NASA/TM–2011-217307

Discretely Conservative Finite-Difference Formulations for Nonlinear Conservation Laws in Split Form: Theory and Boundary Conditions

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November 2011

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November 2011

Acknowledgments

Prof. Nordström contributed to this work while in residence at NASA Langley Research Center, Technical monitor Joseph Morrison. The work of Prof. Yamaleev has been supported in part by NASA under Grant NNX09AV08A and the Army Research Laboratory under Grant W911NF-06-R-006. Dr. Swanson is a Distinguished Research Associate (DRA) at Langley.

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Abstract

Simulations of nonlinear conservation laws that admit discontinuous solutions are typically restricted to discretizations of equations that are explicitly written in divergence form. This restriction is, however, unnecessary. Herein, linear combinations of divergence and product rule forms that have been discretized using diagonal-norm skew-symmetric summation-by-parts (SBP) operators, are shown to satisfy the sufficient conditions of the Lax-Wendroff theorem and thus are appropriate for simulations of discontinuous physical phenomena. Furthermore, special treatments are not required at the points that are near physical boundaries (i.e., discrete conservation is achieved throughout the entire computational domain, including the boundaries). Examples are presented of a fourth-order, SBP finite-difference operator with second-order boundary closures. Sixth- and eighth-order constructions are derived, and included in E. Narrow-stencil difference operators for linear viscous terms are also derived; these guarantee the conservative form of the combined operator.

Contents

1 Introduction

The nonlinear product terms in the continuous Euler equations can be expressed in many forms: 1) conservative, $\frac{\partial}{\partial x_j}(u_i u_j)$; 2) primitive, $u_j \frac{\partial u_i}{\partial x_j}$; 3) skew-symmetric, $\frac{1}{2}\frac{\partial}{\partial x_j}(u_i u_j) + \frac{1}{2}u_j\frac{\partial u_i}{\partial x_j}$; 4) rotational, $u_j(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}) - \frac{\partial}{\partial x_j}(\frac{1}{2}u_i u_j)$; as well as others. Although they are equivalent for smooth flows at the continuous level, each form can exhibit profoundly different discrete accuracy, conservation, and nonlinear stability properties. Growing evidence suggests that the nonconservative forms, particularly the skew-symmetric form, preserve important invariants of the continuous equations (e.g., kinetic energy or entropy) and deliver enhanced accuracy and robustness. This makes the use of split forms an attractive alternative to conventional approaches based on the conservation form of the equations. Evidence that supports this assertion is available in the following references [1–16].

Despite many favorable properties, alternative, nonconservative forms of the Euler equations have historically been viewed as inappropriate for simulations that involve discontinuities, such as supersonic and hypersonic flows. The basis of this misconception likely stems from the Lax-Wendroff theorem [17] which stipulates that the divergence form of the governing equations must be discretized with a conservative operator, a condition that alternative discrete forms apparently lack.[1]

While alternative forms are generally not explicitly conservative, many can be manipulated into an *equivalent, consistent, and conservative* form, thus allowing the Lax-Wendroff theorem to be used to guarantee that the convergent captured discontinuities are weak solutions of the governing equations. For example, Ducros et al. [9] demonstrated that the skew-symmetric form of the convective terms in the Euler equations are discretely conservative (at least for periodic fourth- and sixth-order centered operators), while Jameson [18] used the skew-symmetric form of the Burgers equation and developed discretely conservative second order operators that

[1]Note that Gerritsen and Olssen [6] and, later, Yee and Vinokur [10] showed that correct shock simulations could be achieved using the skew-symmetric form, even before a proof of discrete conservation was identified.

are capable of capturing shocks. Recently Pirozzoli [15, 16] showed that all periodic centered finite-difference operators can be expressed in a conservative form (even on curvilinear grids) if an appropriate splitting of the equations is used.

Several open issues must be resolved before high-fidelity simulations that use alternative forms of the equations become routine. One critical requirement is the need for high-order boundary closures and boundary conditions that maintain the desirable properties of the interior operators. Another is the need for a systematic methodology to extend alternative single-domain operators (e.g., a skew-symmetric discretization) to multiple domains, thereby extending the generality of any alternative approach.

A principle contribution of this work is the recognition that the boundary closures for alternative operators follow immediately if finite-domain summation-by-parts (SBP) discrete operators are used to derive near-wall boundary closures. For example, boundary closures that preserve the skew-symmetric form of the Euler equations up to and including the boundary conditions are achieved by appropriately splitting the equations and using SBP operators for all derivatives. We demonstrate this point by developing three sets of fully conservative, alternative finite-difference operators in this study: (2-4-2), (3-6-3) and (4-8-4).[2]

Another contribution of this work follows from addressing the broader question, "Which discretizations of general conservation laws are discretely conservative?" To this end, we conjecture and support with many examples that *any combination* of the conservative and primitive conservation law operators, $\alpha\mathcal{D}(vw) + (1-\alpha)(v\mathcal{D}w + w\mathcal{D}v)$, when discretized using *any*, diagonal-norm, skew-symmetric, SBP discretization operator, can be expressed in a conservative form after the appropriate discrete manipulations have been made. The generality of this conjecture allows for quite arbitrary splittings of the equations while maintaining discrete conservation.

A final contribution of this work is the recognition that all single-domain alternative operators satisfy all of the conditions that are necessary for extension to general, multiblock discretizations, provided that interface coupling terms that maintain stability, accuracy, and conservation are used [21].

Another critical research element is the need for dissipation operators that are complimentary with the skew-symmetric form, but maintain the design-order accuracy of the finite-domain operator. Rather than include this topic herein, we address it in a companion paper. Because all of the discrete operators that are derived in this paper are nondissipative and have no inherent mechanism to suppress oscillations, we refrain from testing problems that admit strong shocks. As such, we validate the accuracy of the new class of operators by using the viscous Burgers equation.

The layout of the paper is as follows. Section 2 defines SBP operators, briefly summarizes the Lax-Wendroff theorem, and presents a derivation of the discretely conservative split-form finite-difference approximations for conservation laws. Examples of a complete fourth-order operator are presented in section 3. Section 3 also

[2]The (p-$2p$-p) nomenclature denotes that boundary-interior-boundary stencils are p-, $2p$-, p-order accurate, respectively. The resulting operators are (p+1)-order accurate for hyperbolic equations (Euler) and incompletely parabolic equations (Navier-Stokes) and (p+2)-order accurate for parabolic equations (viscous Burgers) [19, 20].

contains a discussion on the relationship between the SBP operators and split-form conservation, as well as a discussion of how alternative operators can be extended to complex geometries. In section 4, several split-form operators are applied to the viscous Burgers equation. Numerical validation of accuracy of an alternative form is given in section 5, with the use of both single and multiple domains. For brevity, the (3-6-3) and (4-8-4) operators appended verbatim in E.

2 Definitions

2.1 Summation-By-Parts Operators

Discretize the physical domain by using N equidistant points that are distributed over the interval $[0, 1]$ as

$$\mathbf{x} = (x_1, x_2, \ldots, x_{N-1}, x_N), \quad x_i = (i-1)\frac{1}{N-1}, \quad i = 1, 2, \ldots, N$$

and define a general class of SBP derivative operators $\mathcal{D}_{sbp}$ that satisfy the following matrix properties:

$$\begin{array}{c} \mathcal{D}_{sbp} \;=\; \mathcal{P}^{-1}\mathcal{Q}, \quad \mathcal{Q} + \mathcal{Q}^T \;=\; \mathcal{B} \;=\; \mathrm{diag}(-1, 0, \cdots, 0, 1) \\ \mathcal{P} = \mathrm{diag}(p_{(1,1)}, \cdots, p_{(N,N)}), \quad \boldsymbol{\zeta}^T \mathcal{P} \boldsymbol{\zeta} > 0, \quad \boldsymbol{\zeta} \neq \mathbf{0} \end{array} \tag{2.1}$$

where $p_{(1,1)} \cdots p_{(N,N)}$ are the diagonal elements of the matrix and $\boldsymbol{\zeta}$ is an arbitrary vector. The SBP operators defined in eq. 2.1 utilize a diagonal norm $\mathcal{P}$ and are, therefore, a subset of the more general SBP form (see ref. [21] or ref. [22]). All are telescoping operators and are conservative in the $\mathcal{P}$ norm. (See A for a proof that all operators that can be expressed in the form of eq. 2.1 are conservative.)

2.2 Complementary Grids

The discrete derivative operator approximating h_x (h sufficiently smooth) is represented herein as a combination of matrix operations that transfer data between two complementary meshes. This nomenclature automatically leads to discrete conservation. The two complementary sets of grid points (or meshes) are defined on the finite interval $0 \leq x \leq 1$, differ in dimension by one and are expressed by using the vectors

$$\mathbf{x} \;=\; [x_1, x_2, \cdots, x_{N-1}, x_N]^T, \quad \bar{\mathbf{x}} \;=\; [\bar{x}_0, \bar{x}_1, \cdots, \bar{x}_{N-1}, \bar{x}_N]^T$$

with $x_1 = \bar{x}_0 = 0$ and $x_N = \bar{x}_N = 1$. Herein, the points $\mathbf{x}$ are referred to as the "solution points," while the points $\bar{\mathbf{x}}$ are referred to as the "flux points." (The overbar nomenclature is reserved herein for those quantities that are defined at the flux points $\bar{\mathbf{x}}$.)

Define the finite-domain conservative difference operator $\frac{\delta h}{\delta x}$ in matrix notation as

$$\frac{\delta h}{\delta x} \;=\; \mathcal{P}^{-1}\Delta\bar{\mathbf{h}}$$

where

$$\Delta = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix} \tag{2.2}$$

is a rectangular $(N+1 \times N)$ matrix. The flux vector $\bar{\mathbf{h}}$ constructed using appropriate interpolation operators on the flux points $\bar{\mathbf{x}}$, is differenced back onto the solution points by Δ. The matrix $\mathcal{P}^{-1}$ accounts for the appropriate grid spacings throughout the domain. Note that discrete conservation follows immediately because the columns of Δ sum to $[-1, 0, \cdots, 0, 1]$. Further details on the definitions of the dual grids, as well as implementation details can be found elsewhere [24].

2.3 The Lax-Wendroff Theorem

Consider a system of conservation laws of the form

$$\begin{array}{l} \frac{\partial v}{\partial t} + \frac{\partial f(v)}{\partial x} = 0, \quad t \geq 0, \quad 0 \leq x \leq 1, \\ v(x,0) = v_0(x)\ ,\ v(0,t) = bc(t) \end{array} \tag{2.3}$$

where $v = v(x,t)$ is the continuous solution vector, $f = f(v)$ is a nonlinear flux vector, and $v_0(x)$ is a bounded piecewise continuous function in L_2. Although smooth solutions to eq. (2.3) do exist, no smoothness can be guaranteed due to the nonlinearity of f. Thus, we seek a weak solution to eq. (2.3) such that the integral relation

$$\int_0^T \int_0^1 [\psi_t v + \psi_x f(v)]\, dx dt = \left[\int_0^1 \psi v dx \right]_0^T + \left[\int_0^T \psi f(v) dt \right]_0^1 \tag{2.4}$$

is satisfied for all smooth test functions $\psi(x,t)$.

Choose a vector-valued function $g(u)$ that is both a function of $2l$ compactly supported vector arguments and related to the flux f with the sole requirement that $g(v, \cdots, v) = f(v)$. Furthermore, define the discrete fluxes g to be of the form

$$g(x + \delta x/2) = g(u_{-l+1}, \cdots, u_l), \quad g(x - \delta x/2) = g(u_{-l}, \cdots, u_{l-1})$$

with suitably modified one-sided expressions near the physical boundaries.

With these definitions, we state here without proof the Lax-Wendroff theorem [17] (originally derived for an infinite domain):

Theorem 2.1. *Consider the conservative discrete analogue of eq.* (2.3)*:*

$$\frac{u(x,t+\delta t) - u(x,t)}{\delta t} + \frac{g(x + \frac{\delta x}{2}) - g(x - \frac{\delta x}{2})}{\delta x} = 0 \tag{2.5}$$

If the discrete solution $u(x,t)$ converges boundedly almost everywhere to $\bar{u}(x,t)$, then $\bar{u}(x,t)$ is a weak solution to the continuous equation.

Finite-domain proofs of the Lax-Wendroff theorem are given in the context of SBP operators in refs. [21, 23].

2.4 Split-Operator Consistency Requirements for Lax-Wendroff

Consider the flux f that is defined in eq. (2.3), which satisfies the property $f(u) = v(u)w(u)$. The general split flux which is composed of a linear combination of the divergence and the product-rule components, can be written as

$$u_t + \alpha f(u)_x + (1-\alpha)\left[v(u)w(u)_x + w(u)v(u)_x\right] = 0 \tag{2.6}$$

Discretizing eq. (2.6) with an SBP spatial operator results in the following semidiscrete equation:

$$\begin{aligned} \mathbf{u}_t + \alpha\mathcal{P}^{-1}\mathcal{Q}W\mathbf{v} + (1-\alpha)\left(V\mathcal{P}^{-1}\mathcal{Q}\mathbf{w} + W\mathcal{P}^{-1}\mathcal{Q}\mathbf{v}\right) &= 0 \\ \mathbf{v} = \left[v(u_1), v(u_2), \ldots, v(u_N)\right]^T, \quad V &= \operatorname{diag}(\mathbf{v}) \\ \mathbf{w} = \left[w(u_1), w(u_2), \ldots, w(u_N)\right]^T, \quad W &= \operatorname{diag}(\mathbf{w}) \end{aligned} \tag{2.7}$$

Equation 2.7 does *not* explicitly lead to a discretely conservative flux form $\frac{\delta g}{\delta x}$. Furthermore, if the resulting discrete flux can be manipulated into a conservation flux that satisfies $g(u, \cdots, u) = f(u)$, then g is not immediately consistent with the physical flux $f = vw$ that satisfies the Rankine-Hugoniot relations. Thus, if the Lax-Wendroff theorem is to be used to guarantee convergence to the weak solution of the continuous equations, then the following conditions must be met:

1. The discrete spatial operator $\alpha\mathcal{P}^{-1}\mathcal{Q}W\mathbf{v} + (1-\alpha)\left(V\mathcal{P}^{-1}\mathcal{Q}\mathbf{w} + W\mathcal{P}^{-1}\mathcal{Q}\mathbf{v}\right)$ must be equivalent to a telescoping form $\frac{\delta g}{\delta x}$.

2. The discrete flux g is Lax-consistent with the physical flux f, where f is the flux that satisfies the Rankine-Hugoniot relations $S[u] - [f] = 0$. (The bracket notation [] denotes the jump in the function.)

3 A Discretely Conservative, Finite-Domain, Split Form

3.1 A Fourth-Order Example

We begin by demonstrating that the fourth-order finite-domain split-form operator is conservative for any value of the parameter α if a diagonal-norm, finite-domain SBP operator is used for discrete differentiation.

The conservation part $\mathcal{P}^{-1}\mathcal{Q}(W\mathbf{v})$ at an interior point i ($s < i < N+1-s$) is obviously conservative, and therefore does not need to be further manipulated. (The number of closure terms that are needed at each boundary is given by the constant s.) The product-rule terms $V\mathcal{P}^{-1}\mathcal{Q}\mathbf{w} + W\mathcal{P}^{-1}\mathcal{Q}\mathbf{v}$, which are discretized by using the conventional centered finite-difference at an interior point i, are

$$\begin{aligned} \left(V\mathcal{P}^{-1}\mathcal{Q}\mathbf{w} + W\mathcal{P}^{-1}\mathcal{Q}\mathbf{v}\right)_i = & \frac{v_i}{12}\left(w_{i-2} - 8w_{i-1} + 8w_{i+1} - w_{i+2}\right) \\ & + \frac{w_i}{12}\left(v_{i-2} - 8v_{i-1} + 8v_{i+1} - v_{i+2}\right) \end{aligned} \tag{3.1}$$

and can be expressed in the following conservative form:

$$\bar{f}_i - \bar{f}_{i-1} = \frac{1}{12}\begin{pmatrix} v_{i-2} \\ v_{i-1} \\ v_i \\ v_{i+1} \\ v_{i+2} \end{pmatrix}^T \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -8 & 0 & 0 \\ 1 & -8 & 0 & 8 & -1 \\ 0 & 0 & 8 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} w_{i-2} \\ w_{i-1} \\ w_i \\ w_{i+1} \\ w_{i+2} \end{pmatrix} \tag{3.2}$$

The individual fluxes in eq. 3.2 are given as

$$\begin{aligned} \bar{f}_i &= \frac{1}{12}\begin{pmatrix} v_{i-2} \\ v_{i-1} \\ v_i \\ v_{i+1} \\ v_{i+2} \end{pmatrix}^T \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 8 & -1 \\ 0 & -1 & 8 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} w_{i-2} \\ w_{i-1} \\ w_i \\ w_{i+1} \\ w_{i+2} \end{pmatrix} \\ \bar{f}_{i-1} &= \frac{1}{12}\begin{pmatrix} v_{i-2} \\ v_{i-1} \\ v_i \\ v_{i+1} \\ v_{i+2} \end{pmatrix}^T \begin{pmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 8 & -1 & 0 \\ -1 & 8 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} w_{i-2} \\ w_{i-1} \\ w_i \\ w_{i+1} \\ w_{i+2} \end{pmatrix} \end{aligned} \tag{3.3}$$

Clearly, the coefficients in the flux matrix simply shift down and to the right for each subsequent flux $\bar{f}_i \quad (s < i < N+1-s)$. The boundary fluxes are more elaborate but telescope in the same manner. The boundary fluxes are completely determined by the form of $\mathcal{Q}$ near the boundaries for both the conservation and product-rule forms.

The discrete split operator that is defined in eq. 2.7 can be written in conservative form by using a linear combination of the conservative divergence and product-rule fluxes as

$$\mathbf{u}_t + \mathcal{P}^{-1}\Delta\overline{\mathbf{f}} = 0, \quad \overline{\mathbf{f}} = \alpha\overline{\mathbf{f}}_c + (1-\alpha)\overline{\mathbf{f}}_e \tag{3.4}$$

The divergence form flux is given by

$$\overline{\mathbf{f}}_c = \begin{pmatrix} w_1 v_1 \\ \frac{1}{96}\left(48 w_1 v_1 + 59 w_2 v_2 - 8 w_3 v_3 - 3 w_4 v_4\right) \\ \frac{1}{96}\left(-11 w_1 v_1 + 59 w_2 v_2 + 51 w_3 v_3 - 3 w_4 v_4\right) \\ \frac{1}{96}\left(-3 w_1 v_1 + 51 w_3 v_3 + 56 w_4 v_4 - 8 w_5 v_5\right) \\ \frac{1}{12}\left(-w_3 v_3 + 7 w_4 v_4 + 7 w_5 v_5 - w_6 v_6\right) \\ \vdots \\ \frac{1}{12}\left(-w_{i-1} v_{i-1} + 7 w_i v_i + 7 w_{i+1} v_{i+1} - w_{i+2} v_{i+2}\right) \\ \vdots \\ \frac{1}{12}\left(-w_{N-5} v_{N-5} + 7 w_{N-4} v_{N-4} + 7 w_{N-3} v_{N-3} - w_{N-2} v_{N-2}\right) \\ \frac{1}{96}\left(-8 u_{N-4} v_{N-4} + 56 u_{N-3} v_{N-3} + 51 u_{N-2} v_{N-2} - 3 u_N v_N\right) \\ \frac{1}{96}\left(-3 w_{N-3} v_{N-3} + 51 w_{N-2} v_{N-2} + 59 w_{N-1} v_{N-1} - 11 w_N v_N\right) \\ \frac{1}{96}\left(-3 w_{N-1} v_{N-1} - 8 w_{N-2} v_{N-2} + 59 w_{N-1} v_{N-1} + 48 w_N v_N\right) \\ w_N v_N \end{pmatrix} \tag{3.5}$$

while the product-rule flux is

$$
\overline{\mathbf{f}}_e = \begin{pmatrix}
w_1 v_1 \\
\frac{1}{96}\left(59u_2v_1 - 8u_3v_1 - 3u_4v_1 + 59u_1v_2 - 8u_1v_3 - 3u_1v_4\right) \\
\frac{1}{96}\left(-8u_3v_1 - 3u_4v_1 + 59u_3v_2 - 8u_1v_3 + 59u_2v_3 - 3u_1v_4\right) \\
\frac{1}{96}\left(-3u_4v_1 + 59u_4v_3 - 8u_5v_3 - 3u_1v_4 + 59u_3v_4 - 8u_3v_5\right) \\
\frac{1}{12}\left(-w_5v_3 - w_3v_5 + 8w_5v_4 + 8w_4v_5 - w_6v_4 - w_6v_4\right) \\
\vdots \\
\frac{1}{12}\big(- w_{i-1}v_{i+1} - w_{i+1}v_{i-1} + 8w_iv_{i+1} \\
+ 8w_{i+1}v_i - w_iv_{i+2} - w_{i+2}v_i \big) \\
\vdots \\
\frac{1}{12}\big(- w_{N-5}v_{N-5} - w_{N-3}v_{N-5} + 8w_{N-4}v_{N-3} \\
+ 8w_{N-3}v_{N-4} - w_{N-3}v_{N-1} - w_{N-1}v_{N-3} \big) \\
\frac{1}{96}\big(- 8u_{N-2}v_{N-4} + 59u_{N-2}v_{N-3} - 3u_Nv_{N-3} \\
- 8u_{N-4}v_{N-2} + 59u_{N-3}v_{N-2} - 3u_{N-3}v_N \big) \\
\frac{1}{96}\big(- 3u_Nv_{N-3} + 59u_{N-1}v_{N-2} - 8u_Nv_{N-2} \\
+ 59u_{N-2}v_{N-1} - 3u_{N-3}v_N - 8u_{N-2}v_N \big) \\
\frac{1}{96}\big(- 3u_Nv_{N-3} - 8u_Nv_{N-2} + 59u_Nv_{N-1} \\
- 3u_{N-3}v_N - 8u_{N-2}v_N + 59u_{N-1}v_N \big) \\
w_Nv_N
\end{pmatrix} \tag{3.6}
$$

We also have constructed operators for $2p = 6$ and $2p = 8$ with boundary closure blocks of dimension $s = 7$ and $s = 9$, respectively. The interior flux forms for the sixth- and eight-order schemes are presented in C. The full flux form operators are appended verbatim in E.

3.2 General SBP Operators

The previous fourth-order finite-domain example (as well as the sixth- and eighth-order examples, which are given in C) suggests that conventional SBP operators lead to conservative finite-domain split-form operators. To this end, we make the following conjecture:

Conjecture 3.1. *The discrete split-form of eq.* (2.7) *can be manipulated into the conservative form*

$$\mathbf{u}_t + \mathcal{P}^{-1}\Delta\overline{\mathbf{f}} = 0 \tag{3.7}$$

for any diagonal-norm SBP operator that can be expressed in the form of eq. (2.1) *and for any value of the parameter α. Furthermore, the resulting local fluxes $\bar{f}_j$ have compact support and are consistent with the original conservative flux $f(u)$.*

The Lax-Wendroff theorem guarantees that discretely captured discontinuities are accurate.

To prove this conjecture for a general banded matrix $\mathcal{Q}$ of halfwidth r, one would need to first prove that the product rule term is conservative. That is,

$$(V\mathcal{Q}\mathbf{w} + W\mathcal{Q}\mathbf{v}) = \Delta\bar{\mathbf{f}}_e$$

Because the term $\mathcal{P}^{-1}\mathcal{Q}\mathbf{f}$ in eq. (2.7) is already discretely conservative by eq. (A2) (i.e., can be manipulated into the form $\mathcal{P}^{-1}\Delta\bar{\mathbf{f}}_c$), the split spatial operator is discretely conservative for any value of the parameter α.

Evidence: Although a formal proof is not forthcoming at this time, considerable evidence suggests that the conjecture is true. Specifically, all of the banded high-order SBP finite-difference operators that are derived for this work (up to order eight) are consistent with this conjecture.

