Duality and conformal twisted boundaries in the Ising model

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Abstract. There has been recent interest in conformal twisted boundary conditions and their realisations in solvable lattice models. For the Ising and Potts quantum chains, these amount to boundary terms that are related to duality, which is a proper symmetry of the model at criticality. Thus, at criticality, the duality-twisted Ising model is translationally invariant, similar to the more familiar cases of periodic and antiperiodic boundary conditions. The complete finite-size spectrum of the Ising quantum chain with this peculiar boundary condition is derived.

Dedicated to the memory of Sonia Stanciu

1. Introduction

Quantum spin chains are one-dimensional models of interacting quantum systems. They have been used to model magnetic properties of materials, in particular when strongly anisotropic behaviour suggests a one-dimensional modelling. They also arise as limits of two-dimensional lattice models of classical statistical mechanics, in an anisotropic limit where the lattice spacing in one space direction vanishes. In particular, the Ising quantum chain discussed below is related to the classical Ising model in this way; and the parameter in the Hamiltonian corresponds to the temperature variable of the two-dimensional classical model.

At criticality, quantum spin chains as well as two-dimensional lattice models possess scaling limits that correspond to $(1+1)$-dimensional conformal field theories. The critical exponents describing the non-analytic behaviour of thermodynamic quantities are given by conformal dimensions of certain conformal operators. For translationally invariant quantum chains, the scaling limit corresponds to a conformal field theory on the torus, and the partition function is a quadratic expression in terms of Virasoro characters, distinguishing left and right moving excitations. For free or fixed boundary conditions, we have a conformal field theory on the half plane, with a partition function which is linear in Virasoro characters.

If the spin chain possesses global symmetries, we can define toroidal boundary conditions, i.e., specific twists at the boundary that do not destroy translational invariance. However, this does not seem to yield all possible boundary conditions one might expect from the conformal field theory. Recently, such "conformal twisted boundary conditions" have attracted growing attention [1, 2], and were realised in solvable lattice models [3]. Here, we consider the simplest possible example of such an exotic boundary condition. At least in this case, it is once more related to a symmetry of the model, which turns out to be duality [4]. Furthermore, the complete spectrum of the Hamiltonian is obtained exactly [5], even for finite chains of length N. The scaling limit partition function is computed, verifying the result expected from a mapping to the XXZ Heisenberg quantum spin chain which originally led to the discovery of these duality twisted boundary conditions in the Ising model [4].

Before we come to the particular example of the Ising quantum chain, we first discuss the construction of toroidal boundary conditions in more generality.

2. Quantum Chains and Translational Invariance

For simplicity, we consider quantum spin chains with nearest neighbour couplings only. In this case, the Hamiltonian for a system of N spins has the form

$$H = \sum_{j=1}^{N} H_{j,j+1}, \qquad H_{j,j+1} = \sum_{a,b} \varepsilon_{a,b} \sigma_j^a \sigma_{j+1}^b + \sum_a \delta_a \sigma_j^a, \tag{1}$$

where $\varepsilon_{a,b}$ and δ_a are arbitrary constants which are independent of j. The Hamiltonian is expressed in terms of local spin operators σ_j^a acting on a tensor product space $\mathscr{V}$,

$$\sigma_j^a = \mathbf{1}_V^{\otimes(j-1)} \otimes \sigma^a \otimes \mathbf{1}_V^{\otimes(N-j)}, \qquad \mathscr{V} = V^{\otimes N} = \underbrace{V \otimes V \otimes \ldots \otimes V}_{N \text{ factors}}, \tag{2}$$

where $\mathbf{1}_V$ denotes the identity operator on the vector space V. Let us assume periodic boundary conditions for the moment, i.e., $H_{N,N+1} \equiv H_{N,1}$, so the last spin couples to the first in the same way as the neighbouring spins along the chain. We define a unitary translation operator T by its action on the local spin operators

$$T\sigma_j^a T^{-1} = \sigma_{j+1}^a,\ 1 \leq j \leq N-1, \qquad T\sigma_N^a T^{-1} = \sigma_1^a, \tag{3}$$

which obviously commutes with the Hamiltonian H, so $THT^{-1} = H$. As $T^N = \mathbf{1}$, the identity operator on $\mathscr{V}$, the eigenvalues of T are of the form $\exp(2\pi i k/N)$, with $k = 0, 1, \ldots, N-1$, and define the lattice momenta of the one-dimensional chain.

