FORCE DENSITY FUNCTION RELATIONSHIPS IN 2-D GRANULAR MEDIA

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Abstract. An integral transform relationship is developed to convert between two important probability density functions (distributions) used in the study of contact forces in granular physics. Developing this transform has now made it possible to compare and relate various theoretical approaches with one another and with the experimental data despite the fact that one may predict the Cartesian probability density and another the force magnitude probability density. Also, the transforms identify which functional forms are relevant to describe the probability density observed in nature, and so the modified Bessel function of the second kind has been identified as the relevant form for the Cartesian probability density corresponding to exponential forms in the force magnitude distribution. Furthermore, it is shown that this transform pair supplies a sufficient mathematical framework to describe the evolution of the force magnitude distribution under shearing. Apart from the choice of several coefficients, whose evolution of values must be explained in the physics, this framework successfully reproduces the features of the distribution that are taken to be an indicator of jamming and unjamming in a granular packing.

Key words. Granular Physics, Probability Density Functions, Fourier Transforms

AMS subject classifications. 60K40, 82D30

1. Introduction. A central topic within modern granular physics research is the study of intergranular force probability densities [1, 2, 3, 4, 5, 6]. The goal being to develop theory—from simplest assumptions—predicting the force density functions seen in simulations and experimental measurements. However, this goal is complicated by the differing forms for the force density functions presented in the literature, two of which are often treated as fundamental.

The first of these force density functions is dependent upon the magnitude and angle of the contact forces between grains. It predicts force chains, the onset of granular jamming [7], strain hardening [1], and fracture of the individual grains. It has several odd and unexplained features, such as a finite, but non-zero, probability at zero force followed by an increasing probability to a peak near the average value of force. It has attracted scientific attention because its exponential tail at high forces and power law at weak forces are reminiscent of the Maxwell-Boltzman distribution of velocities from statistical mechanics. Yet, its overall form is unlike any of the known statistical mechanics distributions and there have been numerous attempts to derive it [8, 9, 10, 11, 12, 13].

The second type of distribution that has been treated as fundamental in the physics literature is a force density function dependent upon the Cartesian components of the contact force vectors. This type of distribution is appealing because it relates to the following conservation law: If external forces are ignored and if the arrangement of grains is static then the total Cartesian force perpendicular to a plane cutting through the medium is conserved for any translation of the plane. Consequently, granular thermodynamic theories have been developed based on this conservation of the total Cartesian forces [14, 15].

The problem addressed by this paper is to clearly define these two types of force density functions and then to establish relationships between them. At first glance this seems straightforward, and it would be if in the literature these densities were handled

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as functions of two dimensions for two-dimensional media and three dimensions for three-dimensional media, but this is usually not the case. Probability force distributions are typically expressed using one of the following variables; the force magnitude, the force angle, or a selected Cartesian component. Because of this contraction to a single variable pertinent information is not available and converting between the two types of densities is not possible. Yet, this conversion is important. Analytical theories usually favor one distribution or the other and empirical investigations typically collect only one type of distribution and without a well established conversion it is not possible to compare the competing theories and their data against one another.

The paper is organized as follows. Random variables are chosen such that the two two-dimensional probability force densities can be defined and the probability theory relations between them shown. In particular, an integral relation is developed whereby the two-dimensional polar force probability density can be converted to a Cartesian force density. From probability theory alone this relationship cannot be inverted but the integral relation can be recast as a set of Fourier transforms. Doing this allows an inverse relation to be found such that if the one-dimensional Cartesian force density function is known for all rotations of the axes, then the two-dimensional polar force probability function can be found. This inversion allows, for the isotropic case, the two force distributions to be treated as a transform pair. The properties of this transform relationship are discussed and significant solutions provided. An example is provided along with data generated from a Monte Carlo process. Then, an example of a pair of force density functions for the anisotropic case is given and compared with published results.

2. Two-Dimensional Force Probability Density Functions. Suppose a large number of grains are placed randomly into a two-dimensional container and that the edges of the container push the grains against each other. These grains are not idealized and may be compressible, have arbitrary shape, be attracted or repulsed by each other, or have frictional contacts. The only restrictions on the grains are that they be static and that at each grain-to-grain contact a force vector be identifiable. Thus, the development presented here is applicable to a wide range of granular media and could be extended to other discrete, static media such as intertwined fibers, foams, glass, and emulsions. Also, the contact forces between the grains and walls may be included in the density functions provided below as long as identifiable force vectors exist and that both force vectors, the grain on the wall and the wall on the grain, are counted in the statistics.

At each grain-to-grain or grain-to-wall contact there are two force vectors, of equal magnitude and opposite direction, according to Newton's third law, as shown in Figure 2.1. Assign the random variable, F, where $0 \leq F < \infty$, to the magnitude of these force vectors and the angle, θ , where $0 \leq \theta < 2\pi$, to the angle between them and the x-axis. A two-dimensional "polar" force probability density can then be defined in terms of the random variables, F and θ, and expressed as $P_{F,\theta}(F,\theta)$, describing the probability of finding a granular contact force with magnitude, F, and angle, θ. An immediate attribute of this function is that

$$P_{F,\theta}(F,\theta) = P_{F,\theta}(F,\theta+\pi), \tag{2.1}$$

a result of there being equal and opposite forces at each granular contact (it is implied in the definition of $P_{F,\theta}(F,\theta)$ that the angle θ is modulo 2π).

