Solving the Dirichlet Problem via Brownian Motion

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1 Introduction

Consider the Dirichlet problem of the following form: Let D be a bounded, connected open set in $\mathbb{R}^d$ and ∂D its boundary. Given any continuous function f defined on the boundary ∂D, one needs to find a function u which is continuous on $\overline{D} = D \cup \partial D$, equal to f on ∂D, and harmonic in D. The problem may be interpreted as that of finding the steady-state temperature distribution u in the heat-conducting region D, given the temperature function on the boundary ∂D. Provided that the boundary ∂D is sufficiently "nice", the problem can be solved in a variety of ways. One of the most intriguing methods (if not the shortest) is via Brownian motion. This method perhaps comes the closest to modeling the physics of heat diffusion. Here we give an outline of this approach which will be greatly amplified in the sections to follow. For simplicity, we temporarily restrict ourselves to the simplest nontrivial case where the dimension $d = 2$.

Before introducing the notion of a standard Brownian motion process, let us review the mean-value property and harmonicity. Recall that a function $u : \mathbb{R}^2 \to \mathbb{C}$ (or identifying $\mathbb{R}^2$ with $\mathbb{C}$, we may write $u : \mathbb{C} \to \mathbb{C}$) has the mean-value property if its average on any circle equals its value at the center of the circle. In other words, for any $(x_1, x_2) \in \mathbb{R}^2$ and $r \geq 0$, we have

$$\frac{1}{2\pi} \int_0^{2\pi} u(x_1 + r\cos\theta, x_2 + r\sin\theta)\, d\theta = u(x),$$

or in terms of complex notation where $x = x_1 + ix_2 \in \mathbb{C}$,

$$\frac{1}{2\pi} \int_0^{2\pi} u(x + re^{i\theta})\, d\theta = u(x). \tag{1}$$

More generally, u has the **mean-value property** on an open subset D of $\mathbb{R}^2$ if (1) holds for all x and $r > 0$, assuming that the closed disk of radius r about x is contained in D. Recall that $u = u(x_1, x_2)$ is **harmonic** if $\Delta u = 0$, where $\Delta u := u_{x_1x_1} + u_{x_2x_2}$. A fact which is crucial to the method is that a continuous function on an open subset O of $\mathbb{R}^2$ (or $\mathbb{R}^d$) is harmonic if and only if it has the mean-value property on O. A statement and proof of this fact is given in Section 2.

We now give a rough description of a standard Brownian motion process in dimension 2. Consider a collection random variables $\{B_t : \Omega \to \mathbb{R}^2 \mid t \geq 0\}$ defined on some probability space $(\Omega, \mathcal{F}, P_x)$, where $\mathcal{F}$ is a sigma algebra of subsets of Ω and P_x is a probability measure, depending on $x \in \mathbb{R}^2$. For $\omega \in \Omega$, one may think of $B_t(\omega)$ as the position of a pollen grain ω at time t. The probability measure P_x satisfies the properties

(i) $P_x[B_0 = x] := P_x[\{\omega \in \Omega \mid B_0(\omega) = x\}] = 1,$ and
(ii) For all real s, t with $0 \leq s < t$, the increment $B_t - B_s$ has probability density $\frac{1}{2\pi(t-s)} \exp\left(-\frac{\|y\|^2}{2(t-s)}\right) dy_1 dy_2$.
(iii) For all real $t_1, \ldots, t_k$ with $0 \leq t_1 < \cdots < t_k,$ we have that $B_{t_1} - x,\ B_{t_2} - B_{t_1}, \ldots, B_{t_k} - B_{t_{k-1}}$ are independent.

The construction of P_x is nontrivial, and for this we refer the reader to [KS88], [La66], and [Va80]. Moreover, it has long been known that with probability 1 (i.e., almost surely, henceforth abbreviated a.s.) relative to P_x, the paths $t \mapsto B_t(\omega)$ are nondifferentiable everywhere. However, (a.s.) the paths are Holder continuous with exponent $\alpha \in (0, \frac{1}{2})$; i.e., $|B_t(\omega) - B_s(\omega)| \leq C_\alpha(\omega) |t - s|^\alpha$ for all $s, t \geq 0$ and some constant $C_\alpha(\omega)$; see [KS88]. For $x \in D$, let the random variable $\tau_D : \Omega \to [0, \infty)$ be defined by

$$\tau_D(\omega) := \inf\{t > 0 : B_t(\omega) \notin D\}.$$

Note that $B_{\tau_D(\omega)}(\omega)$ is the point on ∂D at which the path $B_t(\omega)$ leaves D for the first time. Thus $\tau_D(\omega)$ is the "exit time" of a pollen grain ω. We abbreviate the ∂D-valued random variable $\omega \mapsto B_{\tau_D(\omega)}(\omega)$ by B_{τ_D}; i.e.,

$$(B_{\tau_D})(\omega) := B_{\tau_D(\omega)}(\omega).$$

Our primary goal is to establish

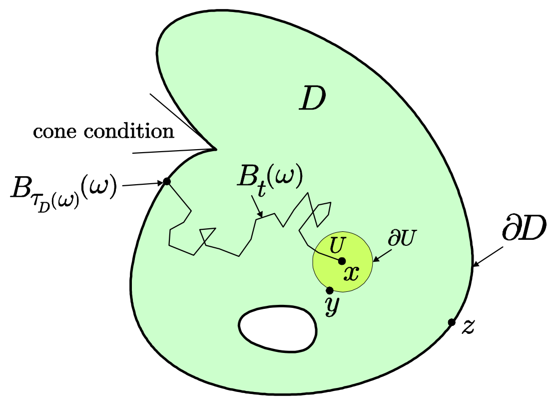
Theorem 1 *For a bounded, open set D with "nice" boundary ∂D, and given continuous function f on D, the solution of the Dirichlet problem*

$$\begin{array}{ll} D.E. & \Delta u = 0 \text{ on } D \\ B.C. & u = f \text{ on } \partial D. \end{array} \tag{2}$$

is given by

$$u\,(x) = E_x[f(B_{\tau_D})] := \int_\Omega f(B_{\tau_D(\omega)}(\omega))\,dP_x(\omega),$$

namely the expectation of the random variable $\omega \mapsto f(B_{\tau_D(\omega)}(\omega))$. In other words $u\,(x)$ is the expected value, with respect to P_x, of the value of the boundary function f at the point where the pollen grain ω (starting at x) first leaves D.



$$u(x) = \int_\Omega f(B_{\tau_D(\omega)}(\omega))\,dP_x(\omega) \overset{?}{=} \frac{1}{m(\partial U)} \int_{\partial U} u(y)\,dy$$

$$= \frac{1}{m(\partial U)} \int_{\partial U} \left(\int_\Omega f(B_{\tau_D(\omega)}(\omega))\,dP_y(\omega) \right) dy$$

$$\lim_{x \to z \in \partial D} u(x) \overset{?}{=} f(z)$$

Proof (outline). First we sketch the proof (due to Kakutani, see [Ka44]) of harmonicity of u. Given $x \in D$, let U be a small open disk centered at x, with $\overline{U} \subset D$. Let $\tau_U(\omega) := \inf\{t > 0 : B_t(\omega) \notin U\}$ be the first time t at which $B_t(\omega)$ reaches boundary of U, denoted by ∂U, starting from x. Then $\tau_U(\omega) < \tau_D(\omega)$ (a.s.), by the continuity (a.s.) of $t \mapsto B_t(\omega)$. By the isotropic nature (or directional independence) of Brownian motion, it follows that B_{τ_U} (where $B_{\tau_U}(\omega) := B_{\tau_U(\omega)}(\omega)$) will be uniformly distributed over ∂U, in the sense that for any subarc A of the boundary circle ∂U,

$$P_x\left(\omega \in \Omega : B_{\tau_U(\omega)}\,(\omega) \in A\right) = \frac{m\,(A)}{m\,(\partial U)},$$

where m denotes the standard Lebesgue measure on the circle ∂U. Intuitively, a Brownian particle has no memory of its past. Hence, the conditional expectation with respect to P_x of $f(B_{\tau_D})$, *given* that $B_{\tau_U} = y \in \partial U$, should be the same as the expectation $E_y\left[f(B_{\tau_D})\right]$, namely $u(y)$. In terms of conditional expectations,

$$E_x[f(B_{\tau_D})|\,(B_{\tau_U} = y \in \partial U)] = E_y[f(B_{\tau_D})] = u(y).$$

This notion of "beginning afresh" after hitting the boundary ∂U is a special case of the SMP "Strong Markov Property" which we shall establish for Brownian motion. The proof of the SMP is surprisingly long for something which is intuitively clear. Accepting the SMP for Brownian motion, and the uniform distribution (with respect to P_x) over ∂U of the hitting point B_{τ_U}, we have

$$\begin{aligned} u(x) &= E_x[f(B_{\tau_D})] \\ &= \frac{1}{m\,(\partial U)} \int_{\partial U} E_x[f(B_{\tau_D})|\,(B_{\tau_U} = y \in \partial U)]\; dy \\ &= \frac{1}{m\,(\partial U)} \int_{\partial U} E_y[f(B_{\tau_D})]\; dy = \frac{1}{m\,(\partial U)} \int_{\partial U} u(y)\; dy\ ; \end{aligned}$$

i.e., $u(x)$ is equal to the average of u over the circle ∂U with center at x. Then u satisfies the mean-value property for all sufficiently small circles centered at points $x \in D$, and u is harmonic in D by Theorem 2.

In the case where D is convex, the argument that $u(x)$ approaches the boundary value $f(x_0)$ as $x \to x_0 \in \partial D$ from within D, runs roughly as follows. Essentially, one must show that given any $\varepsilon > 0$, a pollen grain starting at $x \in D$ first exits D within ε of x_0, with a probability that approaches 1 as $x \to x_0$. Let L be a support line for D at x_0. The component, say B_t^L, of B_t (starting at x) in the direction normal to L is a 1-dimensional $\mathbb{R}$-valued Brownian motion starting at 0. It can be proven that no matter how small positive δ is chosen, the probability that B_t^L is positive at some time $t \in (0, \delta)$ is 1. As a consequence, the probability that $B_t^L(\omega)$ fails to exceed the distance $d(x, L)$ for some $t \in (0, \delta)$ approaches 0 as $x \to x_0$. In other words, the probability that $B_t(\omega)$ crossed L for some $t \in (0, \delta)$ approaches 1 as $x \to x_0$. Since D is convex, $B_t(\omega)$ must leave D no later than it crosses L. Thus, no matter how small $\delta > 0$ is chosen, the probability that $B_t(\omega)$ has left D for some $t \in (0, \delta)$ approaches 1 as $x \to x_0$. Moreover, the probability that, for $t \in (0, \delta)$, $B_t(\omega)$ is within ε of its starting point approaches 1 as $\delta \to 0^+$.

Thus, by choosing δ small enough, we can also assure (with a probability that approaches 1) that when $B_t(\omega)$ (starting at x) leaves D, it does so within distance ε of x_0 as $x \to x_0$. Thus, initially choosing δ small enough, we can assure (with a probability that approaches 1) that $f(\tau_D(x))$ will be arbitrarily close to $f(x_0)$ as $x \to x_0$ using the continuity of f. In Section 7, we relax the convexity assumption to the exterior wedge (or cone) condition, namely that, at each point of ∂D, there is a supporting wedge (instead of line) with positive angle. Also, we will replace the above sketch by a careful proof in this more general case. ■

2 The mean-value property and harmonic functions

Let D be an an open subset set of $\mathbb{R}^2$. We say that a continuous function $\phi : D \to \mathbb{R}$ has the **mean-value property** if for all r with

$$B(\mathbf{x}; r) := \left\{\mathbf{y} \in \mathbb{R}^2 : \|\mathbf{y} - \mathbf{x}\| \leq r\right\} \subseteq D,$$

we have

$$\frac{1}{2\pi} \int_0^{2\pi} \phi(x_1 + r\cos\theta, x_2 + r\sin\theta)\, r d\theta = \phi(x_1, x_2) = \phi(\mathbf{x}). \tag{3}$$

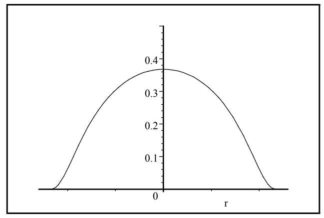
Theorem 2 *If $\phi : D \to \mathbb{R}$ is continuous and has the mean-value property, then ϕ is C^∞ and harmonic.*

Proof. For each $\varepsilon > 0$, let

$$D_\varepsilon := \left\{\mathbf{x} \in D \mid \|\mathbf{x} - \mathbf{z}\| > \varepsilon \text{ for all } \mathbf{z} \in \mathbb{R}^2 \setminus D\right\}.$$

Let $\psi_\varepsilon : \mathbb{R} \to [0, \infty)$ be the C^∞ "bump" function defined by

$$\psi_\varepsilon(r) = \begin{cases} \exp\left(\frac{1}{r^2 - \varepsilon^2}\right) & |r| \leq \varepsilon \\ 0 & |r| \geq \varepsilon. \end{cases}$$



ψ_ε with $\varepsilon = 1$

Let $\mathbf{x} = (x_1, x_2) \in D_\varepsilon$. Using polar coordinates (r, θ) about $\mathbf{x}$ with $r \le \varepsilon$, we have

$$\frac{1}{2\pi}\int_0^{2\pi} \phi(x_1 + r\cos\theta, x_2 + r\sin\theta)\, r d\theta = \phi(x_1, x_2) = \phi(\mathbf{x}),$$

by the mean-value property (3). The function $\mathbf{z} \mapsto \psi_\varepsilon(\|\mathbf{x} - \mathbf{z}\|^2)\phi(\mathbf{z})$ is defined for $\mathbf{z} \in B(\mathbf{x}; \varepsilon)$, and it extends continuously to all of $\mathbb{R}^2$ by taking it to be zero outside of $B(\mathbf{x}; \varepsilon)$. We then have

$$\begin{aligned}
&\int_{\mathbb{R}^2} \psi_\varepsilon(\|\mathbf{x} - \mathbf{z}\|^2)\phi(\mathbf{z})\, dz_1 dz_2 \\
= &\int_0^\infty \int_0^{2\pi} \psi_\varepsilon(r^2)\phi(x_1 + r\cos\theta, x_2 + r\sin\theta)\, r d\theta dr \\
= &\int_0^\infty \psi_\varepsilon(r^2) r \left(\int_0^{2\pi} \phi(x_1 + r\cos\theta, x_2 + r\sin\theta)\, d\theta\right) dr \\
= &\int_0^\infty \psi_\varepsilon(r^2) r \cdot 2\pi\phi(\mathbf{x})\ dr = 2\pi\phi(\mathbf{x}) \int_0^\infty \psi_\varepsilon(r^2) r\ dr.
\end{aligned}$$