Furthermore, consider a banded matrix $\mathcal{Q}$ that satisfies eq. (2.1), with elements $[\mathcal{Q}]_{(i1,i2)} = q_{(i1,i2)}$ and a halfwidth $r \geq 1$. Solving for the conservative flux $\bar{\mathbf{f}}_e$ in the relation $(V\mathcal{Q}\mathbf{w} + W\mathcal{Q}\mathbf{v}) = \Delta\bar{\mathbf{f}}_e$ [25] yields the closed-form expression

$$\begin{aligned} \bar{f}_j &= \sum_{k=1}^{r}\sum_{l=1}^{k}(w_{j+l}v_{j+l-k} + w_{j+l-k}v_{j+l})q_{(j+l-k,j+l)} \\ & 1 \leq j+l, \quad j+l-k \leq N \quad 1 \leq j \leq N-1 \\ & \bar{f}_0 = w_1 v_1, \quad \bar{f}_N = w_N v_N \end{aligned} \tag{3.8}$$

Manipulating the $q_{(j+l-k,j+l)}$ terms produces the consistency condition

$$\begin{aligned} \frac{1}{2} &= \sum_{k=1}^{r}\sum_{l=1}^{k} q_{(j+l-k,j+l)} \\ & 1 \leq j+l, \quad j+l-k \leq N \quad 1 \leq j \leq N-1 \end{aligned} \tag{3.9}$$

Substituting $w_j = w(u)$ and $v_j = v(u)$ into eq. (3.8) and using eq. (3.9) yields

$$\begin{aligned} \bar{f}_j &= 2wv\sum_{k=1}^{r}\sum_{l=1}^{k} q_{(j+l-k,j+l)} = wv \\ & 1 \leq j+l, j+l-k \leq N \quad 1 \leq j \leq N-1, \\ & \bar{f}_0 = wv, \quad \bar{f}_N = wv \end{aligned} \tag{3.10}$$

Thus, the term $\alpha\mathcal{Q}W\mathbf{v} + (1-\alpha)(V\mathcal{Q}\mathbf{w} + W\mathcal{Q}\mathbf{v})$ can be expressed in terms of a telescoping conservative flux, has a compact support of halfwidth r, and is consistent with the original conservative flux in the conservation law given by eq. (2.3). All of the sufficient conditions of the Lax-Wendroff theorem are met, so converged solutions using the above split operators are weak solutions to the conservation law. □

Remark: This conjecture has been verified symbolically for matrices with a halfwidth $1 \leq r \leq 50$.

3.3 A Multidomain Approach for Alternative Operators

While great flexibility is afforded by the potential use of alternative forms of the equations (e.g., skew-symmetric or canonical splittings), a systematic approach is needed to extend alternative uniform-grid single-domain, tensor-product operators to complex geometries. A simple approach is to adopt a general multidomain finite-difference discretization that incorporates a generalized curvilinear mesh in each domain.

3.3.1 Multidomain Operators

A major obstacle in the application of high-order methods to realistic problems is the development a suitable grid around complex geometric features or in regions of strong gradients. (Constraining the grid to design-order smoothness severely complicates grid generation.) s equation is Multidomain techniques greatly simplify the grid generation process for complex configurations by breaking the geometry into the union of piecewise smooth quadrilateral (hexahedral) domains in two (three) dimensions.

Conventional high-order SBP finite-difference techniques naturally extend to multidomain discretizations [21, 23, 26, 27]. Each domain is discretized with a stable tensor-product formulation and then connected to its adjoining neighbors using interface conditions that maintain the stability, accuracy, and conservation of the interior operators. Penalty type interface treatments, which are closely related to those that are used in discontinuous Galerkin and internal penalty finite-element approaches, are most frequently used to join the domains. Domain interfaces need only be C_0 smooth to maintain the stability, conservation and design accuracy of the single-domain operators.

Existing derivations for multidomain high-order finite-difference schemes primarily focus on the divergence form of the equations. Indeed, stability and conservation proofs follow naturally with the use of this form. However, nothing precludes the use of alternative operators in each subdomain. The only additional requirement is the need for domain-interface coupling conditions that retain the desirable interior properties of the alternative operator. These interface operators must be developed on a case-by-case basis; however, as the examples in the next section demonstrate, these operators generally are simple extensions of the original coupling conditions introduced and extended in refs. [21, 23].

3.3.2 Curvilinear Coordinates

Generalized curvilinear coordinate formulations are commonly used in high-order formulations to facilitate the use of nonuniform grid distributions or the definition of "complex" geometries. The likelihood exists that the desirable attributes of a Cartesian grid, alternative formulation will not survive the curvilinear mappings between the physical (x, y, z) and the computational (ξ, η, ζ) space. However, the curvilinear alternative formulation can inherit the desirable Cartesian grid properties. For example, Pirozzoli [16] recently demonstrated that pseudo-kinetic energy can be preserved on curvilinear grids that use an alternative (skew-symmetric) formulation, (i.e., one motivated by formulations that preserve kinetic energy on a Cartesian grid).

4 Examples using Burgers equation

We use the viscous Burgers equation to validate the numerical accuracy of the alternative split operators that are developed in the previous sections. In conservation form, the Burgers equation is

$$
\begin{gathered}
u_t + f(u)_x = \epsilon u_{xx}, \quad f(u) = \left(\frac{u^2}{2}\right), \quad x \in [0,1], \quad t \in [0,\infty) \\
\tilde{a}_0 u(0,t) - \epsilon u_x(0,t) - g_0(t) = 0, \quad \tilde{a}_1 u(1,t) - \epsilon u_x(1,t) - g_1(t) = 0 \\
\tilde{a}_0 = \frac{u(0,t) + |u(0,t)|}{3}, \quad \tilde{a}_1 = \frac{u(1,t) - |u(1,t)|}{3}
\end{gathered}
\tag{4.1}
$$

where ϵ represents the diffusion coefficient. Although the problem is fundamentally different with $\epsilon = 0$, we form the discrete approximation such that when $\epsilon = 0$ the fully inviscid Burgers equation is recovered and only the appropriate boundary conditions are imposed. The boundary conditions that are given in eq. 4.1 ensure that the formulation is strongly well-posed; this is demonstrated further in section 4.1.

The Burgers equation must be split to find continuous and discrete energy estimates. In the next section, we examine numerically the difference between the full conservation form and the split-form by using the discretely conservative method that is developed herein. While our formulation allows for any value of α, only one α yields the energy estimate. This value of α is used in the next section to evaluate advantages, if any, that are gained from using the discrete formulation that satisfies the energy estimate.

4.1 Energy Analysis of the Continuous Problem

The properties and behavior of the inviscid portion of eq. (4.1) are covered in an energy analysis that is presented by Jameson [18]; we reproduce and supplement the essential results here for completeness. The energy that is given in Burgers equation can be analyzed directly without a linearization. If we assume a smooth solution, then can we split the conservation form (eq. (4.1)) by the parameter α to obtain

$$
u_t + \alpha \left(\frac{u^2}{2}\right)_x + (1-\alpha) u u_x = \epsilon u_{xx}, \quad x \in [0,L],\, t \in [0,\infty) \tag{4.2}
$$

This equation is multiplied by u and integrated over the interval $[0, L]$ to obtain the energy

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u\|^2 &= -\int_0^L \left[\alpha u \left(\frac{u^2}{2}\right)_x + (1-\alpha)u^2 u_x\right] \mathrm{d}x + \epsilon \int_0^L u u_{xx} \mathrm{d}x \\
&= -\int_0^L \left[\frac{\alpha}{2}\left(u^3\right)_x + \left(1 - \frac{3}{2}\alpha\right) u^2 u_x\right] \mathrm{d}x + \epsilon \int_0^L \left[(u u_x)_x - u_x u_x\right] \mathrm{d}x \qquad (4.3) \\
&= \frac{\alpha}{2} u^3\Big|_L^0 + \epsilon u u_x|_0^L - \int_0^L \left(1 - \frac{3}{2}\alpha\right) u^2 u_x \mathrm{d}x - \epsilon \|u_x\|^2
\end{aligned}
$$

Equation 4.3 clearly shows that for $\alpha = 2/3$ the integral term disappears and the energy is dependent on the boundary data and the viscous dissipation. This is referred to as the canonical splitting or "skew-symmetric form" of the Burgers equation [18, 28]. For the viscous problem ($\epsilon > 0$), note that any shock that develops is resolvable at the continuous level and that the energy analysis remains valid. For the inviscid problem ($\epsilon = 0$), the integration bounds can be split, and the same canonical splitting parameter is found. See reference [18] for details. The well posedness of the boundary data is evaluated first by substituting $\alpha = 2/3$ into eq. (4.3) and then by analyzing each boundary individually. The left boundary term (scaled by 2 for convenience) is

$$BT_0 = \frac{2}{3}u(0,t)^3 - 2\epsilon u(0,t)u_x(0,t) \qquad (4.4)$$

Completing the squares in eq. 4.4 leads to boundary terms that can be written as

$$BT_0 = \frac{1}{\frac{2}{3}u(0,t)} \left\{ \left[\frac{2}{3}u(0,t)^2 - \epsilon u_x(0,t)\right]^2 - (\epsilon u_x(0,t))^2 \right\} \qquad (4.5)$$

We define

$$a(u) = \frac{2}{3}u(x,t) \qquad (4.6)$$

as a sensor for the hyperbolic part of the equation; $\tilde{a}_0$ and $\tilde{a}_1$ in the boundary conditions of eq. (4.1) serve as switches to eliminate the hyperbolic part of the boundary condition if waves are not propagating toward the interior of the domain from the boundaries. If we use the definition above, the left boundary term is

$$BT_0 = \frac{1}{a_0}\left\{ [a_0 u(0,t) - \epsilon u_x(0,t)]^2 - [\epsilon u_x(0,t)]^2 \right\}, \quad a_0 = a[u(0,t)] \qquad (4.7)$$

We now substitute the left boundary condition from eq. (4.1) for $\epsilon u_x(0,t)$ in eq. (4.7) to obtain

$$BT_0 = \frac{1}{a_0}\left\{ [a_0 u(0,t) + g_0(t) - \tilde{a}_0 u(0,t)]^2 - [\tilde{a}_0 u(0,t) - g_0(t)]^2 \right\}. \qquad (4.8)$$

The condition $a_0 > 0$ in eq. (4.1) produces $\tilde{a}_0 = a$, and we find that

$$BT_0 = \frac{g_0(t)^2}{a_0} - \frac{1}{a}\left[a_0 u(0,t) - g_0(t)\right]^2 \leq \frac{g_0(t)^2}{a_0} \tag{4.9}$$

If $u(0,t) = a[u(0,t)] = 0$, then $BT_0 = 0$, and the energy of the continuous equation is bounded from above by the boundary data. If $a < 0$, then $\tilde{a}_0 = 0$, which yields the condition

$$BT_0 = \frac{1}{a_0}\left[a_0 u(0,t) + g_0(t)\right]^2 - \frac{g_0(t)^2}{a_0} = \frac{g_0(t)^2}{|a_0|} - \frac{1}{|a_0|}\left[a_0 u(0,t) + g_0(t)\right]^2 \tag{4.10}$$

Again, the energy is bounded from above by the boundary data. The right boundary term (again scaled by 2) is

$$BT_1 = -\frac{2}{3}u(L,t)^3 + 2\epsilon u(L,t)u_x(1,t) = -a_1 u(L,t)^2 + 2\tilde{a}_1 - 2u(L,t)g_1(t) \tag{4.11}$$

where $a_1 = a[u(L,t)]$. The same procedure is followed to show that the energy on the right boundary is bounded. We find that

$$BT_1 = \begin{cases} 0 & \text{for} \quad a_1 = 0 \\ \frac{g_1(t)^2}{a_1} - \frac{1}{a_1}\left[a_1 u(L,t) + g_1(t)\right]^2 & \text{for} \quad a_1 > 0 \\ \frac{g_1(t)^2}{|a_1|} - \frac{1}{|a_1|}\left[a_1 u(L,t) - g_1(t)\right]^2 & \text{for} \quad a_1 < 0 \end{cases} \tag{4.12}$$

Since the energy of the continuous equation is bounded by the energy imposed through the boundary terms, the equation and boundary conditions in eq. (4.1) are strongly well-posed.

4.2 Energy Analysis of the Semidiscrete Problem: Single Domain

The canonical splitting can be used to construct a semidiscrete operator that satisfies the semidiscrete analog to eq. (4.3). Using the finite-difference operators $\mathcal{D}$ and $\mathcal{D}_2$ that satisfy the SBP condition on the finite domain, we discretize the skew-symmetric form ($\alpha = 2/3$) of eq. (4.2) as

$$\begin{aligned} \mathbf{u}_t &= -\frac{1}{3}\left[\mathcal{D}U\mathbf{u} + U\mathcal{D}\mathbf{u}\right] + \epsilon\mathcal{D}_2\mathbf{u} \\ &\quad + \sigma_0\mathcal{P}^{-1}\mathbf{e}_0\left[\tilde{a}_0 u_1 - \epsilon\left(\mathcal{S}\mathbf{u}\right)_1 - g_0(t)\right] \\ &\quad + \sigma_1\mathcal{P}^{-1}\mathbf{e}_1\left[\tilde{a}_1 u_N - \epsilon\left(\mathcal{S}\mathbf{u}\right)_N - g_1(t)\right] \\ U &= \text{diag}(\mathbf{u}), \quad \mathbf{e}_0 = (1,0,\dots)^T, \quad \mathbf{e}_1 = (\dots,0,1)^T \\ \tilde{a}_0 &= \frac{u_1 + |u_1|}{3} \quad \tilde{a}_1 = \frac{u_N - |u_N|}{3} \end{aligned} \tag{4.13}$$

The proper values of σ_0 and σ_1 are determined in the following energy analysis. Both forms of $\mathcal{D}_2$ in eq. (D1) (see appendix D) satisfy the energy analysis, but we use the narrow stencil here for illustrative purposes. The simultaneous approximation term (SAT) penalty method [29] is used in eq. (4.13) to satisfy the mixed boundary conditions with specified boundary data. For the finite domain, a diagonal norm $\mathcal{P}$ is chosen to ensure that matrix multiplication of the norm and the nonlinear coefficient matrix commutes. This choice reduces the overall accuracy for hyperbolic problems to $(p+1)$, where $2p$ is the internal accuracy and p is the accuracy at the boundary. For parabolic problems, the narrow-stencil second derivative operator $\mathcal{D}_2 = \mathcal{P}^{-1}(\mathcal{R} + \mathcal{B}\mathcal{S})$ constructed with the diagonal norm, provides a global $(p+2)$ order of accuracy [30]. Premultiplying eq. (4.13) by $\mathbf{u}^T\mathcal{P}$ and using eq. (2.1) yields the discrete energy

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\mathbf{u}\|_{\mathcal{P}}^2 \;=\; & -\frac{1}{3}\mathbf{u}^T\left(\mathcal{Q}U + U\mathcal{Q}\right)\mathbf{u} + \epsilon\mathbf{u}^T\mathcal{R}\mathbf{u} + \epsilon\mathbf{u}^T\mathcal{B}\mathcal{S}\mathbf{u} \\ & + \mathbf{u}^T\mathbf{e}_0\left\{\sigma_0\left[\tilde{a}_0 u_1 - \epsilon\left(\mathcal{S}\mathbf{u}\right)_1 - g_0(t)\right]\right\} \\ & + \mathbf{u}^T\mathbf{e}_1\left\{\sigma_1\left[\tilde{a}_1 u_N - \epsilon\left(\mathcal{S}\mathbf{u}\right)_N - g_1(t)\right]\right\}\end{aligned} \tag{4.14}$$

Adding eq. (4.14) to its transpose yields

$$\begin{aligned}\frac{d}{dt}\|\mathbf{u}\|_{\mathcal{P}}^2 \;=\; & -\frac{2}{3}\mathbf{u}^T\mathcal{B}U\mathbf{u} + 2\epsilon\mathbf{u}^T\mathcal{R}\mathbf{u} + 2\epsilon\mathbf{u}^T\mathcal{B}\mathcal{S}\mathbf{u} \\ & + 2u_1\sigma_0\left(\tilde{a}_0 u_1 - \epsilon\left[\mathcal{S}\mathbf{u}\right)_1 - g_0(t)\right] \\ & + 2u_N\sigma_1\left(\tilde{a}_1 u_N - \epsilon\left[\mathcal{S}\mathbf{u}\right)_N - g_1(t)\right]\end{aligned} \tag{4.15}$$

Simplifying with eq. (2.1) yields

$$\begin{aligned}\frac{d}{dt}\|\mathbf{u}\|_{\mathcal{P}}^2 \; & = 2\epsilon\mathbf{u}^T\mathcal{R}\mathbf{u} \\ & + 2u_1\left\{\sigma_0\left[\tilde{a}_0 u_1 - g_0(t)\right] + \frac{1}{3}u_1^2\right\} - 2\epsilon u_1\left[\left(\sigma_0 + 1\right)\left(\mathcal{S}\mathbf{u}\right)_1\right] \\ & + 2u_N\left\{\sigma_1\left[\tilde{a}_1 u_N - g_1(t)\right] - \frac{1}{3}u_N^2\right\} + 2\epsilon u_N\left[\left(-\sigma_1 + 1\right)\left(\mathcal{S}\mathbf{u}\right)_N\right]\end{aligned} \tag{4.16}$$

Above, $\mathcal{R}$ is negative semidefinite and ensures that the energy only decays. The viscous boundary terms cancel for $\sigma_0 = -1$ and $\sigma_1 = 1$, which leaves

$$\begin{aligned}\frac{d}{dt}\|\mathbf{u}\|_{\mathcal{P}}^2 \; & = 2\epsilon\mathbf{u}^T\mathcal{R}\mathbf{u} + \widetilde{BT}_0 + \widetilde{BT}_1 \\ \widetilde{BT}_0 \; & = u_1\left[a(u_1)u_1 - 2\tilde{a}_0 u_1 + 2g_0(t)\right] \\ \widetilde{BT}_1 \; & = u_N\left[2\tilde{a}_1 u_N - a(u_N)u_N - 2g_1(t)\right]\end{aligned} \tag{4.17}$$

For the discrete equation to be strongly well-posed, the discrete energy must be bounded from above by the imposed boundary data. For the left boundary when $a(u_1) \geq 0$ and using completing the squares,

$$\widetilde{BT}_0 = -au_1^2 + 2u_1 g_0 = \frac{g_0^2}{a} - \frac{1}{a}\left(au_1 - g_0\right)^2. \tag{4.18}$$

The result in eq. 4.18 is equivalent to continuous result found in eq. (4.9).

Completing the squares at the left boundary if $a(u_1) < 0$ yields

$$\widetilde{BT}_0 = au_1^2 + 2u_1 g_0 = \frac{g_0^2}{|a|} - \frac{1}{|a|}\left(au_1 + g_0\right)^2 \tag{4.19}$$

Again, note the similarity between eq. 4.19 and the continuous result given in eq. (4.10). The right boundary also mimics the continuous case, so the semidiscrete equation is strongly well-posed. Note that the same penalty is used for the inviscid and the viscous Burgers equations and that the equation is well-posedness in both cases.

4.3 Energy Analysis of the Semidiscrete Problem: Multidomain

Often multiple semidiscrete domains can be used advantageously to discretize a conservation law. Interface penalties are derived such that the semidiscrete energy is unaffected by the exchange of information between two domains. For a two domain problem on the interval $x \in [-1, 1]$ divided at $x = b$, we use the discretization

$$\begin{aligned}
\mathbf{x}^{(1)} &= \left[x_1^{(1)}, x_2^{(1)}, \ldots, x_{N_1-1}^{(1)}, x_{N_1}^{(1)}\right], \quad x_i^{(1)} = -1 + (i-1)\frac{b+1}{N_1 - 1}, \quad i = 1, 2, \ldots, N_1 \\
\mathbf{x}^{(2)} &= \left[x_1^{(2)}, x_2^{(2)}, \ldots, x_{N_2-1}^{(2)}, x_{N_2}^{(2)}\right], \quad x_i^{(2)} = b + (i-1)\frac{1-b}{N_2 - 1}, \quad i = 1, 2, \ldots, N_2
\end{aligned}$$

where N_1 and N_2 are the number of uniform cells in each of the two domains. The solution on each domain is denoted by $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(2)}$, respectively. Operators that are defined on each domain are denoted similarly. The semidiscrete Burgers equation for the multidomain problem is

$$\begin{aligned}
\mathbf{u}_t^{(1)} &= -\frac{1}{3}\left(\mathcal{D}^{(1)}U^{(1)}\mathbf{u}^{(1)} + U^{(1)}\mathcal{D}^{(1)}\mathbf{u}^{(1)}\right) + \epsilon\mathcal{D}_2^{(1)}\mathbf{u}^{(1)} \\
&\quad + \left(\mathcal{P}^{(1)}\right)^{-1}\mathbf{e}_1^{(1)}\left\{\left(k_{00}u_{N_1}^{(1)} - k_{01}u_1^{(2)}\right) + \epsilon l_{01}\left[\left(\mathcal{S}^{(1)}\mathbf{u}^{(1)}\right)_{N_1} - \left(\mathcal{S}^{(2)}\mathbf{u}^{(2)}\right)_1\right]\right\} \\
&\quad + \left(\mathcal{P}^{(1)}\right)^{-1}\left(\mathcal{S}^{(1)}\right)^T\mathbf{e}_1^{(1)}\epsilon l_{10}\left(u_{N_1}^{(1)} - u_1^{(2)}\right) \\
\mathbf{u}_t^{(2)} &= -\frac{1}{3}\left(\mathcal{D}^{(2)}U^{(2)}\mathbf{u}^{(2)} + U^{(2)}\mathcal{D}^{(2)}\mathbf{u}^{(2)}\right) + \epsilon\mathcal{D}_2^{(2)}\mathbf{u}^{(2)} \\
&\quad + \left(\mathcal{P}^{(2)}\right)^{-1}\mathbf{e}_0^{(2)}\left\{\left(k_{11}u_1^{(2)} - k_{10}u_{N_1}^{(1)}\right) + \epsilon r_{01}\left[\left(\mathcal{S}^{(2)}\mathbf{u}^{(2)}\right)_1 - \left(\mathcal{S}^{(1)}\mathbf{u}^{(1)}\right)_{N_1}\right]\right\} \\
&\quad + \left(\mathcal{P}^{(2)}\right)^{-1}\left(\mathcal{S}^{(2)}\right)^T\mathbf{e}_0^{(2)}\epsilon r_{10}\left(u_1^{(2)} - u_{N_1}^{(1)}\right) \\
U^{(1)} &= \mathrm{diag}(\mathbf{u}^{(1)}), \quad U^{(2)} = \mathrm{diag}(\mathbf{u}^{(2)}), \quad \mathbf{e}_1^{(1)} = (\ldots, 0, 1)^T, \quad \mathbf{e}_0^{(2)} = (1, 0, \ldots)^T
\end{aligned} \tag{4.20}$$

where the boundary condition penalties have been dropped for clarity of the interface treatment. The energy is calculated by premultiplying the two equations by $\left(\mathbf{u}^{(1)}\right)^T P^{(1)}$ and $\left(\mathbf{u}^{(2)}\right)^T P^{(2)}$, respectively. The equations are then added to their transpose as

$$
\begin{aligned}
&\frac{d}{dt}\left(\left\|\mathbf{u}^{(1)}\right\|^2_{P^{(1)}} + \left\|\mathbf{u}^{(2)}\right\|^2_{P^{(2)}}\right) + 2\epsilon\left(\left\|\mathbf{u}_x^{(1)}\right\|^2_{P^{(1)}} + \left\|\mathbf{u}_x^{(2)}\right\|^2_{P^{(2)}}\right) \\
&= \frac{2}{3}\left[\left(u_1^{(2)}\right)^3 - \left(u_{N_1}^{(1)}\right)^3\right] + 2\epsilon\left[\left(S^{(1)}\mathbf{u}^{(1)}\right)_{N_1} - \left(S^{(2)}\mathbf{u}^{(2)}\right)_1\right] \\
&+ 2u_{N_1}^{(1)}\left\{\left(k_{00}u_{N_1}^{(1)} - k_{01}u_1^{(2)}\right) + \epsilon l_{01}\left[\left(\mathcal{S}^{(1)}\mathbf{u}^{(1)}\right)_{N_1} - \left(\mathcal{S}^{(2)}\mathbf{u}^{(2)}\right)_1\right]\right\} \\
&+ 2u_1^{(2)}\left\{\left(k_{11}u_1^{(2)} - k_{10}u_{N_1}^{(1)}\right) + \epsilon r_{01}\left[\left(\mathcal{S}^{(2)}\mathbf{u}^{(2)}\right)_1 - \left(\mathcal{S}^{(1)}\mathbf{u}^{(1)}\right)_{N_1}\right]\right\} \\
&+ 2\left(\mathcal{S}^{(1)}\mathbf{u}^{(1)}\right)_{N_1}\epsilon l_{10}\left(u_{N_1}^{(1)} - u_1^{(2)}\right) + 2\left(\mathcal{S}^{(2)}\mathbf{u}^{(2)}\right)_1\epsilon r_{10}\left(u_1^{(2)} - u_{N_1}^{(1)}\right)
\end{aligned}
\tag{4.21}
$$