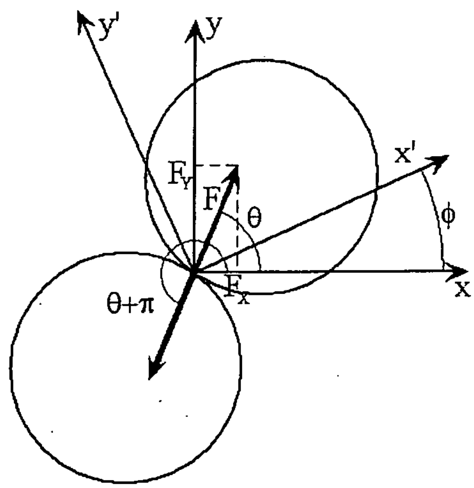

FIG. 2.1. *A sketch showing intergrain forces and the associated random variables.*

A one-dimensional force magnitude probability density function, $P_F(F)$, can be defined by integrating over all values of θ as shown in equation (2.2) below. This equation also shows that the symmetry relation given by equation (2.1) allows the θ integration to occur over any contiguous arc of length π radians.

$$(2.2)\quad P_F(F) \equiv \int_0^{2\pi} P_{F,\theta}(F,\theta)d\theta = 2\int_0^{\pi} P_{F,\theta}(F,\theta)d\theta = 2\int_{\phi-\pi/2}^{\phi+\pi/2} P_{F,\theta}(F,\theta)\ d\theta.$$

By integrating over all values of the force magnitude variable, a one-dimensional force angle probability density function, $P_\theta(\theta)$ is defined by

$$(2.3)\qquad P_\theta(\theta) \equiv \int_0^{\infty} P_{F,\theta}(F,\theta)\ dF.$$

From equations (2.1) and (2.3) the equal and opposite force result, i.e. $P_\theta(\theta) = P_\theta(\theta+\pi)$ can be shown. For frictionless grains (often studied in the physics literature) the direction of the force vector is normal to the contacting surfaces of the grains. In that special case the density function of equation (2.3) is identical to the distribution of contact angles, referred to in the literature as the fabric of the granular material [16].

Not unexpectedly, there exists an alternative pair of random variables, $F_x = F\cos(\theta)$ and $F_y = F\sin(\theta)$, that can be used to describe a Cartesian force probability density function, $P_{F_x,F_y}(F_x, F_y)$, which describes the probability of finding a granular contact force with Cartesian components, F_x and F_y, where $0 \leq F_x,\ F_y < \infty$. The polar and Cartesian force probability densities are related through their corresponding Jacobians yielding

$$P_{F,\theta}(F,\theta) = P_{F_x,F_y}(F\cos(\theta), F\sin(\theta))F \tag{2.4}$$

and

$$P_{F_x,F_y}(F_x,F_y) = P_{F,\theta}((F_x^2+F_y^2)^{1/2}, \arctan(F_y/F_x))/(F_x^2+F_y^2)^{1/2}. \tag{2.5}$$

In equation (2.5), care should be taken to ensure that the arctangent returns an angle in the proper quadrant. Using equations (2.1) and (2.4), a Cartesian force density analogue for the existence of equal and opposite forces is found,

$$P_{F_x,F_y}(F_x,F_y) = P_{F_x,F_y}(-F_x,-F_y). \tag{2.6}$$

Also, by integrating over either of the two Cartesian random variables, a one-dimensional Cartesian force density can be defined. Without loss of generality, integrating over F_y yields $P_{F_x}(F_x)$, which, as shown in equation (2.7) below, can also be found by integrating the polar two-dimensional density function expression from equation (2.5),

$$\begin{aligned} P_{F_x}(F_x) &= \int_{-\infty}^{\infty} P_{F_x,F_y}(F_x,F_y)\,\mathrm{d}F_y \\ &= \int_{-\infty}^{\infty} \frac{P_{F,\theta}((F_x^2+F_y^2)^{1/2}, \arctan(F_y/F_x))}{(F_x^2+F_y^2)^{1/2}}\,\mathrm{d}F_y. \end{aligned} \tag{2.7}$$

Changing variables from F_y to θ via $\tan\theta = F_y/F_x$ yields the following integral relationship between the polar force density function and the one-dimensional Cartesian force density:

$$\begin{aligned} P_{F_x}(F_x) &= \left(\int_0^{\pi/2} + \int_{3\pi/2}^{2\pi}\right) P_{F,\theta}(F_x\sec\theta,\theta)\sec\theta\,\mathrm{d}\theta \qquad \text{if} \qquad F_x \geq 0 \\ P_{F_x}(F_x) &= \int_{\pi/2}^{3\pi/2} P_{F,\theta}(F_x\sec\theta,\theta)\sec\theta\,\mathrm{d}\theta \qquad \text{if} \qquad F_x \leq 0. \end{aligned} \tag{2.8}$$

Equations (2.6) and (2.7) imply that only one of the integrals in (2.8) needs to be calculated.

