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# Optimal routing and control of multiple agents moving in a transportation network and subject to an arrival schedule and separation constraints

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## Abstract

We address the problem of navigating a set of moving agents, e.g. automated guided vehicles, through a transportation network so as to bring each agent to its destination at a specified time. Each pair of agents is required to be separated by a minimal distance, generally agent-dependent, at all times. The speed range, initial position, required destination, and required time of arrival at destination for each agent are assumed provided. The movement of each agent is governed by a controlled differential equation (state equation). The problem consists in choosing for each agent a path and a control strategy so as to meet the constraints and reach the destination at the required time. This problem arises in various fields of transportation, including Air Traffic Management and train coordination, and in robotics. The main contribution of the paper is a model that allows to recast this problem as a decoupled collection of problems in classical optimal control and is easily generalized to the case when inertia cannot be neglected. Some qualitative insight into solution behavior is obtained using the Pontryagin Maximum Principle. Sample numerical solutions are computed using a numerical optimal control solver.


# 1 Introduction

Problems in coordinated motion planning for multiple agents can be roughly classified into two disjoint categories, *decoupled coordination* (each agent's motion is planned separately, then the plans are reconciled), and *centralized coordination* (all the agents' motions are planned simultaneously, with the interaction constraints considered from the start) [1]. The problems considered in this paper fall in the latter category. Centralized coordination of multiple agent motion has been approached using various types of mathematical models, discrete (see, for example, [2] and references therein) and continuous (see, for example, [1, 3–5], and references therein). A review of research on multi-robot coordination problems can be found in [1, section 1.1].

In a number of coordination problems, the moving agents are confined to a *transportation network* (also known as *roadmap coordination space* [1]). A general mathematical model of a transportation network is a *multigraph* [6] with the vertices being points in a Euclidean space and edges being parametrized curves connecting pairs of vertices. Examples of such networks include railroad networks for trains, railroad networks for industrial robots, trolley and tram car networks, and airspaces with fixed nominal routes.

A subclass of network-confined coordination problems consists of those where the agents' paths are not given but sought as part of solving the problem. In such problems, the system exhibits behaviors both continuous (the agent's motion along an edge) and discrete (an agent's choice between two edges emanating from the same vertex). This coupling of the two behaviors suggests *hybrid control systems (HCS)* [7] as a suitable class of models for approaching the problem. Hybrid systems have been applied to various problems of transportation (e.g., highway traffic [8,9]), and in particular to aerospace problems [10–12], and are applied here to the problem of finding routes and speed advisories for a set of agents moving in a transportation network with separation constraints and an arrival schedule imposed. Namely, one is given the following data:

1. A directed multigraph $G = (V, E)$, each vertex $v \in V$ being a point in a Euclidean space $\mathbf{E}$ of dimension 2 or 3. If $e \in E$ is an edge from vertex $v_1$ to vertex $v_2$, then the nominal route segment from waypoint $v_1$ to waypoint $v_2$ is a curve in $\mathbf{E}$, connecting $v_1$ to $v_2$. All such curves will henceforth be assumed *rectifiable* [13] and capable of a parameterization which is continuous and piecewise continuously differentiable. A cusp in the curve can be traversed with the assumption (made throughout this paper, but capable of relaxation) that inertia is neglected, and approximately smoothed if inertia is to be taken into account. A graph-theoretic *path* [6] in $G$ is, therefore, associated (and, henceforth, identified) with a spatial path that can be traversed by a moving agent. A vertex of $G$ of *indegree* $\geq 2$ [6] (respectively, outdegree $\geq 2$) corresponds to two or more route segments merging (respectively, diverging). The modeling framework below imposes no restrictions on the outdegree or indegree of a vertex.

2. A finite set
$$\mathcal{A} = \{1, \dots, A\} \tag{1}$$
of *moving agents* $\alpha \in \mathcal{A}$ in $G$. If agent $\alpha$ is moving along a path in $G$, the agent's position is specified by the arc length coordinate $x^\alpha$ along the path.

3. For each agent $\alpha \in A$, a specification of the agent's initial position $x^{INIT;\alpha}$, required destination $x^{DEST;\alpha}$, and the required time $t^{DEST;\alpha}$ of arriving at the destination. Here $x^{INIT;\alpha}$ and $x^{DEST;\alpha}$ are points in $G$, each point specified, for example, by an edge in $G$ and a fractional distance along that edge.

4. The inertia-free *state equations* [14] (henceforth the dot denotes differentiation with respect to physical time $t$)
$$\dot{x}^\alpha = s^\alpha, \quad \alpha \in \mathcal{A},$$
where the $s^\alpha$'s are the corresponding speeds, describing the motion of those agents $\alpha$ that have not yet reached their destination. In what follows, and with the details provided below, the coordinates

$x^\alpha$ will play the role of *state variables*; the speeds $s^\alpha$, of the *control variables.*

5. *State constraints:* the separation requirement for each pair of agents. This requirement is described mathematically, in terms of the coordinates $x^\alpha$, in section 3.

6. *Control constraints:* bounds on the speeds $s_e^\alpha$.

7. A *cost functional*, specified below.

The problem, defined in detail below (definition 4.1) as the *Scheduled Routing Problem*, consists in finding for each agent $\alpha \in A$ a path $p(\alpha)$ in $G$ from $x^{INIT;\alpha}$ to $x^{DEST;\alpha}$ and a *control strategy* $s^\alpha_{p(\alpha)}(t)$ (i.e., a function from the time domain of the model to the set of admissible controls) such that the resulting movements $x^\alpha(t)$ along the corresponding paths constitute a state trajectory that satisfies the above state and control constraints and that minimizes the cost.

In this paper, we use an HCS framework to formulate a model specialized to the above problem. The main contributions of this model are as follows:

- Reduction of the problem to a special case of an HCS where each solution trajectory lies in only one control mode.

- A clear application of *Depth-First Search* [15] to search through the control modes as economically as possible given the possibly exponential size of the problem. (In the worst case when every agent can be assigned to any of the paths, and when each edge can serve as a path by itself, the number of agent-to-path assignments, i.e. of functions $\mu : \mathcal{A} \to E$, is $|E|^{|\mathcal{A}|}$. This upper bound, however, is a crude overestimate. Better ones are discussed below.)

- Reduction of each control mode to a problem in classical deterministic optimal control, which allows, at least in principle, application of the fundamental results of Pontryagin [16] and Bellman [17], and of the numerical algorithms that have been developed and implemented [18–20].

- A natural way to capture an agent's exiting the system; see Remark 4.1, below.

These contributions together allow for parallel computation of solutions: the classical optimal control problems corresponding to different control modes can be solved in parallel, and their obtained minimal costs values compared. Furthermore, the Depth-First Search algorithm itself admits a parallel implementation [21].

The new hybrid model is formulated in section 2. The classical deterministic optimal control corresponding to a given control mode is formulated in section 4. Numerical solutions to some instances of the problem are given in section 5.

## 2 An HCS formulation

The HCS defined in this section will be instrumental in a precise formulation of the Scheduled Routing Problem. Assume the data 1)-7), listed in the third paragraph of section 1.

- For each agent $\alpha \in \mathcal{A}$, let $e^{INIT;\alpha}$ denote the edge occupied initially by agent $\alpha$. (Two or more agents can occupy the same edge.)

- Let $\mathcal{P}\left(e^{INIT;\alpha}, e^{DEST;\alpha}\right)$ be the set of all paths in the multigraph that begin with the edge $e^{INIT;\alpha}$ and end with the edge $e^{DEST;\alpha}$ that contains $x^{DEST;\alpha}$. The length of a path $p \in \mathcal{P}\left(e^{INIT;\alpha}, e^{DEST;\alpha}\right)$ will be denoted $l(p)$.

- Define a control mode $\mu$ as a mapping that assigns each agent $\alpha$ to a path in $\mathcal{P}\left(e^{INIT;\alpha}, e^{DEST;\alpha}\right)$. In more detail, $\mu$ is a mapping from the set $\mathcal{A}$ of moving agents to the union $\cup_\alpha \mathcal{P}\left(e^{INIT;\alpha}, e^{DEST;\alpha}\right)$ such that

$$\mu(\alpha) \in \mathcal{P}\left(e^{INIT;\alpha}, e^{DEST;\alpha}\right) \text{ for each } \alpha \in \mathcal{A}$$

- For agent $\alpha$, each path $\mu(\alpha) \in \mathcal{P}\left(e^{INIT;\alpha}, e^{DEST;\alpha}\right)$ is parameterized by arc length. For computational convenience, the arc length coordinate increases along the path, with the destination coordinate $x^{DEST;\alpha}$ being zero for each $\alpha$. Thus, $x_\mu^\alpha \in [-l(\mu(\alpha)), 0]$, and

$$x^{DEST;\alpha} = 0$$

  This convention ensures that an agent's destination is a vertex in the transportation network and, furthermore, that it is the same vertex in all control modes.

- Each agent $\alpha$ in each control mode $\mu$ is required to reach its destination $x_\mu^\alpha = x^{DEST;\alpha} = 0$ at a prescribed time $t^{DEST;\alpha}$. Upon reaching destination, the agent is no longer in the model; this is reflected in the restriction on the time domains of the individual equations in the dynamical law (2), stated below.

- In each $\mu$, have the arc length coordinate $x_\mu^\alpha$ evolve according to the dynamical law

$$\dot{y}_\mu^\alpha(t) = s_\mu^\alpha(t) \quad \text{for } 0 \le t \le t^{DEST;\alpha}, \quad \alpha \in \mathcal{A}, \tag{2}$$

where $s_\mu^\alpha$ is the control variable corresponding to the agent's speed of motion along the path. In this formulation, the equations that constitute the dynamical law are imposed over different (albeit overlapping) time domains. The problem, however, will be converted below to one where all dynamical law equations are imposed over the same (rescaled) time domain.