Thus,

$$\phi(\mathbf{x}) = c_\varepsilon^{-1} \int_{\mathbb{R}^2} \psi_\varepsilon(\|\mathbf{x} - \mathbf{z}\|^2)\phi(\mathbf{z})\, dz_1 dz_2, \text{ where } c_\varepsilon = 2\pi \int_0^\infty \psi_\varepsilon(r^2) r\ dr.$$

Since $\psi_\varepsilon(\|\mathbf{x} - \mathbf{z}\|^2)\phi(\mathbf{z})$ is a C^∞ function of $\mathbf{x}$ and all of its partial derivatives are integrable with respect to $\mathbf{z}$, all of the partial derivatives of $\phi(\mathbf{x})$ with respect to x_1 or x_2 can be computed by differentiation under integral (see [Wi89, p.352]):

$$\begin{aligned}
c_\varepsilon \phi_{x_1}(\mathbf{x}) &= \int'_{\mathbb{R}^2} \psi'_\varepsilon(\|\mathbf{x} - \mathbf{z}\|^2) 2\,(x_1 - z_1)\, \phi(\mathbf{z})\, dz_1 dz_2 \\
c_\varepsilon \phi_{x_1 x_1}(\mathbf{x}) &= \int_{\mathbb{R}^2} \left(\psi''_\varepsilon(\|\mathbf{x} - \mathbf{z}\|^2) 4\,(x_1 - z_1)^2 + \psi'_\varepsilon(\|\mathbf{x} - \mathbf{z}\|^2) 2\right) \phi(\mathbf{z})\, dz_1 dz_2.
\end{aligned}$$

To show that ϕ is harmonic on D_ε, we again compute using polar coordinates (r, θ) about $\mathbf{x} \in D_\varepsilon$ and the mean value property:

$$
\begin{aligned}
& c_\varepsilon \left(\phi_{x_1x_1} + \phi_{x_2x_2}\right) = 4 \int_{\mathbb{R}^2} \left(\psi_\varepsilon''(\|\mathbf{x}-\mathbf{z}\|^2)\,\|\mathbf{x}-\mathbf{z}\|^2 + \psi_\varepsilon'(\|\mathbf{x}-\mathbf{z}\|^2)\right) \phi(\mathbf{z})\, dz_1 dz_2 \\
= \; & 4 \int_0^\infty \int_0^{2\pi} \left(\psi_\varepsilon''(r^2) r^2 + \psi_\varepsilon'(r^2)\right) \phi(x_1 + r\cos\theta, x_2 + r\sin\theta)\, r\, d\theta\, dr \\
= \; & 4 \int_0^\infty \left(\psi_\varepsilon''(r^2) r^2 + \psi_\varepsilon'(r^2)\right) r \int_0^{2\pi} \phi(x_1 + r\cos\theta, x_2 + r\sin\theta)\, d\theta\, dr \\
= \; & 4 \cdot 2\pi\phi(\mathbf{x}) \int_0^\infty \tfrac{1}{2}\tfrac{d}{dr}\left(\psi_\varepsilon'(r^2) r^2\right) dr = 4\pi\phi(\mathbf{x}) \lim_{r_0\to\infty} \left(\psi_\varepsilon'(r^2) r^2\right)\Big|_0^{r_0} = 0.
\end{aligned}
$$

Since any $\mathbf{x} \in D$ is in D_ε for some $\varepsilon > 0$, ϕ is harmonic on D. ■

The n-dimensional analog of Theorem 2 and its proof are strictly analogous, with integrations over circles with respect to θ being replaced by integrations over unit spheres S^{n-1} and factors of 2π are replaced by the $(n-1)$-measure of S^{n-1}, namely $\frac{2\pi^{n/2}}{\Gamma(n/2)}$. Observe also the continuity assumption on ϕ can be relaxed to the assumption that ϕ is measurable and integrable on compact subsets (i.e., ϕ is *locally integrable*).

3 Brownian motion and the simple Markov property

Let d be a positive integer. To define Brownian motion on $\mathbb{R}^d$ in a form which is suitable for our purposes, we introduce the following notation:

1.) Ω denotes the set of all continuous functions $\omega : [0, \infty) \to \mathbb{R}^d$.

2.) For each $t \in [0, \infty)$, the function $B_t : \Omega \to \mathbb{R}^d$ is defined by $B_t(\omega) := \omega(t)$.

3.) $\mathcal{F}$ is the σ-algebra of subsets Ω generated by sets of the form $B_t^{-1}(A)$ where $A \in \mathcal{B}(\mathbb{R}^d) :=$ the σ-algebra of generated by the open (or closed) subsets of $\mathbb{R}^d$ and t ranges over $[0, \infty)$.

4.) For each $x \in \mathbb{R}^d$, P_x is the unique probability measure on $(\Omega, \mathcal{F})$ such that

(i) $P_x\left(B_0 = x\right) = 1$,
(ii) For all real s, t with $0 \leq s < t$, the increment $B_t - B_s$
has probability density $p^d(t-s;0,y)dy := \frac{1}{2\pi(t-s)^{d/2}} \exp\left(-\frac{\|y\|^2}{2(t-s)}\right) dy$, and
(iii) For all real $t_1, \ldots, t_k$ with $0 \leq t_1 < \cdots < t_n$, we have that
$B_{t_1} - x,\ B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}$ are independent.

Here $y = (y_1, \ldots, y_d) \in \mathbb{R}^d$ and dy is short for $dy_1 \cdots dy_d$.

Remark 3 *The construction of P_x is nontrivial, and for this we refer the reader to [KS88], [La66], [PS78], and [Va80].*

Definition 4 *In the above notation, we say that the 4-tuple*

$$\left(\Omega, \mathcal{F}, \{P_x\}_{x\in\mathbb{R}^d}, \{B_t\}_{t\in[0,\infty)}\right)$$

*is a d-**dimensional standard Brownian motion process**.*

Remark 5 *One may still refer to $\omega \in \Omega$ as a Brownian particle, but with the above definition, ω is actually the whole path of the particle and $\omega(t)$ is the particle's position at time t. Since $B_t(\omega)$ is simply $\omega(t)$, we could logically abbreviate $\left(\Omega, \mathcal{F}, \{P_x\}_{x\in\mathbb{R}^d}, \{B_t\}_{t\in[0,\infty)}\right)$ by $\left(\Omega, \mathcal{F}, \{P_x\}_{x\in\mathbb{R}^d}\right)$. The item in $\left(\Omega, \mathcal{F}, \{P_x\}_{x\in\mathbb{R}^d}, \{B_t\}_{t\in[0,\infty)}\right)$ which really identifies the process as "Brownian" is the family of measures $\{P_x\}_{x\in\mathbb{R}^d}$.*

Intuitively, $p^d(t;y,z)dz$ is the probability that a particle starting at $y \in \mathbb{R}^d$ at time 0 and "moving randomly" will be in an infinitesimal volume dz about z at time t. The key properties (established in Appendix 8.1) that we need are

$$\int_{\mathbb{R}^d} p^d(t;y,z)\,dz = 1 \text{ and } \int_{\mathbb{R}^d} p^d(s,x,y)p^d(t,y,z)\,dy = p^d\left(s+t,x,z\right). \quad (4)$$

The latter is known as the Chapman-Kolmogorov equation. Intuitively, it says that the probability density that a particle moves from x to z in time $s+t$ is the integral with respect to all intermediate positions y of the density that it moves from x to y in time t multiplied by the density that it moves from y to z in time s. Conditions (i), (ii) and (iii) in 4.) are equivalent to the assertion that for each finite sequence $0 < t_1 < t_2 < \cdots < t_n$, the joint density of $B_{t_1}, B_{t_2}, \ldots, B_{t_n}$ on $\left(\mathbb{R}^d\right)^n$ is given by

$$p^d(t_1; x, x_1) p^d(t_2 - t_1; x_1, x_2) \cdots p^d(t_n - t_{n-1}; x_{n-1}, x_n)\, dx_1 \cdots dx_n. \tag{5}$$

Since d will usually be fixed, we will drop it until further notice, and it will often be convenient to use the notation

$$p_t(y, z) := p(t; y, z).$$

For a d-dimensional Brownian motion $(\Omega, \mathcal{F}, \{P_x\}_{x \in \mathbb{R}^d}, \{B_t\}_{t \in [0,\infty)})$, let $\mathcal{F}_t$ be the sigma field generated by $\cup_{0 \leq s \leq t} \{B_s\}$, and let $\mathcal{F}_\infty$ be $\mathcal{F}$, the sigma field generated by $\cup_{0 \leq s < \infty} \{B_s\}$.

Definition 6 (time shift operator) *For $t \in [0, \infty)$, the time shift operator $\theta_t : \Omega \to \Omega$ is defined by*

$$\theta_t(\omega)(s) = \omega(t + s) \text{ for } s \in [0, \infty).$$

Note that the path $\theta_t(\omega)$ begins at the point $\omega(t)$ and does not involve $\omega|_{[0,t)}$. Thus, for $t > 0$, θ_t is not 1-1, but it is onto. Writing $\omega(s)$ as $B_s(\omega)$, note that $B_s \circ \theta_t = B_{t+s} : \Omega \to \mathbb{R}^d$, since

$$(B_s \circ \theta_t)(\omega) = B_s(\theta_t(\omega)) = \theta_t(\omega)(s) = \omega(t + s) = B_{t+s}(\omega). \tag{6}$$

Thus, via right composition, θ_t maps the collection of random variables $\{B_s\}$ into itself. For $s \geq 0$, we have that $\theta_t : (\Omega, \mathcal{F}_{t+s}) \to (\Omega, \mathcal{F}_s)$ is measurable. Indeed, elements of $\mathcal{F}_s$ are generated by sets of the form $B_r^{-1}(A)$ for $r \leq s$ where $A \in \mathcal{B}(\mathbb{R}^d)$, and

$$\theta_t^{-1}(B_r^{-1}(A)) = (B_r \circ \theta_t)^{-1}(A) = B_{t+r}^{-1}(A) \in \mathcal{F}_{t+s}.$$

Since s can be chosen arbitrarily large, $\theta_t : (\Omega, \mathcal{F}_\infty) \to (\Omega, \mathcal{F}_\infty)$ is measurable.

The Markov Property below intuitively states that after time t, the expected behavior of a Brownian particle ω, beginning at x at time 0, is the same as if it began at $B_t(\omega)$. In other words, other than its current position $B_t(\omega)$, its past, previous to time t, has no effect on its future. Proofs of Markov Property here and the Strong Markov Property in Section 5 are outlined in Chapter 1 of [CZ95], but our other references were essential in verifying many of the details.

Theorem 7 (Markov Property for Brownian motion) *Let*

$$(\Omega, \mathcal{F}, \{P_x\}_{x\in\mathbb{R}^d}, \{B_t\}_{t\in[0,\infty)})$$

be a d-dimensional standard Brownian motion process, let $A \in \mathcal{F}_t$*, and let* $g : \Omega \to \mathbb{R}$ *be an* $\mathcal{F}_\infty$*-measurable function which is positive or bounded. Then, for any* $x \in \mathbb{R}^d$*,*

$$\int_A g(\theta_t(\omega))\, dP_x(\omega) = \int_A \int_\Omega g(\omega')\ dP_{B_t(\omega)}(\omega')\, dP_x(\omega). \tag{7}$$

In other words, if $E_x(X) := \int_\Omega X(\omega) dP_x(\omega)$ *denotes the expectation of a random variable* X*, then* $E_x(1_A g(\theta_t)) = E_x(1_A E_{B_t}(g))$*, where* $E_{B_t}(g)(\omega) := E_{B_t(\omega)}(g(\omega))$*.*

Remark 8 *In the case* $g = 1_C$ *for* $C \in \mathcal{F}_\infty$*, this becomes*

$$\begin{aligned} \int_A 1_C(\theta_t(\omega))\, dP_x(\omega) &= \int_A \int_\Omega 1_C(\omega')\ dP_{B_t(\omega)}(\omega')\, dP_x(\omega), \text{ or} \\ \int_A 1_{\theta_t^{-1}(C)}(\omega)\, dP_x(\omega) &= \int_A P_{B_t(\omega)}(C)\, dP_x(\omega), \text{ or} \\ P_x\left(A \cap \theta_t^{-1}(C)\right) &= \int_A P_{B_t(\omega)}(C)\, dP_x(\omega). \end{aligned}$$

Lemma 9 *For* $A \in \mathcal{F}_t$ *and* $C \in \mathcal{F}_\infty$*, we have*

$$P_x\left(A \cap \theta_t^{-1}(C)\right) = \int_A P_{B_t(\omega)}(C)\, dP_x(\omega). \tag{8}$$

Proof. We first prove this for $A \in \mathcal{F}_t$ of the form

$$A = \{B_{t_1} \in A_1, \ldots, B_{t_k} \in A_k\},\ 0 \le t_1 < \cdots < t_k \le t,$$

where $A_i \in \mathcal{B}(\mathbb{R}^d)$ $(i = 1, \ldots, k)$, and $g = 1_C$ for $C \in \mathcal{F}_\infty$ of the form

$$C = \{B_{s_1} \in C_1, \ldots, B_{s_m} \in C_m\},\ 0 \le s_1 < \cdots < s_m,$$

where $C_j \in \mathcal{B}(\mathbb{R}^d)$ $(j = 1, \ldots, m)$. Since $0 \le t_1 < \cdots < t_k \le t + s_1 < \cdots < t + s_m$, we have (according to (5))

$$
\begin{aligned}
& P_x\left(A \cap \theta_t^{-1}(C)\right) \\
= \; & P_x(B_{t_1} \in A_1, \ldots, B_{t_k} \in A_k, B_{t+s_1} \in C_1, \ldots, B_{t+s_m} \in C_m) \\
= \; & \int_{A_1} \cdots \int_{A_k} \int_{C_1} \cdots \int_{C_m} p_{t_1}(x, y_1) p_{t_2 - t_1}(y_1, y_2) \cdots p_{t_k - t_{k-1}}(y_{k-1}, y_k) \\
& \cdot p_{t+s_1-t_k}(y_k, u_1) p_{t+s_2-(t+s_1)}(u_1, u_2) \cdots p_{t+s_m-(t+s_{m-1})}(u_{m-1}, u_m) \\
& du_1 du_2 \cdots du_m dy_1 dy_2 \cdots dy_k \\
= \; & \int_{A_1} \cdots \int_{A_k} \int_{C_1} \cdots \int_{C_m} p_{t_1}(x, y_1) p_{t_2 - t_1}(y_1, y_2) \cdots p_{t_k - t_{k-1}}(y_{k-1}, y_k) p_{t+s_1-t_k}(y_k, u_1) \\
& \cdot p_{s_2-s_1}(u_1, u_2) \cdots p_{s_m - s_{m-1}}(u_{m-1}, u_m) \, du_1 du_2 \cdots du_m dy_1 dy_2 \cdots dy_k.
\end{aligned}
$$