A more general form for the Cartesian density function can be defined that will prove useful in the analysis that follows. Figure 2.1 shows a coordinate system rotated by angle ϕ, yielding new random variables, $F_{x'}$ and $F_{y'}$. Recalling that rotation does not stretch space, the Jacobian of the transformation between the random variables $F_{x'}, F_{y'}$ and F_x, F_y is unity so their respective probability density functions are equal, i.e. $P_{F_{x'},F_{y'}}(F_{x'},F_{y'}) = P_{F_x,F_y}(F_x(F_{x'},F_{y'}), F_y(F_{x'},F_{y'}))$. Using this result, a one-dimensional Cartesian force density along the x' axis, $P_{F_{x'}}(F_{x'})$, can be defined as

$$P_{F_{x'}}(F_{x'}) = \int_{-\infty}^{\infty} P_{F_{x'},F_{y'}}(F_{x'},F_{y'})\, \mathrm{d}F_{y'}$$

$$(2.9) \qquad = \int_{-\infty}^{\infty} \frac{P_{F,\theta}((F_{x'}^2+F_{y'}^2)^{1/2}, \arctan(F_{y'}/F_{x'})+\phi)}{(F_{x'}^2+F_{y'}^2)^{1/2}}\, \mathrm{d}F_{y'},$$

similar to the result in equation (2.7). Now, changing variables from $F_{y'}$ to θ via $\arctan(F_{y'}/F_{x'})+\phi=\theta$, yields a result similar to that shown in equation (2.8) except that the additional ϕ angle appears in the integrand and in the limits of integration. The variable transformation yields

$$P_{F_{x'}}(F_{x'}) =$$

$$= \left(\int_{\phi}^{\phi+\pi/2} + \int_{\phi+3\pi/2}^{\phi+2\pi}\right) P_{F,\theta}(F_{x'}\sec(\theta-\phi),\theta)\sec(\theta-\phi)\,\mathrm{d}\theta \quad \text{if} \quad F_{x'} \geq 0$$

$$= \int_{\phi+\pi/2}^{\phi+3\pi/2} P_{F,\theta}(F_{x'}\sec(\theta-\phi),\theta)\sec(\theta-\phi)\,\mathrm{d}\theta \qquad \text{if} \qquad F_{x'} \leq 0.$$

(2.10)

Even though the density function $P_{F_{x'}}(F_{x'})$ is well defined by the above equations it suffers from two shortcomings. Firstly, it is an explicit function of the angle ϕ, and this should be reflected in the notation chosen. Secondly, if the angle ϕ is allowed to range from zero to 2π radians there is an ambiguity in the choice of how to represent the domain of this function. Specifically, choosing a rotation angle, ϕ, and a random variable, $F_{x'}$, is identical to rotating by $\phi+\pi$ radians and choosing a value for the random variable of $-F_{x'}$. We chose to resolve the second issue by the allowing the angle ϕ to range from zero to 2π radians and defining a new random variable, F_ϕ, which is equal to $F_{x'}$ but is always positive, i.e. $0 \leq F_\phi < \infty$. The first issue above can then be resolved by adopting the notation $P_{F_\phi}(F_\phi,\phi)$ for the density function associated with this new random variable. The single variable subscript indicates that this is a one-dimensional density function but the two-dimensional domain shows that it is an explicit function of the two variables, F_ϕ and ϕ. So the function $P_{F_\phi}(F_\phi,\phi)$ has the same "polar" domain as the density function $P_{F,\theta}(F,\theta)$. It can be explicitly expressed using the first integral expression above—where $F_{x'}>0$—and merging the integrals by using the 2π periodic nature of the variable θ

$$(2.11) \qquad P_{F_\phi}(F_\phi,\phi) = 2\int_{\phi-\pi/2}^{\phi+\pi/2} P_{F,\theta}(F_\phi\sec(\theta-\phi),\theta)\sec(\theta-\phi)\,\mathrm{d}\theta,$$

where a factor of 2 has been added for normalization. Also, note that the function $P_{F_\phi}(F_\phi,\phi)$, in order to be a density function, must be normalized at each angle, ϕ. In other words, the total number of forces does not change with the choice of ϕ, so the integral of $P_{F_\phi}(F_\phi,\phi)$ over all F_ϕ must always equal 1.

Equation (2.11) is a general integral relation allowing integration of the "polar" two-dimensional probability force density function to yield any desired Cartesian projection density function. This is a useful relationship within granular physics research in that it allows the calculation of a Cartesian density function from a force magnitude density function, but it is not a surprising result. Once the proper definitions are made the derivation is straightforward. The more difficult result is to perform the inverse operation, namely to find the polar form from the Cartesian form, but this is not possible within the realm of probability theory because the function, $P_{F_\phi}(F_\phi, \phi)$, is not a two-dimensional probability density function. Even so, the inverse operation can be performed as demonstrated in the next section.

Finally, by using the variable transformation, $\theta' = \theta + \pi$, it can be shown that it can be shown that

$$P_{F_\phi}(F_\phi, \phi + \pi) = P_{F_\phi}(F_\phi, \phi) \tag{2.12}$$

3. Fourier Transform Representation of Force Density Integrals. In this section it will be shown that equation (2.11) above can be represented as a set of Fourier transforms. Accomplishing this allows equation (2.11) to be inverted by utilizing the Fourier transform inversion properties. This approach is similar to the mathematics used in tomography [17], but the development presented here is distinct for two reasons. First, the symmetry relations of equations (2.1) and (2.3) provide simplification that does not occur in tomography. Second, the emphasis in tomography is to generate three-dimensional functions from a set of two-dimensional images, while in the present development the goal is to obtain two-, or in a future work, three-dimensional representations of the force density functions from one-dimensional Cartesian projections.

