \geq 0 \text{ and } \omega \in \Omega$$

and that $|q^*_t(\omega) \cdot \theta_{i,t}(\omega)| \leq \overline{\varepsilon} < \frac{\underline{z}}{2}$ for all $t \geq 0$ and $\omega \in \Omega$.

Since (C_1, C_2) are feasible consumption processes, (θ_1, θ_2) supports (C_1, C_2) at the price process q^*, $|q^*_t(\omega) \cdot \theta_{i,t}(\omega)| < \frac{\underline{z}}{2}$ for all $i = 1, 2$. This shows that condition (vi A) in Theorem 3A is satisfied. By Proposition 8, condition (v A) in Theorem 3A is satisfied and that completes the proof of Theorem 4 (iii).

PROOF OF COROLLARY 2

By construction $\hat{r}_{2,t}(\omega) = 1$. Since $r_t(\omega) = 1$ for all $t \geq 0$ and $\omega \in \Omega$, then A.4 holds and by Lemma 1 and Proposition 2 (ii), agent 2 vanishes on ω if and only if $\prod_{t=1}^{T} \hat{r}_{1,t}(\omega) \to 0$ on ω. By Propositions 5 and 8, $\prod_{t=1}^{T} \hat{r}_{1,t}(\omega) \to 0$ $P-$a.s. ω if A.7 holds. So it suffices to show that

$$P\left(\omega : \text{limsup} \frac{1}{T} \sum_{t=1}^{T} E_P[\log \hat{r}_{1,t} | \mathcal{F}_{t-1}](\omega) < 0\right) = 1.$$

Let $\Delta^{S-1} = \left\{p \in R^S_+ : \sum_{k=1}^{S} p_k = 1 \text{ and } p_k \geq \underline{p}\right\}$. For any $p \in \Delta^{S-1}$, let $Z_{p,\epsilon}$ be a subset of $[\underline{z}, \overline{z}]$ with S elements satisfying $\sum_{z \in Z_{p,\epsilon}} p_z \cdot \left(z - \sum_{\widetilde{z} \in Z_{p,\epsilon}} p_{\widetilde{z}} \cdot \widetilde{z}\right)^2 \geq \epsilon$, i.e. the variance is at least ϵ. Let $z^*_{p,\epsilon}$ be the smallest element in $Z_{p,\epsilon}$. Let

$$v_\epsilon \equiv \max_{p \in \Delta^{S-1}} \max_{Z_{p,\epsilon} \subset [\underline{z}, \overline{z}]} \max_{c \in \left[0, z^*_{p,\epsilon}\right]} \sum_{z \in Z_{p,\epsilon}} p_z \cdot \log \frac{u_1'(z - c)}{\sum_{\widetilde{z} \in Z_{p,\epsilon}} p_{\widetilde{z}} \cdot u_1'(\widetilde{z} - c)}$$

and let p_ϵ, $Z_{p_\epsilon,\epsilon}$ and c_ϵ be the maximizers.[22]

[22] Note that $\frac{u_1'(z-c)}{\sum_{\tilde{z} \in Z_{p,\epsilon}} p_{\tilde{z}} \cdot u_1'(\tilde{z}-c)}$ is well defined for all $c \in \left[0, z^*_{p,\epsilon}\right]$, $p \in \Delta^{S-1}$ and $Z_{p,\epsilon} \subset [\underline{z}, \overline{z}]$, it is non-negative, bounded and continuous in its arguments.

Note that there exists $z_1, z_2 \in Z_{p_\epsilon,\epsilon}$ such that $z_2 > z_1 + \sqrt{\frac{\epsilon}{S}}$. Note also that

$$\begin{aligned} \log \frac{u_1'(z_1 - c_\epsilon)}{\sum_{z \in Z_{p_\epsilon,\epsilon}} p_{\epsilon,z} \cdot u_1'(z - c_\epsilon)} - \log \frac{u_1'(z_2 - c_\epsilon)}{\sum_{z \in Z_{p_\epsilon,\epsilon}} p_{\epsilon,z} \cdot u_1'(z - c_\epsilon)} &= \log \frac{u_1'(z_1 - c_\epsilon)}{u_1'(z_2 - c_\epsilon)} \\ &\geq \log \frac{u_1'(z_1 - c_\epsilon)}{u_1'(z_1 + \sqrt{\frac{\epsilon}{S}} - c_\epsilon)} > 0 \end{aligned}$$

and it follows by the strict concavity of the function $\log x$ and Jensen's inequality that

$$v_\epsilon = \sum_{z \in Z_{p,\epsilon}} p_{\epsilon,z} \cdot \log \frac{u_1'(z - c_\epsilon)}{\sum_{\widetilde{z} \in Z_{p,\epsilon}} p_{\epsilon,\widetilde{z}} \cdot u_1'(\widetilde{z} - c_\epsilon)} < 0.$$

Notice that since $P_t(\omega) \in \Delta^{S-1}$, $Z_t \in [\underline{z}, \overline{z}]$, $\operatorname{var}[Z_t|\mathcal{F}_{t-1}](\omega) > \epsilon > 0$, $c_{2,t}(\omega) = c_{2,t}(\widetilde{\omega})$ for all $\widetilde{\omega} \in \Omega(s^{t-1}(\omega))$ and $c_{2,t}(\omega) \leq \min_{\widetilde{\omega} \in \Omega(s^{t-1}(\omega))} \{Z_t(\widetilde{\omega})\}$, then

$$E_P[\log \hat{r}_{1,t}|\mathcal{F}_{t-1}](\omega) = E_P\left[\log\left(\frac{u_1'(Z_t - c_{2,t})}{E_P[u_1'(Z_t - c_{2,t})|\mathcal{F}_{t-1}]}\right)\middle|\mathcal{F}_{t-1}\right](\omega) \leq v_\epsilon.$$

