Uncertainty Analysis via Failure Domain Characterization: Unrestricted Requirement Functions

Luis G. Crespo
National Institute of Aerospace, VA, USA

Sean P. Kenny and Daniel P. Giesy
NASA Langley Research Center, Hampton, VA, USA

ABSTRACT: This paper proposes an uncertainty analysis framework based on the characterization of the uncertain parameter space. This characterization enables the identification of worst-case uncertainty combinations and the approximation of the failure and safe domains with a high level of accuracy. Because these approximations are comprised of subsets of readily computable probability, they enable the calculation of arbitrarily tight upper and lower bounds to the failure probability. The methods developed herein, which are based on nonlinear constrained optimization, are applicable to requirement functions whose functional dependency on the uncertainty is arbitrary and whose explicit form may even be unknown. Some of the most prominent features of the methodology are the substantial desensitization of the calculations from the assumed uncertainty model (i.e., the probability distribution describing the uncertainty) as well as the accommodation for changes in such a model with a practically insignificant amount of computational effort.

1 INTRODUCTION

This paper studies the reliability of a system for which a parametric mathematical model is available. The acceptability of the system depends upon its ability to satisfy several design requirements. These requirements, which are represented by a set of inequality constraints on selected output metrics, depend on the uncertain parameter vector $\boldsymbol{p}$. The system is deemed acceptable if all inequalities are satisfied. The requirements/constraints partition the uncertain parameter space into two sets, the failure domain, where at least one of them is violated, and the safe domain, where all of them are satisfied. The reliability analysis of this system consists of assessing its ability to satisfy the requirements when the uncertain parameter $\boldsymbol{p}$ is free to take on any value from a prescribed set. The most common practice in reliability analysis is to assume a probabilistic *uncertainty model* of $\boldsymbol{p}$ (i.e., the random variable that models the uncertainty), and estimate the corresponding probability of failure. Calculating the failure probability is usually difficult since it requires evaluating a multi-dimensional integral over a complex integration domain. Sampling-based approaches (Niederreiter 1992, Kall and Wallace 1994) and methods based on asymptotic approximations of the failure domain (Rackwitz 2001, Royset et al. 2001) are the engines of most (if not all) of the numerical tools used to estimate this probability.

Reliability assessments whose figure of merit is the probability of failure are strongly dependent on the assumed uncertainty model. Quite often this model is created using engineering judgment, expert opinion, and/or limited observations of $\boldsymbol{p}$. The persistent incertitude in the model resulting from this process makes the soundness of the reliability analyses based on failure probabilities questionable. Besides, the uncertainty in the uncertainty model is commonly refined throughout the analysis cycle of the system. This process prevents leveraging the computational effort spent performing previous analyses. Furthermore, in the hypothetical case when the uncertainty model is perfect and final, the failure probability fails to describe practically significant features of the geometry of the failure event. Some of these features are the separation between any given point and the failure domain, the location of worst-case uncertainty combinations, and the geometry of the failure domain boundary.

This paper proposes techniques that characterize the uncertain parameter space with a high level of fidelity. A significant thrust of this research is the generation of sequences of inner approximations to the safe and failure domains by subsets of readily computable probability. These sequences are chosen such that they almost surely fill up the region of interest. The strategies proposed are applicable to requirement functions having arbitrary functional dependencies on the uncertainty whose explicit form may even be unknown. The companion paper (Crespo et al. 2011) proposes strategies with the same goals but restricted to polynomial requirement functions. Overall, the methodology enables the substantial desensitization of the calculations from the assumed uncertainty model as well as the accommodation for changes in such a model with a practically insignificant amount of computational effort.

This paper is organized as follows. Basic concepts are established in Section 2. This is followed by Section 3 where analytical expressions for bounds on the failure probability bounds are derived. Section 4 presents strategies for generating and refining the failure domain approximations that enable calculating the bounds. Finally, a few concluding remarks close the paper. Proofs are omitted due space limitations.

2 BASIC CONCEPTS AND NOTIONS

Uncertainty models of $\boldsymbol{p} \in \mathbb{R}^s$, where s is the number of uncertain parameters, can be probabilistic or non-probabilistic. A set whose members are all possible uncertain parameter realizations is a non-probabilistic model. This set, called the *support set*, will be denoted as $\Delta \subseteq \mathbb{R}^s$. On the other hand, a probabilistic uncertainty model prescribes a measure of probability to each member of this set. This model, in which $\boldsymbol{p}$ is a random vector, is fully prescribed by the joint probability density function $f_{\boldsymbol{p}}(\boldsymbol{p}) : \Delta \to \mathbb{R}$, or equivalently, by the cumulative distribution function $F_{\boldsymbol{p}}(\boldsymbol{p}) : \Delta \to [0, 1]$.

Consider a system that depends on the uncertain parameter $\boldsymbol{p}$. The design requirements imposed upon such a system are given by the vector[1] inequality $\boldsymbol{g}(\boldsymbol{p}) < \boldsymbol{0}$, where $\boldsymbol{g} : \mathcal{D} \to \mathbb{R}^v$, v is the number of constraint functions, and $\Delta \subseteq \mathcal{D} \subseteq \mathbb{R}^s$. The set $\mathcal{D}$, where the constraint functions are defined, will be called the *master domain*.

The *failure domain*, denoted as $\mathcal{F} \subset \mathbb{R}^s$, is comprised of the parameter realizations that fail to satisfy at least one of the requirements. Specifically, the failure domain is given by

$$\mathcal{F} = \bigcup_{i=1}^{v} \{\boldsymbol{p} : \boldsymbol{g}_i(\boldsymbol{p}) \geq 0\}. \quad (1)$$

The *safe domain*, given by $\mathcal{S} = C(\mathcal{F})$, where $C(\cdot)$ denotes the *complement* set operator given by $C(\mathcal{X}) = \mathcal{D} \setminus \mathcal{X}$, consists of the parameter realizations satisfying all the design requirements. The failure probability associated with a probabilistic uncertainty model is given by

$$P[\mathcal{F}] = \int_{\mathcal{F}} f_{\boldsymbol{p}}(\boldsymbol{p}) d\boldsymbol{p}, \quad (2)$$

where $P[\cdot]$ is the probability operator. Techniques for approximating $\mathcal{F}$ and $\mathcal{S}$ will be presented below. The resulting approximations are comprised of hyper-rectangles or quasi-ellipsoids.