- For each $\mu$ and each $\alpha$, impose the arrival requirement
$$x_\mu^\alpha\left(t^{DEST;\alpha}\right) = x^{DEST;\alpha} \tag{3}$$
- For each $\mu$ and each $\alpha$, impose the speed ranges
$$s_\mu^{MIN;\alpha} \leq s_\mu^\alpha \leq s_\mu^{MAX;\alpha} \tag{4}$$

## 3 The geometry of separation constraints

In some transportation types, including aircraft and trains, every pair of moving agents must be be separated by a distance no smaller than a predetermined *minimal separation*. For example, the minimal separation requirements for air traffic in the U.S. are defined in [22] and depend on numerous factors, including airspace type, air traffic automation systems in use, and aircraft weight classes (the classes defined in [22] are: Small, Large, Heavy, B757). Other types of moving agents, e.g. industrial robots, may also be subject to complicated, asymmetric, and anisotropic (e.g., for aircraft, altitude-dependent) separation requirements. The separation requirement will be a key constraint on the state variables in the Scheduled Routing Problem formulated in section 4.2.

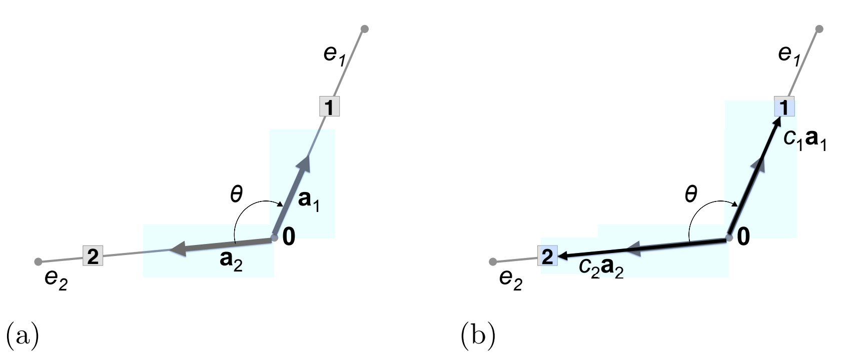



Figure 1. Agents 1, 2 on their respective rectilinear edges $e_1, e_2$, which share a common vertex, taken as the origin $\mathbf{0}$ in $\mathbf{R}^2$. The orientation of the edges is not specified. (a) The unit vectors $\mathbf{a}_1, \mathbf{a}_2$ are collinear with the respective edges, but their directions do not necessary agree with the edges' orientations. (b) With suitably chosen scalar coefficients $c_1, c_2$, the vectors $c_1\mathbf{a}_1$ and $c_2\mathbf{a}_2$ are the respective position vectors of the two agents.

No attempt is made in this paper to capture all such requirements in detail (see, however, section 6.3) for a discussion of anisotropy in separation requirements arising from altitude dependence). Instead, we will use conservative approximations, addressing only the following asymmetry: if two moving agents are in-trail (i.e., one is directly following the other along a route segment which is not necessarily in a horizontal plane), then the minimal separation can depend on the weight class of the leading and trailing agent. To capture this potential asymmetry, for each pair $\alpha_1, \alpha_2$ of agents with the first one leading, we introduce the minimal separation $r_{\alpha_1,\alpha_2}$. If the asymmetry takes place, it can be written

$$r_{\alpha_1,\alpha_2} \neq r_{\alpha_2,\alpha_1} \tag{5}$$

We now calculate the set of all states, in a control mode $\mu$ of a hybrid system described above, where at least two agents violate the separation requirement. The scenario shown in Figure 1A has two agents on two different rectilinear edges, which need not lie in a horizontal plane, with a common vertex and no specified orientation. (If the edges are curvilinear with low curvature near a common vertex or intersection, these portions can be approximated by linear segments; otherwise, the analysis becomes considerably more complicated.)

**Remark 3.1.** *Since edge orientation is not specified, Figure 1 describes four cases: one where both agents are moving toward the common vertex, one where both agents are moving away from the common vertex, and two more cases in which one agent is moving toward, and the other away from, the common vertex.*

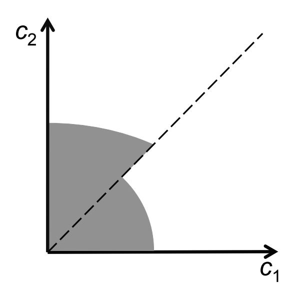



Figure 2. An example of two elliptical sectors in the $c_1c_2$-plane corresponding to conflicting states.

The asymmetry of the gray-shaded region about the dashed diagonal is the asymmetry (5).

We will use the Euclidean inner product $\langle \cdot, \cdot \rangle$ and the corresponding *norm* $|| \cdot ||$ in the 2-D space containing the two edges. Pick the common vertex as the origin and the unit vectors $\mathbf{a}_1, \mathbf{a}_2$ as the basis vectors that, *regardless of the edge orientations*, point from the origin toward the respective agents. With suitable scalars $c_1, c_2$, the vectors $c_1\mathbf{a}_1$ and $c_2\mathbf{a}_2$ are the agents' respective position vectors. The squared distance between the two agents is

$$||c_1\mathbf{a}_1 - c_2\mathbf{a}_2||^2 = (c_1)^2 + (c_2)^2 - 2c_1c_2\langle\mathbf{a}_1, \mathbf{a}_2\rangle \tag{6}$$

Equating the latter expression to the squared minimal separation, say, $r_{1,2}^2$, we obtain the equation

$$(c_1)^2 + (c_2)^2 - 2c_1c_2\langle\mathbf{a}_1, \mathbf{a}_2\rangle = r_{1,2}^2 \tag{7}$$

of an ellipse in the $c_1c_2$-plane. The corresponding set of conflicting sets is described by the elliptical sector obtained by intersecting the ellipse-bound region

$$(c_1)^2 + (c_2)^2 - 2c_1c_2\langle\mathbf{a}_1, \mathbf{a}_2\rangle < r_{1,2}^2$$

with the open octant $c_2 > c_1 > 0$, corresponding to the case when agent 1 is the one closer to the origin. In the other case (agent 2 is closer to the origin), the corresponding elliptical sector is obtained by intersecting

$$(c_1)^2 + (c_2)^2 - 2c_1c_2\langle\mathbf{a}_1, \mathbf{a}_2\rangle < r_{2,1}^2$$

with the octant $c_1 > c_2 > 0$. The role of the angle $\theta$ between the edges $e_1, e_2$ in both sectors is the equality $\langle\mathbf{a}_1, \mathbf{a}_2\rangle = \cos(\theta)$. An example of two such sectors is shown in Figure 2.

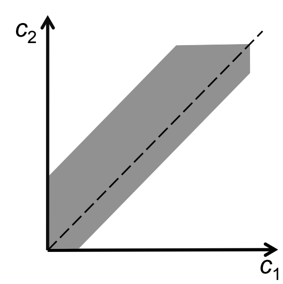



Figure 3. An example of two stripes in the $c_1c_2$-plane corresponding to conflicting states of two agents on the same edge.

In each of the four cases listed in Remark 3.1, the respective continuous state coordinates $x_\mu^1, x_\mu^2$ of agents $1, 2$ in control mode $\mu$ map to the coefficients $c_1, c_2$, as follows:

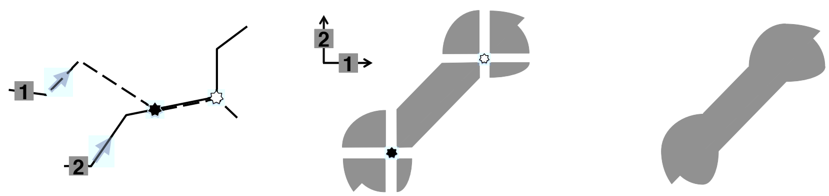
1. If both agents are moving toward the common vertex, then $x_\mu^\alpha = l(e_\alpha) - c_\alpha$ for $\alpha = 1, 2$.

2. If both agents are moving away from the common vertex, then $x_\mu^\alpha = c_\alpha$ for $\alpha = 1, 2$.

3. If agent 1 is approaching, and agent 2 going away from, the common vertex, then $x^1_\mu = l(e_1) - c_1, x^2_\mu = c_2$.

4. If agent 2 is approaching, and agent 1 going away from, the common vertex, then $x^1_\mu = c_1, x^2_\mu = l(e_2) - c_2$.

If $\theta \geq 90°$, then in the last two cases $\mu$ allows only one in-trail sequence, so the minimal separations used for the two sectors in Figure 2 are equal. If the two agents $1, 2$ are on one and the same edge, then the set in the $c_1c_2$-plane of the conflicting states appears as in Figure 3 (the asymmetry about the dashed diagonal corresponds to (5). The mapping from the continuous state coordinates $x^1_\mu, x^2_\mu$ to the coefficients $c_1, c_2$ is constructed analogously to the above four cases.




(a) (b) (c)

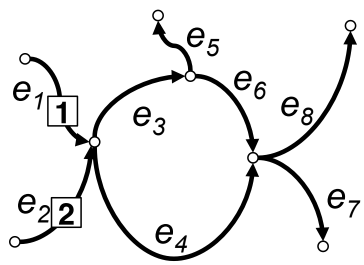
Figure 4. An example of two agents whose paths in the transportation network are prescribed and overlap. The black star shows the beginning of the overlap in (a) and the corresponding state (both agents being at that point) in (b); the white star, the end of the overlap in (a) and the corresponding state (both agents being at that point) in (b). The system, shown in (a), has seven control modes with both agents in the transportation network. Each mode's set of separation-violating states, shown in (b) as a *connected* [23] gray region, is "glued" to some of the others. The result of the gluing is the connected region shown in (c).