The terms that are related to the boundary data have been removed. Because we want to construct the interface penalty such that it conserves the energy, we must choose the free parameters such that the right-hand side of eq. (4.21) vanishes. This results in the following constraints:

$$
\begin{aligned}
k_{00} = \frac{1}{3}u_{N_1}^{(1)}, \quad k_{10} = -k_{01}, \quad k_{11} = -\frac{1}{3}u_1^{(2)} \\
l_{10} = -l_{01} - 1, \quad r_{01} = l_{01} + 1, \quad r_{10} = -l_{01}
\end{aligned}
$$

The formal order of accuracy of the penalty terms must be the same as the order of the boundary finite-difference approximations. This requires that the value of k_{01} be a function of $\mathbf{u}$. We choose an equal weighting of the values on either side of the interface:

$$
k_{01} = \frac{u_{N_1}^{(1)} + u_1^{(2)}}{2} \tag{4.22}
$$

For the viscous terms, $l_{01} = -\frac{1}{2}$ yields an equal weighting of all of the viscous penalty terms; thus, this is the chosen value. The full semidiscrete set of equations for the two-domain problem is

$$
\begin{aligned}
\mathbf{u}_t^{(1)} &= -\frac{1}{3}\left(\mathcal{D}^{(1)}U^{(1)}\mathbf{u}^{(1)} + U^{(1)}\mathcal{D}^{(1)}\mathbf{u}^{(1)}\right) + \epsilon\mathcal{D}_2^{(1)}\mathbf{u}^{(1)} \\
&- \left(\mathcal{P}^{(1)}\right)^{-1}\mathbf{e}_0^{(1)}\left[\tilde{a}_0^{(1)}u_1^{(1)} - \epsilon\left(\mathcal{S}^{(1)}\mathbf{u}^{(1)}\right)_1 - g_0(t)\right] \\
&+ \left(\mathcal{P}^{(1)}\right)^{-1}\mathbf{e}_1^{(1)}\left\{\left(u_{N_1}^{(1)}u_{N_1}^{(1)} - \frac{u_{N_1}^{(1)} + u_1^{(2)}}{2}u_1^{(2)}\right)\right. \\
&\left. -\frac{\epsilon}{2}\left[\left(\mathcal{S}^{(1)}\mathbf{u}^{(1)}\right)_{N_1} - \left(\mathcal{S}^{(2)}\mathbf{u}^{(2)}\right)_1\right]\right\} \\
&- \frac{\epsilon}{2}\left(\mathcal{P}^{(1)}\right)^{-1}\left(\mathcal{S}^{(1)}\right)^T\mathbf{e}_1^{(1)}\left(u_{N_1}^{(1)} - u_1^{(2)}\right), \\
\mathbf{u}_t^{(2)} &= -\frac{1}{3}\left(\mathcal{D}^{(2)}U^{(2)}\mathbf{u}^{(2)} + U^{(2)}\mathcal{D}^{(2)}\mathbf{u}^{(2)}\right) + \epsilon\mathcal{D}_2^{(2)}\mathbf{u}^{(2)} \\
&- \left(\mathcal{P}^{(2)}\right)^{-1}\mathbf{e}_0^{(2)}\left\{\left(u_1^{(2)}u_1^{(2)} - \frac{u_{N_1}^{(1)} + u_1^{(2)}}{2}u_{N_1}^{(1)}\right)\right. \\
&\left. -\frac{\epsilon}{2}\left[\left(\mathcal{S}^{(2)}\mathbf{u}^{(2)}\right)_1 - \left(\mathcal{S}^{(1)}\mathbf{u}^{(1)}\right)_{N_1}\right]\right\} \\
&+ \frac{\epsilon}{2}\left(\mathcal{P}^{(2)}\right)^{-1}\left(\mathcal{S}^{(2)}\right)^T\mathbf{e}_0^{(2)}\left(u_1^{(2)} - u_{N_1}^{(1)}\right) \\
&+ \left(\mathcal{P}^{(2)}\right)^{-1}\mathbf{e}_1^{(2)}\left[\tilde{a}_1^{(2)}u_{N_2}^{(2)} - \epsilon\left(\mathcal{S}^{(2)}\mathbf{u}^{(2)}\right)_{N_2} - g_1(t)\right].
\end{aligned}
\tag{4.23}
$$

In this form, the boundary penalty does not contribute to the energy. The energy changes, however, as a result of the one-sided stencils at the interface. The semidiscrete equations remain strongly well-posed.

5 Numerical Tests

The new discrete operators were implemented for the Burgers equation in a one-dimensional finite-difference solver with a uniform grid. Integration in time was conducted with a five-step, fourth-order Runge-Kutta scheme [31]. The step size in time was chosen such that the temporal error is negligible in comparison with the spatial error.

Two problems are evaluated. For the first problem, the accuracy of the three operator sets that are developed herein is tested by using the skew-symmetric ($\alpha = 2/3$) and the conservation ($\alpha = 1$) forms of the viscous Burgers equation. The second problem tests the accuracy of the multidomain interface closures that are developed based on the new operator sets.

5.1 Accuracy Validation

The viscous Burgers equation is used to test the accuracy of the new operator when it is applied to the nonlinear equation. The test problem is defined as

$$\begin{aligned}
\mathbf{u}_t &= \mathcal{P}^{-1}\Delta\left(\alpha\bar{\mathbf{f}}_c + (1-\alpha)\bar{\mathbf{f}}_e + \bar{\mathbf{f}}_V\right) \\
&\quad + \sigma_0\mathcal{P}^{-1}\mathbf{e}_0\left[-\tilde{a}_0 u_1 - \epsilon\left(\mathcal{S}\mathbf{u}\right)_1 - g_0(t)\right] \\
&\quad + \sigma_1\mathcal{P}^{-1}\mathbf{e}_1\left[-\tilde{a}_1 u_N - \epsilon\left(\mathcal{S}\mathbf{u}\right)_N - g_1(t)\right] \\
g_0(t) &= \frac{v(-1,t) + |v(-1,t)|}{3}v(-1,t) - \epsilon v_x(-1,t) \\
g_1(t) &= \frac{v(1,t) - |v(1,t)|}{3}v(1,t) - \epsilon v_x(1,t) \\
v(x,t) &= \frac{4x}{x^2 + 2t + \frac{1}{40}}, \quad x \in [x_1, x_N], \quad t \in [0,T]
\end{aligned} \tag{5.1}$$

where $\epsilon = 1$ and the initial condition is $\mathbf{u}_0 = v(\mathbf{x}, 0)$. The definition of the viscous flux $\bar{\mathbf{f}}_V$ is given in D. The values for the penalty coefficients σ are those that are identified in section 4. Note that the sign of the convective term has changed for this problem. This is to satisfy the exact solution $v(x,t)$. The problem is simulated on the interval $[-1,1]$ up to $T = 0.05$. Five grids are used to determine the order of accuracy of the set of operators. The L^2 and L^∞ error norms of the conservation form and the canonical split-form are compared in tables 1 through 3. Figure 1 shows the numerical solution for $N = 64$ at different times. All simulations use the narrow-stencil viscous operator.

Table 1. Error Norms and Convergence Rates for Viscous Burgers Equation Using Canonical Split and Conservation Forms, and Conservative Narrow-stencil (2-4-2) Viscous Operator.

	Canonical Split				Conservation			
N	L^2 Error	L^2 Rate	L^∞ Error	L^∞ Rate	L^2 Error	L^2 Rate	L^∞ Error	L^∞ Rate
32	1.64e-03	-	2.70e-03	-	1.29e-03	-	2.44e-03	-
64	1.10e-04	3.90	1.91e-04	3.82	8.86e-05	3.86	1.75e-04	3.80
128	6.97e-06	3.98	1.21e-05	3.97	5.66e-06	3.97	1.17e-05	3.90
256	4.38e-07	3.99	7.66e-07	3.99	3.56e-07	3.99	7.38e-07	3.99
512	2.74e-08	4.00	4.79e-08	4.00	2.23e-08	4.00	4.62e-08	4.00

Table 2. Error Norms and Convergence Rates for Viscous Burgers Equation Using Canonical Split and Conservation Forms, and Conservative Narrow-stencil (3-6-3) Viscous Operator.

	Canonical Split				Conservation			
N	L^2 Error	L^2 Rate	L^∞ Error	L^∞ Rate	L^2 Error	L^2 Rate	L^∞ Error	L^∞ Rate
32	2.34e-04	-	4.62e-04	-	2.83e-04	-	6.24e-04	-
64	4.56e-06	5.69	9.15e-06	5.66	5.61e-06	5.65	1.30e-05	5.58
128	7.52e-08	5.92	1.51e-07	5.92	9.31e-08	5.91	2.19e-07	5.90
256	1.19e-09	5.98	2.40e-09	5.98	1.48e-09	5.98	3.49e-09	5.97
512	2.00e-11	5.90	3.98e-11	5.92	2.33e-11	5.99	5.37e-11	6.02

The expected order of accuracy is $p + 2$, where p is the boundary accuracy. This design order is achieved for both the skew-symmetric and conservation forms

Table 3. Error Norms and Convergence Rates for Viscous Burgers Equation Using Canonical Split and Conservation Forms, and Conservative Narrow-stencil (4-8-4) Viscous Operator.

	Canonical Split				Conservation			
N	L^2 Error	L^2 Rate	L^∞ Error	L^∞ Rate	L^2 Error	L^2 Rate	L^∞ Error	L^∞ Rate
32	1.52e-04	-	3.64e-04	-	1.89e-04	-	4.42e-04	-
64	6.97e-07	7.77	2.17e-06	7.39	8.58e-07	7.78	2.46e-06	7.49
128	3.85e-09	7.50	1.78e-08	6.93	4.45e-09	7.59	1.92e-08	7.00
256	2.74e-11	7.13	1.91e-10	6.55	2.94e-11	7.24	1.98e-10	6.60
512	1.74e-12	3.98	2.65e-12	6.17	1.73e-12	4.09	2.69e-12	6.20

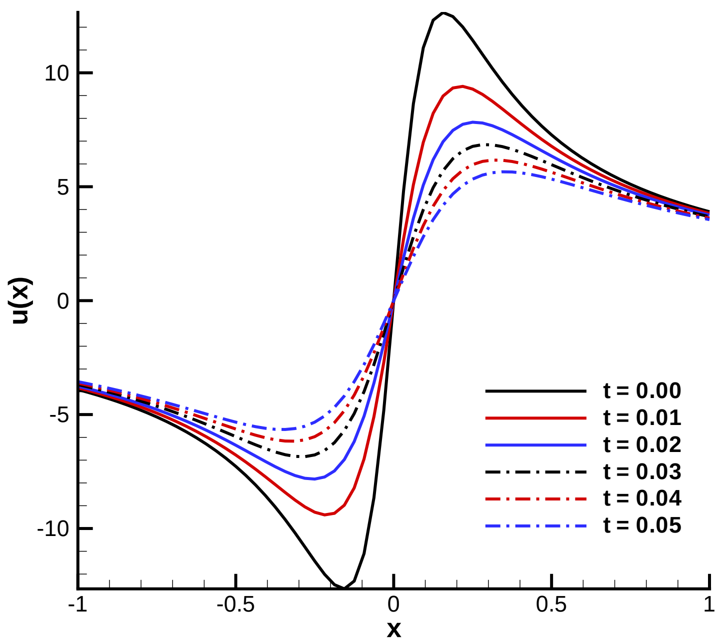

Figure 1. Numerical approximation to skew-symmetric form of viscous Burgers equation with 64 uniform cells and the (4-8-4) narrow stencil conservative operators.

for the (2-4-2) operator. However, the (3-6-3) and (4-8-4) operators exhibit greater accuracy than design order convergence. The (2-4-2) operator does not deviate from design order because the maximum order that can be achieved is equivalent to the design order. This is not the case for the (3-6-3) and (4-8-4) operators. For these operators, the accuracy is at least design order for well resolved solutions. For the (2-4-2) operator, the conservation form is slightly more accurate than the canonical split-form. For the (3-6-3) and (4-8-4) operators, the canonical split-form is more accurate.

5.2 Multidomain Validation

The test problem that was used to test the interfaces is described by the exact solution

$$v(x,t) = 1 - \tanh\left(\frac{x - t - x_0}{2\epsilon}\right), \quad x \in [-10, 10], \quad t \in [0, T] \tag{5.2}$$

where $\epsilon = 0.25$, $x_0 = -5.0$, and the initial condition is $\mathbf{u}_0 = v(\mathbf{x}, 0)$. The problem is simulated up to $T = 10.0$ with four grid resolutions. The skew-symmetric form of the discrete equations is used.

The L^2 and L^∞ errors are tabulated in tables 4 through 6 for each operator on (1) a single domain, (2) two domains with equivalent spacing, and (3) two domains with grid spacing that changes discontinuously at the interface by a factor of $r = 2$. (Herein, the jump in spacing is denoted as the "compression ratio.") The interface between domains is located at $x = 0.0$. The discrete solution with 256 uniform cells on a single domain for the (2-4-2) operator set is shown in figure 2.

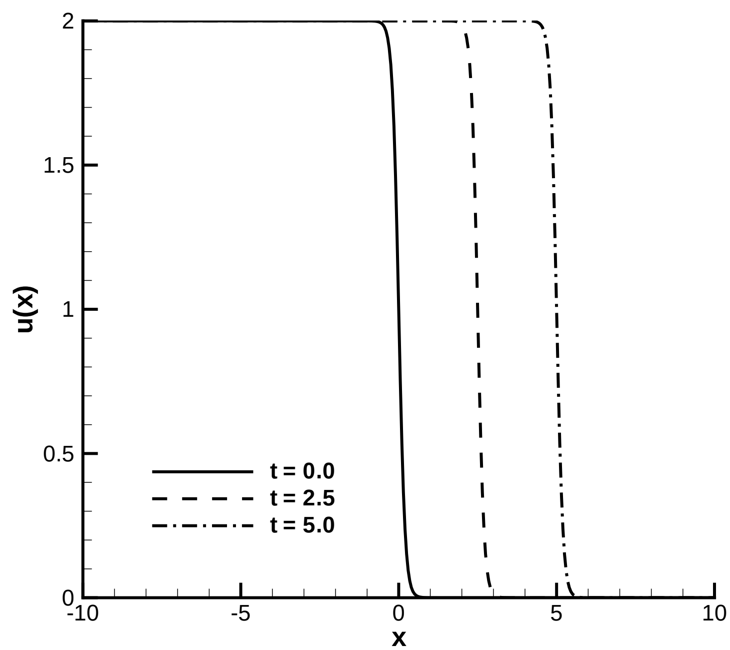

Figure 2. Discrete solution to viscous Burgers equation using 256 equispaced nodes with (4-8-4) narrow operators.

Tables 4 through 6 demonstrate that the interface has a negligible effect on the accuracy for this well-resolved test case. In addition, the treatment has been verified to be design order accurate, even when a discontinuous grid resolution is used across the subdomains.

Table 4. Error Norms and Convergence Rates of (2-4-2) Operator for Single Domain, Two Domains with Uniform Resolution, and Two Domains with compression ratio $r = 2$.

Single Domain

Resolution	Error			
N	L^2 Error	L^2 Rate	L^∞ Error	L^∞ Rate
64	8.89e-03	-	1.13e-02	-
128	6.37e-04	3.80	8.86e-04	3.68
256	4.19e-05	3.93	6.17e-05	3.84
512	2.66e-06	3.98	4.04e-06	3.93

Two Domains, $r = 1$

Resolution		Error			
N_1	N_2	L^2 Error	L^2 Rate	L^∞ Error	L^∞ Rate
32	32	8.91e-03	-	1.14e-02	-
64	64	6.38e-04	3.80	8.87e-04	3.68
128	128	4.19e-05	3.93	6.17e-05	3.85
256	256	2.66e-06	3.98	4.04e-06	3.93

Two Domains, $r = 2$

Resolution		Error			
N_1	N_2	L^2 Error	L^2 Rate	L^∞ Error	L^∞ Rate
32	64	6.26e-04	-	8.65e-04	-
64	128	4.12e-05	3.93	6.08e-05	3.83
128	256	2.61e-06	3.98	3.97e-06	3.94
256	512	1.64e-07	4.00	2.50e-07	3.99

Table 5. Error Norms and Convergence Rates of (3-6-3) Operator for Single Domain, Two Domains with Uniform Resolution, and Two Domains with compression ratio $r = 2$.

Single Domain

Resolution	Error			
N	L^2 Error	L^2 Rate	L^∞ Error	L^∞ Rate
64	3.48e-03	- 3.96e-03	-	
128	9.88e-05	5.14	1.33e-04	4.89
256	1.89e-06	5.71	3.23e-06	5.37
512	3.13e-08	5.92	5.41e-08	5.90

Two Domains, $r = 1$

Resolution		Error			
N_1	N_2	L^2 Error	L^2 Rate	L^∞ Error	L^∞ Rate
32	32	3.47e-03	-	3.91e-03	-
64	64	9.87e-05	5.14	1.33e-04	4.88
128	128	1.89e-06	5.71	3.21e-06	5.37
256	256	3.11e-08	5.92	5.37e-08	5.90

Two Domains, $r = 2$

Resolution		Error			
N_1	N_2	L^2 Error	L^2 Rate	L^∞ Error	L^∞ Rate
32	64	9.82e-05	-	1.34e-04	-
64	128	1.87e-06	5.72	3.15e-06	5.41
128	256	3.11e-08	5.91	5.05e-08	5.96
256	512	5.54e-10	5.81	7.55e-10	6.07

6 Conclusion

We have developed a class of split-form finite-difference operators that satisfy the sufficient conditions of the Lax-Wendroff theorem. These operators are applicable to the conservation or divergence form of the conservation law, but facilitate operator

Table 6. Error Norms and Convergence Rates of (4-8-4) Operator for Single Domain, Two Domains with Uniform Resolution, and Two Domains with compression ratio $r = 2$.

Single Domain

Resolution	Error			
N	L^2 Error	L^2 Rate	L^∞ Error	L^∞ Rate
64	1.97e-03	-	2.00e-03	-
128	2.88e-05	6.10	3.56e-05	5.81
256	1.81e-07	7.31	3.37e-07	6.72
512	8.53e-10	7.73	1.36e-09	7.95

Two Domains, $r = 1$

Resolution		Error			
N_1	N_2	L^2 Error	L^2 Rate	L^∞ Error	L^∞ Rate
32	32	2.16e-03	-	2.70e-03	-
64	64	2.89e-05	6.23	3.79e-05	6.15
128	128	1.84e-07	7.29	2.93e-07	7.02
256	256	1.01e-09	7.51	1.67e-09	7.46

Two Domains, $r = 2$

Resolution		Error			
N_1	N_2	L^2 Error	L^2 Rate	L^∞ Error	L^∞ Rate
32	64	3.01e-05	-	4.44e-05	-
64	128	2.67e-07	6.82	4.36e-07	6.67
128	256	3.43e-09	6.28	4.63e-09	6.56
256	512	1.09e-10	4.98	4.83e-11	6.58

splitting at the discrete level to improve accuracy or robustness, depending on the conservation law. The method for constructing a conservative split operator is illustrated for the (2-4-2) operator with a general splitting parameter. Higher order operators were also derived and are supplied in an accompanying text file. We also developed fully conservative operators for linear viscous terms.

The specific operators that are derived herein were tested on the conservation and canonical split forms of Burgers equation. Energy stability of the discrete form was proven, including boundary and interface closures. The split and conservation forms yield very similar results for Burgers equation. The solutions converged at least at the design order of accuracy in all cases, even when multiple domains were used with discontinuous grid resolutions across interfaces. The interface treatment that was tested herein had a negligible effect on the accuracy of the solution. Nonlinear dissipation operators for split forms will be presented in a future paper.

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Appendix A

Discrete Conservation

The summation-by-parts operators are discretely conservative, as demonstrated in the following lemma.

Lemma A.1. *All differentiation matrices that satisfy the SBP convention given in eq.* (2.1) *are discretely conservative in the $\mathcal{P}$-norm.*

Proof. Consistency of $\mathcal{D}$ implies that all rows in $\mathcal{D}$ the matrix sum to zero. Because $\mathcal{P}$ is nonsingular, the following matrix relations hold:

$$\mathcal{D}_{sbp}\mathbf{1} = \mathcal{P}^{-1}\mathcal{Q}\mathbf{1} = \mathcal{Q}\mathbf{1} = \mathbf{0}, \quad \mathbf{1} = (1, \cdots, 1)^T, \quad \mathbf{0} = (0, \cdots, 0)^T \tag{A1}$$

which demonstrates that all rows of $\mathcal{Q}$ sum to zero. Forming the dotproduct of $\mathcal{Q} + \mathcal{Q}^T$ with the unit vector and combining the result with that obtained in eq. (A1) yields

$$(\mathcal{Q} + \mathcal{Q}^T)\mathbf{1} = \mathcal{Q}^T\mathbf{1} = \mathcal{B}\mathbf{1} = (-1, 0, \ldots, 0, 1)^T \tag{A2}$$

Taking the transpose of eq. (A2) and combining it with the definition of an SBP operator yields

$$\mathbf{1}^T\mathcal{Q} = \mathbf{1}^T\mathcal{P}\mathcal{D}_{sbp} = (-1, 0, \ldots, 0, 1) \tag{A3}$$

Thus, eq. (A3) demonstrates that the columns of $\mathcal{D}_{sbp}$ are conservative in the $\mathcal{P}$ norm (i.e., all interior columns of the matrix $\mathcal{D}$ sum to zero, and the left and right boundary columns sum to -1 and 1, respectively, after premultiplication by $\mathcal{P}$). □

Appendix B

Split Continuous Equations

To utilize a split operator, the flux f in eq. (2.3) must be the product of at least two functions: $f(u) = v(u)w(u)$. Equation (2.3) is split by using the product rule

$$u_t + \alpha f(u)_x + (1-\alpha)\left[v(u)w(u)_x + w(u)v(u)_x\right] = 0. \tag{B1}$$

This splitting does not change the weak solution to the governing equation, nor does it alter the Rankine Hugoniot relation. To show this, we start with the conservation law in one dimension:

$$\frac{d}{dt}\int\limits_{x_L}^{x_R} u dx + f(u)\big|_{x_L}^{x_R} = 0, \quad x \in [x_L, x_R], \quad t \in [0,\infty) \tag{B2}$$

with a discontinuity located at x_d. At this point, note that neither eq. (2.3) nor eq. (B1) is valid across the discontinuity. Both differential forms are valid on either side of the discontinuity, with the *mild* restriction that $w(u)_x$ and $v(u)_x$ exist in all regions where $f(u)_x$ exists. Equation (B2) can be rewritten [32] as

$$\frac{d}{dt}\int\limits_{x_L}^{x_d^-} u\,dx + \frac{d}{dt}\int\limits_{x_d^+}^{x_R} u\,dx + f(u)\big|_{x_L}^{x_R} = 0$$

where u is smooth over the interval of integration. If we apply the Leibniz integral rule, then

$$\int\limits_{x_L}^{x_d^-} u_t\,dx + Su(x_d^-) + \int\limits_{x_d^+}^{x_R} u_t\,dx - Su(x_d^+) + f(u)\big|_{x_L}^{x_R} = 0$$

where $S = \frac{dx_d}{dt}$ is the propagation speed of the discontinuity. Either eq. (2.3) or eq. (B1) can be used to substitute for u_t, because the two forms are equal for smooth solutions. The conservation law becomes

$$\int\limits_{x_L}^{x_d^-} -f(u)_x dx + Su(x_d^-) + \int\limits_{x_d^+}^{x_R} -f(u)_x dx - Su(x_d^+) + f(u)\big|_{x_L}^{x_R} = 0$$

Upon simplification, we get the Rankine Hugoniot relation

$$\left[f(x_d^+) - f(x_d^-)\right] - S\left[u(x_d^+) - u(x_d^-)\right] = 0 \tag{B3}$$

Thus, both eq. (2.3) and eq. (B1) satisfy the sufficient condition of the Lax-Wendroff theorem which specifies that the continuous weak solution of the discretized equations must satisfy the Rankine Hugoniot relations.