Consider now the case where the Hamiltonian has a global symmetry, i.e., it commutes with an operator $Q = g \otimes g \otimes \ldots \otimes g = g_1 g_2 \ldots g_N$, where g belongs to a representation of some group. We can then define a modified Hamilton operator $\tilde{H}$ by setting

$$\tilde{H}_{j,j+1} = H_{j,j+1},\ 1 \leq j \leq N-1, \qquad \tilde{H}_{N,N+1} \equiv \tilde{H}_{N,1} = g_1 H_{N,1} g_1^{-1}, \tag{4}$$

so $\tilde{H}$ differs from H only in the coupling term at the boundary, which is twisted by the local transformation g. On first view, this may appear to be no longer translationally invariant, due to the different coupling between the first and the last spin. However, the transformation $Q = g_1 g_2 \ldots g_N$ is a symmetry of the model, which means that the nearest neighbour coupling $H_{j,j+1}$ commutes with the product $g_j g_{j+1}$. We thus can define a modified translation operator $\tilde{T} = g_1 T$ which commutes with $\tilde{H}$,

$$\begin{aligned}
\tilde{T}\tilde{H}_{j,j+1}\tilde{T}^{-1} &= g_1 T H_{j,j+1} T^{-1} g_1^{-1} = H_{j+1,j+2} = \tilde{H}_{j+1,j+2}, \quad 1 \leq j \leq N-1, \\
\tilde{T}\tilde{H}_{N-1,N}\tilde{T}^{-1} &= g_1 T H_{N-1,N} T^{-1} g_1^{-1} = g_1 H_{N,1} g_1^{-1} = \tilde{H}_{N,1}, \\
\tilde{T}\tilde{H}_{N,1}\tilde{T}^{-1} &= g_1 T g_1 H_{N,1} g_1^{-1} T^{-1} g_1^{-1} = g_1 g_2 T H_{N,1} T^{-1} g_2^{-1} g_1^{-1} \\
&= g_1 g_2 H_{1,2} (g_1 g_2)^{-1} = H_{1,2} = \tilde{H}_{1,2},
\end{aligned} \tag{5}$$

the corresponding boundary conditions are known as toroidal boundary conditions.

The Hamiltonian of the Ising quantum chain is given by

$$H_{j,j+1} = -\frac{1}{4}\left(\sigma_j^z + \sigma_{j+1}^z + 2\lambda \sigma_j^x \sigma_{j+1}^x\right), \qquad H = -\frac{1}{2}\sum_{j=1}^{N} \sigma_j^z + \lambda \sigma_j^x \sigma_{j+1}^x, \tag{6}$$

where σ^x and σ^z are Pauli matrices, so $V \cong \mathbb{C}^2$. It has global spin reversal symmetry, i.e., H commutes with the operator $Q = \sigma^z \otimes \sigma^z \otimes \ldots \otimes \sigma^z = \prod_{j=1}^{N} \sigma_j^z$. Corresponding to this C_2 symmetry we have periodic (H^{P} with $g = \mathbf{1}_V$) and antiperiodic (H^{A} with $g = g^{-1} = \sigma^z$) boundary conditions, the latter yielding a change in sign of the $\sigma_N^x \sigma_1^x$ coupling term, because $\sigma_1^z \sigma_1^x \sigma_1^z = -\sigma_1^x$. It turns out that it is useful to consider the mixed-sector Hamiltonians [7] $H^+ = H^{\mathrm{P}} P_+ + H^{\mathrm{A}} P_-$ and $H^- = H^{\mathrm{A}} P_+ + H^{\mathrm{P}} P_-$ instead, where $P_\pm = (\mathbf{1} \pm Q)/2$ are projectors.