The *hyper-rectangle* having $\boldsymbol{m} > \boldsymbol{0}$ as the vector of half-lengths of the sides and $\bar{\boldsymbol{p}}$ as its geometric center, is given by

$$\mathcal{R}(\bar{\boldsymbol{p}}, \boldsymbol{m}) = \{\boldsymbol{p} : \bar{\boldsymbol{p}} - \boldsymbol{m} < \boldsymbol{p} < \bar{\boldsymbol{p}} + \boldsymbol{m}\}. \quad (3)$$

An alternative representation of this hyper-rectangle is given by

$$\mathcal{R}(\bar{\boldsymbol{p}}, \boldsymbol{m}) = \delta(\bar{\boldsymbol{p}} - \boldsymbol{m}, \bar{\boldsymbol{p}} + \boldsymbol{m}), \quad (4)$$

where

$$\delta(\boldsymbol{x}, \boldsymbol{y}) = [\boldsymbol{x}_1, \boldsymbol{y}_1] \times [\boldsymbol{x}_2, \boldsymbol{y}_2] \times \cdots \times [\boldsymbol{x}_s, \boldsymbol{y}_s], \quad (5)$$

is the Cartesian product of intervals. Note that the first and second argument of δ are the lower and upper limits of the set. The components of $\boldsymbol{l}$ may be real numbers or minus infinity while those of $\boldsymbol{u}$ may be real numbers or infinity. Recall that the ℓ_∞ norm is defined as $\|\boldsymbol{x}\|^\infty = \sup\{|\boldsymbol{x}_i|\}$. Let us define the $\boldsymbol{m}$*-scaled* ℓ_∞ *norm* as $\|\boldsymbol{x}\|_{\boldsymbol{m}}^\infty = \sup\{|\boldsymbol{x}_i|/\boldsymbol{m}_i\}$. A distance between the vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ can be defined as $\|\boldsymbol{x} - \boldsymbol{y}\|_{\boldsymbol{m}}^\infty$. Using this distance, $\mathcal{R}(\bar{\boldsymbol{p}}, \boldsymbol{m})$ is the unit ball centered at $\bar{\boldsymbol{p}}$.

A *subdivision* is the process of dividing a set into subsets. Let $\rho(\cdot)$ be an operator whose input is any given set and its output are the subsets. A bisection-based subdivision in the ith direction is given by

$$\rho(\mathcal{R}) = \{\mathcal{R}(\bar{\boldsymbol{p}} + \boldsymbol{w}, \boldsymbol{m} - \boldsymbol{w}), \mathcal{R}(\bar{\boldsymbol{p}} - \boldsymbol{w}, \boldsymbol{m} - \boldsymbol{w})\},$$

where $\boldsymbol{w} = [0, \ldots, 0, \boldsymbol{m}_i/2, 0, \ldots, 0]$. Alternatively,

$$\rho(\mathcal{R}) = \{\delta(\boldsymbol{v}^1, \boldsymbol{v}^1 + \boldsymbol{m}), \ldots, \delta(\boldsymbol{v}^{2^s}, \boldsymbol{v}^{2^s} + \boldsymbol{m})\},$$

where $\boldsymbol{v}^k$ is a vertex of $\delta(\boldsymbol{l}, \boldsymbol{l} + \boldsymbol{m})$, leads to 2^s rectangular subsets each of volume $\prod_{i=1}^{s} \boldsymbol{m}_i$.

[1] Throughout this paper, it is assumed that vector inequalities hold component-wise, super-indices denote a particular vector or set, and sub-indices refer to vector components; e.g., $\boldsymbol{p}_i^j$ is the ith component of the vector $\boldsymbol{p}^j$.

The *quasi-ellipsoid* having $\boldsymbol{m} > \boldsymbol{0}$ as the semi-principal axes vector and $\bar{\boldsymbol{p}}$ as its geometric center, is given by

$$\mathcal{E}(\bar{\boldsymbol{p}}, \boldsymbol{m}, n) = \left\{ \boldsymbol{p} : \left(\sum_{i=1}^{s} \left(\frac{\boldsymbol{p}_i - \bar{\boldsymbol{p}}_i}{\boldsymbol{m}_i} \right)^n \right)^{\frac{1}{n}} < 1 \right\} \quad (6)$$

where n is an even natural number. Note that $\mathcal{E}$ is a closed set in $\mathbb{R}^s$ having a polynomial boundary of degree n. Further notice that $\mathcal{E}(\bar{\boldsymbol{p}}, \boldsymbol{m}, n)$ approaches $\mathcal{R}(\bar{\boldsymbol{p}}, \boldsymbol{m})$ asymptotically from the inside as $n \to \infty$. Recall that the ℓ_p norm is defined as $\|\boldsymbol{x}\|^p = (\sum |\boldsymbol{x}_i|^p)^{1/p}$. Let us define the *$\boldsymbol{m}$-scaled ℓ_n norm* as $\|\boldsymbol{x}\|^n_{\boldsymbol{m}} = (\sum (\boldsymbol{x}_i/\boldsymbol{m}_i)^n)^{1/n}$. A distance between the vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ can be defined as $\|\boldsymbol{x} - \boldsymbol{y}\|^n_{\boldsymbol{m}}$. Using this distance, $\mathcal{E}(\bar{\boldsymbol{p}}, \boldsymbol{m}, n)$ is the unit ball centered at $\bar{\boldsymbol{p}}$.

The probability of the sets $\mathcal{R}(\bar{\boldsymbol{p}}, \boldsymbol{m}) \subseteq \Delta$ and $\mathcal{E}(\bar{\boldsymbol{p}}, \boldsymbol{m}, n) \subseteq \Delta$ can be analytically calculated or bounded under the following conditions. The probability of a hyper-rectangle can be calculated analytically when the components of $\boldsymbol{p}$ are independent arbitrarily distributed random variables. The probability of a quasi-ellipsoid on the other hand, can be calculated analytically when the components of $\boldsymbol{p}$ are independent, uniformly distributed random variables and $\mathcal{E} \subseteq \Delta$. The probability of the ellipsoid $\mathcal{E}$ can be bounded from below when the components of $\boldsymbol{p}$ are independent arbitrarily distributed random variables. Failure probability bounds result from approximating the failure and safe domains with the union of hyper-rectangles or quasi-ellipsoids and using these analytical expressions. The following section presents the mathematical background required to calculate probability bounds based on the approximations. The sections that follow provide means to generate and sequentially refine these approximations.