Note that since $0 \le t_1 < \cdots < t_k \le t$, using (5) again,

$$
\begin{aligned}
& \int_A g\left(B_t(\omega)\right) \, dP_x(\omega) \\
= \; & \int_{\mathbb{R}^d} \int_{A_1} \cdots \int_{A_k} g\,(u)\, p_{t_1}(x, y_1) \; p_{t_2-t_1}(y_1, y_2) \cdots p_{t_k - t_{k-1}}(y_{k-1}, y_k) p_{t-t_k}(y_k, u) \\
& dy_1 dy_2 \cdots dy_k du
\end{aligned}
$$

Thus,

$$
\begin{aligned}
& \int_A P_{B_t(\omega)}(C) \, dP_x(\omega) \\
= \; & \int_{\mathbb{R}^d} \int_{A_1} \cdots \int_{A_k} P_u(C) p_{t_1}(x, y_1) \; p_{t_2-t_1}(y_1, y_2) \cdots p_{t_k - t_{k-1}}(y_{k-1}, y_k) p_{t-t_k}(y_k, u) \\
& dy_1 dy_2 \cdots dy_k du \\
= \; & \int_{\mathbb{R}^d} \int_{A_1} \cdots \int_{A_k} \\
& \left(\int_{C_1} \cdots \int_{C_m} p_{s_1}(u, u_1) p_{s_2-s_1}(u_1, u_2) \cdots p_{s_m - s_{m-1}}(u_{m-1}, u_m) \, du_1 du_2 \cdots du_m \right) \\
& \cdot p_{t_1}(x, y_1) p_{t_2-t_1}(y_1, y_2) \cdots p_{t_k - t_{k-1}}(y_{k-1}, y_k) p_{t-t_k}(y_k, u) \, dy_1 dy_2 \cdots dy_k du \\
= \; & \int_{\mathbb{R}^d} \int_{A_1} \cdots \int_{A_k} \int_{C_1} \cdots \int_{C_m} p_{s_1}(u, u_1) p_{s_2-s_1}(u_1, u_2) \cdots p_{s_m - s_{m-1}}(u_{m-1}, u_m) \\
& \cdot p_{t_1}(x, y_1) p_{t_2-t_1}(y_1, y_2) \cdots p_{t_k - t_{k-1}}(y_{k-1}, y_k) p_{t-t_k}(y_k, u) \\
& du_1 du_2 \cdots du_m dy_1 dy_2 \cdots dy_k du
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{A_1} \cdots \int_{A_k} \int_{C_1} \cdots \int_{C_m} \int_{\mathbb{R}^d} p_{s_1}(u, u_1) p_{s_2 - s_1}(u_1, u_2) \cdots p_{s_m - s_{m-1}}(u_{m-1}, u_m) \\
&\quad \cdot p_{t_1}(x, y_1) p_{t_2 - t_1}(y_1, y_2) \cdots p_{t_k - t_{k-1}}(y_{k-1}, y_k) p_{t - t_{k-1}}(y_k, u) \\
&\quad du du_1 du_2 \cdots du_m dy_1 dy_2 \cdots dy_k \\
&= \int_{A_1} \cdots \int_{A_k} \int_{C_1} \cdots \int_{C_m} p_{t_1}(x, y_1) p_{t_2 - t_1}(y_1, y_2) \cdots p_{t_k - t_{k-1}}(y_{k-1}, y_k) \\
&\quad \left(\int_{\mathbb{R}^d} p_{s_1}(u, u_1) p_{t - t_{k-1}}(y_k, u) \, du \right) p_{s_2 - s_1}(u_1, u_2) \cdots p_{s_m - s_{m-1}}(u_{m-1}, u_m) \\
&\quad du_1 du_2 \cdots du_m dy_1 dy_2 \cdots dy_k \\
&= \int_{A_1} \cdots \int_{A_k} \int_{C_1} \cdots \int_{C_m} p_{t_1}(x, y_1) p_{t_2 - t_1}(y_1, y_2) \cdots p_{t_k - t_{k-1}}(y_{k-1}, y_k) p_{t + s_1 - t_k}(y_k, u_1) \\
&\quad \cdot p_{s_2 - s_1}(u_1, u_2) \cdots p_{s_m - s_{m-1}}(u_{m-1}, u_m) \, du_1 du_2 \cdots du_m dy_1 dy_2 \cdots dy_k,
\end{aligned}
$$

where we have used the Chapman-Kolmogorov equation (4). This is exactly the same as $P_x\left(A \cap \theta_t^{-1}(C)\right)$ computed above, namely

$$
\begin{aligned}
&P_x\left(A \cap \theta_t^{-1}(C)\right) \\
&= \int_{A_1} \cdots \int_{A_k} \int_{C_1} \cdots \int_{C_m} p_{t_1}(x, y_1) p_{t_2 - t_1}(y_1, y_2) \cdots p_{t_k - t_{k-1}}(y_{k-1}, y_k) p_{t + s_1 - t_k}(y_k, u_1) \\
&\quad \cdot p_{s_2 - s_1}(u_1, u_2) \cdots p_{s_m - s_{m-1}}(u_{m-1}, u_m) \, du_1 du_2 \cdots du_m dy_1 dy_2 \cdots dy_k.
\end{aligned}
$$

Let $\mathcal{C}$ be the collection of $A \in \mathcal{F}_t$ of the form

$$
A = \{B_{t_1} \in A_1, \ldots, B_{t_k} \in A_k\}, \ 0 \le t_1 < \cdots < t_k \le t,
$$

and let $\mathcal{D}$ be the collection of $A \in \mathcal{F}_t$ such that (8) holds for a given $C \in \mathcal{F}_\infty$ of the form $C = \{B_{s_1} \in C_1, \ldots, B_{s_m} \in C_m\}$, $0 \le s_1 < \cdots < s_m$; i.e.,

$$
P_x\left(A \cap \theta_t^{-1}(C)\right) = \int_A P_{B_t(\omega)}(C) \, dP_x(\omega). \tag{9}
$$

Note that $\mathcal{C}$ is closed under finite intersection and we have shown that $\mathcal{D}$ contains $\mathcal{C}$. Theorem 33 in Appendix 8.2 implies that $\mathcal{D} = \mathcal{F}_t$, provided

- (i) $\Omega \in \mathcal{D}$
- (ii) $A, B \in \mathcal{D}$ and $A \subset B \Rightarrow B \setminus A \in \mathcal{D}$
- (iii) $A_n \in \mathcal{D}, \ n = 1, 2, 3 \ldots$ and $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots \Rightarrow A_\infty := \bigcup_{n=1}^\infty A_n \in \mathcal{D}$.

Note that (i) holds, since $\Omega \in \mathcal{C}$. Just take the $k = 1$ and $A_1 = \mathbb{R}^d$, and then $\Omega = A \in \mathcal{C}$, for which (9) holds. For (ii), since $A \subset B$,

$$\begin{aligned} & P_x\left((B \setminus A) \cap \theta_t^{-1}(C)\right) = P_x\left(B \cap \theta_t^{-1}(C)\right) - P_x\left(A \cap \theta_t^{-1}(C)\right) \\ = & \int_B P_{B_t(\omega)}(C)\, dP_x(\omega) - \int_A P_{B_t(\omega)}(C)\, dP_x(\omega) = \int_{B \setminus A} P_{B_t(\omega)}(C)\, dP_x(\omega). \end{aligned}$$

For (iii), since $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$

$$\begin{aligned} P_x\left(A_\infty \cap \theta_t^{-1}(C)\right) &= \lim_{n\to\infty} P_x\left(A_n \cap \theta_t^{-1}(C)\right) \\ &= \lim_{n\to\infty} \int_{A_n} P_{B_t(\omega)}(C)\, dP_x(\omega) = \int_{A_\infty} P_{B_t(\omega)}(C)\, dP_x(\omega). \end{aligned}$$

Thus, $\mathcal{D} = \mathcal{F}_t$.

Now, let $\mathcal{C}$ be the collection of $C \in \mathcal{F}_\infty$ of the form

$$C = \left\{B_{s_1} \in C_1, \ldots, B_{s_m} \in C_m\right\},\ 0 \le s_1 < \cdots < s_m,$$

and let $\mathcal{D}$ be the collection of all $C \in \mathcal{F}_\infty$, such that (8) holds for any fixed $A \in \mathcal{F}_t$. Again, the verification of (i), (ii) and (iii) is easy, and $\mathcal{D} = \mathcal{F}_\infty$. Thus (8) holds for all $A \in \mathcal{F}_t$ and $C \in \mathcal{F}_\infty$. ■

Proof. (of Theorem 7) If $g : \Omega \to \mathbb{R}$ is of the form 1_C for $C \in \mathcal{F}_\infty$. Then, by Lemma 9,

$$\begin{aligned} & \int_A g(\theta_t(\omega))\, dP_x(\omega) = \int_A 1_C(\theta_t(\omega))\, dP_x(\omega) = P_x\left(A \cap \theta_t^{-1}(C)\right) \\ = & \int_A P_{B_t(\omega)}(C)\, dP_x(\omega) = \int_A \int_\Omega 1_C(\omega')\ dP_{B_t(\omega)}(\omega')\, dP_x(\omega) \\ = & \int_A \int_\Omega g(\omega')\ dP_{B_t(\omega)}(\omega')\, dP_x(\omega). \end{aligned}$$

By linearity (7) holds for simple $\mathcal{F}_\infty$-measurable functions. Since any positive $\mathcal{F}_\infty$-measurable function is the pointwise limit of an increasing sequence of simple functions, (7) holds by the Monotone Convergence Theorem. For bounded (and hence integrable on a probability space) $\mathcal{F}_\infty$-measurable functions, we write $g = g_+ - g_-$ for bounded positive functions $g_+ := \max(g, 0)$ and $g_- = \max(-g, 0)$, and apply linearity to get (7). ■

4 Stopping times

Definition 10 (stopping time) *Let $\{\mathcal{F}_t : t \geq 0\}$ be an increasing family of σ-fields on a probability space $(\Omega, \mathcal{F}, P)$. A non-negative random variable τ on Ω (which may assume the value $+\infty$) is called a **stopping time** of $\{\mathcal{F}_t\}$ provided that for every $a > 0$,*

$$\{\tau < a\} \in \mathcal{F}_a. \tag{10}$$

Here and elsewhere we use the abbreviation $\{\tau < a\}$ for $\{\omega \in \Omega : \tau(\omega) < a\}$.

Equivalently, we have

Proposition 11 *A non-negative random variable τ on Ω is a stopping time if and only if for each $a > 0$,*

$$\{\tau \leq a\} \in \mathcal{F}_{a+} := \cap_{t>a} \mathcal{F}_t \tag{11}$$

Proof. If $\{\tau < a\} \in \mathcal{F}_a$, then

$$\{\tau \leq a\} = \cap_{n>1} \left\{ \tau \leq a + \frac{t-a}{n} \right\} \in \mathcal{F}_t \text{ for all } t > a \geq 0.$$

Thus, $\{\tau \leq a\} \in \cap_{t>a} \mathcal{F}_t = \mathcal{F}_{a+}$. Conversely, if $\{\tau \leq a\} \in \mathcal{F}_{a+}$, then

$$\{\tau < a\} = \cup_{n>1} \left\{ \tau \leq a - \frac{1}{n} \right\} \in \cup_{n>1} \mathcal{F}_{\left(a-\frac{1}{n}\right)+} \subset \mathcal{F}_a.$$

■

Note that if $t_0 \geq 0$ is a fixed time and $\tau_{t_0} : \Omega \to [0, \infty)$ is the constant function given by $\tau_{t_0}(\omega) = t_0$, then

$$\{\omega \in \Omega : \tau_{t_0}(\omega) < a\} = \{\omega \in \Omega : t_0 < a\} = \begin{cases} \Omega & \text{if } t_0 < a \\ \phi & \text{if } t_0 \geq a \end{cases} \in \mathcal{F}_a$$

and so τ_{t_0} is an example of a stopping time.