As a result of the symmetries obtained above in equations (2.1) and (2.12) we will require only restricted forms of the Fourier transforms. For example, we require only the Fourier cosine transform instead of the full exponential form. The Fourier cosine transform of a function $f(x)$ is expressed as

$$\mathcal{F}_c[f(x); u] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos(ux)\, \mathrm{d}x,$$

which has the same form as the inverse cosine transform [18]. The Fourier Transform representations are written out in detail in order to resolve notational and normalization variations that appear in the literature. In addition to this one-dimensional transform, we will require a two-dimensional Fourier transform. In Cartesian form this is written as

$$\mathcal{F}_{2D}[f(x,y); (u,v)] = \frac{1}{2\pi} \int_{-\infty}^\infty \int_{-\infty}^\infty f(x,y) \exp\left(i(ux+vy)\right)\, \mathrm{d}x\, \mathrm{d}y, \tag{3.1}$$

but since the density functions are in polar form, this transform will need to be expressed in polar form also. Using the change of variables $x = F\cos(\theta)$, $y = F\sin(\theta)$, $u = G\cos(\phi)$, $v = G\sin(\phi)$, the following form for the two-dimensional, polar, Fourier transform is found,

$$\begin{aligned} &\mathcal{F}_{2D}[f(F,\theta); (G,\phi)] \\ &= \frac{1}{2\pi} \int_0^\infty \int_0^{2\pi} f(F,\theta) \exp\left(iFG(\cos(\theta)\cos(\phi) + \sin(\theta)\sin(\phi))\right) F\, \mathrm{d}F\, \mathrm{d}\theta \end{aligned}$$

$$= \frac{1}{2\pi}\int_0^\infty\int_0^{2\pi} f(F,\theta)\exp\left(iFG\cos(\theta-\phi)\right)F\ \mathrm{d}F\ \mathrm{d}\theta,$$

but we will only be transforming functions where $f(F,\theta) = f(F,\theta+\pi)$. Using this symmetry simplifies this to a two-dimensional form of the Fourier cosine transform

$$\mathcal{F}_{2D,c}[f(F,\theta);(G,\phi)] = \frac{1}{2\pi}\int_0^\infty\int_0^{2\pi} f(F,\theta)\cos(FG\cos(\theta-\phi))F\ \mathrm{d}F\ \mathrm{d}\theta. \tag{3.2}$$

Having established the above notation, now consider the following lemma which is the key result of this paper.

LEMMA 1. *The projected force density function is given by*

$$P_{F_\phi}(F_\phi,\phi) = 2\sqrt{2\pi}\mathcal{F}_c\left[\mathcal{F}_{2D,c}[P_{F,\theta}(F,\theta)/F;(G,\phi)];F_\phi\right]. \tag{3.3}$$

PROOF 1. *Note that*

$$\begin{aligned}
&2\sqrt{2\pi}\mathcal{F}_c\left[\mathcal{F}_{2D,c}[P_{F,\theta}(F,\theta)/F;(G,\phi)];F_\phi\right]\\
&= 2\sqrt{2\pi}\sqrt{\frac{2}{\pi}}\int_0^\infty\left[\frac{1}{2\pi}\int_0^\infty\int_0^{2\pi}P_{F,\theta}(F,\theta)\cos(FG\cos(\theta-\phi))\ \mathrm{d}F\ \mathrm{d}\theta\right]\cos(GF_\phi)\ \mathrm{d}G\\
&= \frac{2}{\pi}\int_0^{2\pi}\int_0^\infty P_{F,\theta}(F,\theta)\left(\int_0^\infty\cos(FG\cos(\theta-\phi))\cos(GF_\phi)\ \mathrm{d}G\right)\mathrm{d}F d\theta\\
&= \int_0^{2\pi}\int_0^\infty P_{F,\theta}(F,\theta)\left(\delta(F\cos(\theta-\phi)-F_\phi)\right)\mathrm{d}F\mathrm{d}\theta.
\end{aligned}$$

Changing the argument of the Dirac delta function, $\delta(x)$, is done using

$$\delta(F\cos(\theta-\phi)-F_\phi) = \delta(F-F_\phi\sec(\theta-\phi))/|\cos(\theta-\phi)|.$$

Thus integration with respect to F is well-defined. The limits of integration with respect to θ can be compacted and shifted so that $\cos(\theta-\phi)$ is always positive, removing the need for the absolute value and introducing a factor of two. Then

$$\begin{aligned}
&2\sqrt{2\pi}\mathcal{F}_c\left[\mathcal{F}_{2D,c}[P_{F,\theta}(F,\theta)/F;(G,\phi)];F_\phi\right]\\
&= 2\int_{\phi-\pi/2}^{\phi+\pi/2}P_{F,\theta}(F_\phi\sec(\theta-\phi),\theta)\sec(\theta-\phi)\ \mathrm{d}\theta\\
&= P_{F_\phi}(F_\phi,\phi).
\end{aligned}$$

The benefit of this form is that equation (3.3) can be immediately inverted because each of the Fourier transforms is its own inverse, yielding the second key result of this paper

LEMMA 2. *The 2-D polar probability force density may be expressed as a function of the projected Cartesian force density by*

$$P_{F,\theta}(F,\theta) = \frac{F}{2\sqrt{2\pi}}\mathcal{F}_{2D,c}[\mathcal{F}_c[P_{F_\phi}(F_\phi,\phi);(G,\phi)];(F,\theta)]. \tag{3.4}$$

Equivalently,

$$P_{F,\theta}(F,\theta) = \frac{F}{(2\pi)^2}\int_0^\infty\int_0^{2\pi}\int_0^\infty P_{F_\phi}(F_\phi,\phi)\cos(F_\phi G)\cos(FG\cos(\phi-\theta))G\ \mathrm{d}G\ \mathrm{d}\phi\ \mathrm{d}F_\phi. \tag{3.5}$$

This integral can be integrated immediately with respect to the variable G, but this leads to an awkward functional form that is not amenable to solving using the standard look-up tables. Instead, we choose to leave it in this form allowing the order of integration to be carried out on a case by case basis. This integral can be further integrated with respect to either F or θ to yield the single variable force density functions, demonstrating that knowledge of the projected Cartesian force density function can yield an explicit form for the force magnitude density.