3 PROBABILITY BOUNDS

The key development in this section is the calculation of the probability of inner approximations to the failure and safe domains. These approximations are comprised of a collection of *almost disjoint* hyper-rectangles or quasi-ellipsoids. Two sets are almost disjoint if they overlap at most in mutual boundary points. Let $\mathcal{F}^{sub}$ and $\mathcal{S}^{sub}$ denote inner approximations (sub-sets) of the failure and safe domains. Thus, $\mathcal{F}^{sup} = C(\mathcal{S}^{sub})$ is an outer approximation (superset) of the failure domain. Because $\emptyset \subseteq \mathcal{F}^{sub} \subseteq \mathcal{F} \subseteq \mathcal{F}^{sup} \subseteq \mathcal{D}$, we have $0 \leq P[\mathcal{F}^{sub}] \leq P[\mathcal{F}] \leq P[\mathcal{F}^{sup}] \leq 1$. Therefore, $P[\mathcal{F}^{sub}]$ and $P[\mathcal{F}^{sup}]$ are lower and upper bounds to the failure probability. Note that the bounds approach the failure probability when $\mathcal{F}^{sub}$ approaches the failure domain and $\mathcal{S}^{sub}$ approaches the safe domain. Further notice that $C(\mathcal{S}^{sub} \cup \mathcal{F}^{sub})$ contains the failure domain boundary $\partial\mathcal{F}$.

The failure domain and its approximations, as well as the worst-case uncertainty combination introduced later, are intrinsic features of the failure event that do not depend on the uncertainty model. While this model affects the failure probability via the integrand of (2), the integration domain $\mathcal{F}$ and its approximations are independent of it. Probability bounds corresponding to a given $\mathcal{F}^{sup}$ are presented next. Extensions corresponding to $\mathcal{F}^{sub}$ follow.

Theorem 1. *Assume that $\boldsymbol{p}$ is an independent random vector with continuous joint cumulative distribution function $F_{\boldsymbol{p}}(\boldsymbol{p})$ supported in Δ. If $\{\mathcal{R}(\bar{\boldsymbol{p}}^1, \boldsymbol{m}^1), \ldots, \mathcal{R}(\bar{\boldsymbol{p}}^k, \boldsymbol{m}^k)\}$ is a collection of hyper-rectangles where each member is a subset of $\mathcal{S}$ and any two members are almost disjoint, then*

$$\mathcal{F}^{sup} = C\left(\bigcup_{i=1}^{k} \mathcal{R}(\bar{\boldsymbol{p}}^i, \boldsymbol{m}^i)\right), \quad (7)$$

is an outer approximation to the failure domain and

$$P[\mathcal{F}^{sup}] = 1 - \sum_{i=1}^{k} \prod_{j=1}^{s} \left\{ F_{\boldsymbol{p}_j}\left(\bar{\boldsymbol{p}}^i_j + \boldsymbol{m}^i_j\right) - F_{\boldsymbol{p}_j}\left(\bar{\boldsymbol{p}}^i_j - \boldsymbol{m}^i_j\right) \right\}, \quad (8)$$

is an upper bound to the failure probability.

The bound is a function of the uncertainty model via $F_{\boldsymbol{p}}(\boldsymbol{p})$, but the the outer approximation $\mathcal{F}^{sup}$ and the containment conditions $\mathcal{F} \subseteq \mathcal{F}^{sup} \subseteq \mathcal{D}$ are not. Note that while most of the computational effort will be devoted to generate $\mathcal{F}^{sup}$, the effort required to evaluate the probability bound is practically insignificant. Furthermore, notice that if additional hyper-rectangles are appended to $\mathcal{S}^{sub}$ until, in the limit; they almost cover $\mathcal{S}$, the upper bound approaches $P[\mathcal{F}]$ from above.

Suppose the uncertainty model of $\boldsymbol{p}$ is changed from $F_{\boldsymbol{p}}(\boldsymbol{p})$ to $\hat{F}_{\boldsymbol{p}}(\boldsymbol{p})$ in $\boldsymbol{p} \in \hat{\Delta}$. If $\hat{\Delta} \subseteq \mathcal{D}$, $\mathcal{F}^{sup}$ still covers the failure domain and a probability bound for the new uncertainty model can be calculated by replacing $F_{\boldsymbol{p}}(\boldsymbol{p})$ by $\hat{F}_{\boldsymbol{p}}(\boldsymbol{p})$ in (8). Therefore, with the outer approximation in hand, we can readily calculate probability bounds corresponding to any uncertainty model supported in the master domain. This enables us to efficiently accommodate for changes in the uncertainty model while leveraging all the computational effort devoted to generate $\mathcal{F}^{sup}$.

The common practice of transforming the probabilistic uncertainty model of $\boldsymbol{p}$ to a space where the joint density function takes on a particular form (Rackwitz 2001) will be used subsequently. One space of interest is the uniform space, where the uncertain parameters become mutually independent uniform random variables with support set $[0,1]$. The corresponding transformation, denoted by $\boldsymbol{u}=U(\boldsymbol{p})$, is a one-to-one mapping of the support set Δ onto the unit cube. Since this is a probability preserving transformation $P[\mathcal{F}]=P[U(\mathcal{F})]$.

Theorem 2. *Assume that* $\mathbf{p}$ *is an independent random variable with joint cumulative distribution function* $F_{\boldsymbol{p}}(\boldsymbol{p})$. *Denote by* $\boldsymbol{u}=U(\boldsymbol{p})$ *a transformation of this distribution to uniform space in the unit cube* $\delta(\mathbf{0},\mathbf{1})$. *If* $\{\mathcal{E}(\bar{\boldsymbol{u}}^1,\boldsymbol{m}^1,n),\ldots,\mathcal{E}(\bar{\boldsymbol{u}}^k,\boldsymbol{m}^k,n)\}$ *is a collection of quasi-ellipsoids of degree* n *where each member is a subset of* $U(\mathcal{S})$ *and any two members are almost disjoint, then*

$$\mathcal{F}^{sup}=C\left(\bigcup_{i=1}^{k}\mathcal{E}(\bar{\boldsymbol{u}}^i,\boldsymbol{m}^i,n)\right), \tag{9}$$

is an outer approximation to the failure domain and

$$P[\mathcal{F}^{sup}]=1-\frac{2^s\Gamma\left(\frac{n+1}{n}\right)}{\Gamma\left(\frac{n+s}{n}\right)}\sum_{j=1}^{k}\prod_{i=1}^{s}\boldsymbol{m}_i^j, \tag{10}$$

where Γ *is the Gamma function, is an upper bound to the failure probability.*

Due to the transformation, $\mathcal{D}=\Delta$. Since the approximation $\mathcal{F}^{sup}$ is a function of the uncertainty model via the transformation U, the bound in (10) does not apply to other uncertainty models. Note however that $\mathcal{F}\subseteq U^{-1}(\mathcal{F}^{sup})$. Unfortunately, the probability bound $P[U^{-1}(\mathcal{F}^{sup})]$ corresponding to other uncertainty models cannot be calculated analytically. Conditional sampling algorithms (Crespo et al. 2009) can be used to approximate this probability.