The above calculation is illustrated, for an example of two moving agents, in Figure 4. In each control mode, the set of conflicting states is shown as a *connected* [23] gray region. For dimension $A$ above 2, one must compute for each pair of agents the set of states violating the separation requirements. Each such set is a cylinder, or union of cylinders, with the base shaped as shown in Figure 4c, in the total state space $\cup_\mu X_\mu$. We note that the set of all separation-violating states in $\cup_\mu X_\mu$ is *cylindrical* in the sense of [5, Definition 2.2], the latter definition a key requirement for the applicability of a number of theoretical results of [5].

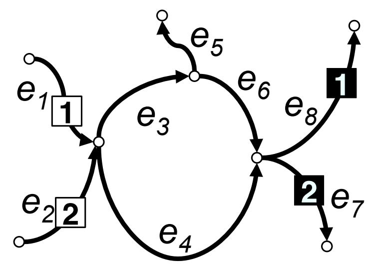
# 4 The Scheduled Routing Problem and the equivalent Stacked Scheduled Routing Problem

For ease of exposition, we precede the general formulation of the problem, suitable for an arbitrary number of moving agents, with a specific, two-agent, example.

## 4.1 An instance of the scheduled routing problem for two agents




(a)




(b)

Figure 5. An initial state (a) and required destinations (b) of a two-agent set.

The initial state of a two-agent scheduled routing is shown in Figure 5a; the required destinations (with possibly different required arrival times) for the two agents, in Figure 5b. In this Figure,

$$\alpha = 1 \;:\; e^{INIT;1} = e_1,\; e^{DEST;1} = e_8,$$

$$\alpha = 2 \;:\; e^{INIT;2} = e_2,\; e^{DEST;2} = e_7$$

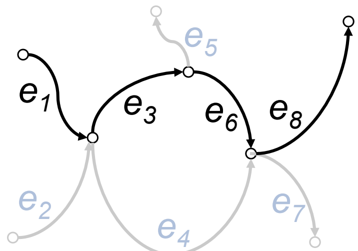
The paths are

$$p_1 : e_1, e_3, e_6, e_8; \quad p_2 : e_1, e_4, e_8; \quad p_3 : e_2, e_3, e_6, e_7; \quad p_4 : e_2, e_4, e_7 \tag{8}$$

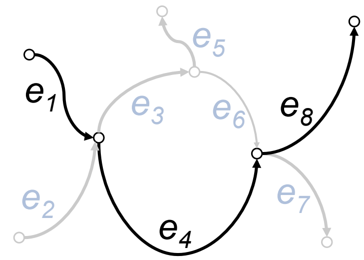
Agent 1 can take either $p_1$ or $p_2$; agent 2, either $p_3$ or $p_4$. Consequently,

$$\mathcal{P}\left(e^{INIT;1}, e^{DEST;1}\right) = \{p_1, p_2\}, \quad \mathcal{P}\left(e^{INIT;2}, e^{DEST;2}\right) = \{p_3, p_4\}$$

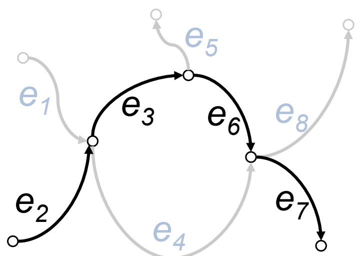
No other paths can be taken.



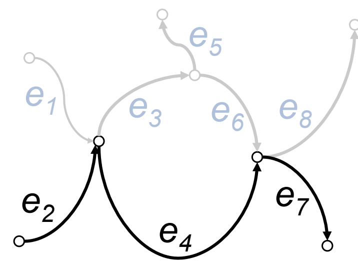

(a)




(b)




(c)




(d)

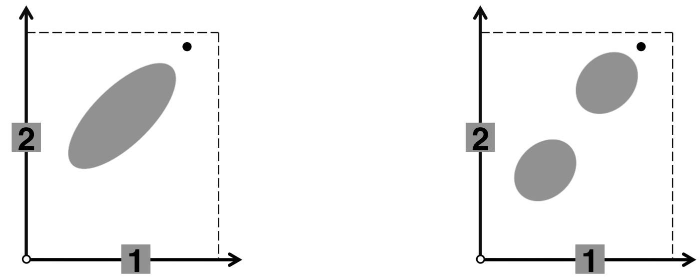
Figure 6. The four paths (8), in that order, for the problem in Figure 5.

The paths $p_1, \ldots, p_4$ are shown, in that order, in Figure 6. We obtain the control modes $\mu_1, \ldots, \mu_4$, defined as follows:

| $\alpha$ | $\mu_1(\alpha)$ | $\mu_2(\alpha)$ | $\mu_3(\alpha)$ | $\mu_4(\alpha)$ |
|---|---|---|---|---|
| 1 | $p_1$ | $p_1$ | $p_2$ | $p_2$ |
| 2 | $p_3$ | $p_4$ | $p_3$ | $p_4$ |

The state space corresponding to each control mode $\mu$ is a rectangle consisting of those state vectors $\left(x^1_\mu, x^2_\mu\right)$ that are compliant with the arc length bounds and separation constraints.

The above system is subject to the operational requirement (3), here for $\alpha \in \mathcal{A} = \{1, 2\}$, that agents 1 and 2 arrive at their destinations at times $t^{DEST;1}, t^{DEST;2}$, respectively.




(a) (b)

Figure 7. Topology of approximated conflict zone in the state spaces (a) $X_{\mu_1}, X_{\mu_4}$ and (b) $X_{\mu_2}, X_{\mu_3}$. The black dot (near top right) denotes the required destination coordinate pair $\left(x_\mu^{DEST;1}, x_\mu^{DEST;2}\right)$.

In each of $\mu_1, \mu_4$, the pairwise conflict zone is *simply connected* [23] (i.e., consists of only one connected component and has no "holes"). Consequently, once each of the conflict zones, of the form shown in Figure 4c, is approximated by an ellipse-bounded region, the state spaces for $\mu_1, \mu_4$ have the topology shown in Figure 7a. In each of $\mu_2, \mu_3$, the two agents' paths have two crossings (regarded here as short overlaps), hence the pairwise conflict zone has two connected components (Figure 7b).

Each control mode $\mu$ is subject to the initial condition

$$x_\mu^\alpha(0) = x^{INIT;\alpha} \text{ for } \alpha \in \mathcal{A} \tag{9}$$

The simplifying assumption underlying (9) is that both agents start their journey simultaneously. This assumption can be relaxed and is used here for mathematical simplicity only.

Finally, for each control mode $\mu$, we can specify a cost functional that suits the goals implied by the context of the specific application. One example is a cost functional that requires the agents (e.g., trains) to move as slowly as possible, for safety, is

$$\sum_{\alpha \in \mathcal{A}} \int_0^{t^{DEST;\alpha}} \left(s_\mu^\alpha\right)^2 dt \tag{10}$$

Thus, for each control mode $\mu$, we have an optimal control problem with dynamical law (2), subject to the initial condition (9), the control constraints that specify permissible value ranges for the speeds $s_\mu^\alpha$, and the following additional constraints:

- The *arrival requirement* (3).
- The *separation requirement*, defined by constructing a function $g_\mu^{SEP;\alpha_1,\alpha_2}((x_\mu^\alpha)_{\alpha\in\mathcal{A}})$ of the state such that the pairwise conflict

zone (shown as a gray-shaded region in the appropriate panel of Figure 7) is exactly the set of states satisfying the functions

$$g_{\mu}^{SEP;\alpha_1,\alpha_2} \geq r^2_{\alpha_1,\alpha_2}, \tag{11}$$

where each $g^{SEP;\alpha_1,\alpha_2}(x^{\alpha_1}, x^{\alpha_2})$ is a quadratic function of the form given by the left-hand side of (7).

For the pair $(\alpha_1 = 1, \alpha_2 = 2)$ (and, in the general scheduled routing problem, for *every* pair $(\alpha_1, \alpha_2)$) of agents, inequality (11) defines a union (denoted $X_{\mu}^{\alpha_1,\alpha_2}$) of regions in the state space of $\mu$, each region bounded by an ellipse-cylindrical ("tube-shaped") hypersurface. The separation requirement thus translates into the requirement that a solution (the state trajectory) is disjoint from the interior of every such region for every pair of agents.

## 4.2 The Scheduled Routing Problem: a general formulation

The central problem of this paper, which will be called the *Scheduled Routing Problem*, can now be stated as follows.

**Definition 4.1.** (Scheduled Routing Problem) *Given a set $\mathcal{A}$ of agents moving on a route network $G = (V, E)$ subject, in each control mode $\mu$, to the dynamical law (2), the initial condition (9), and the control constraints (4), find a control strategy $(s^{\mu}_{\alpha}(\tau))_{\alpha}$ such that the corresponding state trajectory $\mathbf{x}_{\mu}(t) = (x^{\alpha}_{\mu}(t))_{\alpha\in\mathcal{A}}$ that satisfies (3) and minimizes the total cost.*

This is an optimal control problem with a Lagrange cost function. The problem, however, has two non-standard features that hamper application of classical optimal control theory and numerical computation of solutions. One feature is the presence of intermediate constraints. The other is the following non-autonomous behavior: an agent, once at destination, no longer "participates" in the constraints or in the cost. We now use a formalism similar to that in [24] to reduce this problem to a classical, optimal control problem, which is autonomous if its cost functional is. In the rest of this section, the subscript $\mu$ is dropped for brevity.