Appendix C

High-Order Skew-Symmetric Fluxes

C.1 The Fourth-Order Operator: (3-6-3)

The full (3-6-3) operator is available upon request. The use of electronic format necessitates the brevity of this presentation and easier implementation. For reference, the interior flux operator is

$$f_i = \frac{1}{360} \begin{pmatrix} u_{i-2} \\ u_{i-1} \\ u_i \\ u_{i+1} \\ u_{i+2} \\ u_{i+3} \end{pmatrix}^T \begin{pmatrix} 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & -16 & 0 & -9 & 1 & 0 \\ 0 & 0 & 74 & 45 & -9 & 1 \\ 1 & -9 & 45 & 74 & 0 & 0 \\ 0 & 1 & -9 & 0 & -16 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} u_{i-2} \\ u_{i-1} \\ u_i \\ u_{i+1} \\ u_{i+2} \\ u_{i+3} \end{pmatrix} \tag{C1}$$

C.2 The Complete Fifth-Order Operator: (4-8-4)

The full operator can be found in the accompanying text file. The flux for the interior is

$$f_i = \frac{1}{840} \begin{pmatrix} u_{i-3} \\ u_{i-2} \\ u_{i-1} \\ u_i \\ u_{i+1} \\ u_{i+2} \\ u_{i+3} \\ u_{i+4} \end{pmatrix}^T \begin{pmatrix} -1 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{29}{3} & 0 & 0 & \frac{16}{3} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{139}{3} & 0 & -28 & \frac{16}{3} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{533}{3} & 112 & -28 & \frac{16}{3} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{16}{3} & -28 & 112 & \frac{533}{3} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{16}{3} & -28 & 0 & -\frac{139}{3} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & \frac{16}{3} & 0 & 0 & \frac{29}{3} & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} u_{i-3} \\ u_{i-2} \\ u_{i-1} \\ u_i \\ u_{i+1} \\ u_{i+2} \\ u_{i+3} \\ u_{i+4} \end{pmatrix} \tag{C2}$$

Appendix D

Conservative Finite-Difference Viscous Terms

Many conservation laws of interest also have viscous terms. We show that for the linear viscous terms ϵu_{xx} the summation-by-parts narrow-stencil approximation to the second derivative can be cast into a conservative form. The SBP finite-difference operators for the second derivative can be written as

$$\mathcal{D}_2 = \mathcal{D}^2 \quad \text{or} \quad \mathcal{D}_2 = \mathcal{P}^{-1}\left(\mathcal{R} + \mathcal{B}\mathcal{S}\right), \quad \mathcal{R} = \mathcal{R}^T, \quad \mathbf{v}^T\mathcal{R}\mathbf{v} \leq 0 \tag{D1}$$

where $\mathcal{S}$ represents a first derivative operator. In eq. **??**, $\mathcal{D}_2 = \mathcal{D}^2$ is called the wide-stencil approximation to the second derivative, which requires that $\mathcal{S} = \mathcal{D}$; $\mathcal{D}_2 = \mathcal{P}^{-1}\left(\mathcal{R} + \mathcal{B}\mathcal{S}\right)$ is the narrow-stencil approximation to the second derivative and does not impose the same restriction on $\mathcal{S}$ [30]. The narrow-stencil approximation has superior accuracy and stability properties and should be used where possible.

The conservation form of the narrow stencil is found by constructing an approximation to the viscous flux, ϵu_x, at the flux points such that

$$\Delta\bar{\mathbf{f}}_V = -\epsilon\Delta\bar{\mathcal{R}}\mathbf{u} = -\epsilon\left(\mathcal{R} + \mathcal{B}\mathcal{S}\right)\mathbf{u} \tag{D2}$$

This is solved by using a linear system for an appropriate $\mathcal{R}$ and $\mathcal{B}\mathcal{S}$ that satisfies the SBP condition. The viscous flux

$$\bar{\mathbf{f}}_V = -\epsilon\bar{\mathcal{R}}\mathbf{u} \tag{D3}$$

yields a fully conservative discrete approximation to $-\epsilon\mathcal{D}_2\mathbf{u} = \Delta\bar{\mathbf{f}}_V$. The operator $\bar{\mathcal{R}}$ for the $2p = 4$ scheme is presented herein, while the $\bar{\mathcal{R}}$ operators for $2p = 6$ and $2p = 8$ are appended verbatim in E. For the fourth-order case, we begin with the existing fourth-order operator of Mattssön et al. [30], which can be expressed with the relation $\mathcal{D}_2\mathbf{u} = \Delta\bar{\mathcal{R}}\mathbf{u} = \left(\mathcal{R} + \mathcal{B}\mathcal{S}\right)\mathbf{u}$, where

$$\mathcal{R}\delta x = \begin{pmatrix}
-\frac{9}{8} & \frac{59}{48} & -\frac{1}{12} & -\frac{1}{48} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{59}{48} & -\frac{59}{24} & \frac{59}{48} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{12} & \frac{59}{48} & -\frac{55}{24} & \frac{59}{48} & -\frac{1}{12} & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{48} & 0 & \frac{59}{48} & -\frac{59}{24} & \frac{4}{3} & -\frac{1}{12} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{12} & \frac{4}{3} & -\frac{5}{2} & \frac{4}{3} & -\frac{1}{12} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{1}{12} & \frac{4}{3} & -\frac{5}{2} & \frac{4}{3} & -\frac{1}{12} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{1}{12} & \frac{4}{3} & -\frac{59}{24} & \frac{59}{48} & 0 & -\frac{1}{48} \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{12} & \frac{59}{48} & -\frac{55}{24} & \frac{59}{48} & -\frac{1}{12} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{59}{48} & -\frac{59}{24} & \frac{59}{48} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{48} & -\frac{1}{12} & \frac{59}{48} & -\frac{9}{8}
\end{pmatrix} \tag{D4}$$

and

$$\mathcal{BS}\delta x = \begin{pmatrix} \frac{11}{6} & -3 & \frac{3}{2} & -\frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3} & \frac{3}{2} & -3 & \frac{11}{6} \end{pmatrix} \tag{D5}$$

Solving for $\bar{\mathcal{R}}$ in the relation $\Delta\bar{\mathcal{R}}\mathbf{u} = (\mathcal{R} + \mathcal{BS})\,\mathbf{u}$ yields

$$\bar{\mathcal{R}}\delta x = \begin{pmatrix} -\frac{11}{6} & 3 & -\frac{3}{2} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 \\ -\frac{9}{8} & \frac{59}{48} & -\frac{1}{12} & -\frac{1}{48} & 0 & 0 & 0 & 0 & 0 \\ \frac{5}{48} & -\frac{59}{48} & \frac{55}{48} & -\frac{1}{48} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{48} & 0 & -\frac{55}{48} & \frac{29}{24} & -\frac{1}{12} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{12} & -\frac{5}{4} & \frac{5}{4} & -\frac{1}{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & \ddots & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{12} & -\frac{5}{4} & \frac{5}{4} & -\frac{1}{12} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{12} & -\frac{29}{24} & \frac{55}{48} & 0 & -\frac{1}{48} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{48} & -\frac{55}{48} & \frac{59}{48} & -\frac{5}{48} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{48} & \frac{1}{12} & -\frac{59}{48} & \frac{9}{8} \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{3} & \frac{3}{2} & -3 & \frac{11}{6} \end{pmatrix} \tag{D6}$$

Note that this operational form is only design order accurate for a constant diffusivity ϵ.

A similar manipulation can be performed by using the sixth- and eighth-order diffusion operators. These operators are appended verbatim in E.

Appendix E

Stencil Coefficients

The following section includes the coefficients for implementation of the split form conservative finite difference simulations. The given coefficients can be inserted directly into source code using "cut-and-paste".

E.1 Definitions

- First derivative of vector u, $D_1 u = \mathcal{P}^{-1}\mathcal{Q}u$, [see eq. (2.9)] .
- Second derivative of vector u, $D_2 u = \mathcal{P}^{-1}(\mathcal{R} + \mathcal{B}\mathcal{S})u$, [see eq. (3.1)].
- Split first derivative of $w(u)v(u)$,

$$\alpha D_1 W v + (1-\alpha)(W D_1 v + V D_1 w) = \mathcal{P}^{-1}\Delta\bar{f}$$

  , [see eq. (2.24)].
- Conservative Second Derivative, $\mathcal{P}^{-1}\Delta f = -\epsilon\mathcal{P}^{-1}\Delta\bar{\mathcal{R}}u$, [see eq. (3.2)].

For each scheme [(2-4-2), (3-6-3), and (4-8-4)], the following are given:

- The flux points for the local stencil, written in terms of a general flux $f = v * w$.
- The diagonal of the P-norm $\mathcal{P}$ scaled by the grid spacing
- The diagonal of the inverse of the P-norm $\mathcal{P}$ scaled by the inverse of grid spacing
- The nonzero entries of the $\mathcal{Q}$ matrix for the SBP first derivative operator (for reference)
- The nonzero entries of the matrix $\bar{\mathcal{R}}$, used to calculate the conservative flux terms
- The nonzero entries of the matrix $\mathcal{R}$, which approximates the second derivative
- The nonzero entries of the matrix $\mathcal{B} * \mathcal{S}$, which yields the boundary differentiation operator

Note that only the first s+1 entries are shown. The (s+1)th entry is the interior operator, and just shifts indices for subsequent entries in the matrix. The respective symmetry or skew symmetry properties of each matrix is as follows:

- The flux points are per-symmetric. Replace v_i and w_j with $u_{N-(i-1)}$ and $w_{N-(j-1)}$, respectively, at right boundary i=1,2,...,N. Note that for Burgers equation $w_i = u_i$ and $v_j = u_j/2$.

- The P-norm $\mathcal{P}$ is per-symmetric.

$$P_{i,i} = P_{N-(i-1),N-(i-1)}, i = 1, 2, ..., N$$

- The inverse P-norm $\mathcal{P}^{-1}$ is per-symmetric.

$$\mathcal{P}^{-1}_{i,i} = \mathcal{P}^{-1}_{N-(i-1),N-(i-1)}, i = 1, 2, ..., N$$

- The boundary block portion of $\mathcal{Q}$ is per-skew-symmetric.

$$\mathcal{Q}_{N-(i-1),N-(j-1)} = -\mathcal{Q}_{i,j}, i = 1, 2, ..., s, j = 1, 2, ..., s$$

- The boundary block portion of $\bar{\mathcal{R}}$ is per-skew-symmetric.

$$\bar{\mathcal{R}}_{N-(i),N-(j-1)} = -\bar{\mathcal{R}}_{i,j}, i = 0, 2, ..., s-1, j = 1, 2, ..., s$$

- The boundary block portion of $\mathcal{R}$ is per-symmetric.

$$\mathcal{R}_{N-(i-1),N-(j-1)} = \mathcal{R}_{i,j}, i = 1, 2, ..., s, j = 1, 2, ..., s$$

- The boundary contribution $\mathcal{B} * \mathcal{S}$ is per-symmetric.

$$\mathcal{BS}_{N,N-(j-1)} = \mathcal{B} * \mathcal{S}_{1,j}, j = 1, 2, ..., N$$

E.2 Coefficients

E.2.1 (2-4-2) Scheme Skew-Symmetric

Flux point j:

$v_{j-1} {*} w_{j-1} = -0.0833333333333333333333333333{*}\alpha{+}0.0000000000000000000000000000$
$v_{j-1} {*} w_{j+0} = 0.0000000000000000000000000000{*}\alpha{+}0.0000000000000000000000000000$
$v_{j-1} {*} w_{j+1} = 0.0833333333333333333333333333{*}\alpha{-}0.0833333333333333333333333333$
$v_{j-1} {*} w_{j+2} = 0.0000000000000000000000000000{*}\alpha{+}0.0000000000000000000000000000$
$v_{j+0} {*} w_{j-1} = 0.0000000000000000000000000000{*}\alpha{+}0.0000000000000000000000000000$
$v_{j+0} {*} w_{j+0} = 0.5833333333333333333333333333{*}\alpha{+}0.0000000000000000000000000000$
$v_{j+0} {*} w_{j+1} = -0.6666666666666666666666666667{*}\alpha{+}0.6666666666666666666666666667$
$v_{j+0} {*} w_{j+2} = 0.0833333333333333333333333333{*}\alpha{-}0.0833333333333333333333333333$
$v_{j+1} {*} w_{j-1} = 0.0833333333333333333333333333{*}\alpha{-}0.0833333333333333333333333333$
$v_{j+1} {*} w_{j+0} = -0.6666666666666666666666666667{*}\alpha{+}0.6666666666666666666666666667$
$v_{j+1} {*} w_{j+1} = 0.5833333333333333333333333333{*}\alpha{+}0.0000000000000000000000000000$
$v_{j+1} {*} w_{j+2} = 0.0000000000000000000000000000{*}\alpha{+}0.0000000000000000000000000000$
$v_{j+2} {*} w_{j-1} = 0.0000000000000000000000000000{*}\alpha{+}0.0000000000000000000000000000$
$v_{j+2} {*} w_{j+0} = 0.0833333333333333333333333333{*}\alpha{-}0.0833333333333333333333333333$
$v_{j+2} {*} w_{j+1} = 0.0000000000000000000000000000{*}\alpha{+}0.0000000000000000000000000000$
$v_{j+2} {*} w_{j+2} = -0.0833333333333333333333333333{*}\alpha{+}0.0000000000000000000000000000$

Flux point j=3

$v_1*w_1 = -0.0312500000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_1*w_2 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_1*w_3 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_1*w_4 = 0.0312500000000000000000000000*\alpha-0.0312500000000000000000000000$
$v_1*w_5 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_2*w_1 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_2*w_2 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_2*w_3 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_2*w_4 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_2*w_5 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_3*w_1 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_3*w_2 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_3*w_3 = 0.5312500000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_3*w_4 = -0.6145833333333333333333333333*\alpha+0.6145833333333333333333333333$
$v_3*w_5 = 0.0833333333333333333333333333*\alpha-0.0833333333333333333333333333$
$v_4*w_1 = 0.0312500000000000000000000000*\alpha-0.0312500000000000000000000000$
$v_4*w_2 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_4*w_3 = -0.6145833333333333333333333333*\alpha+0.6145833333333333333333333333$
$v_4*w_4 = 0.5833333333333333333333333333*\alpha+0.0000000000000000000000000000$
$v_4*w_5 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_5*w_1 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_5*w_2 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_5*w_3 = 0.0833333333333333333333333333*\alpha-0.0833333333333333333333333333$
$v_5*w_4 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_5*w_5 = -0.0833333333333333333333333333*\alpha+0.0000000000000000000000000000$

Flux point j=2

$v_1*w_1 = -0.1145833333333333333333333333*\alpha+0.0000000000000000000000000000$
$v_1*w_2 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_1*w_3 = 0.0833333333333333333333333333*\alpha-0.0833333333333333333333333333$
$v_1*w_4 = 0.0312500000000000000000000000*\alpha-0.0312500000000000000000000000$
$v_2*w_1 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_2*w_2 = 0.6145833333333333333333333333*\alpha+0.0000000000000000000000000000$
$v_2*w_3 = -0.6145833333333333333333333333*\alpha+0.6145833333333333333333333333$
$v_2*w_4 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_3*w_1 = 0.0833333333333333333333333333*\alpha-0.0833333333333333333333333333$
$v_3*w_2 = -0.6145833333333333333333333333*\alpha+0.6145833333333333333333333333$
$v_3*w_3 = 0.5312500000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_3*w_4 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_4*w_1 = 0.0312500000000000000000000000*\alpha-0.0312500000000000000000000000$
$v_4*w_2 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_4*w_3 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_4*w_4 = -0.0312500000000000000000000000*\alpha+0.0000000000000000000000000000$

Flux point j=1
$v_1 * w_1 = 0.5000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_1 * w_2 = -0.6145833333333333333333333333 * \alpha + 0.6145833333333333333333333333$
$v_1 * w_3 = 0.08333333333333333333333333333 * \alpha - 0.08333333333333333333333333333$
$v_1 * w_4 = 0.03125000000000000000000000000 * \alpha - 0.03125000000000000000000000000$
$v_2 * w_1 = -0.6145833333333333333333333333 * \alpha + 0.6145833333333333333333333333$
$v_2 * w_2 = 0.6145833333333333333333333333 * \alpha + 0.0000000000000000000000000000$
$v_2 * w_3 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_2 * w_4 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_3 * w_1 = 0.08333333333333333333333333333 * \alpha - 0.08333333333333333333333333333$
$v_3 * w_2 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_3 * w_3 = -0.08333333333333333333333333333 * \alpha + 0.0000000000000000000000000000$
$v_3 * w_4 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_4 * w_1 = 0.03125000000000000000000000000 * \alpha - 0.03125000000000000000000000000$
$v_4 * w_2 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_4 * w_3 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_4 * w_4 = -0.03125000000000000000000000000 * \alpha + 0.0000000000000000000000000000$

Flux point j=0
$v_1 * w_1 = 0.0000000000000000000000000000 * \alpha + 1.0000000000000000000000000000$

The P-norm $\mathcal{P}$ is :
$p_{1,1}/dx = 0.3541666666666666666666666666666666667$
$p_{2,2}/dx = 1.229166666666666666666666666666666667$
$p_{3,3}/dx = 0.8958333333333333333333333333333333333$
$p_{4,4}/dx = 1.020833333333333333333333333333333333$
$p_{5,5}/dx = 1.000000000000000000000000000000000000$

The inverse P-norm $\mathcal{P}^{-1}$ is:
$\mathcal{P}^{-1}_{1,1} * dx = 2.823529411764705882352941176470588235 3$
$\mathcal{P}^{-1}_{2,2} * dx = 0.8135593220338983050847457627118644067 8$
$\mathcal{P}^{-1}_{3,3} * dx = 1.116279069767441860465116279069767441 9$
$\mathcal{P}^{-1}_{4,4} * dx = 0.9795918367346938775510204081632653061 2$
$\mathcal{P}^{-1}_{5,5} * dx = 1.000000000000000000000000000000000000$

The $\mathcal{Q}_{i,j} * dx$ nonzero terms are:
$\mathcal{Q}_{1,1} = -0.5000000000000000000000000000000000000$
$\mathcal{Q}_{1,2} = 0.6145833333333333333333333333333333333$
$\mathcal{Q}_{1,3} = -0.08333333333333333333333333333333333333$
$\mathcal{Q}_{1,4} = -0.03125000000000000000000000000000000000$
$\mathcal{Q}_{2,1} = -0.6145833333333333333333333333333333333$
$\mathcal{Q}_{2,3} = 0.6145833333333333333333333333333333333$
$\mathcal{Q}_{3,1} = 0.08333333333333333333333333333333333333$
$\mathcal{Q}_{3,2} = -0.6145833333333333333333333333333333333$

$\mathcal{Q}_{3,4} = 0.61458333333333333333333333333333333333$
$\mathcal{Q}_{3,5} = -0.083333333333333333333333333333333333333$
$\mathcal{Q}_{4,1} = 0.031250000000000000000000000000000000000$
$\mathcal{Q}_{4,3} = -0.61458333333333333333333333333333333333$
$\mathcal{Q}_{4,5} = 0.66666666666666666666666666666666666667$
$\mathcal{Q}_{4,6} = -0.083333333333333333333333333333333333333$
$\mathcal{Q}_{5,3} = 0.083333333333333333333333333333333333333$
$\mathcal{Q}_{5,4} = -0.66666666666666666666666666666666666667$
$\mathcal{Q}_{5,6} = 0.66666666666666666666666666666666666667$
$\mathcal{Q}_{5,7} = -0.083333333333333333333333333333333333333$

The (2-4-2) viscous terms.
The $\bar{\mathcal{R}}_{i,j} * dx$ nonzero terms:
$\bar{\mathcal{R}}_{0,1} = -1.8333333333333333333333333333333333333$
$\bar{\mathcal{R}}_{0,2} = 3.0000000000000000000000000000000000000$
$\bar{\mathcal{R}}_{0,3} = -1.5000000000000000000000000000000000000$
$\bar{\mathcal{R}}_{0,4} = 0.33333333333333333333333333333333333333$
$\bar{\mathcal{R}}_{1,1} = -1.1250000000000000000000000000000000000$
$\bar{\mathcal{R}}_{1,2} = 1.2291666666666666666666666666666666667$
$\bar{\mathcal{R}}_{1,3} = -0.083333333333333333333333333333333333333$
$\bar{\mathcal{R}}_{1,4} = -0.020833333333333333333333333333333333333$
$\bar{\mathcal{R}}_{2,1} = 0.10416666666666666666666666666666666667$
$\bar{\mathcal{R}}_{2,2} = -1.2291666666666666666666666666666666667$
$\bar{\mathcal{R}}_{2,3} = 1.1458333333333333333333333333333333333$
$\bar{\mathcal{R}}_{2,4} = -0.020833333333333333333333333333333333333$
$\bar{\mathcal{R}}_{3,1} = 0.020833333333333333333333333333333333333$
$\bar{\mathcal{R}}_{3,3} = -1.1458333333333333333333333333333333333$
$\bar{\mathcal{R}}_{3,4} = 1.2083333333333333333333333333333333333$
$\bar{\mathcal{R}}_{3,5} = -0.083333333333333333333333333333333333333$
$\bar{\mathcal{R}}_{4,3} = 0.083333333333333333333333333333333333333$
$\bar{\mathcal{R}}_{4,4} = -1.2500000000000000000000000000000000000$
$\bar{\mathcal{R}}_{4,5} = 1.2500000000000000000000000000000000000$
$\bar{\mathcal{R}}_{4,6} = -0.083333333333333333333333333333333333333$

The $\mathcal{R}_{i,j} * dx$ nonzero terms are:
$\mathcal{R}_{1,1} = -1.1250000000000000000000000000000000000$
$\mathcal{R}_{1,2} = 1.2291666666666666666666666666666666667$
$\mathcal{R}_{1,3} = -0.083333333333333333333333333333333333333$
$\mathcal{R}_{1,4} = -0.020833333333333333333333333333333333333$
$\mathcal{R}_{2,1} = 1.2291666666666666666666666666666666667$
$\mathcal{R}_{2,2} = -2.4583333333333333333333333333333333333$
$\mathcal{R}_{2,3} = 1.2291666666666666666666666666666666667$
$\mathcal{R}_{3,1} = -0.083333333333333333333333333333333333333$
$\mathcal{R}_{3,2} = 1.2291666666666666666666666666666666667$
$\mathcal{R}_{3,3} = -2.2916666666666666666666666666666666667$

$\mathcal{R}_{3,4} = 1.22916666666666666666666666666666666667$
$\mathcal{R}_{3,5} = -0.083333333333333333333333333333333333333$
$\mathcal{R}_{4,1} = -0.020833333333333333333333333333333333333$
$\mathcal{R}_{4,3} = 1.22916666666666666666666666666666666667$
$\mathcal{R}_{4,4} = -2.4583333333333333333333333333333333333$
$\mathcal{R}_{4,5} = 1.3333333333333333333333333333333333333$
$\mathcal{R}_{4,6} = -0.083333333333333333333333333333333333333$
$\mathcal{R}_{5,3} = -0.083333333333333333333333333333333333333$
$\mathcal{R}_{5,4} = 1.3333333333333333333333333333333333333$
$\mathcal{R}_{5,5} = -2.5000000000000000000000000000000000000$
$\mathcal{R}_{5,6} = 1.3333333333333333333333333333333333333$
$\mathcal{R}_{5,7} = -0.083333333333333333333333333333333333333$

The $\mathcal{BS}_{i,j}$ nonzero terms are:
$\mathcal{BS}_{1,1} = 1.8333333333333333333333333333333333333$
$\mathcal{BS}_{1,2} = -3.0000000000000000000000000000000000000$
$\mathcal{BS}_{1,3} = 1.5000000000000000000000000000000000000$
$\mathcal{BS}_{1,4} = -0.33333333333333333333333333333333333333$

E.2.2 (3-6-3) Scheme Skew-Symmetric

The flux point coefficients at point j are:
$v_{j-2}*w_{j-2} = 0.016666666666666666666666666667*\alpha+0.0000000000000000000000000000$
$v_{j-2}*w_{j-1} = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_{j-2}*w_{j+0} = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_{j-2}*w_{j+1} = -0.016666666666666666666666666667*\alpha+0.016666666666666666666666666667$
$v_{j-2}*w_{j+2} = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_{j-2}*w_{j+3} = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_{j-1}*w_{j-2} = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_{j-1}*w_{j-1} = -0.1333333333333333333333333333*\alpha+0.0000000000000000000000000000$
$v_{j-1}*w_{j+0} = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_{j-1}*w_{j+1} = 0.1500000000000000000000000000*\alpha-0.1500000000000000000000000000$
$v_{j-1}*w_{j+2} = -0.016666666666666666666666666667*\alpha+0.016666666666666666666666666667$
$v_{j-1}*w_{j+3} = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_{j+0}*w_{j-2} = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_{j+0}*w_{j-1} = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_{j+0}*w_{j+0} = 0.6166666666666666666666666667*\alpha+0.0000000000000000000000000000$
$v_{j+0}*w_{j+1} = -0.7500000000000000000000000000*\alpha+0.7500000000000000000000000000$
$v_{j+0}*w_{j+2} = 0.1500000000000000000000000000*\alpha-0.1500000000000000000000000000$
$v_{j+0}*w_{j+3} = -0.016666666666666666666666666667*\alpha+0.016666666666666666666666666667$
$v_{j+1}*w_{j-2} = -0.016666666666666666666666666667*\alpha+0.016666666666666666666666666667$
$v_{j+1}*w_{j-1} = 0.1500000000000000000000000000*\alpha-0.1500000000000000000000000000$
$v_{j+1}*w_{j+0} = -0.7500000000000000000000000000*\alpha+0.7500000000000000000000000000$
$v_{j+1}*w_{j+1} = 0.6166666666666666666666666667*\alpha+0.0000000000000000000000000000$