Theorem 3. *Assume that* $\mathbf{p}$ *is an independent random variable with joint cumulative distribution function* $F_{\boldsymbol{p}}(\boldsymbol{p})$. *If* $\{\mathcal{E}(\bar{\boldsymbol{p}}^1,\boldsymbol{m}^1,n),\ldots,\mathcal{E}(\bar{\boldsymbol{p}}^k,\boldsymbol{m}^k,n)\}$ *is a collection of quasi-ellipsoids of degree* n *where each member is a subset of* $\mathcal{S}$ *and any two members are almost disjoint, then*

$$\mathcal{F}^{sup}=C\left(\bigcup_{i=1}^{k}\mathcal{E}(\bar{\boldsymbol{p}}^i,\boldsymbol{m}^i,n)\right), \tag{11}$$

is an outer approximation to the failure domain and

$$\psi(\mathcal{F}^{sup})=1-\sum_{i=1}^{k}\prod_{j=1}^{s}\{F_{\boldsymbol{p}_j}\left(\bar{\boldsymbol{p}}_j^i+\eta\boldsymbol{m}_j^i\right)-F_{\boldsymbol{p}_j}\left(\bar{\boldsymbol{p}}_j^i-\eta\boldsymbol{m}_j^i\right)\}, \tag{12}$$

where $\eta=\sqrt[n]{2/(s(s+1))}$, *is an upper bound to the failure probability.*

This bound results from adding the probabilities of the largest hyper-rectangle that fits within each quasi-ellipsoid. These hyper-rectangles are $\mathcal{R}(\bar{\boldsymbol{p}}^i,\eta\boldsymbol{m}^i)$ for $i=1,\ldots,k$. This bound is conservative (i.e., it does not converge to the actual failure probability as $\mathcal{F}^{sup}$ approaches $\mathcal{F}$), since $P[C(\mathcal{R}(\bar{\boldsymbol{p}}^i,\eta\boldsymbol{m}^i))\cap\mathcal{E}(\bar{\boldsymbol{p}}^i,\boldsymbol{m}^i,n)]>0$ in general. Note that the volume of $C(\mathcal{R}(\bar{\boldsymbol{p}}^i,\eta\boldsymbol{m}^i))\cap\mathcal{E}(\bar{\boldsymbol{p}}^i,\boldsymbol{m}^i,n)$ approaches zero as n goes to infinity. As a result, $\psi(\mathcal{F}^{sup})\rightarrow P[\mathcal{F}]$ from above when $\mathcal{F}^{sup}\rightarrow\mathcal{F}$ and $n\rightarrow\infty$. As in Theorem 1, the approximation $\mathcal{F}^{sup}$ can be readily used to estimate the probability bound corresponding to any uncertainty model supported in the master domain.

It is important to notice that the bounds above are probabilities of events and as such they always range from zero to one. This cannot be said of other bounds found in the literature. For example, bounds based on the Markov's and Chebyshev's inequalities (Ross 1998) result from applying the expected value operator to an algebraic inequality and may actually lie outside $[0,1]$, often rendering them impractical.

The probability bounds in Theorems 1-3 can be extended to the case where an inner approximation of the failure domain is available. In such a case, the subsets are in the failure domain, $\mathcal{F}^{sub}$ is given by the complement of the sets at the right hand side of Equations (7), (9), and (11); and the corresponding lower bounds are given by one minus the right hand side of Equations (8), (10) and (12). Therefore, having an inner and an outer approximation of the failure domain enables bounding its probability from below and above. An excessively large lower bound, which will only become larger as $\mathcal{F}^{sub}\rightarrow\mathcal{F}$, can be used as the figure of merit supporting the unacceptability of the system. A sufficiently small upper bound on the other hand, which will only become smaller as $\mathcal{F}^{sup}\rightarrow\mathcal{F}$, can be used as the figure of merit supporting the acceptability of the system. Tighter approximations should only be generated when neither of these two conditions are applicable.

4 REQUIREMENTS WITH ARBITRARY FUNCTIONAL DEPENDENCIES

This section presents a nonlinear optimization-based technique for calculating hyper-rectangular and quasi-ellipsoidal subsets of the safe and failure domains. This technique is applicable to arbitrary functional dependencies of $\boldsymbol{g}$ on $\boldsymbol{p}$. The explicit form of this dependency may even be unknown. This approach relies on the convergence of a nonlinear constrained optimization algorithm to a global minimum. Absolute guarantees of convergence to such a point are not possible from the outset due to the generality in the structure of $\boldsymbol{g}$. However, a variety of algorithmic safeguards can be used to deal with this deficiency (Crespo et al. 2009). This technique should not be used when the dependency of $\boldsymbol{g}$ on $\boldsymbol{p}$ assumes a known polynomial form. In such a case the techniques in (Crespo et al. 2011) are preferred since the correctness of their results is formally verifiable. The notion of homothetic deformations (Crespo et al. 2008, Crespo et al. 2009), of paramount importance for the developments that follow, is briefly introduced next.

4.1 *Homothetic Deformations*

A *homothetic* deformation results from a uniform, radial expansion or contraction of the space about a fixed point. The distance from any point in the space to the fixed point changes by a factor α after the deformation. This factor is called the *similitude ratio* of the homothetic deformation. Note that if α is greater than 1, the deformation is an expansion, while if α is less than 1, the deformation is a contraction. A *reference set*, denoted as $\Omega \subset \mathbb{R}^s$, will be deformed with respect to a fixed point $\bar{\boldsymbol{p}} \in \mathcal{D}$. This point can be an arbitrary parameter realization having no particular significance, or can be our best deterministic estimate of the actual value of $\boldsymbol{p}$. We choose $\bar{\boldsymbol{p}}$ to be the geometric center of the reference set.

Intuitively, one can imagine that Ω is being deformed with respect to $\bar{\boldsymbol{p}}$ until its boundary just touches $\partial\mathcal{F}$. This deformation will be called hereafter the *maximal deformation*. The set resulting from this deformation, denoted as $\mathcal{M}$, is the *maximal set*. A *critical parameter value*, denoted as $\tilde{\boldsymbol{p}}$, is (one of) the point(s) where the maximal set touches $\partial\mathcal{F}$. If $\bar{\boldsymbol{p}}$ is our best estimate of the actual value of $\boldsymbol{p}$, the critical parameter value is the worst-case uncertainty combination associated with the norm that prescribes the boundary of Ω (e.g., the critical parameter value corresponding to the maximal deformation of a hyperrectangle is the worst-case uncertainty combination in the sense of the $\boldsymbol{m}$-scaled ℓ_∞ norm from $\bar{\boldsymbol{p}}$). The *critical similitude ratio*, denoted by $\tilde{\alpha}$, is the similitude ratio of that deformation and is a non-dimensional metric proportional to the separation between $\bar{\boldsymbol{p}}$ and $\partial\mathcal{F}$. Techniques for evaluating set containment, for performing maximal deformations and for generating failure domain approximations are presented next.