## 4.3 The Stacked Scheduled Routing Problem (SSRP), equivalent to the Scheduled Routing Problem

Let $(\alpha_q)_{q=1}^{A}$ be an ordering of the agents by their arrival times (arranged in nondecreasing order). Let $t_0 = 0$, and for each $q > 0$ let

$$t_q = t^{DEST;\alpha_q}$$

Define a new (normalized) time $\tau \in [0, 1]$ and, for each of the intervals $[t_q, t_{q+1}]$, $q = 0, \ldots, A-1$, introduce the following:

- the state variable $\rho_q(\tau)$, which will play the role of "physical time" in the interval $[t_q, t_{q+1}]$:

$$t_q \leq \rho_q(\tau) \leq t_{q+1};$$

- the state variables $y_q^\alpha(\tau), \alpha \in \mathcal{A}$, related to the above $x^\alpha(t)$ by

$$y_q^\alpha(\tau) = x^\alpha(t) \text{ if } \rho_q(\tau) = t; \tag{12}$$

- the vector notation $\mathbf{y}_q(\tau) = (y_q^\alpha(\tau))_{\alpha\in\mathcal{A}}$;

- the control variables $s_q^\alpha(\tau)$, related to the above $s^\alpha(t)$ by

$$s_q^\alpha(\tau) = s^\alpha(t) \text{ if } \rho_q(\tau) = t, \tag{13}$$

- the control variables $z_q > 0$, which represent the rate of flow of physical time with respect to the normalized time $\tau$.

The above definition of the Lagrange cost reflects the assumption that an agent, once at destination, "disappears" from the system, in the sense of being no longer subject to the separation requirement with the other agents. From (2), (12), and (13), one readily obtains the dynamical law equations

$$\left.\begin{array}{rcl} \frac{\mathrm{d}}{\mathrm{d}\tau} y_q^{\alpha_{q'}} & = & z_q s_q^{\alpha_{q'}} \text{ for } q' \geq q+1 \\[2ex] \frac{\mathrm{d}}{\mathrm{d}\tau} \rho_q & = & z_q \end{array}\right\} \quad \alpha \in \mathcal{A} \tag{14}$$

Finally, the initial conditions and arrival requirement, together with the requirement that physical time and state change continuously when passing from one interval $[t_q, t_{q+1}]$ to the next, translate to the endpoint constraints

$$\left.\begin{array}{rcll} y_0^\alpha(0) & = & x^{INIT;\alpha} \text{ (see (9))} & \text{(a)} \\[1.5ex] \rho_q(1) & = & \rho_{q+1}(0) & \text{(b)} \\[1.5ex] \rho_{q+1}(0) & = & t_{q+1} & \text{(c)} \\[1.5ex] y_q^{\alpha_{q'}}(1) & = & y_{q+1}^{\alpha_{q'}}(0) \text{ for } q' \geq q+1 & \text{(d)} \\[1.5ex] y_{q+1}^{\alpha_{q+1}}(1) & = & x^{DEST;\alpha_{q+1}} & \text{(e)} \end{array}\right\} \; 0 \leq q < A \tag{15}$$

Conditions (15.bc) ensure the continuity of physical time flow; conditions (15.d), the continuity of an agent's motion.

**Remark 4.1.** *The state equations (14) and the endpoint constraints (15.d) come with the restriction* $q' \geq q+1$ *because they are imposed, in agreement with (2), only for those agents who have not yet reached their destination: by the end of the* $q$*-th time period, the first* $q$ *agents have reached destination. Thus, upon reaching destination, an agent is excluded from the model, and is no longer represented by a state equation or subject to separation constraints with the other agents. This eliminates the necessity to continue modeling an agent whose role in the model has already been fulfilled.*

Throughout the rest of this paper, the choice (10) of the cost functional is assumed in all the numerical examples. Other choices of the cost functional are discussed in section 6. Since the time intervals between each consecutive pair of arrivals are modeled, in this latter formulation, as if they were occurring simultaneously ("stacked" upon one another), the following definition will be adopted.

**Definition 4.2.** (a) *The newly obtained optimal control problem corresponding to a control mode* $\mu$ *and consisting of the state equations (14), endpoint constraints (15), cost functional (10), the separation constraints on the variables* $y_q^{\alpha_{q'}}$*, the control constraints corresponding to (4) specifying the speed ranges on the variables* $s_q^{\alpha_{q'}}$*, and the positivity constraints*

$$z_q > 0,$$

*will be called a* $\mu$-stacked optimal control problem*.*

(b) *The set of all* $\mu$*-stacked optimal control problems will be called a* Stacked Scheduled Routing Problem (SSRP)*.*

(c) *Of all the optimal solutions to all the* $\mu$*-stacked optimal control problems, a solution achieving a lowest cost is called an* optimal solution to the SSRP*.*

Thus, an SSRP consists of a collection of optimal control problems, and an optimal solution to the SSRP tells not only how quickly the agents are to move, but also how they should be routed. The SSRP is equivalent to the Scheduled Routing Problem (definition 4.1).

### 4.4 Implications of the Pontryagin Maximum Principle for the SSRP

The assumption that we are in a specific control mode $\mu$ is still in force. Denote by $\xi_{y_q}, \xi_{\rho_q}$ the costates for $y_q, \rho_q$. In those states where none of the state constraints is active, the Hamiltonian for each control mode of the SSRP is

$$H = -f_0 + \sum_{q;q' \geq q+1} z_q s_q^{\alpha_{q'}} \xi_{y_q} + \sum_q z_q \xi_{\rho_q},$$

where $f_0$ is the performance index (running cost) corresponding to the cost functional (10). Since $H$ does not explicitly depend on any of the state variables, it follows that the costate variables are constant along the trajectory portions clear of state constraints and, consequently, the maximization of $H$ in each such state is a problem of static maximization. This fact, in turn, implies the existence of an optimal state trajectory in which these portions are rectilinear on each $[t_q, t_{q+1}]$. If, furthermore, there is only one optimal trajectory, then its portions away from the obstacle boundary are necessarily rectilinear.

With the state equations (2) and the set (4) of control constraints, the set of all states reachable from a given state $\mathbf{y}^0_\mu = (y^{0;\alpha}_\mu)_{\alpha\in\mathcal{A}}$ is the pointed polyhedral cone that consists of all states $\mathbf{y}_\mu = (y^{\alpha}_\mu)_{\alpha\in\mathcal{A}}$ satisfying

$$\left.\begin{array}{rcl} 0 & \leq & \left(y^{\alpha_1}_\mu - y^{0;\alpha_1}_\mu\right) \frac{s^{MIN;\alpha_1}_\mu}{s^{MAX;\alpha_2}_\mu} \\ & \leq & y^{\alpha_2}_\mu - y^{0;\alpha_2}_\mu \\ & \leq & \left(y^{\alpha_1}_\mu - y^{0;\alpha_1}_\mu\right) \frac{s^{MAX;\alpha_1}_\mu}{s^{MIN;\alpha_2}_\mu} \end{array}\right\} \quad \alpha_1 \neq \alpha_2$$

This cone has vertex $\mathbf{y}^0_\mu$ and is the intersection of the half-spaces

$$y^{\alpha_2}_\mu - y^{0;\alpha_2}_\mu \leq \frac{s^{MAX;\alpha_1}_\mu}{s^{MIN;\alpha_2}_\mu}\left(y^{\alpha_1}_\mu - y^{0;\alpha_1}_\mu\right), \quad \alpha_1 \neq \alpha_2,$$

$$y^{\alpha_2}_\mu - y^{0;\alpha_2}_\mu \geq \frac{s^{MIN;\alpha_1}_\mu}{s^{MAX;\alpha_2}_\mu}\left(y^{\alpha_1}_\mu - y^{0;\alpha_1}_\mu\right), \quad \alpha_1 \neq \alpha_2,$$

and

$$y^{\alpha}_\mu \geq y^{0;\alpha}_\mu, \quad \alpha \in \mathcal{A}$$

The narrower the speed ranges (4), the narrower the cone, and the smaller the portions of an optimal trajectory that lie on the boundary of the obstacle

$$\cup_{\alpha_1\neq\alpha_2} X^{\alpha_1,\alpha_2}_\mu$$

This suggests that, for narrow speed ranges, optimal trajectories for the Stacked Scheduled Routing Problem can be well approximated by piecewise linear curves.

### 4.5 An approach to computing solutions and an analysis of the associated costs

The number of the control modes $\mu$ is the product

$$\prod_{\alpha\in\mathcal{A}}\left|\mathcal{P}\left(e^{INIT;\alpha}, e^{DEST;\alpha}\right)\right|$$

Paths which are, or contain, cycles are allowed in the model and can be desirable in some applications; e.g., in air traffic models where aircraft may be sent into a holding pattern to absorb delay. Furthermore, the following holds true:

**Remark 4.2.** *Every solution to the control problem described above will correspond to exactly one of the control modes $\mu$. This property of the problem removes two difficulties associated inherently with hybrid systems and absent from classical control systems, the risk of excessively frequent switchings of control mode, and the necessity for "control mode memory," i.e. for keeping track of the control modes entered prior to the current time in the system's evolution.*

Each set $\mathcal{P}\left(e^{INIT;\alpha}, e^{DEST;\alpha}\right)$ can be computed using the Depth-First Search algorithm, whose computational cost in a non-parallel implementation is known [15, section 23.3] to be $O(|V| + |E|)$.