Furthermore, for a frictionless packing, in which the force angles are the same as the contact angles as discussed above, one may integrate out F from equation (3.6) according to Equation (2.3) to obtain the fabric of the packing. This result is striking because it begins with a knowledge of only $P_{F_\phi}(F_\phi,\phi)$, which is a set of contact force distributions that on the surface appear to have no contact angle information. This is the first time that force distributions, alone, have been directly related to the fabric of a packing and this may be important to developing a theory of granular rheology in which changes in the fabric and the forces are coupled. Also, if the empirical studies can determine a simple ϕ-dependence for $P_{F_\phi}(F_\phi,\phi)$, then it may be possible to discern the fabric of a frictionless packing simply by sampling P_{F_ϕ} at only a few orientations of ϕ, or maybe only along the principle stress axes.

4. Force Density Integrals for Isotropic Material. In this section the two integral equations derived above, equations (2.11) and (3.6), are simplified for the isotropic case yielding a pair of integral transform equations. Some of the properties of this transform pair are presented and a list of useful solutions are shown. We start with the following definition:

DEFINITION 4.1. *An isotropic medium is one in which* $P_{F,\theta}(F,\theta) = P_F(F)/(2\pi)$.
Thus, an isotropic medium implies a force density with no angular dependence. As an immediate consequence of this definition, equation (2.11) can be simplified to show that an isotropic material also has no ϕ dependence on its Cartesian projected force density,

$$P_{F_\phi}(F_\phi,\phi) = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} P_F(F_\phi\sec(\theta))\sec(\theta)\ \mathrm{d}\theta. \tag{4.1}$$

So, for the rest of this section we will use the notation $P_{F_\phi}(F_\phi)$ for the Cartesian projected force density, since the function is no longer dependent upon the angle ϕ.

Equation (4.1) can be put into a more useful form by changing variables from the angle θ back to the force magnitude using $F = F_\phi\sec(\theta)$. This yields the integral equation

$$P_{F_\phi}(F_\phi) = \frac{2}{\pi}\int_{F_\phi}^\infty \frac{P_F(F)}{(F^2-F_\phi^2)^{1/2}}\ \mathrm{d}F. \tag{4.2}$$

Equation (3.6) can be simplified as well. Since neither of the density functions have angular dependence, the ϕ integration on the right can be performed yielding the result

$$P_F(F) = F\int_0^\infty\int_0^\infty P_{F_\phi}(F_\phi)\cos(F_\phi G)J_0(FG)G\,\mathrm{d}F_\phi\,\mathrm{d}G \tag{4.3}$$

where J_0 is the zero-order Bessel function [[18], eqn. 3.715.18], where we choose to perform the integration with respect to F_ϕ before G.

Equations (4.2) and (4.3) are inverse transforms relating the force magnitude probability density, $P_F(F)$, to the projected Cartesian force probability density, $P_{F_\phi}(F_\phi)$ for isotropic two-dimensional granular materials. Such materials are of current theoretical and experimental interest and these equations apply to any such material as long as an identifiable force can be assigned to the grain-to-grain, and, if included, grain-to-wall contacts. Consequently, a wide variety of force distributions are expected to result from current research, making it beneficial to discuss some of the properties of this transform pair as well as to present some of the more significant solution pairs. In the discussion below we will use the symbol $\leftrightarrow$ to link transform pairs and we will not always normalize the pairs.

Equations (4.2) and (4.3) are clearly linear in that if $P_{F_\phi}(F_\phi) \leftrightarrow P_F(F)$ and if $R_{F_\phi}(F_\phi) \leftrightarrow R_F(F)$ are two sets of solutions then

$$aP_{F_\phi}(F_\phi) + bR_{F_\phi}(F_\phi) \leftrightarrow aP_F(F) + bR_F(F) \tag{4.4}$$

is a solution pair, where a and b are arbitrary constants. Another method for generating new solutions is to note that if $P_{F_\phi}(F_\phi) \leftrightarrow P_F(F)$ is a solution pair, then

$$F_\phi\frac{\partial P_{F_\phi}(F_\phi)}{\partial F_\phi} \leftrightarrow F\frac{\partial P_F(F)}{\partial F} \tag{4.5}$$

is also a solution pair as a result of properties of the transform. By combining this result with the linearity result it can be shown that

$$F_\phi^2\frac{\partial^2 P_{F_\phi}(F_\phi)}{\partial F_\phi^2} \leftrightarrow F^2\frac{\partial^2 P_F(F)}{\partial F^2} \tag{4.6}$$

is also a solution pair and more generally that

$$F_\phi^n\frac{\partial^n P_{F_\phi}(F_\phi)}{\partial F_\phi^n} \leftrightarrow F^n\frac{\partial^n P_F(F)}{\partial F^n} \tag{4.7}$$

is a solution pair. This result is very useful for obtaining sets of similar solutions when trying to fit experimental or simulation data.

A set of Bessel function integral equations [[18], eqn. 6.592.10, 6.592.12-15] provides a method for obtaining useful solution pairs. Let Z_ν represent any of the Bessel functions, first kind J_ν, second kind Y_ν, third kind H_ν, or modified second kind K_ν. Then after appropriate changes of variable the referenced integral equations can be

TABLE 4.1
Solution pairs for Cartesian and force magnitude functions in isotropic packings generated from equation (4.8).