Definition 12 *Let τ be a stopping time of an increasing family $\{\mathcal{F}_t\}$ of σ-fields. We define*

$$\mathcal{F}_\tau := \{A \in \mathcal{F}_\infty \mid A \cap \{\tau \leq a\} \in \mathcal{F}_a \text{ for all } a \geq 0\} \text{ and}$$
$$\mathcal{F}_{\tau+} := \{A \in \mathcal{F}_\infty \mid A \cap \{\tau < a\} \in \mathcal{F}_a \text{ for all } a \geq 0\}.$$

Remark 13 *We show that the notation is consistent with the special case where* τ *is a constant function, say* $\tau(\omega) := t_0 \geq 0$*; i.e., we have* $\mathcal{F}_\tau = \mathcal{F}_{t_0}$ *and* $\mathcal{F}_{\tau+} = \mathcal{F}_{t_0+}$. *To show that* $\mathcal{F}_\tau = \mathcal{F}_{t_0}$,

$$\begin{aligned} \mathcal{F}_\tau &= \{A \in \mathcal{F}_\infty \mid A \cap \{t_0 \leq a\} \in \mathcal{F}_a \text{ for all } a \geq 0\} \\ &= \{A \in \mathcal{F}_\infty \mid A \in \mathcal{F}_a \text{ for } t_0 \leq a\} = \cap_{a \geq t_0} \mathcal{F}_a = \mathcal{F}_{t_0}. \end{aligned}$$

Moreover,

$$\begin{aligned} \mathcal{F}_{\tau+} &= \{A \in \mathcal{F}_\infty \mid A \cap \{t_0 < a\} \in \mathcal{F}_a \text{ for all } a \geq 0\} \\ &= \{A \in \mathcal{F}_\infty \mid A \in \mathcal{F}_a \text{ for } t_0 < a\} = \cap_{a > t_0} \mathcal{F}_a = \mathcal{F}_{t_0+}. \end{aligned}$$

Proposition 14 *We have that* $\mathcal{F}_\tau$ *and* $\mathcal{F}_{\tau+}$ *are* σ*-fields.*

Proof. We prove this only for $\mathcal{F}_{\tau+}$, since the proof for $\mathcal{F}_\tau$ is analogous. Note that clearly $\phi \in \mathcal{F}_{\tau+}$, and $\Omega \in \mathcal{F}_{\tau+}$, since $\Omega \cap \{\tau < a\} = \{\tau < a\} \in \mathcal{F}_a$. Moreover, since $\mathcal{F}_a$ and $\mathcal{F}_\infty$ are σ-fields, $\mathcal{F}_{\tau+}$ is closed under countable unions. It remains to show that $\mathcal{F}_{\tau+}$ is closed under complementation; i.e., $A \in \mathcal{F}_{\tau+} \Rightarrow A^c \in \mathcal{F}_{\tau+}$. Certainly, $A \in \mathcal{F}_\infty \Rightarrow A^c \in \mathcal{F}_\infty$. Thus, it remains to prove that $A^c \cap \{\tau < a\} \in \mathcal{F}_a$. Note that,

$$A \in \mathcal{F}_{\tau+} \Rightarrow A \cap \{\tau < a\} \in \mathcal{F}_a \Rightarrow (A \cap \{\tau < a\})^c \in \mathcal{F}_a,$$

since $\mathcal{F}_a$ is a σ-field. From the elementary fact $A^c \cap B = B \cap (A \cap B)^c$ with $B = \{\tau < a\}$, we get

$$A \in \mathcal{F}_{\tau+} \Rightarrow A^c \cap \{\tau < a\} = \{\tau < a\} \cap (A \cap \{\tau < a\})^c \in \mathcal{F}_a,$$

since we showed that $(A \cap \{\tau < a\})^c \in \mathcal{F}_a$, and $\{\tau < a\} \in \mathcal{F}_a$ by the definition of stopping time. ■

Definition 15 *If* $\tau : \Omega \to [0, \infty]$ *is a stopping time, then the* ***time shift operator for*** τ *is* $\theta_\tau : \Omega \setminus \{\tau = \infty\} \to \Omega$ *is defined by*

$$\theta_\tau(\omega)(t) = \omega(\tau(\omega) + t).$$

As before Definition 6, we henceforth take $\mathcal{F}_t := \sigma(\{B_s : s \le t\})$. Note that $\mathcal{F}_t$ is strictly contained in $\mathcal{F}_{t+}$. Indeed, for an open ball U in $\mathbb{R}^d$

$$\{\omega \in \Omega \mid \inf\{s > t \mid B_s(\omega) \in U\} = t\} \in \mathcal{F}_{t+} \setminus \mathcal{F}_t,$$

since $\inf\{s > t \mid B_s(\omega) \in U\} = t$ if there is a sequence $\{s_k\}$ with $s_k \downarrow t$ and $B_{s_k}(\omega) \in U$. Thus, this event is determined by knowing $B_s(\omega)$ for s sufficiently close to but bigger than t, whereas $\inf\{s > t \mid B_s(\omega) \in U\} = t$ cannot be determined by $B_{t'}(\omega)$ for $t' \le t$, unless $B_t(\omega) \in U$, but $B_t(\omega) \notin U$ is a possibility.

Definition 16 (exit time) *For a Brownian motion* $(\Omega, \mathcal{F}, \{P_x\}_{x\in\mathbb{R}^d}, \{B_t\}_{t\in[0,\infty)})$ *and* U *an open or closed subset of* $\mathbb{R}^d$*, the* ***exit time from*** U *is the function* $\tau_U : \Omega \to [0, \infty]$ *defined by*

$$\tau_U(\omega) := \inf\{t \in (0, \infty) : B_t(\omega) \in U^c\},$$

where $\inf \phi := \infty$*. Note that* τ_U *is really independent of* $\{P_x\}_{x\in\mathbb{R}^d}$*, but of course its expectation, relative to* P_x*, varies with* x*.*

Proposition 17 *For any open or closed* $U \subseteq \mathbb{R}^d$*, we have that* τ_U *is a stopping time.*

Proof. First suppose that U is closed. We show

$$\{\tau_U < t\} = \bigcup_{r\in\mathbb{Q}\cap(0,t)} \left\{B_r^{-1}(U^c)\right\},$$

which exhibits $\{\tau_U < t\}$ as countable union of members of $\mathcal{F}_t$ and hence a member of $\mathcal{F}_t$ itself. Note that

$$\begin{aligned} & \omega \in \{\tau_U < t\} \Leftrightarrow \tau_U(\omega) < t \\ \Leftrightarrow\ & \inf\{s \in (0, \infty) : B_s(\omega) \in U^c\} < t \\ \Leftrightarrow\ & B_s(\omega) \in U^c \text{ for some } s < t. \end{aligned} \tag{12}$$

Since $s \mapsto B_s(\omega)$ is right continuous, $B_s(\omega) = \lim_{r\downarrow s} B_r(\omega)$, and so since U^c is open, $B_r(\omega) \in U^c$ for sufficiently small rational $r \in (s, t)$. Thus,

$$\begin{aligned} & \omega \in \{\tau_U < t\} \Leftrightarrow B_r(\omega) \in U^c \text{ for some rational } r < t \\ \Leftrightarrow\ & B_r(\omega) \in U^c \text{ for some } r \in \mathbb{Q}\cap[0, t) \\ \Leftrightarrow\ & \omega \in \bigcup_{r\in\mathbb{Q}\cap(0,t)} \{B_r \in U^c\} \Leftrightarrow \omega \in \bigcup_{r\in\mathbb{Q}\cap(0,t)} \left\{B_r^{-1}(U^c)\right\}. \end{aligned}$$

We only used the right continuity of $B_t(\omega)$ for this.

Suppose now that U is open. Let $\{U_n\}_{n=1}^{\infty}$ be a sequence of open sets with $\overline{U_n} \subset U_{n+1}$ and $\cup_{n=1}^{\infty} U_n = U$. Then for each $t > 0$,

$$\begin{aligned}
\{\tau_U > t\} &= \{\omega \in \Omega \mid \forall s \in (0,t],\ B_s(\omega) \in U\} \\
&= \bigcap_{k=1}^{\infty} \left\{\omega \in \Omega \mid \forall s \in \left[\frac{t}{k+1}, t\right],\ B_s(\omega) \in U\right\}.
\end{aligned}$$

Now, since the image of the compact set $\left[\frac{t}{k+1}, t\right]$ under the continuous map $s \mapsto B_s(\omega)$ is compact, it must be in U_n (and hence $\overline{U_n}$) for some n, we have

$$\begin{aligned}
&\left\{\omega \in \Omega \mid \forall s \in \left[\frac{t}{k+1}, t\right],\ B_s(\omega) \in U\right\} \\
&= \bigcup_{n=1}^{\infty} \left\{\omega \in \Omega \mid \forall s \in \left[\frac{t}{k+1}, t\right],\ B_s(\omega) \in \overline{U_n}\right\}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\{\tau_U > t\} &= \bigcap_{k=1}^{\infty} \bigcup_{n=1}^{\infty} \left\{\omega \in \Omega \mid \forall s \in \left[\frac{t}{k+1}, t\right],\ B_s(\omega) \in \overline{U_n}\right\} \\
&= \bigcap_{k=1}^{\infty} \bigcup_{n=1}^{\infty} \left\{\omega \in \Omega \mid \forall s \in \mathbb{Q} \cap \left[\frac{t}{k+1}, t\right],\ B_s(\omega) \in \overline{U_n}\right\},
\end{aligned}$$

where the last equality uses the (left and right) continuity of $s \mapsto B_s(\omega)$. Since each of sets $\left\{\omega \in \Omega \mid \forall s \in \mathbb{Q} \cap \left[\frac{t}{k+1}, t\right],\ B_s(\omega) \in \overline{U_n}\right\}$ is in $\mathcal{F}_t$. Thus, $\{\tau_U > t\} \in \mathcal{F}_t$ and

$$\{\tau_U \le t\} = \{\tau_U > t\}^c \in \mathcal{F}_t.$$

This implies $\{\tau_U < t\} \in \mathcal{F}_t$, since

$$\{\tau_U < t\} = \cup_{k=0}^{\infty} \{\tau_U \le kt/(k+1)\} \subset \cup_{k=0}^{\infty} \mathcal{F}_{kt/(k+1)} \subset \mathcal{F}_t.$$

■

Proposition 18 *For an open or closed subset $U \subset \mathbb{R}^d$ with Lebesgue measure $m(U)$, we have*

$$P_x(\tau_U > t) \le (2\pi t)^{-d/2} m(U).$$

In particular, if $m(U) < \infty$, then $\lim_{t \to \infty} P_x(\tau_U > t) = 0$, from which it follows that $P_x(\tau_U = \infty) = 0$.

Proof. If $\tau_U(\omega) > t$, then $B_t(\omega) \in U$, and so $\{\tau_U > t\} \subseteq \{B_t \in U\}$. Hence

$$\begin{aligned} P_x(\tau_U > t) &\leq P_x(B_t \in U) = \int_U p_t(x,y)\,dy = (2\pi t)^{-d/2} \int_U e^{-\|x-y\|^2/2t}\,dy \\ &\leq (2\pi t)^{-d/2} \int_U 1\,dy = (2\pi t)^{-d/2}\, m(U). \end{aligned}$$

■

5 The Strong Markov Property for Brownian motion

If we replace t in Theorem 7 (Markov property for Brownian motion) by a stopping time τ, we obtain a stronger result, namely

Theorem 19 (Strong Markov Property) *Let* $(\Omega, \mathcal{F}, \{P_x\}_{x\in\mathbb{R}^d}, \{B_t\}_{t\in[0,\infty)})$ *be a Brownian motion. For a stopping time* τ*,*

$$A \in \mathcal{F}_{\tau+} := \{A \in \mathcal{F}_\infty \mid \forall t \geq 0,\ A \cap \{\tau < t\} \in \mathcal{F}_t\},$$

and g *an* $\mathcal{F}_\infty$*-measurable function which is positive or bounded, we have (for any* $x \in \mathbb{R}^d$*)*

$$\int_A g(\theta_{\tau(\omega)}(\omega))\,dP_x(\omega) = \int_A \int_\Omega g(\omega')dP_{B_{\tau(\omega)}(\omega)}(\omega')\,dP_x(\omega). \tag{13}$$

Proof. For the stopping time τ, and integer $n \geq 1$, we define

$$\tau_n := \begin{cases} k2^{-n} & \text{if } (k-1)\,2^{-n} \leq \tau < k2^{-n} \\ \infty & \text{if } \tau = \infty. \end{cases}$$

Note that $\tau_n \downarrow \tau$ as $n \to \infty$. Since τ is a stopping time and $A \in \mathcal{F}_{\tau+}$

$$A \cap \left\{(k-1)\,2^{-n} \leq \tau < k2^{-n}\right\} \in \mathcal{F}_{k2^{-n}}.$$

Thus,

$$
\begin{aligned}
& \int_A \int_\Omega g(\omega') dP_{B_{\tau_n(\omega)}(\omega)}(\omega')\, dP_x(\omega) \\
= \ & \int_{A \cap \left(\cup_{k=1}^\infty \{(k-1)2^{-n} \le \tau < k2^{-n}\} \right)} \int_\Omega g(\omega') dP_{B_{\tau_n(\omega)}(\omega)}(\omega')\, dP_x(\omega) \\
= \ & \sum\nolimits_{k=1}^\infty \int_{A \cap \{(k-1)2^{-n} \le \tau < k2^{-n}\}} \int_\Omega g(\omega') dP_{B_{\tau_n(\omega)}(\omega)}(\omega')\, dP_x(\omega) \\
= \ & \sum\nolimits_{k=1}^\infty \int_{A \cap \{(k-1)2^{-n} \le \tau < k2^{-n}\}} \int_\Omega g(\omega') dP_{B_{k2^{-n}}(\omega)}(\omega')\, dP_x(\omega) \\
= \ & \sum\nolimits_{k=1}^\infty \int_{A \cap \{(k-1)2^{-n} \le \tau < k2^{-n}\}} g(\theta_{k2^{-n}}(\omega))\, dP_x(\omega) \\
= \ & \sum\nolimits_{k=1}^\infty \int_{A \cap \{(k-1)2^{-n} \le \tau < k2^{-n}\}} g(\theta_{\tau_n(\omega)}(\omega))\, dP_x(\omega) \\
= \ & \int_{A \cap \left(\cup_{k=1}^\infty \{(k-1)2^{-n} \le \tau < k2^{-n}\} \right)} g(\theta_{\tau_n(\omega)}(\omega))\, dP_x(\omega) \\
= \ & \int_A g(\theta_{\tau_n(\omega)}(\omega))\, dP_x(\omega).
\end{aligned}
$$

We wish to take a limit as $n \to \infty$, to get

$$
\int_A g(\theta_{\tau(\omega)}(\omega))\, dP_x(\omega) = \int_A \int_\Omega g(\omega') dP_{B_{\tau(\omega)}(\omega)}(\omega')\, dP_x(\omega). \tag{14}
$$

For this it suffices to show that either side of the equation (valid by the simple Markov Property)

$$
\int_A g(\theta_t(\omega))\, dP_x(\omega) = \int_A \int_\Omega g(\omega') dP_{B_t(\omega)}(\omega')\, dP_x(\omega).
$$

is a continuous function of t. This is not easy to prove and may not even be true. However, we can prove the continuity in t for g of the form

$$
g(\omega) = f_1(B_{t_1}(\omega)) \cdots f_r(B_{t_r}(\omega)) \tag{15}
$$

where $0 < t_1 < \cdots < t_r$ and $f_1, \ldots, f_r$ are continuous, bounded functions on $\mathbb{R}^d$. Then

$$
\begin{aligned}
& \int_A g(\theta_t(\omega))\, dP_x(\omega) = \int_A f_1(B_{t_1}(\theta_t(\omega))) \cdots f_r(B_{t_r}(\theta_t(\omega)))\, dP_x(\omega) \\
= \ & \int_A f_1(B_{t+t_1}(\omega)) \cdots f_r(B_{t+t_r}(\omega))\, dP_x(\omega).
\end{aligned}
$$

Since Brownian paths $B_t(\omega)$ are continuous functions of t, and the f_i are assumed continuous, the integrand $f_1(B_{t+t_1}(\omega)) \cdots f_r(B_{t+t_r}(\omega))$ is continuous in t. As the f_i are bounded and P_x is a finite measure, applying the Lebesgue Dominated Convergence Theorem, we have

$$\begin{aligned} & \lim_{t \to t_0} \int_A f_1(B_{t+t_1}(\omega)) \cdots f_r(B_{t+t_r}(\omega))\, dP_x(\omega) \\ = & \int_A \lim_{t \to t_0} \left(f_1(B_{t+t_1}(\omega)) \cdots f_r(B_{t+t_r}(\omega)) \right) dP_x(\omega) \\ = & \int_A f_1(B_{t_0+t_1}(\omega)) \cdots f_r(B_{t_0+t_r}(\omega))\, dP_x(\omega). \end{aligned}$$

Then we will have (13) for such g. We now apply the Dynkin System Theorem to show that having (13) for such g implies that (13) holds for all positive or bounded $\mathcal{F}_\infty$-measurable functions g.