These considerations suggest the following procedure for finding an optimal solution to an SSRP. Upper bounds on the computational cost are provided, in square brackets ([]), where possible.

- Compute all the sets $\mathcal{P}\left(e^{INIT;\alpha}, e^{DEST;\alpha}\right)$. [In a non-parallel implementation, this computation amounts to a Depth-First Search for each agent $\alpha \in \mathcal{A}$, hence the computational cost is $O(|\mathcal{A}|(|V| + |E|))$.]

- For each $\mu$, compute an optimal solution to the corresponding Stacked Scheduled Routing Problem, and the cost $C_\mu$ of that solution. [The computational cost depends on the particular choice of the computational method; see, for example, [18, 25]. The numerical results in this paper were produced using Sequential Quadratic Programming (SQP) [18], chosen for the convenience of available software.]

- Select a control mode $\mu^*$ such that $C_{\mu^*} \leq C_\mu$ for all $\mu$, and declare the corresponding optimal solution the optimal solution for the SSRP (definition 4.2). [$O$(the number of control modes).]

# 5 Sample numerical computations for the Stacked Scheduled Routing Problem

## 5.1 Assumptions and notational conventions

All the transportation networks appearing in the numerical examples of this section are graphs; namely, no two vertices are connected by more than one edge. This allows to specify each path as a sequence of vertices, rather than of edges. To simplify the computations, the separation requirements for each pair of agents are assumed symmetric. The numerical code admits a straightforward, albeit somewhat cumbersome algebraically, generalization that will dispense with this assumption.

In panels (a, b) of Figures 9, 11, 12-17, the computed trajectories and controls are plotted using the symbols described in Table 1. In panels

$$\left.\begin{array}{ll} \textbf{Quantities} & \textbf{Plot symbol} \\ y_q^1, s_q^1 & \text{—} \\ y_q^2, s_q^2 & + \\ y_q^3, s_q^3 & -- \end{array}\right\} \text{for } q = 1, \ldots, A$$

Table 1. The legend used in Figures 9, 11, 12-16, below.

(c) (and, when present, (d)) of these Figures, as well as in Figures 8 and 10, the transportation networks are depicted as directed graphs; moving agents, as points labeled with values of $\alpha$, each point serving as a center of a circle with radius equal to half the required pairwise separation. In all plots, axis label $\rho$ refers, in agreement with the above, to physical time. All plots were generated using the MATLAB software [26].

The first two of the examples in section 5.4 are "abstract," in the sense that no particular application is specified for them. Thus, the units of length and physical time are left unspecified. Application and units are, however, specified for the third example. The required destination $y^{DEST;\alpha}$ is in each case the end of the path assigned to agent $\alpha$ in control mode $\mu$:

$$y^{DEST;\alpha} = 0$$

### 5.2 The cost function

The cost function used in the following examples is (10), which, on converting the scheduled routing problem to a Stacked Scheduled Routing Problem (definition 4.2), takes the form

$$\sum_{q=0}^{A} \sum_{q' \geq q+1} \int_0^1 \left( s_q^{\alpha_{q'}} \right)^2 \rho_q d\tau$$

### 5.3 Computational methods

In each control mode, the corresponding optimal control problem was solved using the OCP solver [20], which uses Sequential Quadratic Programming (SQP) [18].

## 5.4 Numerical examples

### 5.4.1 Three moving agents, one control mode

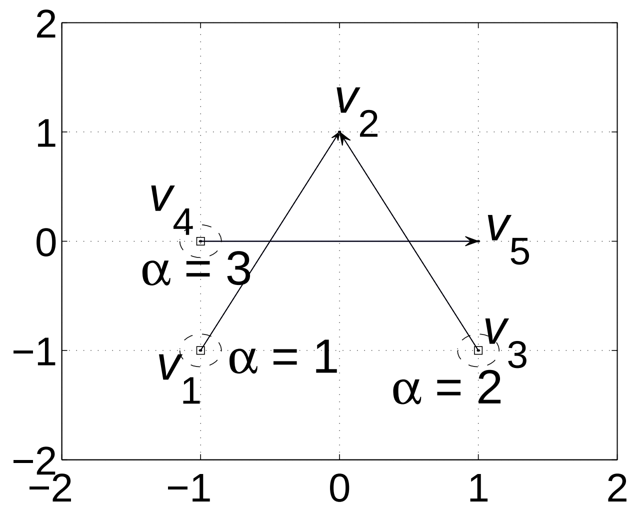



Figure 8. The transportation network and the agents' initial locations in the example of section 5.4.1.

The transportation network for this example is the 2-dimensional directed graph shown, together with the three agents' initial locations, in Figure 8. Each agent can traverse only the edge on which it is positioned initially, hence only one control mode $\mu$ arises.

The speed ranges are given by

| $\alpha$ | 1 | 2 | 3 |
|---|---|---|---|
| $s_\mu^{MIN;\alpha}$ | 0.3 | 0.3 | 0.4 |
| $s_\mu^{MAX;\alpha}$ | 1.5 | 0.8 | 0.9 |

The required times of arrival at destination are

| $\alpha$ | 1 | 2 | 3 |
|---|---|---|---|
| $t^{DEST;\alpha}$ | 2.0 | 3.0 | 4.0 |

The minimal required separation is 0.3.

The numerical solution computed for the control mode $\mu$ is shown in Figure 9.

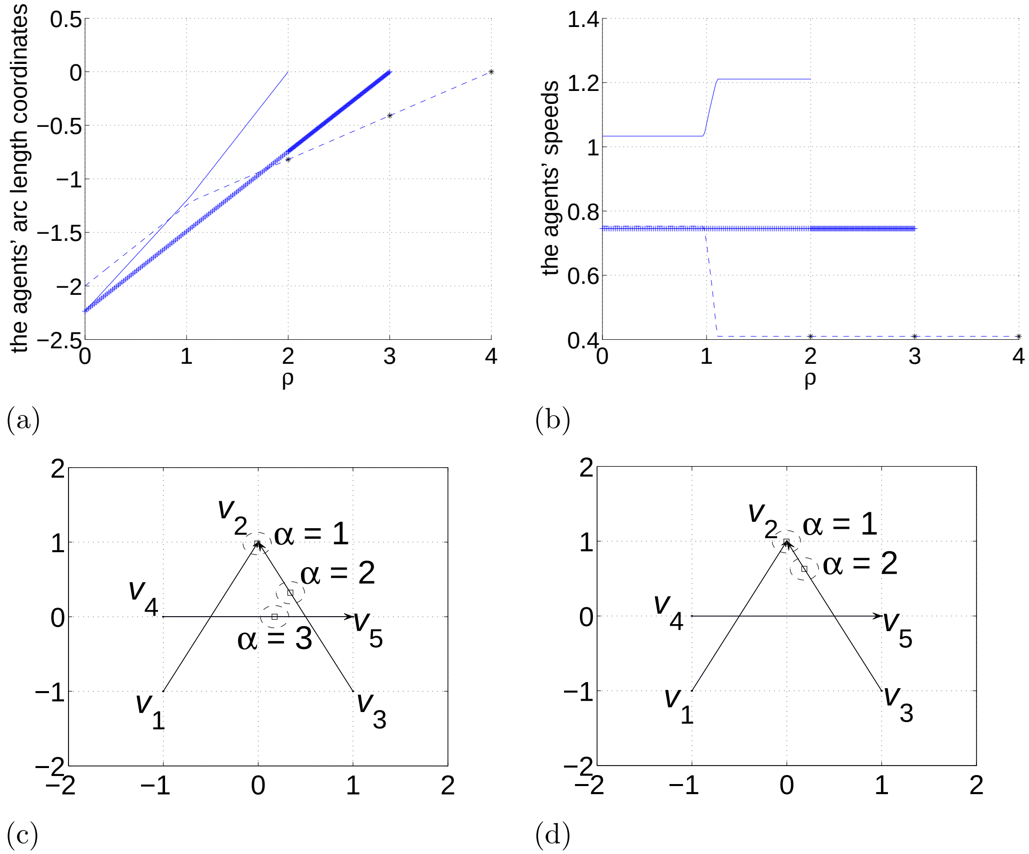



Figure 9. Numerical solutions (for the only possible control mode) in the example of section 5.4.1. (a) State trajectory vs. time. (b) Control strategy vs. time. (c) The positions of the agents in the transportation network at $\rho_q(1)$ for $q = 1$. (d) The positions of the agents in the transportation network at $\rho_q(1)$ for $q = 2$.

### 5.4.2 Two moving agents, four control modes, wide speed ranges

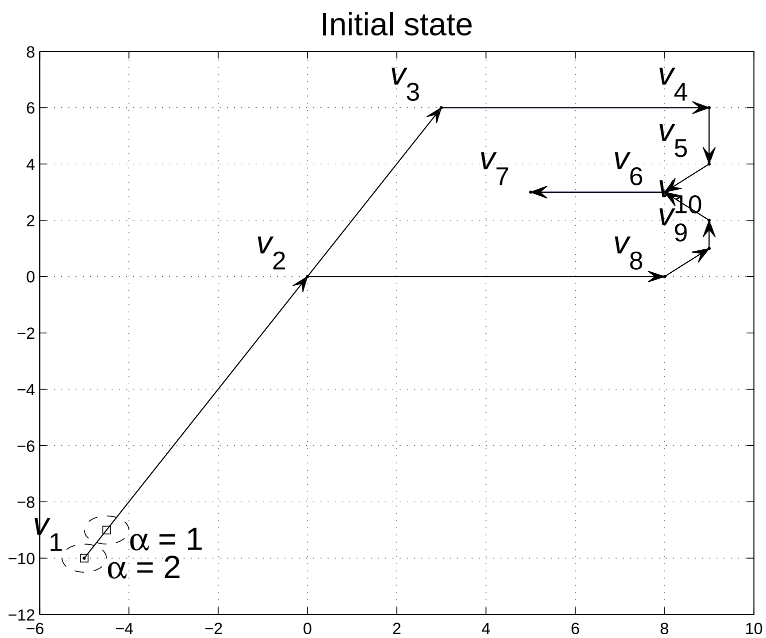



Figure 10. The transportation network and the agents' initial locations in the example of section 5.4.2.