$v_{j+1}*w_{j+2} = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_{j+1}*w_{j+3} = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_{j+2}*w_{j-2} = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_{j+2}*w_{j-1} = -0.01666666666666666666666666667*\alpha+0.01666666666666666666666666667$
$v_{j+2}*w_{j+0} = 0.1500000000000000000000000000*\alpha-0.1500000000000000000000000000$
$v_{j+2}*w_{j+1} = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_{j+2}*w_{j+2} = -0.1333333333333333333333333333*\alpha+0.0000000000000000000000000000$
$v_{j+2}*w_{j+3} = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_{j+3}*w_{j-2} = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_{j+3}*w_{j-1} = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_{j+3}*w_{j+0} = -0.01666666666666666666666666667*\alpha+0.01666666666666666666666666667$
$v_{j+3}*w_{j+1} = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_{j+3}*w_{j+2} = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_{j+3}*w_{j+3} = 0.01666666666666666666666666667*\alpha+0.0000000000000000000000000000$

Flux point j=6
$v_3*w_3 = -0.01396219135802469135802469136*\alpha+0.0000000000000000000000000000$
$v_3*w_4 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_3*w_5 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_3*w_6 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_3*w_7 = 0.01396219135802469135802469136*\alpha-0.01396219135802469135802469136$
$v_3*w_8 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_3*w_9 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_4*w_3 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_4*w_4 = 0.06997685185185185185185185185*\alpha+0.0000000000000000000000000000$
$v_4*w_5 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_4*w_6 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_4*w_7 = -0.06997685185185185185185185185*\alpha+0.06997685185185185185185185185$
$v_4*w_8 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_4*w_9 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_5*w_3 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_5*w_4 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_5*w_5 = -0.2056828703703703703703703704*\alpha+0.0000000000000000000000000000$
$v_5*w_6 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_5*w_7 = 0.2223495370370370370370370370*\alpha-0.2223495370370370370370370370$
$v_5*w_8 = -0.01666666666666666666666666667*\alpha+0.01666666666666666666666666667$
$v_5*w_9 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_6*w_3 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_6*w_4 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_6*w_5 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_6*w_6 = 0.6496682098765432098765432099*\alpha+0.0000000000000000000000000000$
$v_6*w_7 = -0.7830015432098765432098765432*\alpha+0.7830015432098765432098765432$
$v_6*w_8 = 0.1500000000000000000000000000*\alpha-0.1500000000000000000000000000$
$v_6*w_9 = -0.01666666666666666666666666667*\alpha+0.01666666666666666666666666667$
$v_7*w_3 = 0.01396219135802469135802469136*\alpha-0.01396219135802469135802469136$

$v_7 * w_4 = -0.0699768518518518518518518518518518 5 * \alpha + 0.0699768518518518518518518518518518 5$
$v_7 * w_5 = 0.2223495370370370370370370370 * \alpha - 0.2223495370370370370370370370$
$v_7 * w_6 = -0.7830015432098765432098765432 * \alpha + 0.7830015432098765432098765432$
$v_7 * w_7 = 0.6166666666666666666666666667 * \alpha + 0.0000000000000000000000000000$
$v_7 * w_8 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_7 * w_9 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_8 * w_3 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_8 * w_4 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_8 * w_5 = -0.0166666666666666666666666667 * \alpha + 0.0166666666666666666666666667$
$v_8 * w_6 = 0.1500000000000000000000000000 * \alpha - 0.1500000000000000000000000000$
$v_8 * w_7 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_8 * w_8 = -0.1333333333333333333333333333 * \alpha + 0.0000000000000000000000000000$
$v_8 * w_9 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_9 * w_3 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_9 * w_4 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_9 * w_5 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_9 * w_6 = -0.0166666666666666666666666667 * \alpha + 0.0166666666666666666666666667$
$v_9 * w_7 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_9 * w_8 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_9 * w_9 = 0.0166666666666666666666666667 * \alpha + 0.0000000000000000000000000000$

Flux point j=5
$v_2 * w_2 = -0.0227276234567901234567901234 6 * \alpha + 0.0000000000000000000000000000$
$v_2 * w_3 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_2 * w_4 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_2 * w_5 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_2 * w_6 = 0.0227276234567901234567901234 6 * \alpha - 0.0227276234567901234567901234 6$
$v_2 * w_7 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_2 * w_8 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_3 * w_2 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_3 * w_3 = 0.0804822530864197530864197530 9 * \alpha + 0.0000000000000000000000000000$
$v_3 * w_4 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_3 * w_5 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_3 * w_6 = -0.0944444444444444444444444444 4 * \alpha + 0.0944444444444444444444444444 4$
$v_3 * w_7 = 0.0139621913580246913580246913 6 * \alpha - 0.0139621913580246913580246913 6$
$v_3 * w_8 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_4 * w_2 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_4 * w_3 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_4 * w_4 = -0.1737924382716049382716049383 * \alpha + 0.0000000000000000000000000000$
$v_4 * w_5 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_4 * w_6 = 0.2437692901234567901234567901 * \alpha - 0.2437692901234567901234567901$
$v_4 * w_7 = -0.0699768518518518518518518518518518 5 * \alpha + 0.0699768518518518518518518518518518 5$
$v_4 * w_8 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_5 * w_2 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_5 * w_3 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$

$v_5 * w_4 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_5 * w_5 = 0.6160378086419753086419753086 * \alpha + 0.0000000000000000000000000000$
$v_5 * w_6 = -0.8217206790123456790123456790 * \alpha + 0.8217206790123456790123456790$
$v_5 * w_7 = 0.2223495370370370370370370370 * \alpha - 0.2223495370370370370370370370$
$v_5 * w_8 = -0.0166666666666666666666666667 * \alpha + 0.0166666666666666666666666667$
$v_6 * w_2 = 0.0227276234567901234567901234 6 * \alpha - 0.0227276234567901234567901234 6$
$v_6 * w_3 = -0.0944444444444444444444444444 * \alpha + 0.0944444444444444444444444444$
$v_6 * w_4 = 0.2437692901234567901234567901 * \alpha - 0.2437692901234567901234567901$
$v_6 * w_5 = -0.8217206790123456790123456790 * \alpha + 0.8217206790123456790123456790$
$v_6 * w_6 = 0.6496682098765432098765432099 * \alpha + 0.0000000000000000000000000000$
$v_6 * w_7 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_6 * w_8 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_7 * w_2 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_7 * w_3 = 0.0139621913580246913580246913 6 * \alpha - 0.0139621913580246913580246913 6$
$v_7 * w_4 = -0.0699768518518518518518518518 5 * \alpha + 0.0699768518518518518518518518 5$
$v_7 * w_5 = 0.2223495370370370370370370370 * \alpha - 0.2223495370370370370370370370$
$v_7 * w_6 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_7 * w_7 = -0.1663348765432098765432098765 * \alpha + 0.0000000000000000000000000000$
$v_7 * w_8 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_8 * w_2 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_8 * w_3 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_8 * w_4 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_8 * w_5 = -0.0166666666666666666666666667 * \alpha + 0.0166666666666666666666666667$
$v_8 * w_6 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_8 * w_7 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_8 * w_8 = 0.0166666666666666666666666667 * \alpha + 0.0000000000000000000000000000$

Flux point j=4
$v_1 * w_1 = 0.0652777777777777777777777778 * \alpha + 0.0000000000000000000000000000$
$v_1 * w_2 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_1 * w_3 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_1 * w_4 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_1 * w_5 = -0.0652777777777777777777777778 * \alpha + 0.0652777777777777777777777778$
$v_1 * w_6 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_1 * w_7 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_2 * w_1 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_2 * w_2 = -0.1170100308641975308641975309 * \alpha + 0.0000000000000000000000000000$
$v_2 * w_3 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_2 * w_4 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_2 * w_5 = 0.0942824074074074074074074074 1 * \alpha - 0.0942824074074074074074074074 1$
$v_2 * w_6 = 0.0227276234567901234567901234 6 * \alpha - 0.0227276234567901234567901234 6$
$v_2 * w_7 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_3 * w_1 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_3 * w_2 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_3 * w_3 = -0.1723919753086419753086419753 * \alpha + 0.0000000000000000000000000000$

$v_3 * w_4 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_3 * w_5 = 0.2528742283950617283950617284 * \alpha - 0.2528742283950617283950617284$
$v_3 * w_6 = -0.0944444444444444444444444444 * \alpha + 0.0944444444444444444444444444$
$v_3 * w_7 = 0.0139621913580246913580246913 6 * \alpha - 0.0139621913580246913580246913 6$
$v_4 * w_1 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_4 * w_2 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_4 * w_3 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_4 * w_4 = 0.7241242283950617283950617284 * \alpha + 0.0000000000000000000000000000$
$v_4 * w_5 = -0.8979166666666666666666666667 * \alpha + 0.8979166666666666666666666667$
$v_4 * w_6 = 0.2437692901234567901234567901 * \alpha - 0.2437692901234567901234567901$
$v_4 * w_7 = -0.0699768518518518518518518518 5 * \alpha + 0.0699768518518518518518518518 5$
$v_5 * w_1 = -0.0652777777777777777777777777 8 * \alpha + 0.0652777777777777777777777777 8$
$v_5 * w_2 = 0.0942824074074074074074074074 1 * \alpha - 0.0942824074074074074074074074 1$
$v_5 * w_3 = 0.2528742283950617283950617284 * \alpha - 0.2528742283950617283950617284$
$v_5 * w_4 = -0.8979166666666666666666666667 * \alpha + 0.8979166666666666666666666667$
$v_5 * w_5 = 0.6160378086419753086419753086 * \alpha + 0.0000000000000000000000000000$
$v_5 * w_6 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_5 * w_7 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_6 * w_1 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_6 * w_2 = 0.0227276234567901234567901234 6 * \alpha - 0.0227276234567901234567901234 6$
$v_6 * w_3 = -0.0944444444444444444444444444 * \alpha + 0.0944444444444444444444444444$
$v_6 * w_4 = 0.2437692901234567901234567901 * \alpha - 0.2437692901234567901234567901$
$v_6 * w_5 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_6 * w_6 = -0.1720524691358024691358024691 * \alpha + 0.0000000000000000000000000000$
$v_6 * w_7 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_7 * w_1 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_7 * w_2 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_7 * w_3 = 0.0139621913580246913580246913 6 * \alpha - 0.0139621913580246913580246913 6$
$v_7 * w_4 = -0.0699768518518518518518518518 5 * \alpha + 0.0699768518518518518518518518 5$
$v_7 * w_5 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_7 * w_6 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_7 * w_7 = 0.0560146604938271604938271604 9 * \alpha + 0.0000000000000000000000000000$

Flux point j=3
$v_1 * w_1 = -0.0930555555555555555555555555 6 * \alpha + 0.0000000000000000000000000000$
$v_1 * w_2 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_1 * w_3 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_1 * w_4 = 0.1583333333333333333333333333 * \alpha - 0.1583333333333333333333333333$
$v_1 * w_5 = -0.0652777777777777777777777777 8 * \alpha + 0.0652777777777777777777777777 8$
$v_1 * w_6 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_1 * w_7 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_2 * w_1 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_2 * w_2 = 0.2480478395061728395061728395 * \alpha + 0.0000000000000000000000000000$
$v_2 * w_3 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_2 * w_4 = -0.3650578703703703703703703704 * \alpha + 0.3650578703703703703703703704$

$v_2*w_5 = 0.0942824074074074074074074074\,1*\alpha - 0.09428240740740740740740740741$
$v_2*w_6 = 0.02272762345679012345679012346*\alpha - 0.02272762345679012345679012346$
$v_2*w_7 = 0.0000000000000000000000000000*\alpha + 0.0000000000000000000000000000$
$v_3*w_1 = 0.0000000000000000000000000000*\alpha + 0.0000000000000000000000000000$
$v_3*w_2 = 0.0000000000000000000000000000*\alpha + 0.0000000000000000000000000000$
$v_3*w_3 = 0.3450077160493827160493827161*\alpha + 0.0000000000000000000000000000$
$v_3*w_4 = -0.5173996913580246913580246914*\alpha + 0.5173996913580246913580246914$
$v_3*w_5 = 0.2528742283950617283950617284*\alpha - 0.2528742283950617283950617284$
$v_3*w_6 = -0.09444444444444444444444444444*\alpha + 0.09444444444444444444444444444$
$v_3*w_7 = 0.01396219135802469135802469136*\alpha - 0.01396219135802469135802469136$
$v_4*w_1 = 0.1583333333333333333333333333*\alpha - 0.1583333333333333333333333333$
$v_4*w_2 = -0.3650578703703703703703703704*\alpha + 0.3650578703703703703703703704$
$v_4*w_3 = -0.5173996913580246913580246914*\alpha + 0.5173996913580246913580246914$
$v_4*w_4 = 0.7241242283950617283950617284*\alpha + 0.0000000000000000000000000000$
$v_4*w_5 = 0.0000000000000000000000000000*\alpha + 0.0000000000000000000000000000$
$v_4*w_6 = 0.0000000000000000000000000000*\alpha + 0.0000000000000000000000000000$
$v_4*w_7 = 0.0000000000000000000000000000*\alpha + 0.0000000000000000000000000000$
$v_5*w_1 = -0.06527777777777777777777777778*\alpha + 0.06527777777777777777777777778$
$v_5*w_2 = 0.09428240740740740740740740741*\alpha - 0.09428240740740740740740740741$
$v_5*w_3 = 0.2528742283950617283950617284*\alpha - 0.2528742283950617283950617284$
$v_5*w_4 = 0.0000000000000000000000000000*\alpha + 0.0000000000000000000000000000$
$v_5*w_5 = -0.2818788580246913580246913580*\alpha + 0.0000000000000000000000000000$
$v_5*w_6 = 0.0000000000000000000000000000*\alpha + 0.0000000000000000000000000000$
$v_5*w_7 = 0.0000000000000000000000000000*\alpha + 0.0000000000000000000000000000$
$v_6*w_1 = 0.0000000000000000000000000000*\alpha + 0.0000000000000000000000000000$
$v_6*w_2 = 0.02272762345679012345679012346*\alpha - 0.02272762345679012345679012346$
$v_6*w_3 = -0.09444444444444444444444444444*\alpha + 0.09444444444444444444444444444$
$v_6*w_4 = 0.0000000000000000000000000000*\alpha + 0.0000000000000000000000000000$
$v_6*w_5 = 0.0000000000000000000000000000*\alpha + 0.0000000000000000000000000000$
$v_6*w_6 = 0.07171682098765432098765432099*\alpha + 0.0000000000000000000000000000$
$v_6*w_7 = 0.0000000000000000000000000000*\alpha + 0.0000000000000000000000000000$
$v_7*w_1 = 0.0000000000000000000000000000*\alpha + 0.0000000000000000000000000000$
$v_7*w_2 = 0.0000000000000000000000000000*\alpha + 0.0000000000000000000000000000$
$v_7*w_3 = 0.01396219135802469135802469136*\alpha - 0.01396219135802469135802469136$
$v_7*w_4 = 0.0000000000000000000000000000*\alpha + 0.0000000000000000000000000000$
$v_7*w_5 = 0.0000000000000000000000000000*\alpha + 0.0000000000000000000000000000$
$v_7*w_6 = 0.0000000000000000000000000000*\alpha + 0.0000000000000000000000000000$
$v_7*w_7 = -0.01396219135802469135802469136*\alpha + 0.0000000000000000000000000000$

Flux point j=2
$v_1*w_1 = -0.1638888888888888888888888889*\alpha + 0.0000000000000000000000000000$
$v_1*w_2 = 0.0000000000000000000000000000*\alpha + 0.0000000000000000000000000000$
$v_1*w_3 = 0.07083333333333333333333333333*\alpha - 0.07083333333333333333333333333$
$v_1*w_4 = 0.1583333333333333333333333333*\alpha - 0.1583333333333333333333333333$
$v_1*w_5 = -0.06527777777777777777777777778*\alpha + 0.06527777777777777777777777778$

$v_1*w_6 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_2*w_1 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_2*w_2 = 0.6638888888888888888888888889*\alpha+0.0000000000000000000000000000$
$v_2*w_3 = -0.4158410493827160493827160494*\alpha+0.4158410493827160493827160494$
$v_2*w_4 = -0.3650578703703703703703703704*\alpha+0.3650578703703703703703703704$
$v_2*w_5 = 0.09428240740740740740740740741*\alpha-0.09428240740740740740740740741$
$v_2*w_6 = 0.02272762345679012345679012346*\alpha-0.02272762345679012345679012346$
$v_3*w_1 = 0.07083333333333333333333333333*\alpha-0.07083333333333333333333333333$
$v_3*w_2 = -0.4158410493827160493827160494*\alpha+0.4158410493827160493827160494$
$v_3*w_3 = 0.3450077160493827160493827161*\alpha+0.0000000000000000000000000000$
$v_3*w_4 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_3*w_5 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_3*w_6 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_4*w_1 = 0.1583333333333333333333333333*\alpha-0.1583333333333333333333333333$
$v_4*w_2 = -0.3650578703703703703703703704*\alpha+0.3650578703703703703703703704$
$v_4*w_3 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_4*w_4 = 0.2067245370370370370370370370*\alpha+0.0000000000000000000000000000$
$v_4*w_5 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_4*w_6 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_5*w_1 = -0.06527777777777777777777777778*\alpha+0.06527777777777777777777777778$
$v_5*w_2 = 0.09428240740740740740740740741*\alpha-0.09428240740740740740740740741$
$v_5*w_3 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_5*w_4 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_5*w_5 = -0.02900462962962962962962962963*\alpha+0.0000000000000000000000000000$
$v_5*w_6 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_6*w_1 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_6*w_2 = 0.02272762345679012345679012346*\alpha-0.02272762345679012345679012346$
$v_6*w_3 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_6*w_4 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_6*w_5 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_6*w_6 = -0.02272762345679012345679012346*\alpha+0.0000000000000000000000000000$

Flux point j=1
$v_1*w_1 = 0.5000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_1*w_2 = -0.6638888888888888888888888889*\alpha+0.6638888888888888888888888889$
$v_1*w_3 = 0.07083333333333333333333333333*\alpha-0.07083333333333333333333333333$
$v_1*w_4 = 0.1583333333333333333333333333*\alpha-0.1583333333333333333333333333$
$v_1*w_5 = -0.06527777777777777777777777778*\alpha+0.06527777777777777777777777778$
$v_2*w_1 = -0.6638888888888888888888888889*\alpha+0.6638888888888888888888888889$
$v_2*w_2 = 0.6638888888888888888888888889*\alpha+0.0000000000000000000000000000$
$v_2*w_3 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_2*w_4 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_2*w_5 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_3*w_1 = 0.07083333333333333333333333333*\alpha-0.07083333333333333333333333333$
$v_3*w_2 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$

$v_3 * w_3 = -0.0708333333333333333333333333 * \alpha + 0.0000000000000000000000000000$
$v_3 * w_4 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_3 * w_5 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_4 * w_1 = 0.1583333333333333333333333333 * \alpha - 0.1583333333333333333333333333$
$v_4 * w_2 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_4 * w_3 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_4 * w_4 = -0.1583333333333333333333333333 * \alpha + 0.0000000000000000000000000000$
$v_4 * w_5 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_5 * w_1 = -0.0652777777777777777777777778 * \alpha + 0.0652777777777777777777777778$
$v_5 * w_2 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_5 * w_3 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_5 * w_4 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_5 * w_5 = 0.0652777777777777777777777778 * \alpha + 0.0000000000000000000000000000$

Flux point j=0
$v_1 * w_1 = 0.0000000000000000000000000000 * \alpha + 1.0000000000000000000000000000$

The P-norm $\mathcal{P}$ is:
$\mathcal{P}_{1,1}/dx = 0.308333333333333333333333333333333333$
$\mathcal{P}_{2,2}/dx = 1.436087962962962962962962962962962963$
$\mathcal{P}_{3,3}/dx = 0.513310185185185185185185185185185185$
$\mathcal{P}_{4,4}/dx = 1.392824074074074074074074074074074074$
$\mathcal{P}_{5,5}/dx = 0.797453703703703703703703703703703704$
$\mathcal{P}_{6,6}/dx = 1.059606481481481481481481481481481481$
$\mathcal{P}_{7,7}/dx = 0.992384259259259259259259259259259259$
$\mathcal{P}_{8,8}/dx = 1.000000000000000000000000000000000000$

The P-norm inverse $\mathcal{P}^{-1}$ is:
$\mathcal{P}^{-1}_{1,1} * dx = 3.243243243243243243243243243243243243$
$\mathcal{P}^{-1}_{2,2} * dx = 0.696336175631457631489869275778139557$
$\mathcal{P}^{-1}_{3,3} * dx = 1.948139797068771138669673055242390079$
$\mathcal{P}^{-1}_{4,4} * dx = 0.717965763669602792089080937344191457$
$\mathcal{P}^{-1}_{5,5} * dx = 1.253991291727140783744557329462989840$
$\mathcal{P}^{-1}_{6,6} * dx = 0.943746586564718732932823593664664118$
$\mathcal{P}^{-1}_{7,7} * dx = 1.007674185346737888082853210795176226$
$\mathcal{P}^{-1}_{8,8} * dx = 1.000000000000000000000000000000000000$

The $\mathcal{Q}_{i,j} * dx$ nonzero terms are:
$\mathcal{Q}_{1,1} = -0.500000000000000000000000000000000000$
$\mathcal{Q}_{1,2} = 0.663888888888888888888888888888888889$
$\mathcal{Q}_{1,3} = -0.070833333333333333333333333333333333$
$\mathcal{Q}_{1,4} = -0.158333333333333333333333333333333333$
$\mathcal{Q}_{1,5} = 0.065277777777777777777777777777777778$
$\mathcal{Q}_{2,1} = -0.663888888888888888888888888888888889$

$\mathcal{Q}_{2,3} = 0.4158410493827160493827160493827160493 8$
$\mathcal{Q}_{2,4} = 0.3650578703703703703703703703703703703 7$
$\mathcal{Q}_{2,5} = -0.0942824074074074074074074074074074074 07$
$\mathcal{Q}_{2,6} = -0.0227276234567901234567901234567901234 57$
$\mathcal{Q}_{3,1} = 0.0708333333333333333333333333333333333 3$
$\mathcal{Q}_{3,2} = -0.4158410493827160493827160493827160493 8$
$\mathcal{Q}_{3,4} = 0.5173996913580246913580246913580246913 6$
$\mathcal{Q}_{3,5} = -0.2528742283950617283950617283950617284 0$
$\mathcal{Q}_{3,6} = 0.0944444444444444444444444444444444444 4$
$\mathcal{Q}_{3,7} = -0.0139621913580246913580246913580246913 58$
$\mathcal{Q}_{4,1} = 0.1583333333333333333333333333333333333 3$
$\mathcal{Q}_{4,2} = -0.3650578703703703703703703703703703703 7$
$\mathcal{Q}_{4,3} = -0.5173996913580246913580246913580246913 6$
$\mathcal{Q}_{4,5} = 0.8979166666666666666666666666666666666 7$
$\mathcal{Q}_{4,6} = -0.2437692901234567901234567901234567901 2$
$\mathcal{Q}_{4,7} = 0.0699768518518518518518518518518518518 52$
$\mathcal{Q}_{5,1} = -0.0652777777777777777777777777777777777 78$
$\mathcal{Q}_{5,2} = 0.0942824074074074074074074074074074074 07$
$\mathcal{Q}_{5,3} = 0.2528742283950617283950617283950617284 0$
$\mathcal{Q}_{5,4} = -0.8979166666666666666666666666666666666 7$
$\mathcal{Q}_{5,6} = 0.8217206790123456790123456790123456790 1$
$\mathcal{Q}_{5,7} = -0.2223495370370370370370370370370370370 4$
$\mathcal{Q}_{5,8} = 0.0166666666666666666666666666666666666 67$
$\mathcal{Q}_{6,2} = 0.0227276234567901234567901234567901234 57$
$\mathcal{Q}_{6,3} = -0.0944444444444444444444444444444444444 44$
$\mathcal{Q}_{6,4} = 0.2437692901234567901234567901234567901 2$
$\mathcal{Q}_{6,5} = -0.8217206790123456790123456790123456790 1$
$\mathcal{Q}_{6,7} = 0.7830015432098765432098765432098765432 1$
$\mathcal{Q}_{6,8} = -0.1500000000000000000000000000000000000 0$
$\mathcal{Q}_{6,9} = 0.0166666666666666666666666666666666666 67$
$\mathcal{Q}_{7,3} = 0.0139621913580246913580246913580246913 58$
$\mathcal{Q}_{7,4} = -0.0699768518518518518518518518518518518 52$
$\mathcal{Q}_{7,5} = 0.2223495370370370370370370370370370370 4$
$\mathcal{Q}_{7,6} = -0.7830015432098765432098765432098765432 1$
$\mathcal{Q}_{7,8} = 0.7500000000000000000000000000000000000 0$
$\mathcal{Q}_{7,9} = -0.1500000000000000000000000000000000000 0$
$\mathcal{Q}_{7,10} = 0.0166666666666666666666666666666666666 67$
$\mathcal{Q}_{8,5} = -0.0166666666666666666666666666666666666 67$
$\mathcal{Q}_{8,6} = 0.1500000000000000000000000000000000000 0$
$\mathcal{Q}_{8,7} = -0.7500000000000000000000000000000000000 0$
$\mathcal{Q}_{8,9} = 0.7500000000000000000000000000000000000 0$
$\mathcal{Q}_{8,10} = -0.1500000000000000000000000000000000000 0$
$\mathcal{Q}_{8,11} = 0.0166666666666666666666666666666666666 67$

The (3-6-3) viscous terms.