It follows that for every T and every ω

$$\frac{1}{T}\sum_{t=1}^{T} E_P[\log \hat{r}_{1,t}|\mathcal{F}_{t-1}](\omega) \leq v_\epsilon < 0.$$

Therefore, $P-$a.s., $\quad \operatorname{limsup} \frac{1}{T} \sum_{t=1}^{T} E_P[\log \hat{r}_{1,t}|\mathcal{F}_{t-1}](\omega) \leq v_\epsilon < 0.$ ■

REFERENCES

ALCHIAN, A. (1950) "Uncertainty, Evolution, and Economic Theory," *Journal of Political Economy* 58, 211-221.

BECKER, R. (1980) "On the Long Run Steady State in a Simple Dynamic Model of Equilibrium with Heterogeneous Households," *Quarterly Journal of Economics*, 95, 375-382.

BECKER, R., AND C. FOIAS (1987) "A Characterization of Ramsey Equilibrium," *Journal of Economic Theory*, 41, 173-184.

BECKER, R., AND I. ZILCHA (1997) "Stationary Ramsey Equilibria under Uncertainty," *Journal of Economic Theory*, 75, 122-140.

BEKER, P., AND E. ESPINO (2008) "The Dynamics of Efficient Asset Trading with Heterogeneous Beliefs" mimeo (July).

BEWLEY, T. (1982) "An Integration of Equilibrium Theory and Turnpike Theory," *Journal of Mathematical Economics*, 10, 233-267.

BLUME, L., AND D. EASLEY (2006) "If You Are So Smart, Why Aren't You Rich? Belief Selection in Complete and Incomplete Markets," *Econometrica*, 74, 929-966.

CHAMBERLAIN, G., AND C. WILSON (2000) "Optimal Intertemporal Consumption under Uncertainty," *Review of Economic Dynamics*, 3, 365-395.

CHUNG, K. L. (1974) "A Course in Probability Theory," Academic Press, London.
CONSTANTINIDES, G., AND D. DUFFIE (1996) "Asset Pricing with Heterogeneous Consumers," *Journal of Political Economy*, 104, 219-240.
COOTNER, P. (1967) "The Random Character of Stock Prices," MIT Press, Cambridge.
COURY, T., AND E. SCIUBBA (2005) "Belief Heterogeneity and Survival in Incomplete Markets, " mimeo (March).
DUFFIE, D., J.D. GEANAKOPLOS, A. MAS-COLELL, AND A. MCLENNAN (1994) "Stationary Markov Equilibria," *Econometrica*, 62, 745-781.
FAMA, E. (1965) "The Behavior of Stock Market Prices," *Journal of Business*, 38 (1), 34-105.
FREEDMAN, D. (1973) "Another Note on the Borel-Cantelli Lemma and the Strong Law, with the Poisson approximation a by-product ," *Annals of Probability*, 1, 910-925.
FRIEDMAN, M. (1953) "Three Essays on the State of Economic Science," University of Chicago Press, Chicago.
JACKSON, M., E. KALAI, AND R. SMORODINSKY (1999) "Bayesian Representation of Stochastic Processes under Learning: de Finetti Revisited," *Econometrica*, 67, 875-893.
KREBS, T. (2004a) "Non-existence of Recursive Equilibria on Compact State Spaces when Markets are Incomplete," *Journal of Economic Theory*, 115, 134-150.
KREBS, T. (2004b) "Testable Implications of Consumption-Based Asset Pricing Models with Incomplete Markets," *Journal of Mathematical Economics*, 40, 191-206.
KUBLER, F., AND K. SCHMEDDERS (2002) "Recursive Equilibria in Economies with Incomplete Markets," *Macroeconomic Dynamics*, 6, 284-306.
LEVINE, D., AND W. ZAME (1996) "Debt Constraint and Equilibrium in Infinite Horizon Economies with Incomplete Markets," *Journal of Mathematical Economics*, 26, 103-131.
LEVINE, D., AND W. ZAME (2002) "Does Market Incompleteness Matter?," *Econometrica*, 70, 1805-1840.
LJUNGQVIST, L., AND T.J. SARGENT (2004) "Recursive Macroeconomic Theory," Cambridge: MIT.
LUENBERGER, D. (1969) "Optimization by Vector Space Methods," New York: Wiley.
MAGILL, M., AND M. QUINZII (1994) "Infinite Horizon Incomplete Markets," *Econometrica*, 62, 853-880.
PHILLIPS, P.C.B., AND W. PLOBERGER (1996) "An Asymptotic Theory of Bayesian Inference for Time Series," *Econometrica*, 64, 381-412.
RADER, T. (1981) "Utility over Time: The Homothetic Case," *Journal of Economic Theory*, 25, 219-236.
RAMSEY, F. (1928) "A Mathematical Theory of Saving," *Economic Journal*, 38, 543-559.
SANDRONI, A. (2000) "Do Markets Favour Agents Able to Make Accurate Predictions," *Econometrica*, 68, 1303-1341.
SANDRONI, A. (2005) "Market Selection when Markets Are Incomplete," *Journal of Mathematical Economics*, 41, 91-104.

SANTOS, M., AND M. WOODFORD (1997) "Rational Asset Pricing Bubbles," *Econometrica*, 65, 19-57.