The transportation network for this example is the 2-dimensional directed graph shown, together with the two agents' initial locations, in Figure 10.

The two paths considered here are

$$p_1 : v_1, v_2, v_8, v_9, v_{10}, v_6; \quad p_2 : v_1, v_2, v_3, v_4, v_5, v_6$$

The speed ranges are given by

| $\alpha$ | 1 | 2 |
|---|---|---|
| $s_\mu^{MIN;\alpha}$ | 0.6 | 0.6 |
| $s_\mu^{MAX;\alpha}$ | 1.4 | 1.4 |

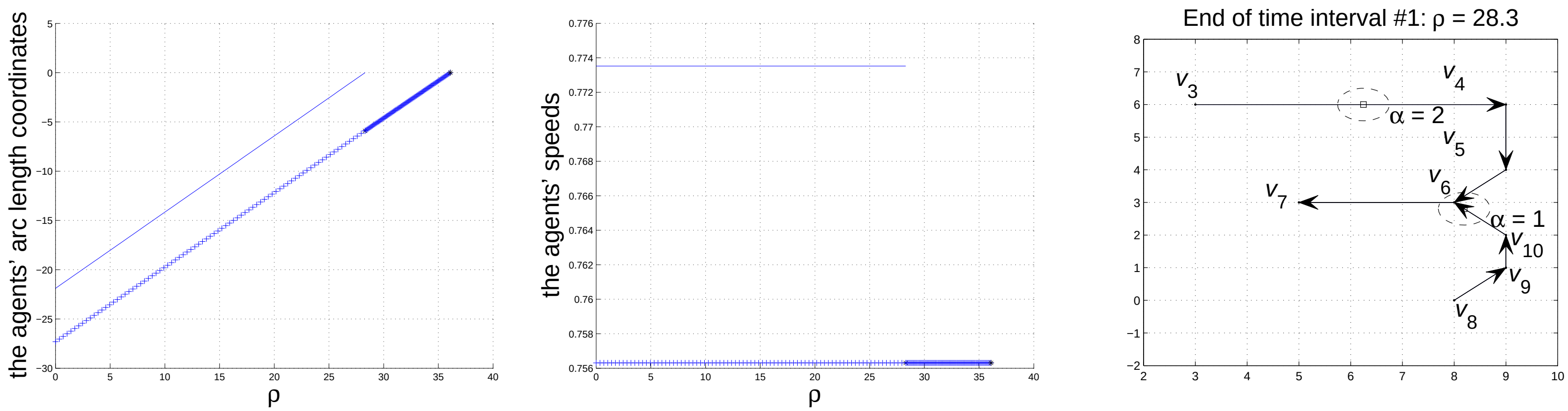
The required times of arrival at destination are

| $\alpha$ | 1 | 2 |
|---|---|---|
| $t^{DEST;\alpha}$ | 28.3 | 36.1 |

The minimal required separation is 1.0.

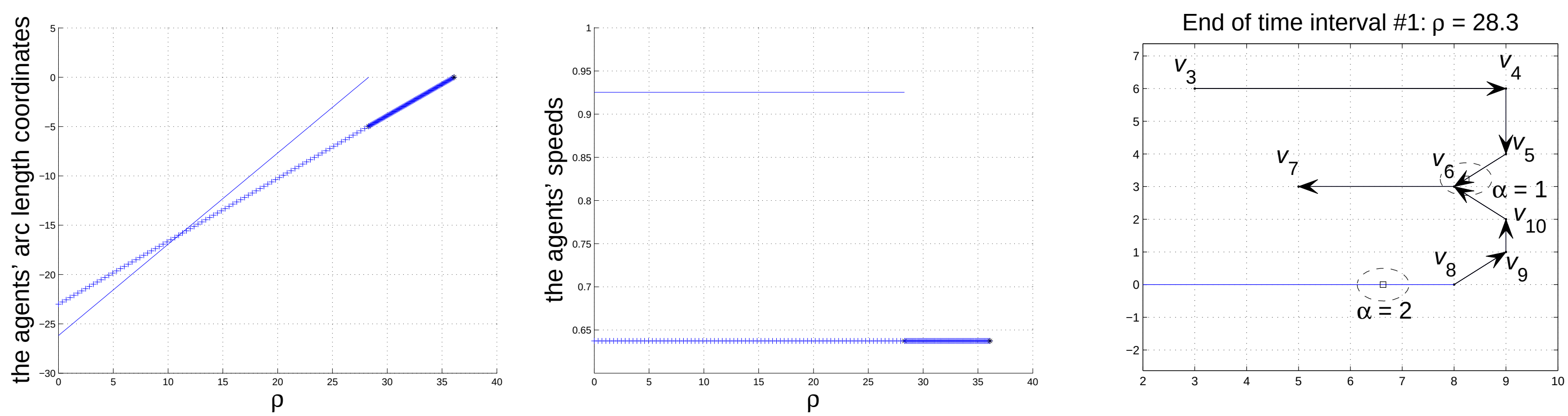
The control modes are as follows.

| $\alpha$ | $\mu_1(\alpha)$ | $\mu_2(\alpha)$ | $\mu_3(\alpha)$ | $\mu_4(\alpha)$ |
|---|---|---|---|---|
| 1 | $p_1$ | $p_2$ | $p_2$ | $p_1$ |
| 2 | $p_2$ | $p_1$ | $p_2$ | $p_1$ |



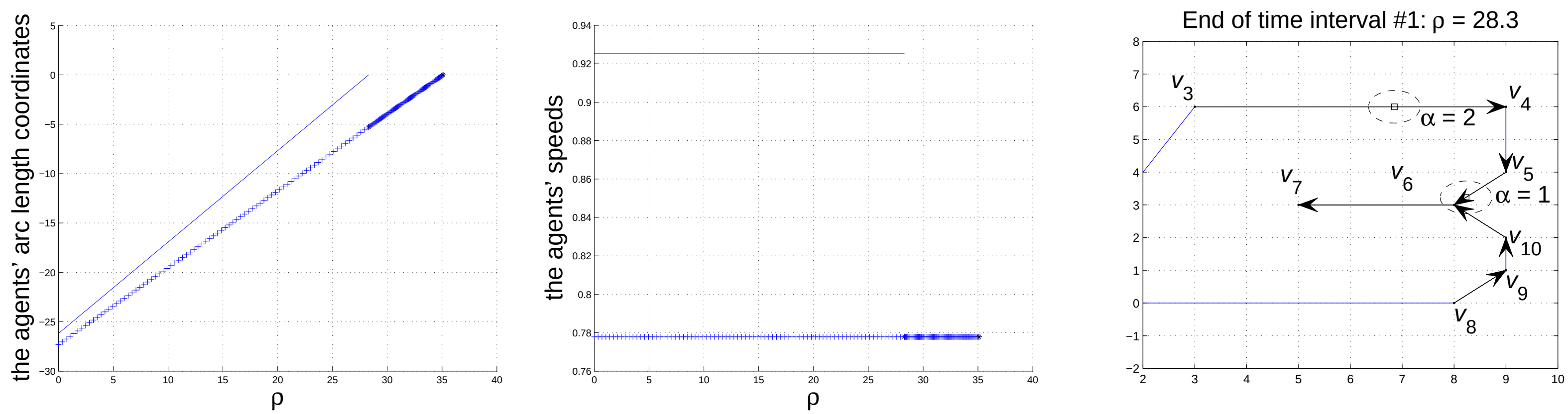

(a) (b) (c)

Figure 11. Numerical solutions for control mode $\mu_1$ in the example of section 5.4.2. (a) State trajectory vs. time. (b) Control strategy vs. time. (c) The positions of the agents in the transportation network at $\rho_q(1)$ for $q = 1$.



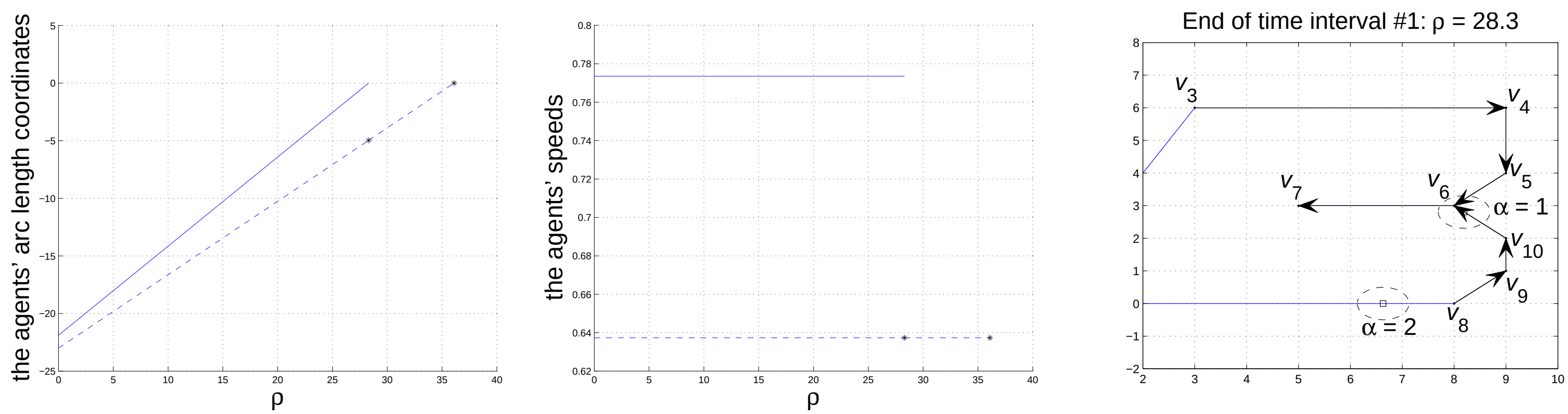

(a) (b) (c)

Figure 12. Numerical solutions for control mode $\mu_2$ in the example of section 5.4.2. (a) State trajectory vs. time. (b) Control strategy vs. time. (c) The positions of the agents in the transportation network at $\rho_q(1)$ for $q = 1$.