4.2 *Set Containment*

We want to determine if the reference set Ω, having one of the geometries in (3) or (6), is fully contained in the safe or failure domains. This determination will be based on the calculation of the critical similitude ratio $\tilde{\alpha}$. The set containment condition can be stated as follows. Let $\sigma = 1$ when $\bar{\boldsymbol{p}} \in \mathcal{S}$ and $\sigma = -1$ otherwise. $\Omega \subseteq \mathcal{S}$ if and only if $\sigma = 1$ and $\tilde{\alpha} \geq 1$. Likewise, $\Omega \subseteq \mathcal{F}$ if and only if $\sigma = -1$ and $\tilde{\alpha} \geq 1$. The formulation required to calculate $\tilde{\alpha}$ is presented next.

4.3 *Maximal Deformation*

The means for calculating $\mathcal{M}$, $\tilde{\boldsymbol{p}}$ and $\tilde{\alpha}$ are presented next. Let the master domain be $\mathcal{D} = \mathcal{R}(\boldsymbol{a}, \boldsymbol{b})$.

The maximal deformation of the reference set $\Omega = \mathcal{R}(\bar{\boldsymbol{p}}, \boldsymbol{m})$ leads to

$$\tilde{\boldsymbol{p}} = \underset{\boldsymbol{p}}{\operatorname{argmin}} \left\{ \|\boldsymbol{p} - \bar{\boldsymbol{p}}\|_{\boldsymbol{m}}^{\infty} : \sigma \max_j \boldsymbol{g}_j(\boldsymbol{p}) \geq 0 \right\}, \quad (13)$$

$$\tilde{\alpha} = \frac{\|\tilde{\boldsymbol{p}} - \bar{\boldsymbol{p}}\|_{\boldsymbol{m}}^{\infty}}{\|\boldsymbol{m}\|}, \quad (14)$$

$$\mathcal{M} = \mathcal{R}(\bar{\boldsymbol{p}}, \tilde{\alpha}\boldsymbol{m}) \cap \mathcal{D}. \quad (15)$$

Therefore, when $\sigma = 1$, the problem of finding the critical parameter value becomes the problem of finding a vector $\tilde{\boldsymbol{p}}$ in $\partial\mathcal{F}$ of minimal distance in the $\boldsymbol{m}$-scaled ℓ_∞ norm from $\bar{\boldsymbol{p}}$. Notice that $\mathcal{R}(\bar{\boldsymbol{p}}, \tilde{\alpha}\boldsymbol{m})$ may not be contained in the master domain. This possibility is allowed for two reasons. First, because the maximal set corresponding to the case where $\mathcal{R}(\bar{\boldsymbol{p}}, \tilde{\alpha}\boldsymbol{m}) \not\subseteq \mathcal{D}$ is larger than it would be if we require $\mathcal{R}(\bar{\boldsymbol{p}}, \tilde{\alpha}\boldsymbol{m}) \subseteq \mathcal{D}$. Second, because $\mathcal{D} \cap \mathcal{R}(\bar{\boldsymbol{p}}, \tilde{\alpha}\boldsymbol{m})$ will remain hyper-rectangular, and therefore, we can calculate its probability analytically.

Now consider the deformation of $\Omega = \mathcal{E}(\bar{\boldsymbol{p}}, \boldsymbol{m}, n)$. In this case, we have

$$\tilde{\boldsymbol{p}} = \underset{\boldsymbol{p}}{\operatorname{argmin}} \left\{ \|\boldsymbol{p} - \bar{\boldsymbol{p}}\|_{\boldsymbol{m}}^{n} : \sigma \max_j \boldsymbol{g}_j(\boldsymbol{p}) \geq 0 \right\}, \quad (16)$$

$$\tilde{\alpha} = \min \left\{ \frac{\|\tilde{\boldsymbol{p}} - \bar{\boldsymbol{p}}\|_{\boldsymbol{m}}^{n}}{\|\boldsymbol{m}\|}, \min_i \left\{ \frac{\boldsymbol{b}_i - |\bar{\boldsymbol{p}}_i - \boldsymbol{a}_i|}{\boldsymbol{m}_i} \right\} \right\} \quad (17)$$

$$\mathcal{M} = \mathcal{E}(\bar{\boldsymbol{p}}, \tilde{\alpha}\boldsymbol{m}, n). \quad (18)$$

As before, when $\sigma = 1$, the problem of finding the critical parameter value becomes the problem of finding a vector $\tilde{\boldsymbol{p}}$ in the failure domain of minimal distance in the $\boldsymbol{m}$-scaled ℓ_n norm from $\bar{\boldsymbol{p}}$. In contrast to Equation (13), Equation (16) ensures the containment of $\mathcal{E}(\bar{\boldsymbol{p}}, \tilde{\alpha}\boldsymbol{m}, n)$ by the master domain. This is required since in the case where the deformation extends beyond $\mathcal{D}$, $P[\mathcal{D} \cap \mathcal{E}(\bar{\boldsymbol{p}}, \tilde{\alpha}\boldsymbol{m}, n)]$ cannot be calculated analytically. Note that $\mathcal{M} \subset \mathcal{S}$ when $\sigma = 1$, and $\mathcal{M} \subset \mathcal{F}$ when $\sigma = -1$.

4.4 *Failure Domain Approximations*

In this section we generate a sequence of failure domain approximations using the developments above. These sequences, given by

$\{\mathcal{F}_1^{sub}, \mathcal{F}_2^{sub}, \ldots\}$ and $\{\mathcal{F}_1^{sup}, \mathcal{F}_2^{sup}, \ldots\}$ (or equivalently $\{C(\mathcal{S}_1^{sub}), C(\mathcal{S}_2^{sub}), \ldots\}$) approach the failure domain from inside and outside as their number of terms increase. Note that the sequences $\{\mathcal{S}_1^{sub}, \mathcal{S}_2^{sub}, \ldots\}$ and $\{C(\mathcal{F}_1^{sub}), C(\mathcal{F}_2^{sub}), \ldots\}$ approach the safe domain in the same fashion. Two algorithms for calculating these sequences are presented next.

4.4.1 *Algorithm 1*

This algorithm generates the approximations by uniting maximal sets that satisfy the almost disjoint condition of the Theorems. This condition is attained by making the maximal set contained in the safe (failure) domain at any given iteration a part of the failure (safe) domain in subsequent iterations. The additional constrained functions $\boldsymbol{g}_{\mathcal{F}}$ and $\boldsymbol{g}_{\mathcal{S}}$, yet to be defined, are used to implement this idea. The algorithm's setup is as follows.