	$\mathbf{P_{F_\phi}(F_\phi)}$	$\mathbf{P_F(F)}$
1.	$(2\alpha/\pi)K_0(\alpha F_\phi)$	$\alpha\exp(-\alpha F)$
2.	$(2\alpha^2/\pi)F_\phi K_1(\alpha F_\phi)$	$\alpha^2 F\exp(-\alpha F)$
3.	$(-F_\phi)^n(2\alpha/\pi)\frac{\partial^n}{\partial F_\phi^n}(K_0(\alpha F_\phi))$	$\alpha^{n+1}F^n\exp(-\alpha F)$
4.	$(2\alpha^3/\pi)F_\phi^2 K_2(\alpha F_\phi)$	$\alpha^2 F(1+\alpha F)\exp(-\alpha F)$
5.	$-Y_0(\alpha F_\phi)$	$\cos(\alpha F)$
6.	$J_0(\alpha F_\phi)$	$\sin(\alpha F)$
7.	$\alpha\exp(-\alpha F_\phi)$	$\alpha^2 F K_0(\alpha F)$
8.	$\alpha^2 F_\phi\exp(-\alpha F_\phi)$	$\alpha^2 F(\alpha F K_1(\alpha F) - K_0(\alpha F))$
9.	$\alpha^{n+1}F_\phi^n\exp(-\alpha F_\phi)$	$\alpha^2(-F)^n\frac{\partial^n}{\partial F^n}(FK_0(\alpha F))$
10.	$(2/\pi)\cos(\alpha F_\phi)$	$\alpha F J_0(\alpha F)$
11.	$(2/\pi)\sin(\alpha F_\phi)$	$\alpha F Y_0(\alpha F)$

put into the form of either equation (4.2) or (4.3) yielding the following transform pair

$$\sqrt{\frac{2}{\pi\alpha}}\frac{Z_{\nu-1/2}(\alpha F_\phi)}{F_\phi^{\nu-1/2}} \leftrightarrow \frac{Z_\nu(\alpha F)}{F^{\nu-1}}. \tag{4.8}$$

Using this result and the differential generation result of equation (4.7), the solution pairs listed in table 4.1 can be found. Not unexpectedly, since the transforms derived above are between polar and Cartesian representations, the solution pairs are often a Cartesian function, i.e. a sine, cosine, or exponential; and a polar function, i.e. a Bessel function. Probability densities are positive functions with finite total integrals, so the exponential and modified Bessel function pairs are especially useful. Since the transform relationships are linear, it is worthwhile to show the cosine and sine solutions (both as $P_F(F)$ and $P_{F_\phi}(F_\phi)$) so that, if desired, the Fourier components of a solution can be considered.

Other solution pairs to equations (4.2) and (4.3) exist and can be found in the standard integral tables, but many of them can not be normalized. Table 4.2 shows five normalized solution pairs involving Modified Bessel Functions and Gaussians, which may be useful in granular physics applications.

To close this section a normalized pair of solutions is shown that have been fit to the results of a Monte Carlo granular force model [12]. In this case, the grains are round, hard, frictionless disks in an isotropic packing. Over 20,000 grain forces were calculated, and the Cartesian force density function and the force magnitude density

TABLE 4.2
More solution pairs for Cartesian and force magnitude distributions in isotropic packings.

	$\mathbf{P_{F_\phi}(F_\phi)}$	$\mathbf{P_F(F)}$
1.	$2\sqrt{\alpha/\pi}\exp(-\alpha F_\phi^2)$	$2\alpha F\exp(-\alpha F^2)$
2.	$2\alpha F_\phi \exp(-\alpha F_\phi^2)$	$2\alpha F(2\alpha F^2-1)\exp(-\alpha F^2)$
3.	$2\sqrt{\alpha/\pi^3}\exp(-\alpha F_\phi^2/2)K_0(\alpha F_\phi^2/2)$	$2\sqrt{\alpha/\pi}\exp(-\alpha F^2)$
5.	$(2\alpha/\pi)K_0^2(\alpha F_\phi/2)$	$(2\alpha/\pi)K_0(\alpha F)$
6.	$(2\alpha^2 F_\phi/\pi^2)K_0(\alpha F_\phi/2)K_1(\alpha F_\phi/2)$	$(2\alpha^2 F/\pi)K_1(\alpha F)$

function were found empirically. A fit to the Cartesian force density function was then found using a three term modified Bessel function summation as shown below. The solution pairs given above were then used to determine the force magnitude density function. The expansion coefficients can be chosen so that both distributions provide an excellent fit to the empirical data, as seen in Figure 4.1, thus demonstrating the success of the transformation developed in this paper. The plotted density functions are

$$P_{F_\phi}(F_\phi) = C\left[11F_\phi^2 K_2\left(\frac{\pi}{2}F_\phi\right) - 2F_\phi K_1\left(\frac{\pi}{2}F_\phi\right) + 2K_0\left(\frac{\pi}{2}F_\phi\right)\right] \tag{4.9}$$

and

$$P_F(F) = C\left[(11\pi/2)F^2 + (11-\pi)F + \pi\right]\exp\left(-\frac{\pi}{2}F\right) \tag{4.10}$$

where the normalization constant $C = \pi^2/(128 - 6\pi + 2\pi^2)$.