(a) (b) (c)

Figure 13. Numerical solutions for control mode $\mu_3$ in the example of section 5.4.2. (a) State trajectory vs. time. (b) Control strategy vs. time. (c) The positions of the agents in the transportation network at $\rho_q(1)$ for $q = 1$.




(a) (b) (c)

Figure 14. Numerical solutions for control mode $\mu_4$ in the example of section 5.4.2. (a) State trajectory vs. time. (b) Control strategy vs. time. (c) The positions of the agents in the transportation network at $\rho_q(1)$ for $q = 1$.

The respective numerical solutions for these modes are shown in Figures 11-14.

The total costs for each control mode are as follows:

| control mode | $\mu_1$ | $\mu_2$ | $\mu_3$ | $\mu_4$ |
|---|---|---|---|---|
| total cost | 37.5823 | 38.8925 | 45.4652 | 31.5979 |

Therefore, the optimal routing and control strategy are achieved in control mode $\mu_4$.

### 5.4.3 Two moving agents, four control modes, narrow speed ranges

In this example, the transportation network, 3-dimensional, models a terminal airspace, which consists of two arrival paths merging into the same final approach segment. The transportation network and the initial positions of the two aircraft are shown in Figure 15.

Here the edge $(v_6, v_7)$ is the final approach to a runway, and the moving agents are aircraft. For realism, one unit of arc length is taken here to be 3 nautical miles, and the unit of speed is taken to be 200 knots, a typical speed allowed in U.S. terminal airspaces at altitudes below 10,000 feet.

The two paths considered here are

$$p_1 : v_1, v_2, v_8, v_9, v_{10}, v_6; \quad p_2 : v_1, v_2, v_3, v_4, v_5, v_6$$

The control modes are as follows.

| $\alpha$ | $\mu_1(\alpha)$ | $\mu_2(\alpha)$ | $\mu_3(\alpha)$ | $\mu_4(\alpha)$ |
|---|---|---|---|---|
| 1 | $p_1$ | $p_2$ | $p_2$ | $p_1$ |
| 2 | $p_2$ | $p_1$ | $p_2$ | $p_1$ |

The speed ranges are given by

| $\alpha$ | 1 | 2 |
|---|---|---|
| $s_\mu^{MIN;\alpha}$ | 0.8 (= 160 kts) | 0.8 |
| $s_\mu^{MAX;\alpha}$ | 1.2 (= 240 kts) | 1.2 |

The required times of arrival at destination are

| $\alpha$ | 1 | 2 |
|---|---|---|
| $t^{DEST;\alpha}$ | 21.2 (= 0.03 hrs) | 28.1 (= 0.04 hrs) |

The minimal required separation is 1.0 (= 3 nmi).

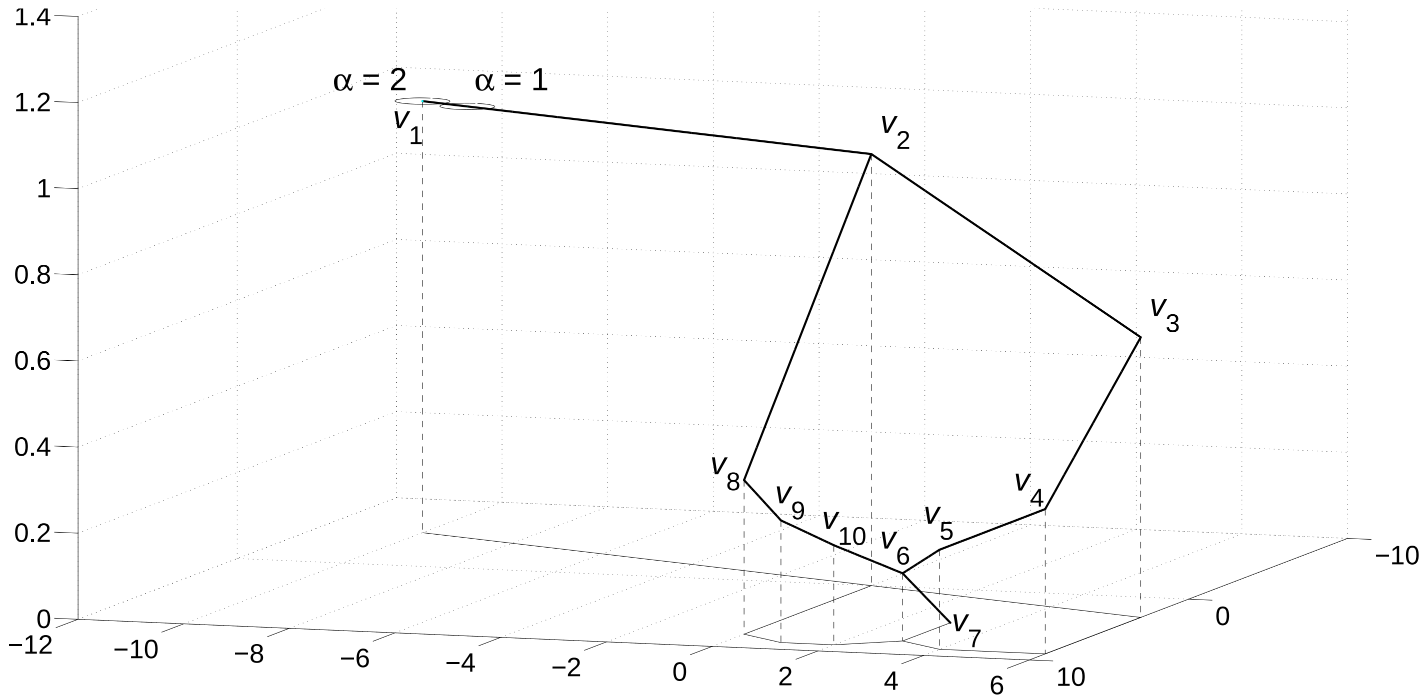
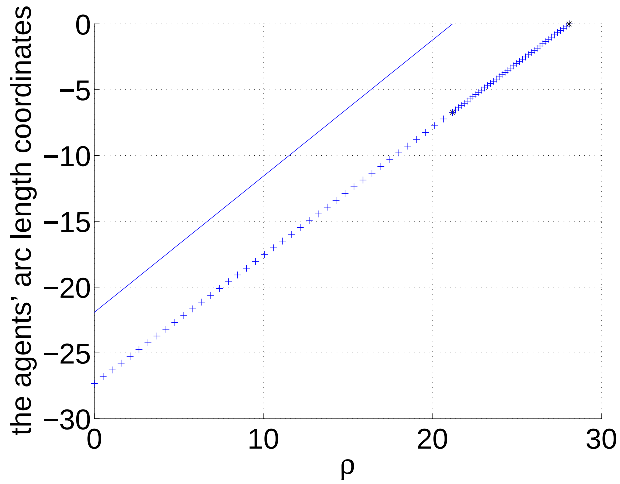
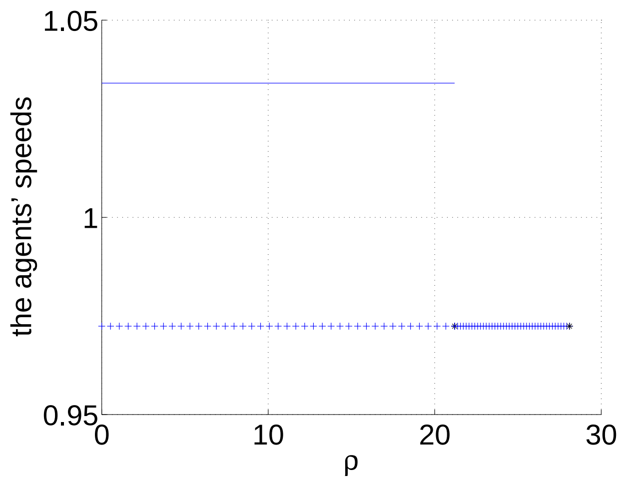


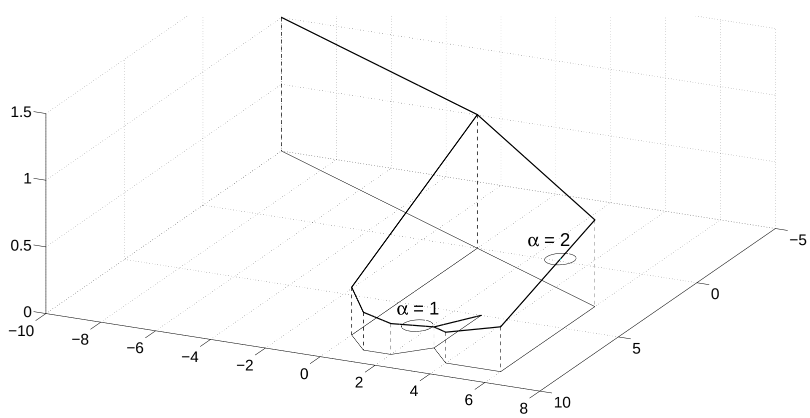

Figure 15. The 3-dimensional route network and the moving agents' initial positions (centers of the circles in horizontal planes), for the numerical example of section 5.4.3. Here the route network is a terminal airspace, and the agents are arriving aircraft. The radius of each circle is half the minimal required pairwise separation of 3 nmi.