For a product of nonempty open intervals $J = (a_1, b_1) \times \cdots \times (a_d, b_d) \subset \mathbb{R}^d$ (the collection of which we denote by $\mathcal{J}$), the characteristic function 1_J is the limit of an increasing sequence of continuous, nonnegative functions. Explicitly, let $a < b \in \mathbb{R} \cup \{-\infty, +\infty\}$. For a positive integer n, let $h_n(\cdot\,; a, b) : \mathbb{R} \to [0, \infty)$ be the continuous function defined by

$$h_n(x; a, b) = \begin{cases} 0 & x \notin (a, b) \\ 1 & x \in (a + \frac{b-a}{3n}, b - \frac{b-a}{3n}), \end{cases}$$

and interpolating linearly to define $h_n(\cdot\,; a, b)$ on $(a, a + \frac{b-a}{3n})$ and $(b - \frac{b-a}{3n}, b)$. Then

$$J^n(\cdot) := h_n(\cdot\,; a_1, b_1) \times \cdots \times h_n(\cdot\,; a_d, b_d) \uparrow 1_J$$

Let $J_1, \ldots, J_r \in \mathcal{J}$ and let $H_n : \Omega \to \mathbb{R}$ be defined by

$$H_n(\omega) = J_1^n(B_{t_1}(\omega)) \cdots J_r^n(B_{t_r}(\omega)).$$

Since $J_1^n, \ldots, J_r^n$ are continuous, the result (14), for g of the form (15), yields

$$\int_A H_n(\theta_{\tau(\omega)}(\omega))\, dP_x(\omega) = \int_A \int_\Omega H_n(\omega') dP_{B_{\tau(\omega)}(\omega)}(\omega')\, dP_x(\omega).$$

Since $H_n \uparrow (1_{J_1} \circ B_{t_1}) \times \cdots \times (1_{J_r} \circ B_{t_r})$, by the Lebesgue Dominated Convergence Theorem, we may take the limit of both sides under the integral(s) to get that (14) holds for $g = (1_{J_1} \circ B_{t_1}) \times \cdots \times (1_{J_r} \circ B_{t_r})$. Observe that

$$\left((1_{J_1} \circ B_{t_1}) \times \cdots \times (1_{J_r} \circ B_{t_r}) \right)(\omega) = 1_{A_J(t_1, \ldots, t_r)}(\omega),$$

where

$$A_J(t_1,\ldots,t_r) = \{\omega : B_{t_1}(\omega) \in J_1, \ldots, B_{t_r}(\omega) \in J_r\}\,.$$

Let $\mathcal{C}$ be the collection of $C \in \mathcal{F}_\infty$ of the form $A_J(t_1,\ldots,t_r)$, and let $\mathcal{D}$ be the collection of $D \in \mathcal{F}_\infty$ such that

$$\int_A 1_D(\theta_t(\omega))\, dP_x(\omega) = \int_A \int_\Omega 1_D(\omega') dP_{B_t(\omega)}(\omega')\, dP_x(\omega). \tag{16}$$

Note that $\mathcal{C}$ is closed under finite intersection and generates $\mathcal{F}_\infty$, and we have shown that $\mathcal{D}$ contains $\mathcal{C}$. Theorem 33 implies that $\mathcal{D} = \mathcal{F}_\infty$, provided

- (i) $\Omega \in \mathcal{D}$
- (ii) $D_1, D_2 \in \mathcal{D}$ and $D_1 \subset D_2 \Rightarrow D_2 \setminus D_1 \in \mathcal{D}$
- (iii) $A_n \in \mathcal{D},\ n = 1,2,3\ldots$ and $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots \Rightarrow A_\infty := \bigcup_{n=1}^\infty A_n \in \mathcal{D}$.

Note that (i) holds, since $\Omega \in \mathcal{C}$. Just take the $r = 1$ and $J_1 = \mathbb{R}^d$. Alternatively, for $D = \Omega$, both sides of (16) reduce to $P_x(A)$; recall that $\omega \mapsto \omega \circ \theta_t$ is onto. For (ii), since $D_1 \subset D_2$,

$$\begin{aligned} &\int_A 1_{D_2\setminus D_1}(\theta_t(\omega))\, dP_x(\omega) = \int_A 1_{D_2}(\theta_t(\omega))\, dP_x(\omega) - \int_A 1_{D_1}(\theta_t(\omega))\, dP_x(\omega) \\ = &\int_A \int_\Omega 1_{D_2}(\omega') dP_{B_t(\omega)}(\omega')\, dP_x(\omega) - \int_A \int_\Omega 1_{D_1}(\omega') dP_{B_t(\omega)}(\omega')\, dP_x(\omega) \\ = &\int_A \int_\Omega 1_{D_2\setminus D_1}(\omega') dP_{B_t(\omega)}(\omega')\, dP_x(\omega). \end{aligned}$$

For (iii), since $D_1 \subseteq D_2 \subseteq D_3 \subseteq \cdots$, we have $1_{D_n} \uparrow 1_{D_\infty}$ and applying the Lebesgue Dominated Convergence Theorem, we get (16) for $D = D_\infty$. Thus, $\mathcal{D} = \mathcal{F}_\infty$. Having (13) for $g = 1_D$ for all $D \in \mathcal{F}_\infty$, by linearity (13) holds for simple $\mathcal{F}_\infty$-measurable functions. Since any positive $\mathcal{F}_\infty$-measurable function is the pointwise limit of an increasing sequence of simple functions, (13) holds by the Monotone Convergence Theorem. For bounded (and hence integrable on a probability space) $\mathcal{F}_\infty$-measurable functions, we write $g = g_+ - g_-$ for bounded positive functions $g_+ := \max(g, 0)$ and $g_- = \max(-g, 0)$, and apply linearity to get (7). ■

Corollary 20 *The Markov Property* (*Theorem 7*) *holds for* $A \in \mathcal{F}_{t+}$, *not only* $A \in \mathcal{F}_t$.

Proof. We apply Theorem 19 to the in the case τ is constant with value t; i.e., $\tau = t$. In this case,

$$
\begin{aligned}
& \mathcal{F}_{\tau+} := \{A \in \mathcal{F}_\infty \mid \forall s \geq 0,\ A \cap \{\tau < s\} \in \mathcal{F}_s\} \\
= \ & \{A \in \mathcal{F}_\infty \mid \forall s \geq 0,\ A \cap \{t < s\} \in \mathcal{F}_s\} \\
= \ & \{A \in \mathcal{F}_\infty \mid A \in \mathcal{F}_s \text{ for } t < s\} = \mathcal{F}_{t+}.
\end{aligned}
$$

■

We will eventually use the following result to establish the correct boundary behavior for the proposed Brownian solution of the Dirichlet problem.

Corollary 21 (Blumenthal's zero-one law) *If $A \in \mathcal{F}_{0+} = \cap_{t>0} \mathcal{F}_t$, then $P_x(A) = 0$ or $P_x(A) = 1$.*

Proof. We apply

$$\int_A g(\theta_{\tau(\omega)}(\omega))\, dP_x(\omega) = \int_A \int_\Omega g(\omega') dP_{B_{\tau(\omega)}(\omega)}(\omega')\, dP_x(\omega)$$

with $\tau = 0$, $A \in \mathcal{F}_{0+}$ and $g = 1_A$. Then

$$
\begin{aligned}
P_x(A) &= \int_A 1_A\, dP_x(\omega) = \int_A \int_\Omega 1_A(\omega') dP_{B_0(\omega)}(\omega')\, dP_x(\omega) \\
&= \int_A P_x(A)\, dP_x(\omega) = P_x(A)^2.
\end{aligned}
$$

Thus, $P_x(A) = 0$ or $P_x(A) = 1$. ■

6 Proof of the mean-value property of the Brownian solution

Let D be a connected, open, bounded subset of $\mathbb{R}^d$. We seek a solution of the Dirichlet problem

$$\Delta u = 0,\ u = f \text{ on } \partial D$$

where ∂D is sufficiently nice (e.g., ∂D is smooth or D is convex). Let $\tau_D : \Omega \to [0, \infty]$ be the exit time from D. Since D is bounded, the measure of D is finite, in which case by Proposition 18, $P_x(\tau_D = \infty) = 0$. In this section, our goal is to show that the proposed solution

$$u(x) := E_x\left(f(B_{\tau_D})\right) = \int_\Omega f(B_{\tau_{D}(\omega)}(\omega))\, dP_x(\omega) \ \text{ for } x \in D,$$

has the mean-value property, namely

Theorem 22 *For a ball $U = U(x,r) := \left\{y \in \mathbb{R}^d \mid \|y-x\| \le r\right\}$ centered at $x \in D$ with radius r and with $U \subset D$, we have*

$$u(x) = \frac{1}{|\partial U|}\int_{\partial U} u(y)\, d_{\partial B} y.$$

where $\partial U := \left\{y \in \mathbb{R}^d \mid \|y-x\| = r\right\}$ is the boundary of U, $d_{\partial B}y$ is the $(d-1)$-dimensional "volume element" of ∂U, and $|\partial U| := \int_{\partial U} 1\, d_{\partial B} y = \frac{2\pi^{d/2}}{\Gamma(d/2)} r^{d-1}$ is the $(d-1)$-dimensional measure of ∂U.

Proof. We apply Theorem 19 with $A = \Omega$, $\tau = \tau_U$ (not τ_D) and $g : \Omega \to [0,\infty]$ given by $g = f \circ B_{\tau_D}$ or more precisely,

$$g(\omega) := f((B_{\tau_D})(\omega)) := f(B_{\tau_D(\omega)}(\omega))).$$

In this case,

$$\int_A g(\theta_{\tau(\omega)}(\omega))\, dP_x(\omega) = \int_A \int_\Omega g(\omega') dP_{B_{\tau(\omega)}(\omega)}(\omega')\, dP_x(\omega)$$

becomes

$$\int_\Omega f(B_{\tau_D(\omega)}(\theta_{\tau_U(\omega)}(\omega)))\, dP_x(\omega) = \int_\Omega \int_\Omega f(B_{\tau_D(\omega)}(\omega'))) dP_{B_{\tau_U(\omega)}(\omega)}(\omega')\, dP_x(\omega) \tag{17}$$

Note that on $\{\tau_D > s \in [0,\infty)\}$, we have (by definition of τ_D) $\tau_D = s + \tau_D \circ \theta_s$, since

$$\tau_D\left(\theta_s(\omega)\right) = \tau_D(\omega_{s+\cdot}) = \tau_D(\omega) - s. \tag{18}$$

Similarly, on $\{\tau_D > \tau_U\}$, we have

$$\tau_D = \tau_U + \tau_D \circ \theta_{\tau_U}.$$

Note that

$$\begin{aligned}
B_{\tau_D}(\theta_{\tau_U}(\omega)) &= B_{\tau_D(\theta_{\tau_U}(\omega))}(\theta_{\tau_U}(\omega)) = B_{\tau_D(\theta_{\tau_U}(\omega))}(\omega(\tau_U(\omega) + \cdot)) \\
&= \omega(\tau_U(\omega) + \tau_D(\theta_{\tau_U}(\omega))) = B_{\tau_U(\omega)+\tau_D(\theta_{\tau_U(\omega)})}(\omega) \\
&= B_{\tau_D(\omega)}(\omega) = \left(B_{\tau_D}\right)(\omega).
\end{aligned}$$

Hence, (17) becomes

$$\begin{aligned} u(x) &= \int_\Omega f(B_{\tau_D(\omega)}(\omega))\, dP_x(\omega) = \int_\Omega \int_\Omega f(B_{\tau_D(\omega')}(\omega')) dP_{B_{\tau_U(\omega)}(\omega)}(\omega')\, dP_x(\omega) \\ &= \int_\Omega u(B_{\tau_U(\omega)}(\omega))\, dP_x(\omega). \end{aligned}$$

Let $Y : \Omega \to \partial U$ be given by $Y(\omega) := B_{\tau_U(\omega)}(\omega)$. Let $P_x \circ Y^{-1}$ be the measure on ∂U defined by $(P_x \circ Y^{-1})\,(C) = P_x(Y^{-1}(C))$, where C is a Borel subset of ∂U. Let $\mathrm{O}_x(d) :=$ the group of orthogonal transformations of $\mathbb{R}^d$ about $x \in \mathbb{R}^d$. Because of the directional independence of Brownian motion, we expect intuitively that the probability measure $P_x \circ Y^{-1}$ on ∂U is invariant under the action of $\mathrm{O}_x(d)$ on ∂U, and hence is the standard normalized measure on the sphere. In other words, it is expected that for $R \in \mathrm{O}_x(d)$, we have

$$\left(P_x \circ Y^{-1}\right)(R(C)) = \left(P_x \circ Y^{-1}\right)(C);\ \text{i.e., } P_x(Y^{-1}(R(C))) = P_x(Y^{-1}(C)). \tag{19}$$

We show this as follows. For $R \in \mathrm{O}_x(d)$, define $\widetilde{R} : \Omega \to \Omega$ by $\widetilde{R}(\omega) := R \circ \omega$. It follows from the continuity of R and R^{-1} that

$$A \in \mathcal{F} \Leftrightarrow \widetilde{R}^{-1}(A) := \left\{\omega \in \Omega \mid \widetilde{R}(\omega) = R \circ \omega \in A\right\} \in \mathcal{F}.$$