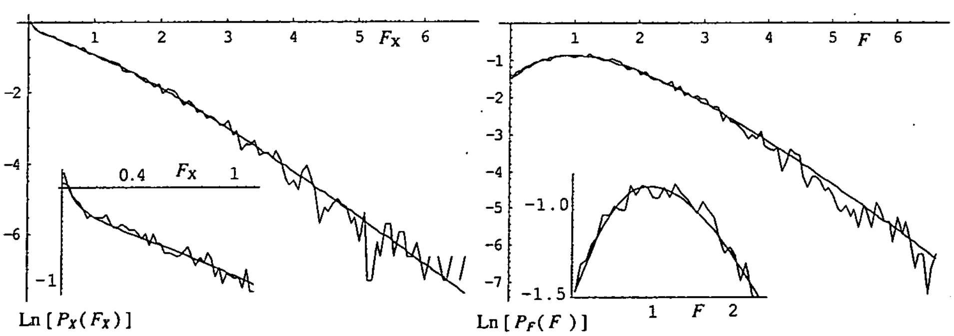

FIG. 4.1. *(Left) Graph of Cartesian distribution from Monte Carlo simulation with analytical fit. The inset shows behavior $F_x = 1$. (Right) Graph of force magnitude distribution from Monte Carlo simulation with analytical fit. The inset shows behavior $F = 2$.*

It should be helpful to further granular research that the modified Bessel Function of the second kind has been identified as the naturally-occurring form for the Cartesian distribution, corresponding to the exponential forms of the polar distributions. The two "knees" in the curve that are visible in the Cartesian distribution of Figure 4.1 seem to be indicated in the Cartesian distributions of Bagi [14], as well, although this identification has not been previously made.

5. An Example of an Anisotropic Solution Pair. In most real world cases granular media are subjected to greater total compressional force along one axis than the other, for example, in a gravity dominated situation. Consequently, the anisotropic case is of interest, although it is significantly more complicated. Furthermore, the physics of jamming and unjamming have emerged as possibly the key concepts in granula media. The anisotropic case is relevant to this because shear stress (an aspect of anisotropy in the stress state) is one of the three ways to unjam granular media as represented by the jamming phase diagram [19]. It has been proposed that the evolution of $P_F(F)$—from having a peak as shown in figure 2.1 to being monotonically decreasing—serves as an indicator of unjamming [7]. Numerical simulations have indeed shown that this evolution occurs when the material is unjammed through stress anisotropy [1]. Therefore, an explanation for the evolution of $P_F(F)$ is central to the aims of granular physics. In this section a reasonable form is selected for the density function $P_{F_\phi}(F_\phi, \phi)$ for a sample anisotropic case and then the integrals evaluated to yield $P_F(F)$. This solution is discussed and compared against published data [1]. The purpose is to demonstrate that this mathematical framework is a *sufficient* framework to include the evolution of $P_F(F)$.

The reason we start with $P_{F_\phi}(F_\phi, \phi)$ instead of $P_{F,\theta}(F, \theta)$ in this demonstration is because the Cartesian distribution is the one associated with the force conservation law, and it is through that conservation law that the anisotropy is injected into the problem. It is well known that the normal components (diagonal elements) in the stress tensor scale according to $\sigma_{xx} = a + b\cos(2\phi)$ as the coordinate system is rotated through angle ϕ. Hence, the quantity of conserved force normal to the layers of a granular material will also scale according to this form when the layer is oriented at angle ϕ. The values of a and b are determined by the forces applied along the principle stress axes of the system. Based on the successful fits presented at the end of the previous section we choose to let $P_{F_\phi}(F_\phi, \phi)$ be represented as a sum of the first three modified Bessel functions, but where this explicit angular dependence is added.

$$(5.1)\ P_{F_\phi}(F_\phi, \phi) = \sum_{n=0}^{2} a_n \left(\frac{a-b}{a+b}\right)^{n-1} (a + b\cos(2\phi))^{n+1} F_\phi^n K_n((a + b\cos(2\phi))F_\phi).$$

The parameter b determines the amount of variation in force with angle, equaling zero for the isotropic case and approaching a for extreme anisotropy. Thus the force density is shifted towards higher forces along the Y-axis with b nonzero. The $(a+b\cos(2\phi))^{n+1}$ factor has been included to normalize the distribution at every particular value of ϕ as required from the discussion above. The $(a + b)/(a - b)$ factor is conjectural, to increase the weighting of the lower order terms as the anisotropy increases. Doing this yields results that correspond to dynamic simulations as seen in Figure 5.1, but whose basis is unclear. The point being that choices can be made for the Cartesian form of the force density functions, which can then be converted to force magnitude or force angle density functions (i.e. fabric) and compared to published data.

The force magnitude density function is found by using this form for $P_{F_\phi}(F_\phi, \phi)$ in equation (3.6) and integrating with respect to θ (see equation (2.2)) yielding

$$P_F(F) = \frac{F}{(2\pi)^2}\sum_{n=0}^{2} a_n \left(\frac{a-b}{a+b}\right)^{n-1} \int_0^{2\pi}\int_0^{2\pi}\int_0^{\infty}\int_0^{\infty} (a+b\cos(2\phi))^{n+1} F_\phi^n$$
$$K_n((a+b\cos(2\phi))F_\phi)\cos(F_\phi G)\cos(FG\cos(\phi-\theta))G \, \mathrm{d}G \, \mathrm{d}F_\phi \, \mathrm{d}\phi \, \mathrm{d}\theta.$$

The integration with respect to θ can be performed immediately yielding $2\pi J_0(FG)$, and the integration with respect to F_ϕ can be performed via [18, eqn. 6.699.12]. Then the integration with respect to G can be performed via [18, eqn. 6.565.4] yielding the partial result

$$P_F(F) =$$
$$\sum_{n=0}^{2} \frac{a_n F^{n+1/2}}{4}\sqrt{\frac{2}{\pi}}\left(\frac{a-b}{a+b}\right)^{n-1} \int_0^{2\pi} (a+b\cos(2\phi))^{n+3/2} K_{n-1/2}(F(a+b\cos(2\phi))) \, \mathrm{d}\phi.$$