Let $\boldsymbol{g}(\boldsymbol{p}) < \boldsymbol{0}$ denote the set of system requirements and $\hat{f}_{\boldsymbol{p}}(\boldsymbol{p})$ be a joint density function of uniform random variables supported in $\mathcal{D}$. Let P_{max} be the largest admissible failure probability associated with the system for a given uncertainty model $f_{\boldsymbol{p}}(\boldsymbol{p})$, for all $\boldsymbol{p} \in \Delta \subseteq \mathcal{D}$. If the reference set Ω is chosen to be the hyper-rectangle let $n = \infty$. If the reference set Ω is chosen to be the quasi-ellipsoid make n an even natural number. Set $i = 1$, $\mathcal{F}_i^{sub} = \emptyset$, $\mathcal{S}_i^{sub} = \emptyset$, $\boldsymbol{g}_{\mathcal{F}} = \emptyset$ and $\boldsymbol{g}_{\mathcal{S}} = \emptyset$.

1. Find a sample $\hat{\boldsymbol{p}}$ of $\hat{f}_{\boldsymbol{p}}(\boldsymbol{p})$ conditional on $\hat{\boldsymbol{p}} \in C(\mathcal{F}^{sub} \cup \mathcal{S}^{sub})$. Let $\bar{\boldsymbol{p}} = \hat{\boldsymbol{p}}$ and calculate σ.
2. If $\sigma = 1$, calculate the maximal set $\mathcal{M}$ using the inequality constraint $[\boldsymbol{g}, \boldsymbol{g}_{\mathcal{F}}] \leq \boldsymbol{0}$, let $\mathcal{S}_{i+1}^{sub} = \mathcal{S}_i^{sub} \cup \mathcal{M}$ and $\mathcal{F}_{i+1}^{sub} = \mathcal{F}_i^{sub}$; and redefine $\boldsymbol{g}_{\mathcal{F}}$ as $[\boldsymbol{g}_{\mathcal{F}}, 1 - \|\boldsymbol{p} - \bar{\boldsymbol{p}}\|_{\boldsymbol{m}}^n]$. If $\sigma = -1$, calculate $\mathcal{M}$ using the inequality constraint $[\min_j(-\boldsymbol{g}_j), \boldsymbol{g}_{\mathcal{S}}] \leq \boldsymbol{0}$, let $\mathcal{F}_{i+1}^{sub} = \mathcal{F}_i^{sub} \cup \mathcal{M}$ and $\mathcal{S}_{i+1}^{sub} = \mathcal{S}_i^{sub}$; and redefine $\boldsymbol{g}_{\mathcal{S}}$ as $[\boldsymbol{g}_{\mathcal{S}}, 1 - \|\boldsymbol{p} - \bar{\boldsymbol{p}}\|_{\boldsymbol{m}}^n]$.
3. Let $\mathcal{F}_{i+1}^{sup} = C(\mathcal{S}_{i+1}^{sub})$. Evaluate $P[\mathcal{F}_{i+1}^{sub}]$ and $P[\mathcal{F}_{i+1}^{sup}]$, or the lower bounds $\psi(\mathcal{F}_{i+1}^{sub})$ and $\psi(\mathcal{F}_{i+1}^{sup})$, according to $f_{\boldsymbol{p}}(\boldsymbol{p})$ and the applicable Theorem.
4. If $P[\mathcal{F}_{i+1}^{sub}] \geq 1 - P_{max}$ declare the system acceptable and stop. If $P[\mathcal{F}_{i+1}^{sup}] \leq P_{max}$ declare the system unacceptable and stop. Otherwise increase i by one go to Step (1).

As i increases, the failure domain approximations approach the failure domain (i.e., $\mathcal{F}^{sub}$ and $\mathcal{S}^{sub}$ expand by the addition of new reference sets while $\mathcal{F}^{sup}$ contracts by the removal of new reference sets). Note that $P[\mathcal{F}_i^{sub}]$ and $\psi(\mathcal{F}_i^{sub})$ are monotonically increasing functions, while $P[\mathcal{F}_i^{sup}]$ and $\psi(\mathcal{F}_i^{sup})$ are monotonically decreasing functions. Further notice that a good coverage of the master domain may require an impractically large number of deformations. Conversely, depending upon the problem, convergence may be achieved in relatively few iterations.

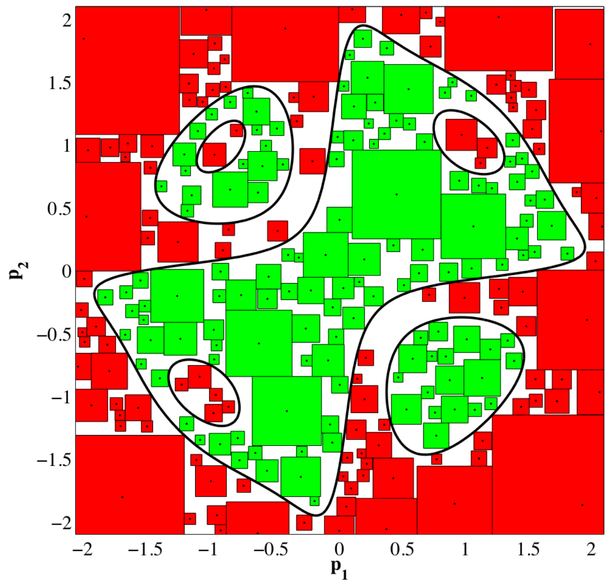

Figure 1: $\mathcal{F}^{sub}$ (red), $\mathcal{S}^{sub}$ (green) and $\partial\mathcal{F}$ (line).

Example 1: Consider the constraint functions

$$\boldsymbol{g}_1 = \boldsymbol{p}_1^2\boldsymbol{p}_2^4 + \boldsymbol{p}_1^4\boldsymbol{p}_2^2 - 3\boldsymbol{p}_1^2\boldsymbol{p}_2^2 - \boldsymbol{p}_1\boldsymbol{p}_2 + \frac{\boldsymbol{p}_1^6 + \boldsymbol{p}_2^6}{200}$$

$$-\frac{7}{100} + \sin(\boldsymbol{p}_1\boldsymbol{p}_2)^3, \tag{19}$$

$$\boldsymbol{g}_2 = -\boldsymbol{p}_1^2\boldsymbol{p}_2^4 - \boldsymbol{p}_1^4\boldsymbol{p}_2^2 + 3\boldsymbol{p}_1^2\boldsymbol{p}_2^2 + \frac{\boldsymbol{p}_1^5\boldsymbol{p}_2^3}{10} - 0.9$$

$$-\frac{\tanh(\boldsymbol{p}_1 - \boldsymbol{p}_2)}{10}, \tag{20}$$

for $\mathcal{D} = \mathcal{R}(\bar{\boldsymbol{p}}, \boldsymbol{m})$ where $\bar{\boldsymbol{p}} = [0, 0]^\top$ and $\boldsymbol{m} = [2.1, 2.1]^\top$. These constraint functions were chosen so the failure and safe domains are multiply connected. Figure 1 shows the hyper-rectangular maximal sets that constitute the approximations $\mathcal{F}^{sub}$ (red) and $\mathcal{S}^{sub}$ (green). At this particular step of the sequence, $i = 250$ subsets cover 70% of the master domain. Note that, by construction, none of the subsets composing the approximations cross $\partial\mathcal{F}$. Besides, each subset either touches this boundary or touches another subset. Further notice that the number of subsets required to well cover the master domain is a function of the geometry of the failure domain and not necessarily of the size of such a set.