(a)

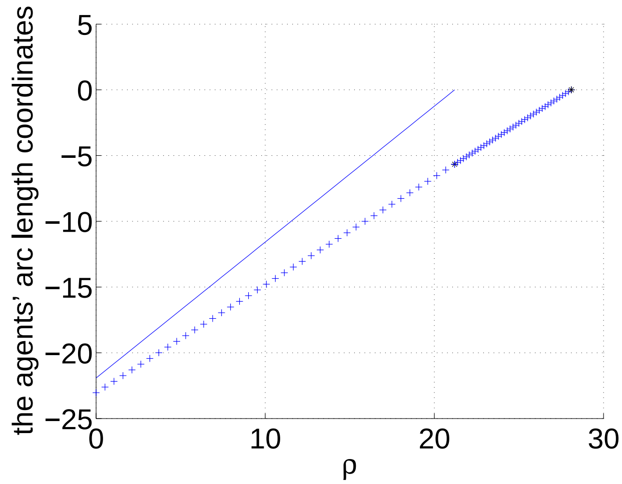



(b)

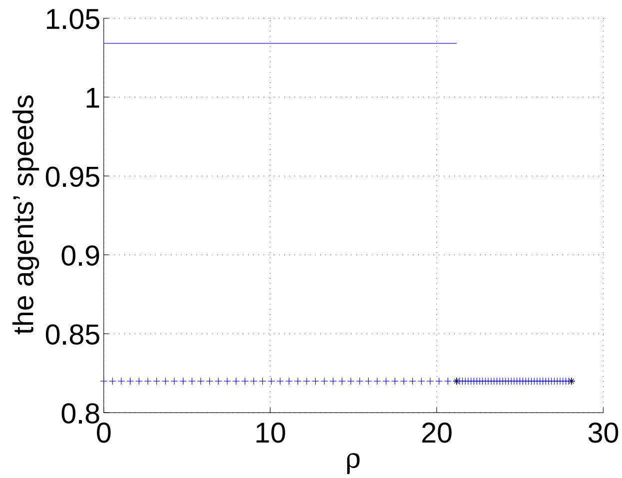



(c)

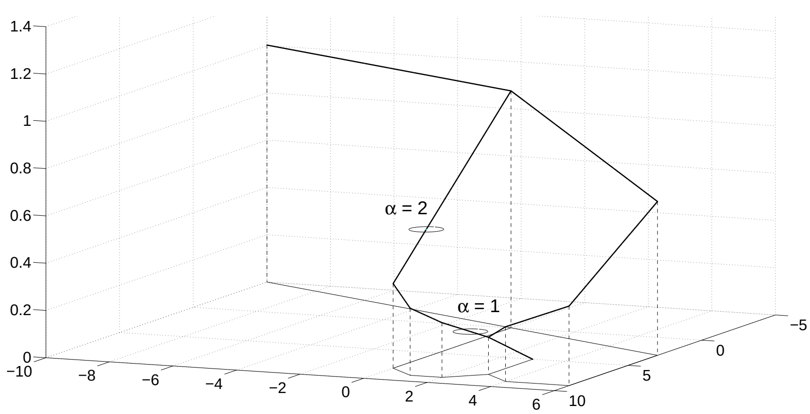
Figure 16. Numerical solutions for control mode $\mu_1$ in the example of section 5.4.3. (a) State trajectory vs. time. (b) Control strategy vs. time. (c) The positions of the agents in the transportation network at $\rho_q(1)$ for $q = 1$.




(a)




(b)




(c)

Figure 17. Numerical solutions for control mode $\mu_4$ in the example of section 5.4.3. (a) State trajectory vs. time. (b) Control strategy vs. time. (c) The positions of the agents in the transportation network at $\rho_q(1)$ for $q = 1$.

In control modes $\mu_2, \mu_3$, the control problem has no solution, since the path lengths are such that the imposed speed ranges prevent at least one agent from reaching its destination at time. The numerical solution for control mode $\mu_1$ is shown in Figure 16 and incurs a total cost of 49.2396. In $\mu_4$, the computed optimal control strategies prescribe the constant speeds

$$s^1_{\mu_4} = 1.03, \quad s^2_{\mu_4} = 0.82$$

and the solution (Figure 17) incurs a total cost of 41.5594. Therefore, the optimal routing and control strategy are achieved in control mode $\mu_4$.

# 6 Discussion

The above modeling framework addresses the problem of navigating a set of moving agents in a transportation network, with constraints on the agents' initial locations, required destinations, and required times of arrival at destination, as well as on the agents' minimal pairwise distances. We now discuss several directions in which the above model can be varied and generalized.

## 6.1 A model that includes inertia

Inertia can be included by treating both the $y^\alpha_\mu$'s and the $s^\alpha_\mu$'s as state variables, and the accelerations $a^\alpha_\mu$ as the control variables. The corresponding new state equations would assume the form

$$\left.\begin{array}{l} \dot{y}^\alpha_\mu = s^\alpha_\mu \\ \dot{s}^\alpha_\mu = a^\alpha_\mu \end{array}\right\}, \quad \alpha \in \mathcal{A}$$

The rescaling of physical time to normalized time $\tau \in [0, 1]$ formulated in section 4.2 would be carried out analogously. The resulting problem falls in the class of the *kinodynamic motion planning* problems; some related theoretical results can be found in [5] and in references therein.

## 6.2 Different choices of the cost function

Of the vast number of possible choices of cost function, we briefly discuss two.

- In situations where inertia cannot be neglected and acceleration must be the control variable (or among the control variables), it may be desirable to keep the movement as smooth as possible, e.g., for passenger comfort or cargo safety. A cost function that would serve this goal is the integral of the sum of the squared accelerations:

$$\sum_{\alpha \in \mathcal{A}} \int_0^{t^{DEST;\alpha}} \left(a^\alpha_\mu\right)^2 dt$$

- In situations where it is undesirable to impose times $t^{DEST;\alpha}$ of arrival at destination as rigid constraints, and preferable to minimize the absolute differences between the actual arrival times $t^{AAT;\alpha}$. This goal would be served by taking the sum

$$\sum_{\alpha \in \mathcal{A}} \left( t^{DEST;\alpha} - t^{AAT;\alpha} \right)^2$$

  as the cost.

### 6.3 Asymmetric and anisotropic pairwise separation requirements

In some applications, the pairwise separation requirements for the moving agents can be asymmetric or anisotropic, or both. Such is the case when the moving agents in question are aircraft traveling along their nominal routes. An example of asymmetry is as follows. If two aircraft are consecutively in-trail and at the same altitude, and the leader's and follower's respective weight classes [22] are Heavy and Small, the required separation is considerably larger than if the two engine types were in the opposite order. An example of anisotropy is the requirement of vertical separation between two aircraft, where aircraft are required to maintain either 1000 ft vertical separation or the prescribed lateral separation previously discussed; the resultant shape of an aircraft's safety envelope is a cylinder. A mathematical form for such an anisotropic constraint would use, not the Euclidean norm, but one of the following form: putting $\mathbf{a}_k = (a_k^x, a_k^y, a_k^z)$, the norm in the left-hand side of (6) would be replaced by a "mixed" norm, Euclidean in the $xy$-plane, and the max-norm along the height $z$. If $r$ were the minimal horizontal separation, imposed when the two aircrafts' altitudes differ by less than a quantity $h$, the mixed norm and the corresponding separation constraint would be

$$\begin{aligned}
&||c_1\mathbf{a}_1 - c_2\mathbf{a}_2||_{\text{mixed}} \\
&:= \max\left\{ \tfrac{1}{r}\sqrt{(c_1 a_1^x - c_2 a_2^x)^2 + (c_1 a_1^y - c_2 a_2^y)^2},\ \tfrac{1}{h}\left|c_1 a_1^z - c_2 a_2^z\right| \right\} \\
&\geq 1
\end{aligned}$$

The computations in section 3 would have to be modified accordingly and would no longer have the closed quadratic form.

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**14. ABSTRACT**
We address the problem of navigating a set of moving agents, e.g. automated guided vehicles, through a transportation network so as to bring each agent to its destination at a specified time. Each pair of agents is required to be separated by a minimal distance, generally agent-dependent, at all times. The speed range, initial position, required destination, and required time of arrival at destination for each agent are assumed provided. The movement of each agent is governed by a controlled differential equation (state equation). The problem consists in choosing for each agent a path and a control strategy so as to meet the constraints and reach the destination at the required time. This problem arises in various fields of transportation, including Air Traffic Management and train coordination, and in robotics. The main contribution of the paper is a model that allows to recast this problem as a decoupled collection of problems in classical optimal control and is easily generalized to the case when inertia cannot be neglected. Some qualitative insight into solution behavior is obtained using the Pontryagin Maximum Principle. Sample numerical solutions are computed using a numerical optimal control solver.

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