Using the identities for the half order modified Bessel functions the integrations with respect to ϕ can be made yielding the result,

$$\begin{aligned} P_F(F) = \frac{\pi}{2} \exp(-aF)(a_0\left(\frac{a+b}{a-b}\right)(aI_0(bF) - bI_1(bF)) + a_1((a^2+b^2)FI_0(bF) \\ -(b+2abF)I_1(bF)) + a_2\left(\frac{a-b}{a+b}\right)(F(a^2+2b^2+a^3F+3ab^2F)I_0(bF) \\ -(3b+5abF+3a^2bF^2+b^3F^2)I_1(bF))) \end{aligned} \tag{5.2}$$

where the exponential dependence on the force is expected from the isotropic case, but the Bessel function dependence on the parameter b is novel ($I(x)$ is the modified Bessel function of the first kind). Using the values $a_0 = \pi^2/2$, $a_1 = -\pi$, $a_2 = 11$, and $a = \pi/2$ figure 5.1 shows a plot of equation (5.3) for various degrees of anisotropy.

For $b = 0$, the isotropic case, the plot is identical to that shown in Figure 4.1 and equation (5.3) reduces to equation (4.10), but as b increases the shape of the curve changes, slowly moving towards a pure exponential. This is in agreement with published simulation data where the force magnitude density function evolves in a similar fashion with increasing anisotropy [1]. This demonstrates that the mathematical framework can produce this evolution naturally; nothing more exotic than the relative weighting of the Bessel terms need be invoked to produce it.

6. Summary and Conclusions. It is possible within the straightforward techniques of probability theory to convert from $P_{F,\theta}(F,\theta)$ to $P_{F_\phi}(F_\phi,\phi)$. Unfortunately, those techniques cannot provide a conversion in the opposite direction. However, we may recognize that the conversion in the forward direction is equivalent to the composition of Fourier cosine transforms of the function. Since these transforms have their own well-defined inverses, the conversion from $P_{F_\phi}(F_\phi,\phi)$ to $P_{F,\theta}(F,\theta)$ can likewise be expressed.

This inverse conversion is interesting for several reasons. First, it allows theoretical models that only predict Cartesian force distributions (such as the q model [15]), to be directly compared against the force magnitude distributions, which have been more important to granular physics. Second, the inverse conversion indicates a previously unrecognized relationship between the Cartesian force component distributions

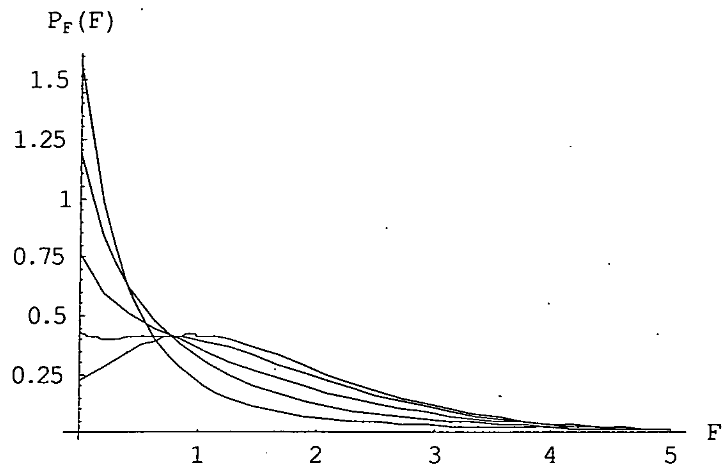

FIG. 5.1. *Evolution of force magnitude distribution with increasing anisotropy. The curve with the lowest probability density for $F = 0$ is the $b = 0$ isotropic case. The other curves, in order, correspond to $b = 0.3, 0.6, 0.9$, and 1.57.*

$P_{F_\phi}(F_\phi, \phi)$ and the fabric of the granular packing, which may be important in future theoretical developments. Third, for the special case of isotropic granular packings in which the ϕ and θ dependencies may be eliminated, the transform pair reduces to a simple form that can be solved for a wide range of functions. This indicates which functional forms for the Cartesian components correspond to particular functional forms for the force magnitudes. Since it is well-known that the distribution of the latter has an exponential tail, the corresponding form for $P_{F_X}(F_X)$ ought to be modified Bessel functions of the second kind. Expansion in a series of such functions (of increasing order) display two characteristic "knees" when graphed, and indeed it turns out that such knees are observed in the empirical Cartesian distributions. Thus, the natural form for $P_{F_X}(F_X)$ appears to have been identified, and this should provide insight into the physical mechanisms that produce the distributions. Fourth, treating these modified Bessel functions with increasing anisotropic stress naturally produces an evolution of $P_F(F)$ that depends upon the choice of coefficients in the series expansion. Prior research has associated this evolution with the occurrence of jamming and unjamming in granular packings, and so the inverse transform indicates that jamming may be described as in increase in weighting of the zeroth-order modified Bessel function. This insight should be helpful to explain the physics of jamming and unjamming, which are important concepts in granular physics. Finally, we have noted that this inverse transform proves some of the integrals listed in Gradshteyn and Ryzhik to be valid over parameter domains larger than those indicated by Gradshteyn and Ryzhik. Because this inverse conversion identifies these relationships and natural functional forms for granular force distributions, its derivation should be a helpful contribution in future research into the physics of granular jamming and unjamming.

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