For $A \in \mathcal{F}$, we have

$$P_x(A) = P_x(\widetilde{R}^{-1}(A))$$

(i.e., P_x is invariant under $\widetilde{R}$). Indeed, for the generating subsets of the form $\{\omega \in \Omega \mid B_{t_1}(\omega) \in A_1, \ldots, B_{t_n}(\omega) \in A_n\}$ where $A_i \in \mathcal{B}(\mathbb{R}^d)$, this follows ultimately from properties (i), (ii) and (iii) of Definition 4, or more directly from the rotational invariance of the joint density (5). In the case

$$A = Y^{-1}(R(C)) = \left\{\omega \in \Omega \mid B_{\tau_U(\omega)}(\omega) \in R(C)\right\},$$

we have

$$P_x(Y^{-1}(R(C))) = P_x(A) = P_x(\widetilde{R}^{-1}(A)) = P_x(\widetilde{R}^{-1}(Y^{-1}(R(C)))).$$

Then (19) follows from $\widetilde{R}^{-1}(Y^{-1}(R(C))) = Y^{-1}(C)$, which we now show:

$$
\begin{aligned}
& \widetilde{R}^{-1}(Y^{-1}(R(C))) \\
= & \left\{\omega \in \Omega \mid \widetilde{R}(\omega) \in Y^{-1}(R(C))\right\} = \left\{\omega \in \Omega \mid Y(\widetilde{R}(\omega)) \in R(C)\right\} \\
= & \{\omega \in \Omega \mid Y(R \circ \omega) \in R(C)\} = \{\omega \in \Omega \mid (R \circ \omega)(\tau_U(R \circ \omega)) \in R(C)\} \\
= & \{\omega \in \Omega \mid \omega(\tau_U(R \circ \omega)) \in C\} = \{\omega \in \Omega \mid \omega(\tau_U(\omega)) \in C\} \\
= & \left\{\omega \in \Omega \mid B_{\tau_U(\omega)}(\omega) \in C\right\} = Y^{-1}(C).
\end{aligned}
$$

Any rotationally invariant probability measure on ∂U must be the standard measure σ normalized so that ∂U has unit measure. Thus,

$$
\begin{aligned}
u(x) &= \int_\Omega u(B_{\tau_U(\omega)}(\omega))\, dP_x(\omega) = \int_\Omega u(Y(\omega))\, dP_x(\omega) \\
&= \int_{\partial U} u(y)\, d\left(P_x \circ Y^{-1}\right)(y) = \int_{\partial U} u(y)\, d\sigma(y) = \frac{1}{|\partial U|} \int_{\partial U} u(y)\, d_{\partial U} y,
\end{aligned}
$$

as required. ■

7 Boundary behavior of the Brownian solution

Here we verify that for bounded open domains D satisfying the exterior cone condition, the boundary condition for the Dirichlet problem is satisfied by the hypothetical Brownian solution.

Definition 23 *Let $D \subseteq \mathbb{R}^d$ be an open set and let $y \in \partial D$, then y is called a **regular boundary point of** D if $P_y(\tau_D = 0) = 1$; i.e., a Brownian particle is almost sure to leave D before any positive time relative to P_y. The set of regular boundary points of D is denoted by $(\partial D)_r$.*

Remark 24 *Since $\{\tau_D = 0\}$ is in $\mathcal{F}_{0+}$, by Corollary 21 (Blumenthal's zero-one law) we have $P_y(\tau_D = 0) = 1$ or 0. Hence, if $P_y(\tau_D = 0) > 0$, then $P_y(\tau_D = 0) = 1$, and the condition $P_y(\tau_D = 0) = 1$ in Definition 23 can be replaced by $P_y(\tau_D = 0) > 0$.*

Definition 25 *Let $D \subseteq \mathbb{R}^d$ be an open set. Then D* ***satisfies a cone condition at*** *$y \in \partial D$ if there is an open circular cone V with vertex at y such that $V \cap U(y,r) \subset D^c$ for some ball $U(y,a)$ of some radius $a > 0$ about y. In particular, if D is convex, then each $y \in \partial D$ is regular, since D^c is contained in a half-space of a hyperplane through y.*

Proposition 26 *If D satisfies a cone condition at $y \in \partial D$, then $y \in (\partial D)_r$.*

Proof. Note that

$$\begin{aligned} P_y(\tau_D &< t) \geq P_y(B_t \in D^c) \geq P_y(B_t \in V \cap U(y,r)) \\ &= \frac{m\left(V \cap \partial U(y,r)\right)}{m\left(\partial U(y,r)\right)} P_y(B_t \in U(y,r)) = C(V) P_y(B_t \in U(y,r)), \end{aligned}$$

where $C(V)$ is the solid angle of the cone. For $s_d = 2\pi^{d/2}/\Gamma(d/2) = (d-1)$-measure of the unit sphere in dimension d, we have

$$\begin{aligned} P_y(B_t &\in U(y,r)) = \int_0^r \frac{1}{(2\pi t)^{d/2}} \exp\left(-\frac{\rho^2}{2t}\right) s_d \rho^{d-1} d\rho \\ (\text{with } u &= \rho/\sqrt{t}) = \int_0^{\frac{r}{\sqrt{t}}} \frac{1}{(2\pi t)^{d/2}} \exp\left(-\frac{1}{2}u^2\right) s_d \left(u\sqrt{t}\right)^{d-1} \sqrt{t} du \\ &= \frac{s_d}{(2\pi)^{d/2}} \int_0^{\frac{r}{\sqrt{t}}} \exp\left(-\frac{1}{2}u^2\right) u^{d-1} du \to 1 \text{ as } t \to 0^+. \end{aligned}$$

Thus,

$$\begin{aligned} P_y(\tau_D &= 0) = P_y(\cap_{n=1}^{\infty} \{\tau_D < 1/n\}) = \lim_{n\to\infty} P_y(\tau_D < 1/n) \\ &\geq \lim_{n\to\infty} C(V) P_y(B_{1/n} \in U(y,r)) = C(V) > 0. \end{aligned}$$

Then by Remark 24, $P_y(\tau_D = 0) = 1$ and $y \in (\partial D)_r$.

Definition 27 *A real-valued function $h : X \to \mathbb{R}$ on a metric space X is* ***upper semicontinuous*** *at $y \in X$ if $\lim_{x\to y} h(x) \leq h(y)$.*

■

Proposition 28 *If $h_n : X \to \mathbb{R}$ is upper semicontinuous at $y \in X$ for $n = 1, 2, 3, \ldots$, $h_1 \geq h_2 \geq \cdots$, and $h_n(x) \to h(x) \in \mathbb{R}$ for each x, then $h(x)$ is upper semicontinuous at y.*

Proof. Since $h_n(y) \downarrow h(y)$, for any $\varepsilon > 0$, there is an integer $N = N(\varepsilon)$, such that for $n \geq N$, we have

$$h_N(y) + \tfrac{1}{2}\varepsilon \leq h(y) + \varepsilon.$$

Moreover, since h_N is upper semicontinuous at y, there is $\delta_N > 0$ such that $h_N(x) \leq h_N(y) + \frac{1}{2}\varepsilon$ for $d_N(x, y) < \delta$. As $h_n(x) \downarrow h(x)$, for all $n > N$, we have

$$h_n(x) \leq h_N(y) + \tfrac{1}{2}\varepsilon \leq h(y) + \varepsilon \quad \text{for } d_N(x, y) < \delta.$$

Hence, for arbitrary $\varepsilon > 0$,

$$h(x) = \lim_{n\to\infty} h_n(x) \leq h(y) + \varepsilon \quad \text{for } d_N(x, y) < \delta.$$

Thus, $\lim_{x\to y} h(x) \leq h(y) + \varepsilon$ for arbitrary $\varepsilon > 0$, and so $\lim_{x\to y} h(x) \leq h(y)$. ■

Lemma 29 *For an open set $D \subseteq \mathbb{R}^d$ and fixed $t > 0$, the function $h : \mathbb{R}^d \to \mathbb{R}$, given by $h(x) = P_x(\tau_D < t)$, is upper semicontinuous on $\mathbb{R}^d$.*

Proof. We will show that h is the limit of a decreasing sequence of continuous (and hence upper semicontinuous) functions. While h may not be continuous, by Proposition 28, h is upper semicontinuous. Recall from (18) that if $s < \tau_D(\omega)$, we have $\tau_D(\omega) = s + \tau_D(\theta_s(\omega))$; i.e., on $\{s < \tau_D\}$, we have $\tau_D = s + \tau_D \circ \theta_s$. Note that

$$t < \tau_D(\omega) \Leftrightarrow \omega((0, t)) \subseteq D,$$

and so

$$\begin{aligned} t - s &< \tau_D(\theta_s(\omega)) \Leftrightarrow \theta_s(\omega)(u) \in D \text{ for } 0 < u < t - s \\ &\Leftrightarrow \omega(s + u) \subseteq D \text{ for } 0 < u < t - s \Leftrightarrow \omega((s, t)) \subseteq D. \end{aligned}$$

For $0 < s < s' < t$, we have

$$t - s < \tau_D(\theta_s(\omega)) \Leftrightarrow \omega((s, t)) \subseteq D \Rightarrow \omega((s', t)) \subseteq D \Leftrightarrow t - s' < \tau_D(\theta_{s'}(\omega)).$$

Thus,

$$\{t - s < \tau_D \circ \theta_s\} \subseteq \{t - s' < \tau_D \circ \theta_{s'}\},$$

so that $\{t - s < \tau_D \circ \theta_s\}$ decreases as $s \downarrow 0$, and

$$\{t < \tau_D\} = \cap_{0<s<t} \{t - s < \tau_D \circ \theta_s\} = \cap_{n=1}^{\infty} \{t - \tfrac{t}{n} < \tau_D \circ \theta_{t/n}\}.$$

Hence

$$h(x) = P_x(t < \tau_D) = \lim_{n \to \infty} P_x(t - \frac{t}{n} < \tau_D \circ \theta_{t/n}),$$

and so h is the limit of a decreasing sequence of functions h_n, given by

$$h_n(x) = P_x\left(t - \frac{t}{n} < \tau_D \circ \theta_{t/n}\right) = P_x\left(\theta_{t/n}^{-1}\left(t - \frac{t}{n} < \tau_D\right)\right).$$

If we can show that the h_n are continuous, then $h(x)$ is upper semicontinuous on $\mathbb{R}^d$ by Proposition 28. The next result establishes the desired continuity of h_n. ■

Proposition 30 *For any fixed $t > 0$ and any $C \in \mathcal{F}_\infty$, the function $x \mapsto P_x(\theta_t^{-1}(C))$ is continuous on $\mathbb{R}^d$.*

Proof. Let $g : \mathbb{R}^d \to \mathbb{R}$ be given by $g(y) = P_y(C) \le 1$. We will show below that g is $\mathcal{F}_\infty$-measurable. Then by Theorem 7 with $A = \Omega$, we have

$$\begin{aligned} P_x\left(\theta_t^{-1}(C)\right) &= \int_\Omega P_{B_t(\omega)}(C)\, dP_x(\omega) = \int_\Omega g(B_t(\omega))\, dP_x(\omega) \\ &= \int_{\mathbb{R}^d} p_t(x,y) g(y) dy = \int_{\mathbb{R}^d} \frac{1}{(2\pi t)^{d/2}} \exp\left(-\frac{1}{2t} \|y - x\|^2\right) g(y) dy, \end{aligned}$$

which is a continuous (indeed, C^∞) function of x. Thus, it remains to show that g is $\mathcal{F}_\infty$-measurable. Let $\mathcal{C}$ be the collection of $C \in \mathcal{F}_\infty$ of the form

$$C = \{B_{t_1} \in C_1, \ldots, B_{t_k} \in C_k\},\ 0 \le t_1 < t_2 < \cdots < t_k,$$

where $C_i \in \mathcal{B}(\mathbb{R}^d)$, $i = 1, \ldots, k$, we have

$$P_y(C) = \begin{cases} \int_{C_1} \cdots \int_{C_m} p_{t_1}(y, y_1) p_{t_2 - t_1}(y_1, y_2) \cdots p_{t_k - t_{k-1}}(y_{k-1}, y_k) dy_1 \ldots dy_k & t_1 > 0 \\ \int_{C_2} \cdots \int_{C_m} p_{t_2}(y, y_2) \cdots p_{t_k - t_{k-1}}(y_{k-1}, y_k) dy_2 \ldots dy_k & t_1 = 0, \end{cases}$$

which is a continuous (and hence measurable) function of y. Let $\mathcal{D}$ be the collection of $D \in \mathcal{F}_\infty$ for which $y \mapsto P_y(D)$ is $\mathcal{F}_\infty$-measurable. Note that

$\mathcal{C}$ is closed under finite intersection and we have shown that $\mathcal{D}$ contains $\mathcal{C}$. Theorem 33 implies that $\mathcal{D} = \mathcal{F}_\infty$, provided

(i) $\Omega \in \mathcal{D}$
(ii) $D_1, D_2 \in \mathcal{D}$ and $D_1 \subset D_2 \Rightarrow D_2 \setminus D_1 \in \mathcal{D}$
(iii) $D_n \in \mathcal{D},\ n = 1, 2, 3\ldots$ and $D_1 \subseteq D_2 \subseteq D_3 \subseteq \cdots \Rightarrow D_\infty := \bigcup_{n=1}^\infty D_n \in \mathcal{D}$.