The $\bar{\mathcal{R}}_{i,j} * dx$ nonzero terms:
$\bar{\mathcal{R}}_{0,1} = -2.28333333333333333333333333333333333333$
$\bar{\mathcal{R}}_{0,2} = 5.00000000000000000000000000000000000000$
$\bar{\mathcal{R}}_{0,3} = -5.00000000000000000000000000000000000000$
$\bar{\mathcal{R}}_{0,4} = 3.33333333333333333333333333333333333333$
$\bar{\mathcal{R}}_{0,5} = -1.25000000000000000000000000000000000000$
$\bar{\mathcal{R}}_{0,6} = 0.200000000000000000000000000000000000000$
$\bar{\mathcal{R}}_{1,1} = -1.18402777777777777777777777777777777778$
$\bar{\mathcal{R}}_{1,2} = 1.32777777777777777777777777777777777778$
$\bar{\mathcal{R}}_{1,3} = -0.0708333333333333333333333333333333333333$
$\bar{\mathcal{R}}_{1,4} = -0.105555555555555555555555555555555555556$
$\bar{\mathcal{R}}_{1,5} = 0.0326388888888888888888888888888888888889$
$\bar{\mathcal{R}}_{2,1} = 0.143750000000000000000000000000000000000$
$\bar{\mathcal{R}}_{2,2} = -1.12252121913580246913580246913580246914$
$\bar{\mathcal{R}}_{2,3} = 0.760848765432098765432098765432098765432$
$\bar{\mathcal{R}}_{2,4} = 0.259502314814814814814814814814814814815$
$\bar{\mathcal{R}}_{2,5} = -0.0302160493827160493827160493827160493827$
$\bar{\mathcal{R}}_{2,6} = -0.0113638117283950617283950617283950617284$
$\bar{\mathcal{R}}_{3,1} = 0.0729166666666666666666666666666666666667$
$\bar{\mathcal{R}}_{3,2} = -0.290839120370370370370370370370370370370$
$\bar{\mathcal{R}}_{3,3} = -0.837907021604938271604938271604938271605$
$\bar{\mathcal{R}}_{3,4} = 1.29430169753086419753086419753086419753$
$\bar{\mathcal{R}}_{3,5} = -0.283090277777777777777777777777777777778$
$\bar{\mathcal{R}}_{3,6} = 0.0515991512345679012345679012345679012346$
$\bar{\mathcal{R}}_{3,7} = -0.00698109567901234567901234567901234567901$
$\bar{\mathcal{R}}_{4,1} = -0.0326388888888888888888888888888888888889$
$\bar{\mathcal{R}}_{4,2} = 0.0742187500000000000000000000000000000000$
$\bar{\mathcal{R}}_{4,3} = 0.196892361111111111111111111111111111111$
$\bar{\mathcal{R}}_{4,4} = -1.59871527777777777777777777777777777778$
$\bar{\mathcal{R}}_{4,5} = 1.51274305555555555555555555555555555556$
$\bar{\mathcal{R}}_{4,6} = -0.192170138888888888888888888888888888889$
$\bar{\mathcal{R}}_{4,7} = 0.0396701388888888888888888888888888888889$
$\bar{\mathcal{R}}_{5,2} = 0.0113638117283950617283950617283950617284$
$\bar{\mathcal{R}}_{5,3} = -0.0559818672839506172839506172839506172840$
$\bar{\mathcal{R}}_{5,4} = 0.197118055555555555555555555555555555556$
$\bar{\mathcal{R}}_{5,5} = -1.43220293209876543209876543209876543210$
$\bar{\mathcal{R}}_{5,6} = 1.45127121913580246913580246913580246914$
$\bar{\mathcal{R}}_{5,7} = -0.182679398148148148148148148148148148148$
$\bar{\mathcal{R}}_{5,8} = 0.0111111111111111111111111111111111111111$
$\bar{\mathcal{R}}_{6,3} = 0.00698109567901234567901234567901234567901$
$\bar{\mathcal{R}}_{6,4} = -0.0466512345679012345679012345679012345679$
$\bar{\mathcal{R}}_{6,5} = 0.211238425925925925925925925925925925926$
$\bar{\mathcal{R}}_{6,6} = -1.42711419753086419753086419753086419753$
$\bar{\mathcal{R}}_{6,7} = 1.38332368827160493827160493827160493827$
$\bar{\mathcal{R}}_{6,8} = -0.138888888888888888888888888888888888889$

$\bar{\mathcal{R}}_{6,9} = 0.01111111111111111111111111111111111111111$
$\bar{\mathcal{R}}_{7,5} = -0.01111111111111111111111111111111111111111$
$\bar{\mathcal{R}}_{7,6} = 0.1388888888888888888888888888888888888889$
$\bar{\mathcal{R}}_{7,7} = -1.361111111111111111111111111111111111111$
$\bar{\mathcal{R}}_{7,8} = 1.361111111111111111111111111111111111111$
$\bar{\mathcal{R}}_{7,9} = -0.1388888888888888888888888888888888888889$
$\bar{\mathcal{R}}_{7,10} = 0.01111111111111111111111111111111111111111$

The $\mathcal{R}_{i,j} * dx$ nonzero terms:
$\mathcal{R}_{1,1} = -1.184027777777777777777777777777777777778$
$\mathcal{R}_{1,2} = 1.327777777777777777777777777777777777778$
$\mathcal{R}_{1,3} = -0.07083333333333333333333333333333333333333$
$\mathcal{R}_{1,4} = -0.1055555555555555555555555555555555555556$
$\mathcal{R}_{1,5} = 0.03263888888888888888888888888888888888889$
$\mathcal{R}_{2,1} = 1.327777777777777777777777777777777777778$
$\mathcal{R}_{2,2} = -2.450298996913580246913580246913580246913580246$
$\mathcal{R}_{2,3} = 0.8316820987654320987654320987654320987654320987$
$\mathcal{R}_{2,4} = 0.3650578703703703703703703703703703703703$
$\mathcal{R}_{2,5} = -0.06285493827160493827160493827160493827$
$\mathcal{R}_{2,6} = -0.01136381172839506172839506172839506172$
$\mathcal{R}_{3,1} = -0.07083333333333333333333333333333333333333$
$\mathcal{R}_{3,2} = 0.8316820987654320987654320987654320987654320987$
$\mathcal{R}_{3,3} = -1.598755787037037037037037037037037037037$
$\mathcal{R}_{3,4} = 1.034799382716049382716049382716049382716$
$\mathcal{R}_{3,5} = -0.2528742283950617283950617283950617284$
$\mathcal{R}_{3,6} = 0.0629629629629629629629629629629629629629$
$\mathcal{R}_{3,7} = -0.006981095679012345679012345679012345679$
$\mathcal{R}_{4,1} = -0.1055555555555555555555555555555555555556$
$\mathcal{R}_{4,2} = 0.3650578703703703703703703703703703703703$
$\mathcal{R}_{4,3} = 1.034799382716049382716049382716049382716$
$\mathcal{R}_{4,4} = -2.893016975308641975308641975308641975$
$\mathcal{R}_{4,5} = 1.795833333333333333333333333333333333333$
$\mathcal{R}_{4,6} = -0.2437692901234567901234567901234567901$
$\mathcal{R}_{4,7} = 0.04665123456790123456790123456790123457$
$\mathcal{R}_{5,1} = 0.03263888888888888888888888888888888888889$
$\mathcal{R}_{5,2} = -0.06285493827160493827160493827160493827$
$\mathcal{R}_{5,3} = -0.2528742283950617283950617283950617284$
$\mathcal{R}_{5,4} = 1.795833333333333333333333333333333333333$
$\mathcal{R}_{5,5} = -2.944945987654320987654320987654320987654320987$
$\mathcal{R}_{5,6} = 1.643441358024691358024691358024691358$
$\mathcal{R}_{5,7} = -0.2223495370370370370370370370370370370370$
$\mathcal{R}_{5,8} = 0.01111111111111111111111111111111111111111$
$\mathcal{R}_{6,2} = -0.01136381172839506172839506172839506172$
$\mathcal{R}_{6,3} = 0.0629629629629629629629629629629629629629$
$\mathcal{R}_{6,4} = -0.2437692901234567901234567901234567901$

$\mathcal{R}_{6,5} = 1.6434413580246913580246913580246913580$
$\mathcal{R}_{6,6} = -2.8783854166666666666666666666666666667$
$\mathcal{R}_{6,7} = 1.5660030864197530864197530864197530864$
$\mathcal{R}_{6,8} = -0.15000000000000000000000000000000000000$
$\mathcal{R}_{6,9} = 0.011111111111111111111111111111111111111$
$\mathcal{R}_{7,3} = -0.0069810956790123456790123456790123456790$
$\mathcal{R}_{7,4} = 0.046651234567901234567901234567901234568$
$\mathcal{R}_{7,5} = -0.22234953703703703703703703703703703704$
$\mathcal{R}_{7,6} = 1.5660030864197530864197530864197530864$
$\mathcal{R}_{7,7} = -2.7444347993827160493827160493827160494$
$\mathcal{R}_{7,8} = 1.5000000000000000000000000000000000000$
$\mathcal{R}_{7,9} = -0.15000000000000000000000000000000000000$
$\mathcal{R}_{7,10} = 0.011111111111111111111111111111111111111$
$\mathcal{R}_{8,5} = 0.011111111111111111111111111111111111111$
$\mathcal{R}_{8,6} = -0.15000000000000000000000000000000000000$
$\mathcal{R}_{8,7} = 1.5000000000000000000000000000000000000$
$\mathcal{R}_{8,8} = -2.7222222222222222222222222222222222222$
$\mathcal{R}_{8,9} = 1.5000000000000000000000000000000000000$
$\mathcal{R}_{8,10} = -0.15000000000000000000000000000000000000$
$\mathcal{R}_{8,11} = 0.011111111111111111111111111111111111111$

The $\mathcal{BS}_{i,j}$ nonzero terms
$\mathcal{BS}_{1,1} = 2.2833333333333333333333333333333333333$
$\mathcal{BS}_{1,2} = -5.0000000000000000000000000000000000000$
$\mathcal{BS}_{1,3} = 5.0000000000000000000000000000000000000$
$\mathcal{BS}_{1,4} = -3.3333333333333333333333333333333333333$
$\mathcal{BS}_{1,5} = 1.2500000000000000000000000000000000000$
$\mathcal{BS}_{1,6} = -0.20000000000000000000000000000000000000$

E.2.3 (4-8-4) Scheme Skew-Symmetric

The coefficients for flux point j are :
$v_{j-3}{*}w_{j-3} = -0.003571428571428571428571428571{*}\alpha{+}0.0000000000000000000000000000$
$v_{j-3}{*}w_{j-2} = 0.0000000000000000000000000000{*}\alpha{+}0.0000000000000000000000000000$
$v_{j-3}{*}w_{j-1} = 0.0000000000000000000000000000{*}\alpha{+}0.0000000000000000000000000000$
$v_{j-3}{*}w_{j+0} = 0.0000000000000000000000000000{*}\alpha{+}0.0000000000000000000000000000$
$v_{j-3}{*}w_{j+1} = 0.003571428571428571428571428571{*}\alpha{-}0.003571428571428571428571428571$
$v_{j-3}{*}w_{j+2} = 0.0000000000000000000000000000{*}\alpha{+}0.0000000000000000000000000000$
$v_{j-3}{*}w_{j+3} = 0.0000000000000000000000000000{*}\alpha{+}0.0000000000000000000000000000$
$v_{j-3}{*}w_{j+4} = 0.0000000000000000000000000000{*}\alpha{+}0.0000000000000000000000000000$
$v_{j-2}{*}w_{j-3} = 0.0000000000000000000000000000{*}\alpha{+}0.0000000000000000000000000000$
$v_{j-2}{*}w_{j-2} = 0.03452380952380952380952380952{*}\alpha{+}0.0000000000000000000000000000$
$v_{j-2}{*}w_{j-1} = 0.0000000000000000000000000000{*}\alpha{+}0.0000000000000000000000000000$
$v_{j-2}{*}w_{j+0} = 0.0000000000000000000000000000{*}\alpha{+}0.0000000000000000000000000000$

$v_{j-2}*w_{j+1} = -0.03809523809523809523809523810*\alpha+0.03809523809523809523809523810$
$v_{j-2}*w_{j+2} = 0.003571428571428571428571428571*\alpha-0.003571428571428571428571428571$
$v_{j-2}*w_{j+3} = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_{j-2}*w_{j+4} = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_{j-1}*w_{j-3} = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_{j-1}*w_{j-2} = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_{j-1}*w_{j-1} = -0.1654761904761904761904761905*\alpha+0.0000000000000000000000000000$
$v_{j-1}*w_{j+0} = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_{j-1}*w_{j+1} = 0.2000000000000000000000000000*\alpha-0.2000000000000000000000000000$
$v_{j-1}*w_{j+2} = -0.03809523809523809523809523810*\alpha+0.03809523809523809523809523810$
$v_{j-1}*w_{j+3} = 0.003571428571428571428571428571*\alpha-0.003571428571428571428571428571$
$v_{j-1}*w_{j+4} = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_{j+0}*w_{j-3} = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_{j+0}*w_{j-2} = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_{j+0}*w_{j-1} = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_{j+0}*w_{j+0} = 0.6345238095238095238095238095*\alpha+0.0000000000000000000000000000$
$v_{j+0}*w_{j+1} = -0.8000000000000000000000000000*\alpha+0.8000000000000000000000000000$
$v_{j+0}*w_{j+2} = 0.2000000000000000000000000000*\alpha-0.2000000000000000000000000000$
$v_{j+0}*w_{j+3} = -0.03809523809523809523809523810*\alpha+0.03809523809523809523809523810$
$v_{j+0}*w_{j+4} = 0.003571428571428571428571428571*\alpha-0.003571428571428571428571428571$
$v_{j+1}*w_{j-3} = 0.003571428571428571428571428571*\alpha-0.003571428571428571428571428571$
$v_{j+1}*w_{j-2} = -0.03809523809523809523809523810*\alpha+0.03809523809523809523809523810$
$v_{j+1}*w_{j-1} = 0.2000000000000000000000000000*\alpha-0.2000000000000000000000000000$
$v_{j+1}*w_{j+0} = -0.8000000000000000000000000000*\alpha+0.8000000000000000000000000000$
$v_{j+1}*w_{j+1} = 0.6345238095238095238095238095*\alpha+0.0000000000000000000000000000$
$v_{j+1}*w_{j+2} = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_{j+1}*w_{j+3} = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_{j+1}*w_{j+4} = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_{j+2}*w_{j-3} = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_{j+2}*w_{j-2} = 0.003571428571428571428571428571*\alpha-0.003571428571428571428571428571$
$v_{j+2}*w_{j-1} = -0.03809523809523809523809523810*\alpha+0.03809523809523809523809523810$
$v_{j+2}*w_{j+0} = 0.2000000000000000000000000000*\alpha-0.2000000000000000000000000000$
$v_{j+2}*w_{j+1} = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_{j+2}*w_{j+2} = -0.1654761904761904761904761905*\alpha+0.0000000000000000000000000000$
$v_{j+2}*w_{j+3} = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_{j+2}*w_{j+4} = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_{j+3}*w_{j-3} = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_{j+3}*w_{j-2} = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_{j+3}*w_{j-1} = 0.003571428571428571428571428571*\alpha-0.003571428571428571428571428571$
$v_{j+3}*w_{j+0} = -0.03809523809523809523809523810*\alpha+0.03809523809523809523809523810$
$v_{j+3}*w_{j+1} = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_{j+3}*w_{j+2} = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_{j+3}*w_{j+3} = 0.03452380952380952380952380952*\alpha+0.0000000000000000000000000000$
$v_{j+3}*w_{j+4} = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_{j+4}*w_{j-3} = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$

$v_{j+4}*w_{j-2} = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_{j+4}*w_{j-1} = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_{j+4}*w_{j+0} = 0.0035714285714285714285714285714285714285571*\alpha-0.0035714285714285714285714285571$
$v_{j+4}*w_{j+1} = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_{j+4}*w_{j+2} = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_{j+4}*w_{j+3} = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_{j+4}*w_{j+4} = -0.0035714285714285714285714285571*\alpha+0.0000000000000000000000000000$

Flux point j=8
$v_4*w_4 = -0.0096027579627530024355542118082*\alpha+0.0000000000000000000000000000$
$v_4*w_5 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_4*w_6 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_4*w_7 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_4*w_8 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_4*w_9 = 0.0096027579627530024355542118082*\alpha-0.0096027579627530024355542118082$
$v_4*w_{10} = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_4*w_{11} = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_4*w_{12} = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_5*w_4 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_5*w_5 = 0.0432900302868060804568741 0767*\alpha+0.0000000000000000000000000000$
$v_5*w_6 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_5*w_7 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_5*w_8 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_5*w_9 = -0.0432900302868060804568741 0767*\alpha+0.0432900302868060804568741 0767$
$v_5*w_{10} = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_5*w_{11} = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_5*w_{12} = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_6*w_4 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_6*w_5 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_6*w_6 = -0.0553580050075585789871504 1572*\alpha+0.0000000000000000000000000000$
$v_6*w_7 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_6*w_8 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_6*w_9 = 0.0517865764361300075585789 8715*\alpha-0.0517865764361300075585789 8715$
$v_6*w_{10} = 0.0035714285714285714285714285571*\alpha-0.0035714285714285714285714285571$
$v_6*w_{11} = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_6*w_{12} = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_7*w_4 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_7*w_5 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_7*w_6 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_7*w_7 = -0.0832765823150247753422356 5970*\alpha+0.0000000000000000000000000000$
$v_7*w_8 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$
$v_7*w_9 = 0.1178003918388342991517594 692*\alpha-0.1178003918388342991517594 692$
$v_7*w_{10} = -0.0380952380952380952380952 3810*\alpha+0.0380952380952380952380952 3810$
$v_7*w_{11} = 0.0035714285714285714285714285571*\alpha-0.0035714285714285714285714285571$
$v_7*w_{12} = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$

$v_8 * w_4 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_8 * w_5 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_8 * w_6 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_8 * w_7 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_8 * w_8 = 0.6049473149985302763080540858 * \alpha + 0.0000000000000000000000000000$
$v_8 * w_9 = -0.7704235054747207524985302763 * \alpha + 0.7704235054747207524985302763$
$v_8 * w_{10} = 0.2000000000000000000000000000 * \alpha - 0.2000000000000000000000000000$
$v_8 * w_{11} = -0.03809523809523809523809523810 * \alpha + 0.03809523809523809523809523810$
$v_8 * w_{12} = 0.003571428571428571428571428571 * \alpha - 0.003571428571428571428571428571$
$v_9 * w_4 = 0.009602757962753002435542118082 * \alpha - 0.009602757962753002435542118082$
$v_9 * w_5 = -0.04329003028680608045687410767 * \alpha + 0.04329003028680608045687410767$
$v_9 * w_6 = 0.05178657643613000755857898715 * \alpha - 0.05178657643613000755857898715$
$v_9 * w_7 = 0.1178003918388342991517594692 * \alpha - 0.1178003918388342991517594692$
$v_9 * w_8 = -0.7704235054747207524985302763 * \alpha + 0.7704235054747207524985302763$
$v_9 * w_9 = 0.6345238095238095238095238095 * \alpha + 0.0000000000000000000000000000$
$v_9 * w_{10} = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_9 * w_{11} = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_9 * w_{12} = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_{10} * w_4 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_{10} * w_5 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_{10} * w_6 = 0.003571428571428571428571428571 * \alpha - 0.003571428571428571428571428571$
$v_{10} * w_7 = -0.03809523809523809523809523810 * \alpha + 0.03809523809523809523809523810$
$v_{10} * w_8 = 0.2000000000000000000000000000 * \alpha - 0.2000000000000000000000000000$
$v_{10} * w_9 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_{10} * w_{10} = -0.1654761904761904761904761905 * \alpha + 0.0000000000000000000000000000$
$v_{10} * w_{11} = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_{10} * w_{12} = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_{11} * w_4 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_{11} * w_5 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_{11} * w_6 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_{11} * w_7 = 0.003571428571428571428571428571 * \alpha - 0.003571428571428571428571428571$
$v_{11} * w_8 = -0.03809523809523809523809523810 * \alpha + 0.03809523809523809523809523810$
$v_{11} * w_9 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_{11} * w_{10} = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_{11} * w_{11} = 0.03452380952380952380952380952 * \alpha + 0.0000000000000000000000000000$
$v_{11} * w_{12} = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_{12} * w_4 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_{12} * w_5 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_{12} * w_6 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_{12} * w_7 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_{12} * w_8 = 0.003571428571428571428571428571 * \alpha - 0.003571428571428571428571428571$
$v_{12} * w_9 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_{12} * w_{10} = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_{12} * w_{11} = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_{12} * w_{12} = -0.003571428571428571428571428571 * \alpha + 0.0000000000000000000000000000$

Flux point j=7

$v_3*w_3 = -0.09050595238095238095238095238*\alpha+0.0000000000000000000000000000$

$v_3*w_4 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$

$v_3*w_5 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$

$v_3*w_6 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$

$v_3*w_7 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$

$v_3*w_8 = 0.09050595238095238095238095238*\alpha-0.09050595238095238095238095238$

$v_3*w_9 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$

$v_3*w_{10} = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$

$v_3*w_{11} = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$

$v_4*w_3 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$

$v_4*w_4 = 0.4166642632747543461829176115*\alpha+0.0000000000000000000000000000$

$v_4*w_5 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$

$v_4*w_6 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$

$v_4*w_7 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$

$v_4*w_8 = -0.4262670212375073486184597296*\alpha+0.4262670212375073486184597296$

$v_4*w_9 = 0.009602757962753002435542118082*\alpha-0.009602757962753002435542118082$

$v_4*w_{10} = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$

$v_4*w_{11} = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$

$v_5*w_3 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$

$v_5*w_4 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$

$v_5*w_5 = -0.7125127557581674645166708659*\alpha+0.0000000000000000000000000000$

$v_5*w_6 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$

$v_5*w_7 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$

$v_5*w_8 = 0.7558027860449735449735449735*\alpha-0.7558027860449735449735449735$

$v_5*w_9 = -0.04329003028680608045687410767*\alpha+0.04329003028680608045687410767$

$v_5*w_{10} = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$

$v_5*w_{11} = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$

$v_6*w_3 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$

$v_6*w_4 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$

$v_6*w_5 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$

$v_6*w_6 = 0.4365587627026119089611153103*\alpha+0.0000000000000000000000000000$

$v_6*w_7 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$

$v_6*w_8 = -0.4919167677101704879482657260*\alpha+0.4919167677101704879482657260$

$v_6*w_9 = 0.05178657643613000755857898715*\alpha-0.05178657643613000755857898715$

$v_6*w_{10} = 0.003571428571428571428571428571*\alpha-0.003571428571428571428571428571$

$v_6*w_{11} = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$

$v_7*w_3 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$

$v_7*w_4 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$

$v_7*w_5 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$

$v_7*w_6 = 0.0000000000000000000000000000*\alpha+0.0000000000000000000000000000$

$v_7*w_7 = 0.4497956821617535903250188964*\alpha+0.0000000000000000000000000000$

$v_7*w_8 = -0.5330722644767783656672545561*\alpha+0.5330722644767783656672545561$