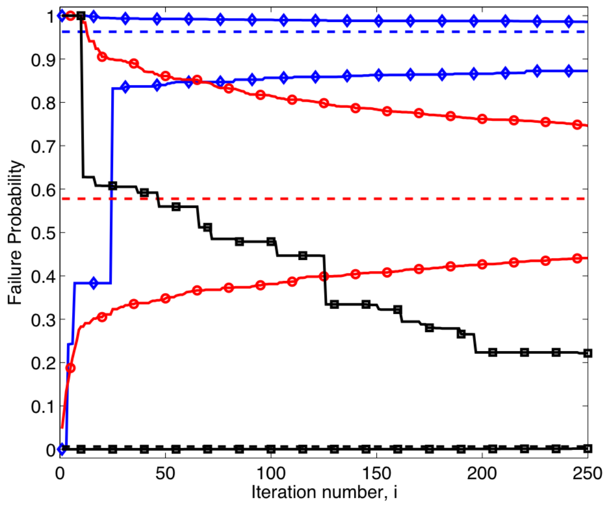

Figure 2: Probability bounds for several uncertainty models.

Figure 2 displays the failure probability bounds corresponding to several uncertainty models as a function of the iteration number i. A uniform distribution (red circle line), a generalized beta with parameters $[10, 1]$ (blue diamond line), and a generalized beta with parameters $[2, 3]$ (black square line) are considered. The support set of these three models is the master domain. The horizontal lines correspond to high-fidelity Monte Carlo approximations to the failure probability. Additional iterations lead to the enlargement of the approximations and consequently to the tightening of the bounds. In the limit, they converge to $P[\mathcal{F}]$. The size of the subset being annexed to the approximation as well as its probability tend to decrease with i. Therefore, the generation of arbitrarily tight bounds may require an impractically large number of subsets where many of them will have very small probability. Note however that for any uncertainty model satisfying $\Delta \subseteq \mathcal{F}^{sub} \cup \mathcal{F}^{sup}$, both bounds take on the exact failure probability value. Recall that the calculation of probability bounds shown in Figure 2 and those corresponding to any uncertainty model supported in $\mathcal{D}$ require a practically insignificant amount of computational effort.

4.4.2 *Algorithm 2*

The algorithm below iteratively generates the indexed sets $\Lambda_i, \mathcal{S}_i^{sub}$, and $\mathcal{F}_i^{sub}$ where $\mathcal{S}_i^{sub}$ is an inner approximation to the safe domain, $\mathcal{F}_i^{sub}$ is an inner approximation to the failure domain, and Λ_i is a region whose containment in $\mathcal{F}$ or $\mathcal{S}$ is to be determined. The terms in the inner approximations are reference sets of various homothetic deformations. At any given iteration we first chose a hyper-rectangle from those in Λ_i. By the means presented in Section 4.2 we determine if the reference set inscribed in this hyper-rectangle is contained in the safe or failure domains. If the reference set is contained in the safe domain, the inner approximation to the safe domain is expanded with this element. If the reference is contained into the failure domain, the inner approximation to the failure domain is expanded with this element. Otherwise, the rectangle is subdivided into smaller subsets (see section 2 for two subdividing logics), and these subsets are appended to Λ_i. The algorithm terminates when the bounds to the failure probability exceeds a prescribed limit. The algorithmic representation of this procedure is as follows.

Use the same setup of Algorithm 1. Furthermore, set $i = 1$, $\Lambda_i = \{\mathcal{D}\}$, $\mathcal{F}_i^{sub} = \emptyset$ and $\mathcal{S}_i^{sub} = \emptyset$.

1. Let $\mathcal{R}(\bar{\boldsymbol{p}}, \boldsymbol{m})$ be a largest element of Λ. Let $\Omega = \mathcal{R}(\bar{\boldsymbol{p}}, \boldsymbol{m})$ for hyper-rectangles and $\Omega = \mathcal{E}(\bar{\boldsymbol{p}}, \boldsymbol{m}, n)$ for quasi-ellipsoids.

2. Calculate σ and $\tilde{\alpha}$.

3. If $\tilde{\alpha} < 1$, set $\Lambda_{i+1} = (\Lambda_i \setminus \mathcal{R}) \cup \rho(\mathcal{R})$, $\mathcal{S}_{i+1}^{sub} = \mathcal{S}_i^{sub}$, and $\mathcal{F}_{i+1}^{sub} = \mathcal{F}_i^{sub}$. If $\tilde{\alpha} \geq 1$ and $\sigma = 1$, let $\Lambda_{i+1} = \Lambda_i \setminus \mathcal{R}$, $\mathcal{S}_{i+1}^{sub} = \mathcal{S}_i^{sub} \cup \Omega$ and $\mathcal{F}_{i+1}^{sub} = \mathcal{F}_i^{sub}$. If $\tilde{\alpha} \geq 1$ and $\sigma = -1$ let $\Lambda_{i+1} = \Lambda_i \setminus \mathcal{R}$, $\mathcal{F}_{i+1}^{sub} = \mathcal{F}_i^{sub} \cup \Omega$ and $\mathcal{S}_{i+1}^{sub} = \mathcal{S}_i^{sub}$.

4. Let $\mathcal{F}_{i+1}^{sup} = C(\mathcal{S}_{i+1}^{sub})$. Evaluate $P[\mathcal{F}_{i+1}^{sub}]$ and $P[\mathcal{F}_{i+1}^{sup}]$ or their lower bounds $\psi(\mathcal{F}_{i+1}^{sub})$ and $\psi(\mathcal{F}_{i+1}^{sup})$ depending upon the applicable Theorem.

5. If $P[\mathcal{F}_{i+1}^{sub}] \geq 1 - P_{max}$ declare the system acceptable and stop. If $P[\mathcal{F}_{i+1}^{sup}] \leq P_{max}$ declare the system unacceptable and stop. Otherwise increase i by one, and go to Step (1).