Note that (i) holds, since the constant function $y \mapsto P_y(\Omega) = 1$ is $\mathcal{F}_\infty$-measurable. For (ii), since $D_1 \subset D_2$, we have $y \mapsto P_y\left(D_2 \setminus D_1\right) = P_y\left(D_2\right) - P_y\left(D_1\right)$ is $\mathcal{F}_\infty$-measurable. For (iii), since $D_1 \subseteq D_2 \subseteq D_3 \subseteq \cdots$, we have that $y \mapsto P_y(\bigcup_{n=1}^\infty D_n) = \lim_{n\to\infty} P_y(D_n)$ is the limit of $\mathcal{F}_\infty$-measurable functions and hence is $\mathcal{F}_\infty$-measurable. Thus, $\mathcal{D} = \mathcal{F}_\infty$. ■

Theorem 31 *Let D be a connected, open, bounded subset of $\mathbb{R}^d$, and let $f : \partial D \to \mathbb{R}$ be a continuous function. Let*

$$u(x) := E_x\left(f(B_{\tau_D})\right) = \int_\Omega f(B_{\tau_{D(\omega)}}(\omega))\, dP_x(\omega) \quad \text{for } x \in \overline{D}.$$

If $y \in (\partial D)_r$, then

$$\lim_{x\to y,\ x\in\overline{D}} u(x) = f(y). \tag{20}$$

Proof. Since $y \in (\partial D)_r$, we have $P_y(\tau_D = 0) = 1$, and so

$$u(y) = E_y\left(f(B_{\tau_D})\right) = E_y\left(f(B_0)\right) = E_y\left(f(y)\right) = f(y).$$

To verify $\lim_{x\to y,\ x\in\overline{D}} u(x) = u(y)$, it suffices to show that

$$\lim_{x\to y,\ x\in\overline{D}} \left|E_x\left(f(B_{\tau_D})\right) - f(y)\right| = 0.$$

Since y is fixed $f(y)$ is a constant, and so $E_x(f(y)) = f(y)$. Hence,

$$E_x\left(f(B_{\tau_D})\right) - f(y) = E_x\left(f(B_{\tau_D}) - f(y)\right).$$

Moreover,

$$\left|E_x\left(f(B_{\tau_D}) - f(y)\right)\right| \le E_x\left(\left|f(B_{\tau_D}) - f(y)\right|\right).$$

Thus, it suffices to show that

$$\lim_{x\to y,\ x\in\overline{D}} E_x\left(\left|f(B_{\tau_D}) - f(y)\right|\right) = 0.$$

Given $\varepsilon_1 > 0$, by the continuity of f, we can find $\delta > 0$ such that for all $x \in \partial D \cap U(y,\delta)$, $|f(x) - f(y)| \leq \varepsilon_1$. Let $M = \max_{x \in \partial D} \{|f(x)|\}$. Note that $\tau_D(\omega) < \tau_{U(y,\delta)}(\omega) \Leftrightarrow B_{\tau_D}(\omega) \in U(y,\delta) \Rightarrow |f(B_{\tau_D}(\omega)) - f(y)| \leq \varepsilon_1$. Hence,

$$\begin{aligned} E_x\left(|f(B_{\tau_D}) - f(y)|\right) &\leq P_x(\tau_D < \tau_{U(y,\delta)})\varepsilon_1 + P_x(\tau_D \geq \tau_{U(y,\delta)})2M \\ &\leq \varepsilon_1 + P_x(\tau_{U(y,\delta)} \leq \tau_D)2M \end{aligned}$$

It remains to show that $P_x(\tau_{U(y,\delta)} \leq \tau_D)$ can be made small by choosing δ small and x sufficiently close to y. Suppose that $\delta < \frac{1}{2}\|x-y\|$ so that $x \in U(y,\delta/2)$. Then $U(x,\delta/2) \subseteq U(y,\delta)$, and so

$$\tau_{U(x,\delta/2)} \leq \tau_{U(y,\delta)}, \text{ a.s. } P_x.$$

Thus, if $\delta < \frac{1}{2}|x-y|$, then

$$P_x(\tau_{U(y,\delta)} \leq \tau_D) \leq P_x(\tau_{U(x,\delta/2)} \leq \tau_D).$$

For any $t \in (0,\infty)$ and $\omega \in \Omega$, we have $\tau_D(\omega) \leq t$ or $\tau_D(\omega) > t$. Hence,

$$\tau_{U(x,\delta/2)}(\omega) \leq \tau_D(\omega) \Rightarrow \tau_{U(x,\delta/2)}(\omega) \leq t \text{ or } \tau_D(\omega) > t, \text{ and so}$$

$$P_x(\tau_{U(x,\delta/2)} \leq \tau_D) \leq P_x(\tau_{U(x,\delta/2)} \leq t) + P_x(\tau_D > t).$$

Let $s_{d-1} = 2\pi^{d/2}/\Gamma(d/2) = (d-1)$-dimensional measure of the unit sphere in $\mathbb{R}^d$. Making the change of variable $u = \rho/\sqrt{t}$, we have

$$\begin{aligned} P_x(\tau_{U(x,\delta/2)} &\leq t) \geq P_y(B_t \in U(x,\delta/2)^c) = \int_{\delta/2}^{\infty} \frac{1}{(2\pi t)^{d/2}} e^{-\frac{\rho^2}{2t}} s_d \rho^{d-1} d\rho \\ &= \int_{\frac{\delta}{2\sqrt{t}}}^{\infty} \frac{1}{(2\pi t)^{d/2}} \exp\left(e^{-\frac{u^2}{2}}\right) s_d \left(u\sqrt{t}\right)^{d-1} \sqrt{t} du \\ &= \frac{s_d}{(2\pi)^{d/2}} \int_{\frac{\delta}{2\sqrt{t}}}^{\infty} \exp\left(-\tfrac{1}{2}u^2\right) u^{d-1} du \to 0 \text{ as } t \to 0^+ \ (\delta \text{ fixed}). \end{aligned}$$

Thus, if $\delta < \frac{1}{2}\|x-y\|$, given any $\varepsilon_2 > 0$, for $t > 0$ sufficiently small we have

$$P_x(\tau_{U(x,\delta/2)} \leq \tau_D) \leq P_x(\tau_{U(x,\delta/2)} \leq t) + P_x(\tau_D > t) \leq \varepsilon_2 + P_x(\tau_D > t).$$

In summary, given $\varepsilon_1 > 0$, by the continuity of f, and choosing $\delta > 0$ such that for all $z \in \partial D \cap U(y,\delta)$, $|f(z) - f(y)| \leq \varepsilon_1$, choosing $x \in U(y,\delta/2)$, and choosing $t > 0$ sufficiently small, we have

$$\begin{aligned} E_x\left(|f(B_{\tau_D}) - f(y)|\right) &\leq P_x(\tau_D < \tau_{U(y,\delta)})\varepsilon_1 + P_x(\tau_{U(y,\delta)} \leq \tau_D)2M \\ &\leq \varepsilon_1 + (\varepsilon_2 + P_x(\tau_D > t))2M. \end{aligned}$$

It remains to show that for any $t > 0$, $\lim_{x\to y} P_x(\tau_D > t) = 0$. Since $P_x(\tau_D > t)$ is upper semicontinuous in x by Lemma 29, we have

$$\lim_{x\to y} P_x(\tau_D > t) \le P_y(\tau_D > t) = 1 - P_y(\tau_D \le t) \le 1 - P_y(\tau_D = 0) = 0,$$

since $P_y(\tau_D = 0)$ by the assumption $y \in (\partial D)_r$. Thus, given $\varepsilon_3 > 0$, we could choose x sufficiently close to y so that $P_x(\tau_D > t) < \varepsilon_3$, and (with the choices already made)

$$E_x\left(|f(B_{\tau_D}) - f(y)|\right) \le \varepsilon_1 + (\varepsilon_2 + \varepsilon_3)2M,$$

for x sufficiently close to y. With hindsight, for any $\varepsilon > 0$, we could choose $\varepsilon_1 = \varepsilon/2$ and $(\varepsilon_2 + \varepsilon_3)2M = \varepsilon/2$ (e.g., $\varepsilon_2 = \varepsilon_3 = \frac{\varepsilon}{8M}$) to get

$$E_x\left(|f(B_{\tau_D}) - f(y)|\right) \le \varepsilon,$$

for x sufficiently close to y, and hence as required

$$\lim_{x\to y,\ x\in\overline{D}} E_x\left(f(B_{\tau_D})\right) = u\,(y) = f(y).$$

■

8 Appendix

8.1 The Chapman-Kolmogorov equation

Recall that for $y, z \in \mathbb{R}^d$ and $t > 0$,

$$p(t; y, z) = p^d(t; y, z) := \frac{1}{(2\pi t)^{d/2}} \exp\left(-\frac{\|z-y\|^2}{2t}\right).$$

We use the abbreviation $dz = dz_1 dz_2 \cdots dz_d$ for the volume element of $\mathbb{R}^d$. Intuitively, $p(t; y, z)dz$ is the probability that a particle starting at $y \in \mathbb{R}^d$ at time 0 and "moving randomly" will be in an infinitesimal volume dz about z at time t. The key properties that we need are

$$\int_{\mathbb{R}^d} p(t; y, z)\, dz = 1 \text{ and } \int_{\mathbb{R}^d} p\,(s, x, y)\, p\,(t, y, z)\; dy = p\,(s+t, x, z)\,.$$

The latter is known as the Chapman-Kolmogorov equation. Intuitively, it says that the probability density that a particle moves from x to z in time $s+t$ is the integral with respect to all intermediate positions y of the density that it moves from x to y in time in time t multiplied by the density that it moves from y to z in time s.

We first verify that $\int_{\mathbb{R}^d} p(t;y,z)\,dz = 1$. For fixed $y \in \mathbb{R}^d$, under the change of variables $w = z - y$

$$\int_{\mathbb{R}^d} p(t;y,z)\,dz = \frac{1}{(2\pi t)^{d/2}} \int_{\mathbb{R}^d} \exp\left(-\frac{\|z-y\|^2}{2t}\right) dz$$
$$= \frac{1}{(2\pi t)^{d/2}} \int_{\mathbb{R}^d} \exp\left(-\frac{\|w\|^2}{2t}\right) dw = \frac{1}{(2\pi t)^{d/2}} \int_{\mathbb{R}^d} \exp\left(-\frac{w_1^2+\cdots+w_d^2}{2t}\right) dw$$
$$= \prod_{i=1}^{d} \left(\frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} \exp\left(-\frac{w_i^2}{2t}\right) dw_i\right) = \left(\frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} \exp\left(-\frac{s^2}{2t}\right) ds\right)^d$$

With $x = s/\sqrt{t}$,

$$\frac{1}{(2\pi t)^{1/2}} \int_{\mathbb{R}} \exp\left(-\frac{s^2}{2t}\right) dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{1}{2}x^2}\,dx$$

Thus, it suffices to show that $\left(\int_{\mathbb{R}} e^{-\frac{1}{2}x^2}\,dx\right)^2 = 2\pi$. For this, note that

$$\begin{aligned}
\left(\int_{\mathbb{R}} e^{-\frac{1}{2}x^2}\,dx\right)^2 &= \int_{\mathbb{R}} e^{-\frac{1}{2}x^2}\,dx \int_{\mathbb{R}} e^{-\frac{1}{2}y^2}\,dy = \int_{\mathbb{R}} e^{-\frac{1}{2}y^2} \left(\int_{\mathbb{R}} e^{-\frac{1}{2}x^2}\,dx\right) dy \\
&= \int_{\mathbb{R}^2} e^{-\frac{1}{2}\left(x^2+y^2\right)}\,dxdy = \int_0^{2\pi}\int_0^{\infty} e^{-\frac{1}{2}r^2}\,rdrd\theta \\
&= -2\pi \int_0^{\infty} \frac{d}{dr}\left(e^{-\frac{1}{2}r^2}\right) dr = -2\pi \lim_{r\to\infty}\left(e^{-\frac{1}{2}r^2} - e^0\right) = 2\pi.
\end{aligned}$$

Now we check the Chapman-Kolmogorov equation

$$p\,(s+t,x,z) = \int_{\mathbb{R}^d} p\,(s,x,y)\,p\,(t,y,z)\;dy.$$

While this is accomplished via Fourier transforms with relative ease (see [BC96], Chapter 7) , for those unfamiliar with Fourier transforms, it can be done directly as follows.

$$
\begin{aligned}
& (2\pi s)^{d/2} (2\pi t)^{d/2} \int_{\mathbb{R}^d} p(s,x,y)\, p(t,y,z)\ dy \\
= & \int_{\mathbb{R}^d} \exp\left(-\frac{\|x-y\|^2}{2s}\right) \exp\left(-\frac{\|y-z\|^2}{2t}\right) dy \\
= & \int_{\mathbb{R}^d} \exp\left(-\left(\frac{1}{2s}\|x-y\|^2 + \frac{1}{2t}\|y-z\|^2\right)\right) dy \\
= & \int_{\mathbb{R}^d} \exp\left(-\left(\frac{1}{2s}\left(\|x\|^2+\|y\|^2-2x\cdot y\right) + \frac{1}{2t}\left(\|y\|^2+\|z\|^2-2y\cdot z\right)\right)\right) dy \\
= & \int_{\mathbb{R}^d} \exp\left(-\left(\left(\frac{1}{2s}+\frac{1}{2t}\right)\|y\|^2 - \left(\frac{1}{2s}\|x\|^2+\frac{1}{2t}\|z\|^2\right) + \frac{1}{s}x\cdot y + \frac{1}{t}y\cdot z\right)\right) dy
\end{aligned}
$$

$$
\begin{aligned}
= & \exp\left(-\frac{1}{2s}\|x\|^2 - \frac{1}{2t}\|z\|^2\right)\cdot \\
& \cdot \int_{\mathbb{R}^d} \exp\left(-\tfrac{1}{2}\left(\frac{1}{s}+\frac{1}{t}\right)\left(\|y\|^2 + y\cdot 2\frac{\frac{1}{s}x+\frac{1}{t}z}{\frac{1}{s}+\frac{1}{t}}\right)\right) dy \\
= & \exp\left(-\frac{1}{2s}\|x\|^2 - \frac{1}{2t}\|z\|^2\right)\cdot \\
& \cdot \int_{\mathbb{R}^d} \exp\left(-\tfrac{1}{2}\left(\frac{s+t}{st}\right)\left(\|y\|^2 + y\cdot 2\frac{tx+sz}{s+t}\right)\right) dy \\
= & \exp\left(-\frac{1}{2s}\|x\|^2 - \frac{1}{2t}\|z\|^2\right)\cdot \\
& \cdot \int_{\mathbb{R}^d} \exp\left(-\tfrac{1}{2}\left(\frac{s+t}{st}\right)\left(\left\|y+\frac{tx+sz}{s+t}\right\|^2 - \left\|\frac{tx+sz}{s+t}\right\|^2\right)\right) dy \\
= & \exp\left(-\frac{1}{2s}\|x\|^2 - \frac{1}{2t}\|z\|^2 + \tfrac{1}{2}\left(\frac{s+t}{st}\right)\left\|\frac{tx+sz}{s+t}\right\|^2\right)\cdot \\
& \int_{\mathbb{R}^d} \exp\left(-\tfrac{1}{2}\left(\frac{s+t}{st}\right)\left\|y+\frac{tx+sz}{s+t}\right\|^2\right) dy.
\end{aligned}
$$