$v_7*w_9 = 0.1178003918388342991517594692*\alpha-0.1178003918388342991517594692$

$v_7{*}w_{10} = -0.03809523809523809523809523810{*}\alpha{+}0.03809523809523809523809523810$
$v_7{*}w_{11} = 0.003571428571428571428571428571{*}\alpha{-}0.003571428571428571428571428571$
$v_8{*}w_3 = 0.09050595238095238095238095238{*}\alpha{-}0.09050595238095238095238095238$
$v_8{*}w_4 = -0.4262670212375073486184597296{*}\alpha{+}0.4262670212375073486184597296$
$v_8 * w_5 = 0.7558027860449735449735449735 * \alpha - 0.7558027860449735449735449735$
$v_8{*}w_6 = -0.4919167677101704879482657260{*}\alpha{+}0.4919167677101704879482657260$
$v_8{*}w_7 = -0.5330722644767783656672545561{*}\alpha{+}0.5330722644767783656672545561$
$v_8 * w_8 = 0.6049473149985302763080540858 * \alpha + 0.0000000000000000000000000000$
$v_8 * w_9 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_8 * w_{10} = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_8 * w_{11} = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_9 * w_3 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_9{*}w_4 = 0.009602757962753002435542118082{*}\alpha{-}0.009602757962753002435542118082$
$v_9{*}w_5 = -0.04329003028680608045687410767{*}\alpha{+}0.04329003028680608045687410767$
$v_9{*}w_6 = 0.05178657643613000755857898715{*}\alpha{-}0.05178657643613000755857898715$
$v_9 * w_7 = 0.1178003918388342991517594692 * \alpha - 0.1178003918388342991517594692$
$v_9 * w_8 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_9{*}w_9 = -0.1358996959509112286890064668{*}\alpha{+}0.0000000000000000000000000000$
$v_9 * w_{10} = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_9 * w_{11} = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_{10} * w_3 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_{10} * w_4 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_{10} * w_5 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_{10}{*}w_6 = 0.003571428571428571428571428571{*}\alpha{-}0.003571428571428571428571428571$
$v_{10}{*}w_7 = -0.03809523809523809523809523810{*}\alpha{+}0.03809523809523809523809523810$
$v_{10} * w_8 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_{10} * w_9 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_{10}{*}w_{10} = 0.03452380952380952380952380952{*}\alpha{+}0.0000000000000000000000000000$
$v_{10}{*}w_{11} = 0.0000000000000000000000000000{*}\alpha{+}0.0000000000000000000000000000$
$v_{11} * w_3 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_{11} * w_4 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_{11} * w_5 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_{11} * w_6 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_{11}{*}w_7 = 0.003571428571428571428571428571{*}\alpha{-}0.003571428571428571428571428571$
$v_{11} * w_8 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_{11} * w_9 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_{11}{*}w_{10} = 0.0000000000000000000000000000{*}\alpha{+}0.0000000000000000000000000000$
$v_{11}{*}w_{11} = -0.003571428571428571428571428571{*}\alpha{+}0.0000000000000000000000000000$

Flux point j=6
$v_2{*}w_2 = -0.2107209027515327118501721676{*}\alpha{+}0.0000000000000000000000000000$
$v_2 * w_3 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_2 * w_4 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_2 * w_5 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_2 * w_6 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$

$v_2 * w_7 = 0.2107209027515327118501721676 * \alpha - 0.2107209027515327118501721676$
$v_2 * w_8 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_2 * w_9 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_2 * w_{10} = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_3 * w_2 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_3 * w_3 = 1.0905082763815402704291593 18 * \alpha + 0.0000000000000000000000000000$
$v_3 * w_4 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_3 * w_5 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_3 * w_6 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_3 * w_7 = -1.1810142287624926513815402 70 * \alpha + 1.181014228762492651381540270$
$v_3 * w_8 = 0.0905059523809523809523809523 8 * \alpha - 0.09050595238095238095238095238$
$v_3 * w_9 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_3 * w_{10} = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_4 * w_2 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_4 * w_3 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_4 * w_4 = -2.220792581963340891912320484 * \alpha + 0.0000000000000000000000000000$
$v_4 * w_5 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_4 * w_6 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_4 * w_7 = 2.637456845238095238095238095 * \alpha - 2.637456845238095238095238095$
$v_4 * w_8 = -0.4262670212375073486184597296 * \alpha + 0.4262670212375073486184597296$
$v_4 * w_9 = 0.009602757962753002435542118082 * \alpha - 0.009602757962753002435542118082$
$v_4 * w_{10} = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_5 * w_2 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_5 * w_3 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_5 * w_4 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_5 * w_5 = 2.067331343301209372637944067 * \alpha + 0.0000000000000000000000000000$
$v_5 * w_6 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_5 * w_7 = -2.779844099059376837154614932 * \alpha + 2.779844099059376837154614932$
$v_5 * w_8 = 0.7558027860449735449735449735 * \alpha - 0.7558027860449735449735449735$
$v_5 * w_9 = -0.04329003028680608045687410767 * \alpha + 0.04329003028680608045687410767$
$v_5 * w_{10} = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_6 * w_2 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_6 * w_3 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_6 * w_4 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_6 * w_5 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_6 * w_6 = -0.2263261349678760393046107332 * \alpha + 0.0000000000000000000000000000$
$v_6 * w_7 = 0.6628848976704879482657260435 * \alpha - 0.6628848976704879482657260435$
$v_6 * w_8 = -0.4919167677101704879482657260 * \alpha + 0.4919167677101704879482657260$
$v_6 * w_9 = 0.05178657643613000755857898715 * \alpha - 0.05178657643613000755857898715$
$v_6 * w_{10} = 0.003571428571428571428571428571 * \alpha - 0.003571428571428571428571428571$
$v_7 * w_2 = 0.2107209027515327118501721676 * \alpha - 0.2107209027515327118501721676$
$v_7 * w_3 = -1.181014228762492651381540270 * \alpha + 1.181014228762492651381540270$
$v_7 * w_4 = 2.637456845238095238095238095 * \alpha - 2.637456845238095238095238095$
$v_7 * w_5 = -2.779844099059376837154614932 * \alpha + 2.779844099059376837154614932$
$v_7 * w_6 = 0.6628848976704879482657260435 * \alpha - 0.6628848976704879482657260435$

$v_7 * w_7 = 0.4497956821617535903250188964 * \alpha + 0.0000000000000000000000000000$
$v_7 * w_8 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_7 * w_9 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_7 * w_{10} = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_8 * w_2 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_8 * w_3 = 0.0905059523809523809523809523 8 * \alpha - 0.09050595238095238095238095238$
$v_8 * w_4 = -0.4262670212375073486184597296 * \alpha + 0.4262670212375073486184597296$
$v_8 * w_5 = 0.7558027860449735449735449735 * \alpha - 0.7558027860449735449735449735$
$v_8 * w_6 = -0.4919167677101704879482657260 * \alpha + 0.4919167677101704879482657260$
$v_8 * w_7 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_8 * w_8 = 0.07187505052175191064079952969 * \alpha + 0.0000000000000000000000000000$
$v_8 * w_9 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_8 * w_{10} = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_9 * w_2 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_9 * w_3 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_9 * w_4 = 0.009602757962753002435542118082 * \alpha - 0.009602757962753002435542118082$
$v_9 * w_5 = -0.04329003028680608045687410767 * \alpha + 0.04329003028680608045687410767$
$v_9 * w_6 = 0.05178657643613000755857898715 * \alpha - 0.05178657643613000755857898715$
$v_9 * w_7 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_9 * w_8 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_9 * w_9 = -0.01809930411207692953724699756 * \alpha + 0.0000000000000000000000000000$
$v_9 * w_{10} = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_{10} * w_2 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_{10} * w_3 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_{10} * w_4 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_{10} * w_5 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_{10} * w_6 = 0.003571428571428571428571428571 * \alpha - 0.003571428571428571428571428571$
$v_{10} * w_7 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_{10} * w_8 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_{10} * w_9 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_{10} * w_{10} = -0.003571428571428571428571428571 * \alpha + 0.0000000000000000000000000000$

Flux point j=5
$v_1 * w_1 = -0.1239583333333333333333333333 * \alpha + 0.0000000000000000000000000000$
$v_1 * w_2 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_1 * w_3 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_1 * w_4 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_1 * w_5 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_1 * w_6 = 0.1239583333333333333333333333 * \alpha - 0.1239583333333333333333333333$
$v_1 * w_7 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_1 * w_8 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_1 * w_9 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_2 * w_1 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_2 * w_2 = 1.048880436009280255312001344 * \alpha + 0.0000000000000000000000000000$
$v_2 * w_3 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$

$v_2 * w_4 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_2 * w_5 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_2 * w_6 = -1.2596013387608129671621735111 * \alpha + 1.2596013387608129671621735111$
$v_2 * w_7 = 0.2107209027515327118501721676 * \alpha - 0.2107209027515327118501721676$
$v_2 * w_8 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_2 * w_9 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_3 * w_1 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_3 * w_2 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_3 * w_3 = -3.0836807530496766607877718999 * \alpha + 0.0000000000000000000000000000$
$v_3 * w_4 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_3 * w_5 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_3 * w_6 = 4.1741890294312169312169312170 * \alpha - 4.1741890294312169312169312170$
$v_3 * w_7 = -1.1810142287624926513815402700 * \alpha + 1.1810142287624926513815402700$
$v_3 * w_8 = 0.0905059523809523809523809524 * \alpha - 0.0905059523809523809523809524$
$v_3 * w_9 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_4 * w_1 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_4 * w_2 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_4 * w_3 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_4 * w_4 = 3.9400002237391870328378264890 * \alpha + 0.0000000000000000000000000000$
$v_4 * w_5 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_4 * w_6 = -6.1607928057025279247501469720 * \alpha + 6.1607928057025279247501469720$
$v_4 * w_7 = 2.6374568452380952380952380950 * \alpha - 2.6374568452380952380952380950$
$v_4 * w_8 = -0.4262670212375073486184597296 * \alpha + 0.4262670212375073486184597296$
$v_4 * w_9 = 0.0096027579627530024355421180 82 * \alpha - 0.0096027579627530024355421180 82$
$v_5 * w_1 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_5 * w_2 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_5 * w_3 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_5 * w_4 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_5 * w_5 = -1.2812415733654572940287226000 * \alpha + 0.0000000000000000000000000000$
$v_5 * w_6 = 3.3485729166666666666666666667 * \alpha - 3.3485729166666666666666666667$
$v_5 * w_7 = -2.7798440990593768371546149320 * \alpha + 2.7798440990593768371546149320$
$v_5 * w_8 = 0.7558027860449735449735449735 * \alpha - 0.7558027860449735449735449735$
$v_5 * w_9 = -0.0432900302868060804568741076 7 * \alpha + 0.0432900302868060804568741076 7$
$v_6 * w_1 = 0.1239583333333333333333333333 * \alpha - 0.1239583333333333333333333333$
$v_6 * w_2 = -1.2596013387608129671621735111 * \alpha + 1.2596013387608129671621735111$
$v_6 * w_3 = 4.1741890294312169312169312170 * \alpha - 4.1741890294312169312169312170$
$v_6 * w_4 = -6.1607928057025279247501469720 * \alpha + 6.1607928057025279247501469720$
$v_6 * w_5 = 3.3485729166666666666666666667 * \alpha - 3.3485729166666666666666666667$
$v_6 * w_6 = -0.2263261349678760393046107332 * \alpha + 0.0000000000000000000000000000$
$v_6 * w_7 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_6 * w_8 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_6 * w_9 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_7 * w_1 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_7 * w_2 = 0.2107209027515327118501721676 * \alpha - 0.2107209027515327118501721676$
$v_7 * w_3 = -1.1810142287624926513815402700 * \alpha + 1.1810142287624926513815402700$

$v_7 * w_4 = 2.6374568452380952380952380 95 * \alpha - 2.637456845238095238095238095$
$v_7 * w_5 = -2.779844099059376837154614932 * \alpha + 2.779844099059376837154614932$
$v_7 * w_6 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_7 * w_7 = 1.112680579832241538590744940 * \alpha + 0.0000000000000000000000000000$
$v_7 * w_8 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_7 * w_9 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_8 * w_1 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_8 * w_2 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_8 * w_3 = 0.09050595238095238095238095238 * \alpha - 0.09050595238095238095238095238$
$v_8 * w_4 = -0.4262670212375073486184597296 * \alpha + 0.4262670212375073486184597296$
$v_8 * w_5 = 0.7558027860449735449735449735 * \alpha - 0.7558027860449735449735449735$
$v_8 * w_6 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_8 * w_7 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_8 * w_8 = -0.4200417171884185773074661964 * \alpha + 0.0000000000000000000000000000$
$v_8 * w_9 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_9 * w_1 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_9 * w_2 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_9 * w_3 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_9 * w_4 = 0.009602757962753002435542118082 * \alpha - 0.009602757962753002435542118082$
$v_9 * w_5 = -0.04329003028680608045687410767 * \alpha + 0.04329003028680608045687410767$
$v_9 * w_6 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_9 * w_7 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_9 * w_8 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_9 * w_9 = 0.03368727232405307802133198959 * \alpha + 0.0000000000000000000000000000$

Flux point j=4
$v_1 * w_1 = 0.4209583333333333333333333333 * \alpha + 0.0000000000000000000000000000$
$v_1 * w_2 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_1 * w_3 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_1 * w_4 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_1 * w_5 = -0.5449166666666666666666666667 * \alpha + 0.5449166666666666666666666667$
$v_1 * w_6 = 0.1239583333333333333333333333 * \alpha - 0.1239583333333333333333333333$
$v_1 * w_7 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_1 * w_8 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_1 * w_9 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_2 * w_1 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_2 * w_2 = -1.964242213745065927605610145 * \alpha + 0.0000000000000000000000000000$
$v_2 * w_3 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_2 * w_4 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_2 * w_5 = 3.013122649754346182917611489 * \alpha - 3.013122649754346182917611489$
$v_2 * w_6 = -1.259601338760812967162173511 * \alpha + 1.259601338760812967162173511$
$v_2 * w_7 = 0.2107209027515327118501721676 * \alpha - 0.2107209027515327118501721676$
$v_2 * w_8 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_2 * w_9 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_3 * w_1 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$

$v_3 * w_2 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_3 * w_3 = 3.2075498006687242798353909465 * \alpha + 0.0000000000000000000000000000$
$v_3 * w_4 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_3 * w_5 = -6.2912305537184009406231628450 * \alpha + 6.2912305537184009406231628450$
$v_3 * w_6 = 4.1741890294312169312169312169 * \alpha - 4.1741890294312169312169312169$
$v_3 * w_7 = -1.1810142287624926513815402704 * \alpha + 1.1810142287624926513815402704$
$v_3 * w_8 = 0.0905059523809523809523809523 * \alpha - 0.0905059523809523809523809523$
$v_3 * w_9 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_4 * w_1 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_4 * w_2 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_4 * w_3 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_4 * w_4 = -1.1642659202569916855631141350 * \alpha + 0.0000000000000000000000000000$
$v_4 * w_5 = 5.1042661439961787184009406231 * \alpha - 5.1042661439961787184009406231$
$v_4 * w_6 = -6.1607928057025279247501469724 * \alpha + 6.1607928057025279247501469724$
$v_4 * w_7 = 2.6374568452380952380952380952 * \alpha - 2.6374568452380952380952380952$
$v_4 * w_8 = -0.4262670212375073486184597296 * \alpha + 0.4262670212375073486184597296$
$v_4 * w_9 = 0.0096027579627530024355421180820 * \alpha - 0.0096027579627530024355421180820$
$v_5 * w_1 = -0.5449166666666666666666666667 * \alpha + 0.5449166666666666666666666667$
$v_5 * w_2 = 3.0131226497543461829176114890 * \alpha - 3.0131226497543461829176114890$
$v_5 * w_3 = -6.2912305537184009406231628450 * \alpha + 6.2912305537184009406231628450$
$v_5 * w_4 = 5.1042661439961787184009406231 * \alpha - 5.1042661439961787184009406231$
$v_5 * w_5 = -1.2812415733654572940287226000 * \alpha + 0.0000000000000000000000000000$
$v_5 * w_6 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_5 * w_7 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_5 * w_8 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_5 * w_9 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_6 * w_1 = 0.1239583333333333333333333333 * \alpha - 0.1239583333333333333333333333$
$v_6 * w_2 = -1.2596013387608129671621735110 * \alpha + 1.2596013387608129671621735110$
$v_6 * w_3 = 4.1741890294312169312169312169 * \alpha - 4.1741890294312169312169312169$
$v_6 * w_4 = -6.1607928057025279247501469724 * \alpha + 6.1607928057025279247501469724$
$v_6 * w_5 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_6 * w_6 = 3.1222467816987906273620559340 * \alpha + 0.0000000000000000000000000000$
$v_6 * w_7 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_6 * w_8 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_6 * w_9 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_7 * w_1 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_7 * w_2 = 0.2107209027515327118501721676 * \alpha - 0.2107209027515327118501721676$
$v_7 * w_3 = -1.1810142287624926513815402704 * \alpha + 1.1810142287624926513815402704$
$v_7 * w_4 = 2.6374568452380952380952380952 * \alpha - 2.6374568452380952380952380952$
$v_7 * w_5 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_7 * w_6 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_7 * w_7 = -1.6671635192271352985638699920 * \alpha + 0.0000000000000000000000000000$
$v_7 * w_8 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_7 * w_9 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_8 * w_1 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$

$v_8 * w_2 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_8 * w_3 = 0.0905059523809523809523809523 8 * \alpha - 0.0905059523809523809523809523 8$
$v_8 * w_4 = -0.4262670212375073486184597296 * \alpha + 0.4262670212375073486184597296$
$v_8 * w_5 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_8 * w_6 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_8 * w_7 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_8 * w_8 = 0.3357610688565549676660787772 * \alpha + 0.0000000000000000000000000000$
$v_8 * w_9 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_9 * w_1 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_9 * w_2 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_9 * w_3 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_9 * w_4 = 0.0096027579627530024355421180 82 * \alpha - 0.0096027579627530024355421180 82$
$v_9 * w_5 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_9 * w_6 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_9 * w_7 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_9 * w_8 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_9 * w_9 = -0.0096027579627530024355421180 82 * \alpha + 0.0000000000000000000000000000$

Flux point j=3
$v_1 * w_1 = -0.4192916666666666666666666667 * \alpha + 0.0000000000000000000000000000$
$v_1 * w_2 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_1 * w_3 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_1 * w_4 = 0.8402500000000000000000000000 * \alpha - 0.8402500000000000000000000000$
$v_1 * w_5 = -0.5449166666666666666666666667 * \alpha + 0.5449166666666666666666666667$
$v_1 * w_6 = 0.1239583333333333333333333333 * \alpha - 0.1239583333333333333333333333$
$v_1 * w_7 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_1 * w_8 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_2 * w_1 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_2 * w_2 = 1.4585120471413874191651969 43 * \alpha + 0.0000000000000000000000000000$
$v_2 * w_3 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_2 * w_4 = -3.4227542608864533467708070 88 * \alpha + 3.4227542608864533467708070 88$
$v_2 * w_5 = 3.0131226497543461829176114 89 * \alpha - 3.0131226497543461829176114 89$
$v_2 * w_6 = -1.2596013387608129671621735 11 * \alpha + 1.2596013387608129671621735 11$
$v_2 * w_7 = 0.2107209027515327118501721676 * \alpha - 0.2107209027515327118501721676$
$v_2 * w_8 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_3 * w_1 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_3 * w_2 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_3 * w_3 = -0.5392203804747207524985302763 * \alpha + 0.0000000000000000000000000000$
$v_3 * w_4 = 3.7467701811434450323339212 23 * \alpha - 3.7467701811434450323339212 23$
$v_3 * w_5 = -6.2912305537184009406231628 45 * \alpha + 6.2912305537184009406231628 45$
$v_3 * w_6 = 4.1741890294312169312169312 17 * \alpha - 4.1741890294312169312169312 17$
$v_3 * w_7 = -1.1810142287624926513815402 70 * \alpha + 1.1810142287624926513815402 70$
$v_3 * w_8 = 0.0905059523809523809523809523 8 * \alpha - 0.0905059523809523809523809523 8$
$v_4 * w_1 = 0.8402500000000000000000000000 * \alpha - 0.8402500000000000000000000000$
$v_4 * w_2 = -3.4227542608864533467708070 88 * \alpha + 3.4227542608864533467708070 88$

$v_4 * w_3 = 3.7467701811434450323339212223 * \alpha - 3.7467701811434450323339212223$
$v_4 * w_4 = -1.1642659202569916855631141135 * \alpha + 0.0000000000000000000000000000$
$v_4 * w_5 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_4 * w_6 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_4 * w_7 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_4 * w_8 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_5 * w_1 = -0.5449166666666666666666666667 * \alpha + 0.5449166666666666666666666667$
$v_5 * w_2 = 3.0131226497543461829176114489 * \alpha - 3.0131226497543461829176114489$
$v_5 * w_3 = -6.2912305537184009406231628445 * \alpha + 6.2912305537184009406231628445$
$v_5 * w_4 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_5 * w_5 = 3.8230245706307214243722180023 * \alpha + 0.0000000000000000000000000000$
$v_5 * w_6 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_5 * w_7 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_5 * w_8 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_6 * w_1 = 0.1239583333333333333333333333 * \alpha - 0.1239583333333333333333333333$
$v_6 * w_2 = -1.2596013387608129671621735111 * \alpha + 1.2596013387608129671621735111$
$v_6 * w_3 = 4.1741890294312169312169312217 * \alpha - 4.1741890294312169312169312217$
$v_6 * w_4 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_6 * w_5 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_6 * w_6 = -3.0385460240037372973880910039 * \alpha + 0.0000000000000000000000000000$
$v_6 * w_7 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_6 * w_8 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_7 * w_1 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_7 * w_2 = 0.2107209027515327118501721676 * \alpha - 0.2107209027515327118501721676$
$v_7 * w_3 = -1.1810142287624926513815402270 * \alpha + 1.1810142287624926513815402270$
$v_7 * w_4 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_7 * w_5 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_7 * w_6 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_7 * w_7 = 0.9702933260109599395313681028 * \alpha + 0.0000000000000000000000000000$
$v_7 * w_8 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_8 * w_1 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_8 * w_2 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_8 * w_3 = 0.0905059523809523809523809523 8 * \alpha - 0.0905059523809523809523809523 8$
$v_8 * w_4 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_8 * w_5 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_8 * w_6 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_8 * w_7 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_8 * w_8 = -0.0905059523809523809523809523 8 * \alpha + 0.0000000000000000000000000000$

Flux point j=2
$v_1 * w_1 = -0.0782083333333333333333333333 * \alpha + 0.0000000000000000000000000000$
$v_1 * w_2 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_1 * w_3 = -0.3410833333333333333333333333 * \alpha + 0.3410833333333333333333333333$
$v_1 * w_4 = 0.8402500000000000000000000000 * \alpha - 0.8402500000000000000000000000$
$v_1 * w_5 = -0.5449166666666666666666666667 * \alpha + 0.5449166666666666666666666667$

$v_1 * w_6 = 0.1239583333333333333333333333 * \alpha - 0.1239583333333333333333333333$
$v_1 * w_7 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_2 * w_1 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_2 * w_2 = 0.5782083333333333333333333333 * \alpha + 0.0000000000000000000000000000$
$v_2 * w_3 = 0.8803037138080540858318636096 * \alpha - 0.8803037138080540858318636096$
$v_2 * w_4 = -3.4227542608864533467708070888 * \alpha + 3.4227542608864533467708070888$
$v_2 * w_5 = 3.0131226497543461829176114889 * \alpha - 3.0131226497543461829176114889$
$v_2 * w_6 = -1.2596013387608129671621735111 * \alpha + 1.2596013387608129671621735111$
$v_2 * w_7 = 0.2107209027515327118501721676 * \alpha - 0.2107209027515327118501721676$
$v_3 * w_1 = -0.3410833333333333333333333333 * \alpha + 0.3410833333333333333333333333$
$v_3 * w_2 = 0.8803037138080540858318636096 * \alpha - 0.8803037138080540858318636096$
$v_3 * w_3 = -0.5392203804747207524985302763 * \alpha + 0.0000000000000000000000000000$
$v_3 * w_4 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_3 * w_5 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_3 * w_6 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_3 * w_7 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_4 * w_1 = 0.8402500000000000000000000000 * \alpha - 0.8402500000000000000000000000$
$v_4 * w_2 = -3.4227542608864533467708070888 * \alpha + 3.4227542608864533467708070888$
$v_4 * w_3 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_4 * w_4 = 2.582504260886453346770807088 * \alpha + 0.0000000000000000000000000000$
$v_4 * w_5 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_4 * w_6 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_4 * w_7 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_5 * w_1 = -0.5449166666666666666666666667 * \alpha + 0.5449166666666666666666666667$
$v_5 * w_2 = 3.0131226497543461829176114889 * \alpha - 3.0131226497543461829176114889$
$v_5 * w_3 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_5 * w_4 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_5 * w_5 = -2.4682059830876795162509448222 * \alpha + 0.0000000000000000000000000000$
$v_5 * w_6 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_5 * w_7 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_6 * w_1 = 0.1239583333333333333333333333 * \alpha - 0.1239583333333333333333333333$
$v_6 * w_2 = -1.2596013387608129671621735111 * \alpha + 1.2596013387608129671621735111$
$v_6 * w_3 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_6 * w_4 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_6 * w_5 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_6 * w_6 = 1.1356430054274796338288401778 * \alpha + 0.0000000000000000000000000000$
$v_6 * w_7 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_7 * w_1 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_7 * w_2 = 0.2107209027515327118501721676 * \alpha - 0.2107209027515327118501721676$
$v_7 * w_3 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_7 * w_4 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_7 * w_5 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_7 * w_6 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_7 * w_7 = -0.2107209027515327118501721676 * \alpha + 0.0000000000000000000000000000$