3. Duality Twist in the Ising Quantum Chain

Duality is a symmetry that relates the ordered and disordered phases of the classical Ising model. It provides an equality between the partition functions at two different temperatures, the critical temperature being mapped onto itself. In the quantum chain language, duality relates the Hamiltonians H with parameters λ and $1/\lambda$; the critical point corresponds to $\lambda = 1$.

In order to understand the duality transformation, it is advantageous to rewrite the mixed-sector Hamiltonians $H^{\pm}$ as follows

$$H^{\pm}(\lambda) = -\sum_{j=1}^{2N-1}[(e_{2j-1}-\tfrac{1}{2})+\lambda(e_{2j}-\tfrac{1}{2})] - [(e_{2N-1}-\tfrac{1}{2})-\lambda(e_{2N}^{\pm}-\tfrac{1}{2})], \quad (7)$$

where the Temperley-Lieb operators e_j are given by

$$e_{2j-1} = \frac{1}{2}(\mathbf{1}+\sigma_j^z), \quad e_{2j} = \frac{1}{2}(\mathbf{1}+\sigma_j^x\sigma_{j+1}^x), \quad e_{2N}^{\pm} = \frac{1}{2}(\mathbf{1}\pm Q\sigma_N^x\sigma_1^x). \quad (8)$$

Defining invertible operators $g_j = (1+\mathrm{i})e_j - \mathbf{1}$, with $g_j^{-1} = g_j^*$ and $\mathrm{i}^2 = -1$, the appropriate duality transformations are $D^+ = g_1 g_2 \ldots g_{2N-1}$ and $D^- = D^+\sigma_N^x$ [4]. The corresponding duality maps are $D^{\pm}H^{\pm}(\lambda) = \lambda H^{\pm}(1/\lambda)D^{\pm}$. Evidently, $D^{\pm}$ act on the operators e_j like a translation, i.e., $D^{\pm}e_j = e_{j+1}D^{\pm}$ for $1 \leq j \leq 2N-2$, and at the boundary $D^{\pm}e_{2N-1} = e_{2N}^{\pm}D^{\pm}$ and $D^{\pm}e_{2N}^{\pm} = e_1 D^{\pm}$. Thus the squares of the duality transformations $D^{\pm}$ commute with the corresponding Hamiltonians $H^{\pm}$ and are nothing but the appropriate translation operators $T^{\pm} = (D^{\pm})^2$ of the mixed-sector Hamiltonians [4].

At criticality, when $\lambda = 1$, duality itself becomes a symmetry, as $D^{\pm}H^{\pm}(1) = H^{\pm}(1)D^{\pm}$. Thus we can define corresponding twisted boundary conditions. This works in a slightly different way as for the periodic and antiperiodic boundary conditions discussed above, as we have to consider an odd number of generators e_j. The corresponding mixed-sector Hamiltonians are given by [4]

$$\tilde{H}^{\pm} = -\sum_{j=1}^{2N-2}(e_j - \tfrac{1}{2}) - (e_{2N-1}^{\pm} - \tfrac{1}{2}), \quad (9)$$

where the operators e_j, for $1 \leq j \leq 2N-2$, are defined as in equation (8) above, and where $e_{2N-1}^{\pm} = (\mathbf{1}\pm Q\sigma_N^y\sigma_1^x)/2$. So the duality-twisted Ising Hamiltonian contains coupling terms of the type $\pm\sigma_N^y\sigma_1^x$ at the boundary, and, in particular, does *not* contain a term σ_N^z.