For $w = y + \frac{tx+sz}{s+t}$,

$$\begin{aligned}
&\int_{\mathbb{R}^d} \exp\left(-\tfrac{1}{2}\left(\frac{s+t}{st}\right)\left\|y+\frac{tx+sz}{s+t}\right\|^2\right) dy \\
= &\int_{\mathbb{R}^d} \exp\left(-\tfrac{1}{2}\left(\frac{s+t}{st}\right)\|w\|^2\right) dw = \left(2\pi\frac{st}{s+t}\right)^{d/2}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\int_{\mathbb{R}^d} p(s,x,y)\,p(t,y,z)\;dy \\
= \;& (2\pi s)^{-d/2}(2\pi t)^{-d/2}\left(2\pi\frac{st}{s+t}\right)^{d/2}\cdot \\
& \cdot \exp\left(-\frac{1}{2s}\|x\|^2-\frac{1}{2t}\|z\|^2+\tfrac{1}{2}\left(\frac{s+t}{st}\right)\left\|\frac{tx+sz}{s+t}\right\|^2\right) \\
= \;& (2\pi(s+t))^{-d/2}\exp\left(-\frac{1}{2s}\|x\|^2-\frac{1}{2t}\|z\|^2+\tfrac{1}{2}\left(\frac{s+t}{st}\right)\left\|\frac{tx+sz}{s+t}\right\|^2\right) \\
= \;& (2\pi(s+t))^{-d/2}\exp\left(-\frac{(x-z)^2}{2(s+t)}\right) = p(s+t,x,z),
\end{aligned}$$

where we have used the result

$$\begin{aligned}
&-\frac{1}{2s}\|x\|^2-\frac{1}{2t}\|z\|^2+\tfrac{1}{2}\left(\frac{s+t}{st}\right)\left\|\frac{tx+sz}{s+t}\right\|^2 \\
= \;& -\frac{1}{2}\left(\frac{\|x\|^2}{s}+\frac{\|z\|^2}{t}-\left(\frac{1}{st(s+t)}\right)\left(\|x\|^2t^2+2ts(x\cdot z)+\|z\|^2s^2\right)\right) \\
= \;& -\frac{1}{2}\left(\frac{\|x\|^2}{s}+\frac{\|z\|^2}{t}-\left(\frac{1}{s+t}\right)\frac{\|x\|^2t^2+2ts(x\cdot z)+\|z\|^2s^2}{st}\right) \\
= \;& -\frac{1}{2}\left(\frac{\|x\|^2}{s}+\frac{\|z\|^2}{t}-\left(\frac{1}{s+t}\right)\left(\frac{\|x\|^2t}{s}+2x\cdot z+\frac{z^2s}{t}\right)\right) \\
= \;& -\frac{1}{2}\left(\left(\frac{1}{s}-\left(\frac{t}{s(s+t)}\right)\right)\|x\|^2+\left(\frac{1}{t}-\left(\frac{1}{s+t}\right)\frac{s}{t}\right)\|z\|^2-\frac{2x\cdot z}{s+t}\right) \\
= \;& -\frac{1}{2}\left(\frac{1}{s+t}\|x\|^2+\frac{1}{s+t}\|z\|^2-\frac{2x\cdot z}{s+t}\right) = -\frac{\|x-z\|^2}{2(s+t)}.
\end{aligned}$$

8.2 Dynkin systems

Much of this treatment of Dynkin systems was inspired by [Ash72].

Definition 32 (Dynkin system) *A collection of subsets $\mathcal{D}$ of a set Ω is a **Dynkin system** if*

$$\begin{array}{ll} (i) & \Omega \in \mathcal{D} \\ (ii) & A, B \in \mathcal{D} \text{ and } A \subset B \Rightarrow B \setminus A \in \mathcal{D} \\ (iii) & A_n \in \mathcal{D},\ n = 1, 2, 3 \ldots \text{ and } A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{D}. \end{array}$$

Note that the intersection of any collection of Dynkin systems of a set Ω is a Dynkin system.

Theorem 33 (Dynkin System Theorem) *Let $\mathcal{C}$ be a collection of subsets of Ω which is closed under pairwise intersection (i.e., closed under finite intersection). If $\mathcal{D}$ is a Dynkin system which contains $\mathcal{C}$, then $\mathcal{D}$ contains the sigma field $\sigma(\mathcal{C})$ generated by by $\mathcal{C}$.*

Proof. Let $\mathcal{D}_{\mathcal{C}}$ be the smallest Dynkin system which contains $\mathcal{C}$ (i.e., the intersection of all Dynkin systems containing $\mathcal{C}$. We will show that $\mathcal{D}_{\mathcal{C}} = \sigma(\mathcal{C})$. Then $\mathcal{D} \supseteq \mathcal{D}_{\mathcal{C}} = \sigma(\mathcal{C})$. We have that $\mathcal{D}_{\mathcal{C}} \subseteq \sigma(\mathcal{C})$, since $\sigma(\mathcal{C})$ is a Dynkin system containing $\mathcal{C}$. It suffices to show that $\sigma(\mathcal{C}) \subseteq \mathcal{D}_{\mathcal{C}}$. Let

$$\mathcal{B} := \{A \in \mathcal{D}_{\mathcal{C}} : A \cap B \in \mathcal{D}_{\mathcal{C}} \text{ for all } B \in \mathcal{C}\}.$$

By definition, $\mathcal{B} \subseteq \mathcal{D}_{\mathcal{C}}$. We show that $\mathcal{D}_{\mathcal{C}} \subseteq \mathcal{B}$ and hence $\mathcal{D}_{\mathcal{C}} = \mathcal{B}$, by showing that $\mathcal{C} \subseteq \mathcal{B}$ and that $\mathcal{B}$ is a Dynkin system. Note that $\mathcal{C} \subseteq \mathcal{B}$ since $\mathcal{C} \subseteq \mathcal{D}_{\mathcal{C}}$ and $\mathcal{C}$ is closed under intersection. Moreover, we verify that $\mathcal{B}$ is a Dynkin system by giving the proof of each of the required properties $(i), (ii)$ and (iii), as follows:

$$(i) \quad \begin{cases} \Omega \in \mathcal{B} : \\ \Omega \in \mathcal{D}_{\mathcal{C}} \text{ and } \Omega \cap B = B \in \mathcal{D}_{\mathcal{C}} \text{ for all } B \in \mathcal{C} \subseteq \mathcal{D}_{\mathcal{C}} \end{cases}$$

$$(ii) \quad \begin{cases} A_1, A_2 \in \mathcal{B} \text{ and } A_2 \subseteq A_1 \Rightarrow A_1 \setminus A_2 \in \mathcal{B} : \\ A_1, A_2 \in \mathcal{B} \text{ and } A_2 \subseteq A_1 \\ \Rightarrow A_1 \setminus A_2 \in \mathcal{D}_{\mathcal{C}},\ A_1 \cap B \in \mathcal{D}_{\mathcal{C}},\ A_2 \cap B \in \mathcal{D}_{\mathcal{C}} \\ \qquad \text{and } A_2 \cap B \subseteq A_1 \cap B \text{ for all } B \in \mathcal{C} \\ \Rightarrow A_1 \setminus A_2 \in \mathcal{D}_{\mathcal{C}}, \text{ and } (A_1 \setminus A_2) \cap B = (A_1 \cap B) \setminus (A_2 \cap B) \in \mathcal{D}_{\mathcal{C}} \\ \qquad \text{for all } B \in \mathcal{C} \\ \Rightarrow A_1 \setminus A_2 \in \mathcal{B} \end{cases}$$

$$
(iii)\quad \begin{cases}
A_n \in \mathcal{B},\ n = 1,2,3\ldots \text{ and } A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{B} : \\
A_n \in \mathcal{B},\ n = 1,2,3\ldots \text{ and } A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots \\
\Rightarrow A_n \in \mathcal{D}_{\mathcal{C}},\ n = 1,2,3\ldots \text{ and } A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots,\ \bigcup_{n=1}^{\infty} A_n \in \mathcal{D}_{\mathcal{C}}, \\
A_n \cap B \in \mathcal{D}_{\mathcal{C}},\ n = 1,2,3\ldots \text{ and } A_1 \cap B \subseteq A_2 \cap B \subseteq A_3 \cap B \subseteq \cdots \\
\qquad \text{for all } B \in \mathcal{C} \\
\Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{D}_{\mathcal{C}} \text{ and } \left(\bigcup_{n=1}^{\infty} A_n\right) \cap B = \bigcup_{n=1}^{\infty} (A_n \cap B) \in \mathcal{D}_{\mathcal{C}} \\
\Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{B}.
\end{cases}
$$

Thus, $\mathcal{D}_{\mathcal{C}} = \mathcal{B}$; i.e.,

$$
\{A \in \mathcal{D}_{\mathcal{C}} : A \cap B \in \mathcal{D}_{\mathcal{C}} \text{ for all } B \in \mathcal{C}\} = \mathcal{B} = \mathcal{D}_{\mathcal{C}},
$$

or

$$
A \in \mathcal{D}_{\mathcal{C}} \Leftrightarrow A \cap B \in \mathcal{D}_{\mathcal{C}} \text{ for all } B \in \mathcal{C}. \tag{21}
$$

Let

$$
\mathcal{D}'_{\mathcal{C}} := \{A \in \mathcal{D}_{\mathcal{C}} : A \cap D \in \mathcal{D}_{\mathcal{C}} \text{ for all } D \in \mathcal{D}_{\mathcal{C}}\}.
$$

We show that $\mathcal{D}_{\mathcal{C}} = \mathcal{B} \Rightarrow \mathcal{C} \subseteq \mathcal{D}'_{\mathcal{C}}$. If $A \in \mathcal{C}$ and $D \in \mathcal{D}_{\mathcal{C}}$, we need to show that $A \cap D \in \mathcal{D}_{\mathcal{C}}$, but $D \in \mathcal{D}_{\mathcal{C}} \Rightarrow A \cap D \in \mathcal{D}_{\mathcal{C}}$ by (21). Now we check that $\mathcal{D}'_{\mathcal{C}}$ is a Dynkin system:

$$
(i)\quad \begin{cases}
\Omega \in \mathcal{D}'_{\mathcal{C}} : \\
\Omega \in \mathcal{D}_{\mathcal{C}} \text{ and } \Omega \cap D = D \in \mathcal{D}_{\mathcal{C}} \text{ for all } D \in \mathcal{D}_{\mathcal{C}}
\end{cases}
$$

$$
(ii)\quad \begin{cases}
A_1, A_2 \in \mathcal{D}'_{\mathcal{C}} \text{ and } A_2 \subseteq A_1 \Rightarrow A_1 \setminus A_2 \in \mathcal{D}'_{\mathcal{C}} : \\
A_1, A_2 \in \mathcal{D}'_{\mathcal{C}} \text{ and } A_2 \subseteq A_1 \\
\Rightarrow A_1 \setminus A_2 \in \mathcal{D}_{\mathcal{C}}, \text{ and } A_1 \cap D \in \mathcal{D}_{\mathcal{C}},\ A_2 \cap D \in \mathcal{D}_{\mathcal{C}} \\
\qquad \text{and } A_2 \cap D \subseteq A_1 \cap D \text{ for all } D \in \mathcal{D}_{\mathcal{C}} \\
\Rightarrow A_1 \setminus A_2 \in \mathcal{D}_{\mathcal{C}}, \text{ and } (A_1 \setminus A_2) \cap D = (A_1 \cap D) \setminus (A_2 \cap D) \in \mathcal{D}_{\mathcal{C}} \\
\qquad \text{for all } D \in \mathcal{D}_{\mathcal{C}} \\
\Rightarrow A_1 \setminus A_2 \in \mathcal{D}'_{\mathcal{C}}
\end{cases}
$$

$$
(iii)\quad \begin{cases}
A_n \in \mathcal{D}'_{\mathcal{C}},\ n = 1,2,3\ldots \text{ and } A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{D}'_{\mathcal{C}} : \\
A_n \in \mathcal{D}'_{\mathcal{C}},\ n = 1,2,3\ldots \text{ and } A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots \\
\Rightarrow A_n \in \mathcal{D}_{\mathcal{C}},\ n = 1,2,3\ldots \text{ and } A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots,\ \bigcup_{n=1}^{\infty} A_n \in \mathcal{D}_{\mathcal{C}}, \\
A_n \cap D \in \mathcal{D}_{\mathcal{C}},\ n = 1,2,3\ldots \text{ and } A_1 \cap D \subseteq A_2 \cap D \subseteq A_3 \cap D \subseteq \cdots \\
\qquad \text{for all } D \in \mathcal{D}_{\mathcal{C}} \\
\Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{D}_{\mathcal{C}} \text{ and } \left(\bigcup_{n=1}^{\infty} A_n\right) \cap D = \bigcup_{n=1}^{\infty} (A_n \cap D) \in \mathcal{D}_{\mathcal{C}} \\
\Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{D}'_{\mathcal{C}}.
\end{cases}
$$

Since $\mathcal{D}'_{\mathcal{C}}$ is a Dynkin system and we have shown that $\mathcal{C} \subseteq \mathcal{D}'_{\mathcal{C}}$, we have $\mathcal{D}_{\mathcal{C}} \subseteq \mathcal{D}'_{\mathcal{C}} \subseteq \mathcal{D}_{\mathcal{C}}$ and so $\mathcal{D}_{\mathcal{C}} = \mathcal{D}'_{\mathcal{C}}$. Since

$$
\mathcal{D}_{\mathcal{C}} = \mathcal{D}'_{\mathcal{C}} = \{A \in \mathcal{D}_{\mathcal{C}} : A \cap D \in \mathcal{D}_{\mathcal{C}} \text{ for all } D \in \mathcal{D}_{\mathcal{C}}\},
$$

we have that $\mathcal{D}_{\mathcal{C}}$ is closed under intersection. Then $\mathcal{D}_{\mathcal{C}}$ is a field. It follows from property (*iii*) of a Dynkin system that $\mathcal{D}_{\mathcal{C}}$ is closed under countable unions since a countable union

$$\bigcup_{n=1}^{\infty} B_n$$

can be expressed as a union of an *increasing* sequence of partial unions, namely

$$\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} \left(\bigcup_{k=1}^{n} B_k \right).$$

Thus, $\mathcal{D}_{\mathcal{C}} = \sigma(\mathcal{C})$. ■

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