Flux point j=1

$v_1 * w_1 = 0.5000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_1 * w_2 = -0.5782083333333333333333333333 * \alpha + 0.5782083333333333333333333333$
$v_1 * w_3 = -0.3410833333333333333333333333 * \alpha + 0.3410833333333333333333333333$
$v_1 * w_4 = 0.8402500000000000000000000000 * \alpha - 0.8402500000000000000000000000$
$v_1 * w_5 = -0.5449166666666666666666666667 * \alpha + 0.5449166666666666666666666667$
$v_1 * w_6 = 0.1239583333333333333333333333 * \alpha - 0.1239583333333333333333333333$
$v_2 * w_1 = -0.5782083333333333333333333333 * \alpha + 0.5782083333333333333333333333$
$v_2 * w_2 = 0.5782083333333333333333333333 * \alpha + 0.0000000000000000000000000000$
$v_2 * w_3 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_2 * w_4 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_2 * w_5 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_2 * w_6 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_3 * w_1 = -0.3410833333333333333333333333 * \alpha + 0.3410833333333333333333333333$
$v_3 * w_2 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_3 * w_3 = 0.3410833333333333333333333333 * \alpha + 0.0000000000000000000000000000$
$v_3 * w_4 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_3 * w_5 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_3 * w_6 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_4 * w_1 = 0.8402500000000000000000000000 * \alpha - 0.8402500000000000000000000000$
$v_4 * w_2 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_4 * w_3 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_4 * w_4 = -0.8402500000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_4 * w_5 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_4 * w_6 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_5 * w_1 = -0.5449166666666666666666666667 * \alpha + 0.5449166666666666666666666667$
$v_5 * w_2 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_5 * w_3 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_5 * w_4 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_5 * w_5 = 0.5449166666666666666666666667 * \alpha + 0.0000000000000000000000000000$
$v_5 * w_6 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_6 * w_1 = 0.1239583333333333333333333333 * \alpha - 0.1239583333333333333333333333$
$v_6 * w_2 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_6 * w_3 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_6 * w_4 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_6 * w_5 = 0.0000000000000000000000000000 * \alpha + 0.0000000000000000000000000000$
$v_6 * w_6 = -0.1239583333333333333333333333 * \alpha + 0.0000000000000000000000000000$

Flux point j=0

$v_1 * w_1 = 0.0000000000000000000000000000 * \alpha + 1.0000000000000000000000000000$

The P-norm $\mathcal{P}$ is:

$\mathcal{P}_{1,1}/dx = 0.29950000000000000000000000000000000000$
$\mathcal{P}_{2,2}/dx = 1.4888460333207357016880826404635928445$
$\mathcal{P}_{3,3}/dx = 0.38651394400352733686067019400352733686$

$\mathcal{P}_{4,4}/dx = 1.5399915674603174603174603174603174603$
$\mathcal{P}_{5,5}/dx = 0.7353607253086419753086419753086419753 1$
$\mathcal{P}_{6,6}/dx = 1.0203624889770723104056437389770723104$
$\mathcal{P}_{7,7}/dx = 1.0523566468253968253968253968253968254$
$\mathcal{P}_{8,8}/dx = 0.9724592702821869488536155202821869488 5$
$\mathcal{P}_{9,9}/dx = 1.0046093238221214411690602166792642983$
$\mathcal{P}_{10,10}/dx = 1.0000000000000000000000000000000000000$

The P-norm inverse $\mathcal{P}^{-1}$ is:
$\mathcal{P}^{-1}_{1,1} * dx = 3.3388981636060100166944908180300500835$
$\mathcal{P}^{-1}_{2,2} * dx = 0.6716611238635541960980036120083054181 0$
$\mathcal{P}^{-1}_{3,3} * dx = 2.5872287805246011319981479868054754454$
$\mathcal{P}^{-1}_{4,4} * dx = 0.6493542050033128015865372381767448092 7$
$\mathcal{P}^{-1}_{5,5} * dx = 1.3598768136281481908969048637106484335$
$\mathcal{P}^{-1}_{6,6} * dx = 0.9800438675499664919415104375158026942 2$
$\mathcal{P}^{-1}_{7,7} * dx = 0.9502481910639904484973493448896585564 1$
$\mathcal{P}^{-1}_{8,8} * dx = 1.0283207025316559720897982867102527893$
$\mathcal{P}^{-1}_{9,9} * dx = 0.9954118245642147959964174014896896604 1$
$\mathcal{P}^{-1}_{10,10} * dx = 1.0000000000000000000000000000000000000$

The $\mathcal{Q}_{i,j} * dx$ nonzero terms are:
$\mathcal{Q}_{1,1} = -0.5000000000000000000000000000000000000$
$\mathcal{Q}_{1,2} = 0.5782083333333333333333333333333333333$
$\mathcal{Q}_{1,3} = 0.3410833333333333333333333333333333333$
$\mathcal{Q}_{1,4} = -0.8402500000000000000000000000000000000$
$\mathcal{Q}_{1,5} = 0.5449166666666666666666666666666666667$
$\mathcal{Q}_{1,6} = -0.1239583333333333333333333333333333333$
$\mathcal{Q}_{2,1} = -0.5782083333333333333333333333333333333$
$\mathcal{Q}_{2,3} = -0.8803037138080540858318636096413874191 7$
$\mathcal{Q}_{2,4} = 3.4227542608864533467708070882674057277$
$\mathcal{Q}_{2,5} = -3.0131226497543461829176114890400604686$
$\mathcal{Q}_{2,6} = 1.2596013387608129671621735113798605862$
$\mathcal{Q}_{2,7} = -0.2107209027515327118501721676324850928 0$
$\mathcal{Q}_{3,1} = -0.3410833333333333333333333333333333333$
$\mathcal{Q}_{3,2} = 0.8803037138080540858318636096413874191 7$
$\mathcal{Q}_{3,4} = -3.7467701811434450323339212228101116990$
$\mathcal{Q}_{3,5} = 6.2912305537184009406231628453850676073$
$\mathcal{Q}_{3,6} = -4.1741890294312169312169312169312169312$
$\mathcal{Q}_{3,7} = 1.1810142287624926513815402704291593181$
$\mathcal{Q}_{3,8} = -0.0905059523809523809523809523809523809 52$
$\mathcal{Q}_{4,1} = 0.8402500000000000000000000000000000000$
$\mathcal{Q}_{4,2} = -3.4227542608864533467708070882674057277$
$\mathcal{Q}_{4,3} = 3.7467701811434450323339212228101116990$
$\mathcal{Q}_{4,5} = -5.1042661439961787184009406231628453851$

$\mathcal{Q}_{4,6} = 6.1607928057025279247501469723691945914$
$\mathcal{Q}_{4,7} = -2.6374568452380952380952380952380952381$
$\mathcal{Q}_{4,8} = 0.4262670212375073486184597295708406819$5
$\mathcal{Q}_{4,9} = -0.0096027579627530024355421180818006214832$
$\mathcal{Q}_{5,1} = -0.54491666666666666666666666666666666667$
$\mathcal{Q}_{5,2} = 3.0131226497543461829176114890400604686$
$\mathcal{Q}_{5,3} = -6.2912305537184009406231628453850676073$
$\mathcal{Q}_{5,4} = 5.1042661439961787184009406231628453851$
$\mathcal{Q}_{5,6} = -3.3485729166666666666666666666666666667$
$\mathcal{Q}_{5,7} = 2.7798440990593768371546149323927101705$
$\mathcal{Q}_{5,8} = -0.75580278604497354497354497354497354498$
$\mathcal{Q}_{5,9} = 0.043290030286806080456874107667758461409$
$\mathcal{Q}_{6,1} = 0.12395833333333333333333333333333333333$
$\mathcal{Q}_{6,2} = -1.2596013387608129671621735113798605862$
$\mathcal{Q}_{6,3} = 4.1741890294312169312169312169312169312$
$\mathcal{Q}_{6,4} = -6.1607928057025279247501469723691945914$
$\mathcal{Q}_{6,5} = 3.3485729166666666666666666666666666667$
$\mathcal{Q}_{6,7} = -0.66288489767048794826572604350382128160$
$\mathcal{Q}_{6,8} = 0.49191676771017048794826572604350382128$
$\mathcal{Q}_{6,9} = -0.051786576436130007558578987150415721844$
$\mathcal{Q}_{6,10} = -0.0035714285714285714285714285714285714286$
$\mathcal{Q}_{7,2} = 0.21072090275153271185017216763248509280$
$\mathcal{Q}_{7,3} = -1.1810142287624926513815402704291593181$
$\mathcal{Q}_{7,4} = 2.6374568452380952380952380952380952381$
$\mathcal{Q}_{7,5} = -2.7798440990593768371546149323927101705$
$\mathcal{Q}_{7,6} = 0.66288489767048794826572604350382128160$
$\mathcal{Q}_{7,8} = 0.53307226447677836566725455614344503233$
$\mathcal{Q}_{7,9} = -0.11780039183883429915175946921978668010$
$\mathcal{Q}_{7,10} = 0.038095238095238095238095238095238095238$
$\mathcal{Q}_{7,11} = -0.0035714285714285714285714285714285714286$
$\mathcal{Q}_{8,3} = 0.090505952380952380952380952380952380952$
$\mathcal{Q}_{8,4} = -0.42626702123750734861845972957084068195$
$\mathcal{Q}_{8,5} = 0.75580278604497354497354497354497354498$
$\mathcal{Q}_{8,6} = -0.49191676771017048794826572604350382128$
$\mathcal{Q}_{8,7} = -0.53307226447677836566725455614344503233$
$\mathcal{Q}_{8,9} = 0.77042350547472075249853027630805408583$
$\mathcal{Q}_{8,10} = -0.20000000000000000000000000000000000000$
$\mathcal{Q}_{8,11} = 0.038095238095238095238095238095238095238$
$\mathcal{Q}_{8,12} = -0.0035714285714285714285714285714285714286$
$\mathcal{Q}_{9,4} = 0.0096027579627530024355421180818006214832$
$\mathcal{Q}_{9,5} = -0.043290030286806080456874107667758461409$
$\mathcal{Q}_{9,6} = 0.051786576436130007558578987150415721844$
$\mathcal{Q}_{9,7} = 0.11780039183883429915175946921978668010$
$\mathcal{Q}_{9,8} = -0.77042350547472075249853027630805408583$
$\mathcal{Q}_{9,10} = 0.80000000000000000000000000000000000000$

$\mathcal{Q}_{9,11} = -0.20000000000000000000000000000000000$
$\mathcal{Q}_{9,12} = 0.03809523809523809523809523809523809523 8$
$\mathcal{Q}_{9,13} = -0.0035714285714285714285714285714285714286$
$\mathcal{Q}_{10,6} = 0.0035714285714285714285714285714285714286$
$\mathcal{Q}_{10,7} = -0.038095238095238095238095238095238095238$
$\mathcal{Q}_{10,8} = 0.20000000000000000000000000000000000$
$\mathcal{Q}_{10,9} = -0.80000000000000000000000000000000000$
$\mathcal{Q}_{10,11} = 0.80000000000000000000000000000000000$
$\mathcal{Q}_{10,12} = -0.20000000000000000000000000000000000$
$\mathcal{Q}_{10,13} = 0.038095238095238095238095238095238095238$
$\mathcal{Q}_{10,14} = -0.0035714285714285714285714285714285714286$

The (4-8-4) viscous terms.

The $\bar{\mathcal{R}}_{i,j} * dx$ nonzero terms are:
$\bar{\mathcal{R}}_{0,1} = -2.4500000000000000000000000000000000$
$\bar{\mathcal{R}}_{0,2} = 6.0000000000000000000000000000000000$
$\bar{\mathcal{R}}_{0,3} = -7.5000000000000000000000000000000000$
$\bar{\mathcal{R}}_{0,4} = 6.6666666666666666666666666666666667$
$\bar{\mathcal{R}}_{0,5} = -3.7500000000000000000000000000000000$
$\bar{\mathcal{R}}_{0,6} = 1.2000000000000000000000000000000000$
$\bar{\mathcal{R}}_{0,7} = -0.16666666666666666666666666666666667$
$\bar{\mathcal{R}}_{1,1} = -1.1602083333333333333333333333333333$
$\bar{\mathcal{R}}_{1,2} = 1.1564166666666666666666666666666667$
$\bar{\mathcal{R}}_{1,3} = 0.34108333333333333333333333333333333$
$\bar{\mathcal{R}}_{1,4} = -0.56016666666666666666666666666666667$
$\bar{\mathcal{R}}_{1,5} = 0.27245833333333333333333333333333333$
$\bar{\mathcal{R}}_{1,6} = -0.049583333333333333333333333333333333$
$\bar{\mathcal{R}}_{2,1} = 0.093973038486880787332060994325567220421$
$\bar{\mathcal{R}}_{2,2} = -0.78549893930185917599538939754826348881$
$\bar{\mathcal{R}}_{2,3} = 0.046946483020436971650521028934066801314$
$\bar{\mathcal{R}}_{2,4} = 0.90729349114883760012958720175606131931$
$\bar{\mathcal{R}}_{2,5} = -0.26981952253301897863082607780986533944$
$\bar{\mathcal{R}}_{2,6} = -0.0063708948742115737446125435968063627529$
$\bar{\mathcal{R}}_{2,7} = 0.013476344052934369258658793939239849966$
$\bar{\mathcal{R}}_{3,1} = 0.078746435191005348453800791189557639584$
$\bar{\mathcal{R}}_{3,2} = -0.35162113760879514268346232087309277418$
$\bar{\mathcal{R}}_{3,3} = -0.78713436791133549855006393774752877572$
$\bar{\mathcal{R}}_{3,4} = 1.3893360044549165552850116426510172239$
$\bar{\mathcal{R}}_{3,5} = -0.45575020893114105530342782230270280369$
$\bar{\mathcal{R}}_{3,6} = 0.19808018162235668097172402106009418736$
$\bar{\mathcal{R}}_{3,7} = -0.092080135398545507498723070166851812875$
$\bar{\mathcal{R}}_{3,8} = 0.020423228581538619325140696189507115568$
$\bar{\mathcal{R}}_{4,1} = 0.032926148937526826310667271834304379849$
$\bar{\mathcal{R}}_{4,2} = -0.25191802597411192851286613885119855940$
$\bar{\mathcal{R}}_{4,3} = 0.86404038300564659933759665503107585561$

$\bar{\mathcal{R}}_{4,4} = -2.270573437204335813417518462110016052$9

$\bar{\mathcal{R}}_{4,5} = 1.905603640280185014154434005565036415$2

$\bar{\mathcal{R}}_{4,6} = -0.4354424241479112822731821154034676769$8

$\bar{\mathcal{R}}_{4,7} = 0.201031977711462766941895564896291987$97

$\bar{\mathcal{R}}_{4,8} = -0.04950936579356338351524362819474659793$8

$\bar{\mathcal{R}}_{4,9} = 0.003841103185101200974216847232720248593$3

$\bar{\mathcal{R}}_{5,1} = -0.049583333333333333333333333333333333333$3

$\bar{\mathcal{R}}_{5,2} = 0.24538547964509061280525769040437984250$

$\bar{\mathcal{R}}_{5,3} = -0.46041931266967499182355759462391059263$

$\bar{\mathcal{R}}_{5,4} = 0.46614273043637761869004215693100311426$

$\bar{\mathcal{R}}_{5,5} = -1.3962385850218969553701412457089944128$

$\bar{\mathcal{R}}_{5,6} = 1.3149492005817200178282234837505867257$

$\bar{\mathcal{R}}_{5,7} = -0.13710317505277669221310366487172969582$

$\bar{\mathcal{R}}_{5,8} = 0.025083621435175386943099478252132619196$

$\bar{\mathcal{R}}_{5,9} = -0.0082166260206816635264869708001342671278$

$\bar{\mathcal{R}}_{6,2} = 0.013476344052934369258658793939239849966$

$\bar{\mathcal{R}}_{6,3} = -0.095466650296571124068030350198485565808$

$\bar{\mathcal{R}}_{6,4} = 0.28303601526589854838736615694592248492$

$\bar{\mathcal{R}}_{6,5} = -0.37189053652589385913924939420370302618$

$\bar{\mathcal{R}}_{6,6} = -0.87133674643234682661783233764660965659$

$\bar{\mathcal{R}}_{6,7} = 1.1363503084863723276987369694549351287$

$\bar{\mathcal{R}}_{6,8} = -0.10962852877844960704052530455432144803$

$\bar{\mathcal{R}}_{6,9} = 0.017245508513770457235161180548736518789$

$\bar{\mathcal{R}}_{6,10} = -0.0017857142857142857142857142857142857143$

$\bar{\mathcal{R}}_{7,3} = 0.020423228581538619325140696189507115568$

$\bar{\mathcal{R}}_{7,4} = -0.11461749320859847500014290897545707151$

$\bar{\mathcal{R}}_{7,5} = 0.24553622332394432725008432375299558042$

$\bar{\mathcal{R}}_{7,6} = -0.14002362171685620274536032506670231808$

$\bar{\mathcal{R}}_{7,7} = -1.2086425337884685391310360256213220479$

$\bar{\mathcal{R}}_{7,8} = 1.3257988165251715762302257817106939321$

$\bar{\mathcal{R}}_{7,9} = -0.15030001654212813132573693881511201603$

$\bar{\mathcal{R}}_{7,10} = 0.023611111111111111111111111111111111111$

$\bar{\mathcal{R}}_{7,11} = -0.0017857142857142857142857142857142857143$

$\bar{\mathcal{R}}_{8,4} = 0.0038411031851012009742168472327202485933$

$\bar{\mathcal{R}}_{8,5} = -0.021645015143403040228437053833879230705$

$\bar{\mathcal{R}}_{8,6} = 0.036310098576467624086671705719324766944$

$\bar{\mathcal{R}}_{8,7} = 0.094189280727723188040648358108675568993$

$\bar{\mathcal{R}}_{8,8} = -1.3644581220605526161081716637272192828$

$\bar{\mathcal{R}}_{8,9} = 1.4063261467781557067271352985638699924$

$\bar{\mathcal{R}}_{8,10} = -0.17638888888888888888888888888888888889$

$\bar{\mathcal{R}}_{8,11} = 0.023611111111111111111111111111111111111$

$\bar{\mathcal{R}}_{8,12} = -0.0017857142857142857142857142857142857143$

$\bar{\mathcal{R}}_{9,6} = 0.0017857142857142857142857142857142857143$

$\bar{\mathcal{R}}_{9,7} = -0.023611111111111111111111111111111111111$

$\bar{\mathcal{R}}_{9,8} = 0.17638888888888888888888888888888888889$

$\bar{\mathcal{R}}_{9,9} = -1.4236111111111111111111111111111111111$
$\bar{\mathcal{R}}_{9,10} = 1.4236111111111111111111111111111111111$
$\bar{\mathcal{R}}_{9,11} = -0.17638888888888888888888888888888888889$
$\bar{\mathcal{R}}_{9,12} = 0.023611111111111111111111111111111111111$
$\bar{\mathcal{R}}_{9,13} = -0.0017857142857142857142857142857142857143$

The $\mathcal{R}_{i,j} * dx$ nonzero terms are:
$\mathcal{R}_{1,1} = -1.1602083333333333333333333333333333333$
$\mathcal{R}_{1,2} = 1.1564166666666666666666666666666666667$
$\mathcal{R}_{1,3} = 0.34108333333333333333333333333333333333$
$\mathcal{R}_{1,4} = -0.56016666666666666666666666666666666667$
$\mathcal{R}_{1,5} = 0.27245833333333333333333333333333333333$
$\mathcal{R}_{1,6} = -0.049583333333333333333333333333333333333$
$\mathcal{R}_{2,1} = 1.2541813718202141206653943276589005538$
$\mathcal{R}_{2,2} = -1.9419156059685258426620560642149301555$
$\mathcal{R}_{2,3} = -0.29413685031289636168281230439926653202$
$\mathcal{R}_{2,4} = 1.4674601578155042667962538684227279860$
$\mathcal{R}_{2,5} = -0.54227785586635231196415941114319867278$
$\mathcal{R}_{2,6} = 0.043212438459121759588720789736526970580$
$\mathcal{R}_{2,7} = 0.013476344052934369258658793939239849966$
$\mathcal{R}_{3,1} = -0.015226603295875438878260203136009580837$
$\mathcal{R}_{3,2} = 0.43387780169306403331192707667517071464$
$\mathcal{R}_{3,3} = -0.83408085093177247020058496668159557703$
$\mathcal{R}_{3,4} = 0.48204251330607895515542444089495590463$
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REPORT DOCUMENTATION PAGE	*Form Approved* *OMB No. 0704-0188*

1. REPORT DATE *(DD-MM-YYYY)*
01- 11 - 2011

2. REPORT TYPE
Technical Memorandum

3. DATES COVERED *(From - To)*
July 2010 to October 2011

4. TITLE AND SUBTITLE
Discretely Conservative Finite-Difference Formulations for Nonlinear Conservation Laws in Split Form: Theory and Boundary Conditions

5a. CONTRACT NUMBER

5b. GRANT NUMBER

5c. PROGRAM ELEMENT NUMBER

6. AUTHOR(S)
Fisher, Travis C.; Carpenter, Mark H.; Nordstrom, Jan; Yamaleev, Nail K.; Swanson, R. Charles

5d. PROJECT NUMBER

5e. TASK NUMBER

5f. WORK UNIT NUMBER
599489.02.07.07.03.13.01

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
NASA Langley Research Center
Hampton, VA 23681-2199

8. PERFORMING ORGANIZATION REPORT NUMBER
L-20084

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)
National Aeronautics and Space Administration
Washington, DC 20546-0001

10. SPONSOR/MONITOR'S ACRONYM(S)
NASA

11. SPONSOR/MONITOR'S REPORT NUMBER(S)
NASA/TM-2011-217307

12. DISTRIBUTION/AVAILABILITY STATEMENT
Unclassified - Unlimited
Subject Category 64
Availability: NASA CASI (443) 757-5802

13. SUPPLEMENTARY NOTES

14. ABSTRACT
Simulations of nonlinear conservation laws that admit discontinuous solutions are typically restricted to discretizations of equations that are explicitly written in divergence form. This restriction is, however,unnecessary. Herein,linear combinations of divergence and product rule forms that have been discretized using diagonal-norm skew-symmetric summation-by-parts(SBP) operators, are shown to satisfy the sufficient conditions of the Lax-Wendroff theorem and thus are appropriate for simulations of discontinuous physical phenomena. Furthermore, special treatments are not required at the points that are near physical boundaries (i.e., discrete conservation is achieved throughout the entire computational domain, including the boundaries). Examples are presented of a fourth-order, SBP finite-difference operator with second-order boundary closures.Sixth- and eighth-order constructions are derived, and included in E. Narrow-stencil difference operators for linear viscous terms are also derived; these guarantee the conservative form of the combined operator.

15. SUBJECT TERMS
Burgers Equation; Lax-Wendroff; energy estimate; high-order finite difference methods; numerical stability; skew-symmetric splitting

16. SECURITY CLASSIFICATION OF: a. REPORT	b. ABSTRACT	c. THIS PAGE	17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
					STI Help Desk (email: help@sti.nasa.gov)
					19b. TELEPHONE NUMBER *(Include area code)*
U	U	U	UU	73	(443) 757-5802