The Hamiltonians $\tilde{H}^{\pm}$ are translationally invariant [4], the corresponding translation operators $\tilde{T}^{\pm} = (\tilde{D}^{\pm})^2$ can be constructed as above as the squares of the appropriate duality transformations $\tilde{D}^+ = g_1 g_2 \ldots g_{2N-2}$ and $\tilde{D}^- = \tilde{D}^+\sigma_N^z$, which commute with the critical Hamiltonians $\tilde{H}^+$ and $\tilde{H}^-$, respectively, of equation (9).

4. Spectrum and Partition Function

The spectrum of the duality-twisted Ising quantum chain can be calculated by a modified version [5] of the standard approach. Essentially, the Hamiltonians $\tilde{H}^{\pm}$ (9) are rewritten in terms of fermionic operators by means of a Jordan-Wigner transformation, and the resulting bilinear expressions in fermionic operators are subsequently diagonalised by a Bogoliubov-Valatin transformation, see [5] for details. The diagonal form of the Hamiltonian is

$$\tilde{H}^{\pm} = \sum_{k=0}^{N-1}\Lambda_k \eta_k^{\dagger}\eta_k + E_0\mathbf{1} \quad (10)$$

where $\eta_k^\dagger$ and η_k are fermionic creation and annihilation operators, respectively. The energies of the elementary fermionic excitations are given by

$$\Lambda_k = 2|\sin(\tfrac{p_k}{2})|, \qquad p_k = \frac{4k\pi}{2N-1}, \qquad k = 0,1,2,\ldots,N-1. \tag{11}$$

The ground-state energy E_0 is

$$-E_0 = \sum_{k=0}^{N-1} \sin(\tfrac{k\pi}{\tilde{N}}) = \frac{1+\cos(\frac{\pi}{2\tilde{N}})}{2\sin(\frac{\pi}{2\tilde{N}})} = \frac{2\tilde{N}}{\pi} - \frac{\pi}{24(\tilde{N})} + \mathrm{O}[(\tilde{N})^{-3}] \tag{12}$$

which shows the expected finite-size corrections of a translational invariant critical quantum chain with an effective number of sites of $\tilde{N} = N - 1/2$, reminiscent of the fact that it is related to the XXZ Heisenberg quantum chain with an odd number $2\tilde{N} = 2N-1$ of sites [4]. With the appropriate finite-size scaling, the linearised low-energy spectrum in the infinite system is

$$\frac{\tilde{N}}{2\pi}\,(\tilde{H}^{\pm} - \frac{\tilde{N}}{N}E_0^{\mathrm{P}}\mathbf{1}) \stackrel{N\to\infty}{\longrightarrow} \sum_{r=0}^{\infty}[ra_r^\dagger a_r + (r+\tfrac{1}{2})b_r^\dagger b_r] + \frac{1}{16} \tag{13}$$

where the fermionic operators a_k and b_k follow from the η_k by suitable renumbering, and where $E_0^{\mathrm{P}} = 1/\sin(\frac{\pi}{2N})$ denotes the ground-state energy of the N-site Ising quantum chain with periodic boundary conditions. The conformal partition functions are given by the combinations $(\chi_0 + \chi_{1/2})\bar{\chi}_{1/16}$ and $\chi_{1/16}(\bar{\chi}_0 + \bar{\chi}_{1/2})$, respectively, of characters χ_Δ of irreducible representations with highest weight Δ of the $c = 1/2$ Virasoro algebra, corresponding to operators with conformal spin $1/16$ and $7/16$.

5. Concluding Remarks

For the Ising and Potts quantum chains [4], “exotic” conformal twisted boundary conditions can be realised by means of twists related to duality, which is a symmetry of the model at criticality. It would be interesting to know whether this is a more general feature. If this is the case, this observation might help to identify non-trivial symmetries in quantum chains or two-dimensional solvable lattice models of statistical mechanics.

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