Note that the subdividing logic used to generate reference sets ensures the almost disjoint condition required by the Theorems. As i increases, the approximations approach the failure domain (i.e., $\mathcal{F}^{sub}$ and $\mathcal{S}^{sub}$ expand by the addition of new reference sets while $\mathcal{F}^{sup}$ contracts by the removal of new reference sets). As before, $P[\mathcal{F}_i^{sub}]$ and $\psi(\mathcal{F}_i^{sub})$ are monotonically increasing functions of i, while $P[\mathcal{F}_i^{sup}]$ and $\psi(\mathcal{F}_i^{sup})$ are monotonically decreasing functions of the same variable. Note that the elements left in Λ_i are an approximation of $\partial\mathcal{F}$. The larger the value of i the smaller the volume of this approximation.

Example 2: Figure 3 shows the failure domain approximations resulting from applying Algorithm 2 to the same requirement functions in Example 1. Note that the approximation of the failure domain boundary (white) is significantly better than that of Algorithm 1. As a result, the approximations give a better sense of the connectedness of the actual failure domain than those resulting from Algorithm 1. A

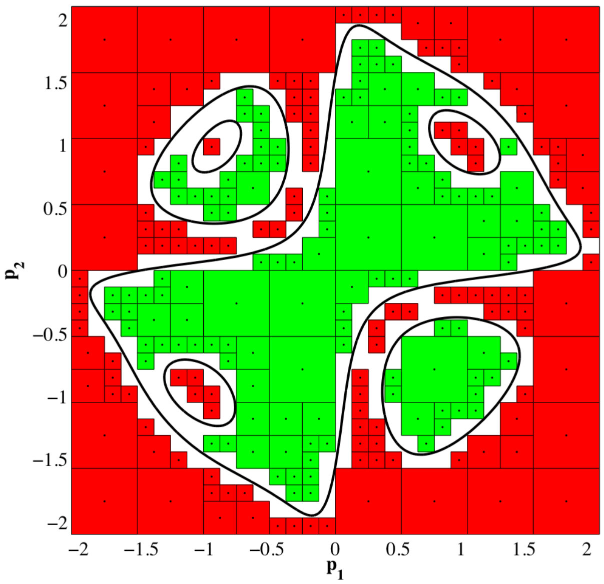

Figure 3: $\mathcal{F}^{sub}$ (red), $\mathcal{S}^{sub}$ (green), and $\partial\mathcal{F}$ (line).

better coverage of the master domain is attained because $\mathcal{F}^{sub}$ and $\mathcal{S}^{sub}$ grow from the inside out. In this particular case, $i = 250$ subsets cover 78% of $\mathcal{D}$. This improved coverage comes at the expense of having to perform many deformations whose maximal sets are not ultimately annexed to the approximations. In this example, 654 deformations were required to generate these 250 sets. This is the basis that makes Algorithm 1 more computationally efficient than Algorithm 2 in general. Figure 4 shows the probability bounds corresponding to the same uncertainty models used in Figure 2. These bounds are tighter than those from Algorithm 1 because the approximations are improved by appending/removing the largest subset among those available.

5 CONCLUSIONS

This paper proposes an uncertainty analysis framework for characterizing the failure and safe domains of a system whose design requirements have an arbitrary functional dependency on the uncertainty. The characteristics of interest are worst-case uncertainty combinations, metrics that evaluate the separation between any given point and the failure domain, approximations to the failure and safe domains; as well as lower and upper bounds to the failure probability. A nonlinear constrained optimization-based approach is proposed. This and all other methods requiring the exploration of the uncertain parameter space suffer from the curse of dimensionality, and as such, their computational demands grow exponentially with the number of uncertain parameters. Unfortunately only this

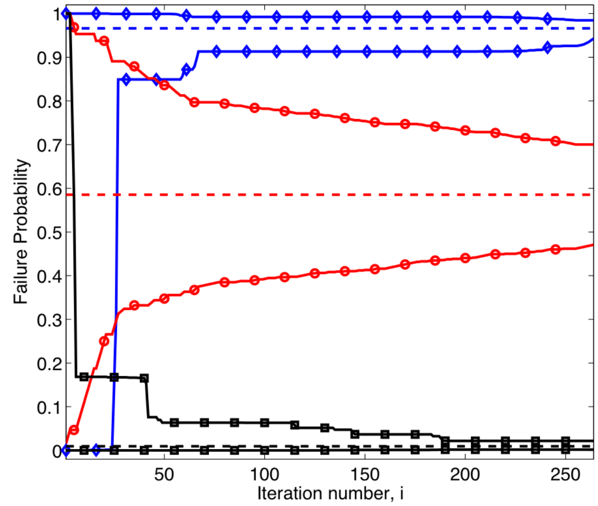

Figure 4: Probability bounds for several uncertainty models.

space can provide the sense of causality required to understand and prevent failure. The high dimensionality of this space along with the inability to guarantee that optimization problems posed there will converge to the global optimum are the main liability of the engineering decisions supported by the outcomes of these methods. A significant feature of the methodology proposed is that it allows accommodating for changes in the uncertainty model with practically insignificant computational effort. Furthermore, the algorithms proposed allow for data parallelism (i.e., perform computations simultaneously on elements of a subdivision of the master domain). This will help to mitigate the formidable challenges of having a large number of uncertain parameters.

REFERENCES

Crespo, L. G., D. P. Giesy, and S. P. Kenny (2008). Robust analysis and robust design of uncertain systems. *AIAA Journal 46*(2), 1–10.

Crespo, L. G., D. P. Giesy, and S. P. Kenny (2009). Reliability-based analysis and design via failure domain bounding. *Structural Safety 31*, 306–315.

Crespo, L. G., C. A. Munoz, A. J. Narkawicz, S. P. Kenny, and D. P. Giesy (2011, 18-22 September). Uncertainty analysis via failure domain characterization: Polynomial requirement functions. In *ESREL 2011*, Number 1, Trojes, France.

Kall, P. and S. Wallace (1994). *Stochastic Programming*. New York: Wiley.

Niederreiter, H. (1992). *Random Number Generation and Quasi-Monte Carlo Methods*. Philadelphia, PA: Society for Industrial and Applied Mathematics.

Rackwitz, R. (2001). Reliability analysis, a review and some perspectives. *Structural Safety 23*, 365–395.

Ross, S. (1998). *A First Course in Probability* (Fifth ed.). Upper Saddle River, NJ 07458: Simon & Schuster.

Royset, J., A. D. Kiureghian, and E. Polak (2001). Reliability-based optimal structural design by the decoupling approach. *Reliability Engineering and System Safety 73*, 213–221.